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On Matroids of the Greatest W-Connectivity

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The Whitney connectivity (W-connectivity) of a matroid M is defined by T. Inukai and L. Weinberg as the least integer k for which there exists a subset S of the ground set E of M such that $\rho(S) \geqslant k$, $\rho(E-S) \geqslant k$, and

$$\rho(S) + \rho(E-S) - \rho M + 1 = k,$$

where ρ is the rank function of M. M is called a Whitney matroid if there exists no such integer. In this case, the W-connectivity of M to be the rank of M is defined. In this paper, several properties of Whitney matroids are demonstrated. In addition, the Whitney matroids whose duals are also Whitney matroids are characterized, and an interpretation of binary W-matroids is given.

1. ELEMENTARY RESULTS

In this paper, M is a matroid with the rank function ρ on a finite set E of n elements. For an element e of E, the restriction of M to $E-e$ will be denoted by $M-e$, and the contraction of M to $E-e$ will be denoted by M/e. For the terminology and notation not specified here, see [4].

According to Inukai and Weinberg [3], the Whitney connectivity (W-connectivity) of M, denoted by $\lambda(M)$, is defined as the least integer k for which there is a subset S of E such that $\rho(S) \geqslant k$, $\rho(E-S) \geqslant k$, and

$$\rho(S) + \rho(E-S) - \rho M + 1 = k.$$

If no such integer exists, then the W-connectivity of M is defined in [3] to be infinite. However, for reasons explained below, we prefer to define $\lambda(M) = \rho M$ in this case. For convenience, a matroid with $\lambda(M) = \rho M$ will be called a Whitney matroid (W-matroid). By [3, Lemma 6] W-matroids are just those whose W-connectivity is defined to be infinite in [3].

In this section, we will deduce several elementary properties of W-

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matroids. In Section 2, we characterize the W-matroids whose dual matroids are also W-matroids, and in Section 3, an interpretation of a binary W-matroid is given.

The following theorem given in [3] demonstrates some necessary and sufficient conditions for a matroid to be a W-matroid.

THEOREM 1 [3, Theorem 5]. *The following statements are equivalent:*

(a) *M is a W-matroid.*

(b) *For each nonnull proper subset S of E, either S or $E - S$ contains a base of M.*

(c) *For each nonnull proper subset S of E, either $\rho(M \cdot S) = 0$, or $\rho(M \cdot (E - S)) = 0$.*

(d) *For each pair of cocircuits C^* and C_1^* of M, $C^* \cap C_1^* \neq \varnothing$.*

By Theorem 1, we can easily give another alternative definition of a W-matroid.

LEMMA 2. *M is a W-matroid if and only if each cocircuit of M contains a base of M.*

Proof. Suppose that C^* is a cocircuit of the W-matroid M. Since $E - C^*$ is a hyperplane of M, $E - C^*$ does not contain a base of M. Hence, by Theorem 1(b), C^* contains a base of M.

Conversely, suppose that each cocircuit of M contains a base. Let C^* and C_1^* be cocircuits of M, and let B be a base of M contained in C^*. Then $C^* \cap C_1^* \neq \varnothing$, because $B \cap C_1^* \neq \varnothing$. Hence, by Theorem 1(d), M is a W-matroid.

Combining this result and the assertion of [3, Lemma 6] that, if M is not a W-matroid, then $\rho(C^*) \geqslant \lambda(M)$ for any cocircuit C^* of M, we have

COROLLARY 3. *For any matroid M,*

$$\lambda(M) \leqslant \min\{|C^*|; C^* \text{ is a cocircuit of } M\}.$$

Since the W-connectivity of a matroid corresponds to the vertex connectivity of a graph [3, Theorem 1], and the minimum cardinality of cocircuits of a matroid corresponds to the edge connectivity, Corollary 3 is a natural extension of the well-known result in graph theory that the vertex connectivity is less than or equal to the edge connectivity of a graph. It is just for this reason that we define the W-connectivity of the matroids having the greatest W- connectivity to be ρM and not infinite.

The following lemma shows the connection between a W-matroid and its minors.

LEMMA 4. *Let $e, e' \in E$. Then*

(a) *$M - e$ is a W-matroid implies that M is a W-matroid;*

(b) *M is a W-matroid implies that M/e is a W-matroid;*

(c) *if e and e' are parallel elements, or e is a loop, then M is a W-matroid implies that $M - e$ is a W-matroid.*

These results follow readily by Theorem 1(d).

By Lemma 4(a) and (c), we can restrict ourselves to considering simple matroids in studying W-matroids. In view of Lemma 4(a), we define a W-matroid M to be minimal if for any element e of M, $M - e$ is not a W-matroid.

THEOREM 5. *A W-matroid M is minimal if and only if, for any element e of M, there are cocircuits C^* and C_1^* of M such that $C^* \cap C_1^* = e$.*

This result is an immediate consequence of Theorem 1(d).

A uniform matroid $U_{r,n}$ is a matroid on a set E of n elements such that every subset of E with r elements is a base.

THEOREM 6. *Denote ρM by r. Then M is a W-matroid implies that $n \geqslant 2r - 1$. The equality holds if and only if $M = U_{r,2r-1}$.*

Proof. If M is a uniform matroid, then it is easy to verify that M is a W-matroid if and only if $n \geqslant 2r - 1$. When M is not a uniform matroid, let S be a subset of E such that $|S| = r$, and S is not a base of M. Now S does not contain a base of M, so, by Theorem 1(b), $E - S$ contains a base of M. Thus $|E - S| \geqslant r$. Hence, $|E| = |S| + |E - S| \geqslant r + r = 2r$.

2. DUAL WHITNEY MATROIDS

In [2], Inukai and Weinberg identify the matroids having the greatest Tutte connectivity as being a class of uniform matroids. But in the case of Whitney connectivity, the structure of a W-matroid is not as simple as it first appears. So we try to consider some more specific cases. First, we consider the W-matroids whose duals are also Whitney matroids. The next theorem characterizes these matroids:

THEOREM 7. *Both M and M^* are W-matroids if and only if one of the following conditions is satisfied:*

(a) *n is odd, and $M = U_{r,n}$, where $r = \frac{1}{2}(n+1)$ or $\frac{1}{2}(n-1)$.*

(b) *n is even, and*

(b1) *every subset of E of $\frac{1}{2}n$ elements is either a base or a cobase of M, and*

(b2) *every subset of E of $\frac{1}{2}n+1$ elements contains a base and a cobase of M.*

Proof. Necessity. Suppose that both M and M^* are W-matroids. If M is a uniform matroid, then it is easy to see that $\rho M = \frac{1}{2}(n-1)$, or $\frac{1}{2}n$, or $\frac{1}{2}(n+1)$. Conditions (a) and (b) are satisfied.

If M is not a W-matroid, let S be a subset of E such that $|S| = \rho M$, and S is not a base of M. Now S contains a circuit C of M. Since M^* is a W-matroid, C contains a cobase B^* of M. If B^* is a proper subset of S, then S is a dependent set of M^*. Thus there is a cocircuit C^* of M contained in S. Since M is a W-matroid, C^* contains a base B of M. By $B \subseteq C^* \subseteq S$, and $|B| = |S| = \rho M$, we conclude that $B = S$, and then S itself is a base of M, contradicting the hypothesis. Hence, $B^* = S$; i.e., S is a cobase of M. Accordingly, $\rho M^* = \rho M = \frac{1}{2}n$.

On the other hand, let T be a subset of E of $\frac{1}{2}n+1$ elements. Since M is a W-matroid and $E-T$ contains no base of M, by Theorem 1(b), T contains a base of M. Similarly, T contains a cobase of M.

Sufficiency. When n is odd, $r = \frac{1}{2}(n+1)$ or $\frac{1}{2}(n-1)$, and $M = U_{r,n}$, it is obvious that M and M^* are W-matroids. When n is even, and conditions (b1) and (b2) are satisfied, we prove that M is a W-matroid. (The same argument serves to prove that M^* is also a W-matroid.) By Theorem 1(b), we need only to show that, for every nunnull proper subset S of E, either S or $E-S$ contains a base of M. By (b1), we see that $\rho M = \frac{1}{2}n$.

Let S be a subset of E. If $|S| = \frac{1}{2}n$, by (b1), S or $E-S$ is then a base of M. Otherwise, without loss of generality, we may assume that $n > |S| \geqslant \frac{1}{2}n+1$. By (b2), we see that S contains a base of M. The proof is complete.

We note that conditions (b1) and (b2) are independent of each other. Let $E = \{a, b, c, d\}$. The family

$$\{\{a, b\}, \{a, c\}, \{a, d\}\}$$

of subsets of E is the base set of a matroid on E which satisfies (b1) but not (b2), and the family

$$\{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$$

of subsets of E is the base set of a matroid on E which satisfies (b2) but not (b1).

In Theorem 7, when n is odd, it is obvious that $U_{r,2r+1}$ is not a minimal W-matroid, and $U_{r,2r-1}$ is a minimal W-matroid; when n is even, we have the following result:

THEOREM 8. *If both M and M^* are W-matroids and n is even, then M is a minimal W-matroid if and only if, for every element e of E, there is a base B of M containing e which is not a cobase of M.*

Proof. *Necessity*. Suppose that M and M^* are W-matroids, and $e \in E$. If every base of M containing e is a cobase of M, then e is in the intersection of all the cobases of M which are not bases of M. Consider a subset S of E which has exactly $\frac{1}{2}n$ elements and does not contain e. By Theorem 7(b1), S is either a base or a cobase of M. Since $e \notin S$, by the hypothesis, S is a cobase of M implies that S is a base of M. Hence, S is a base of M. Consequently, every subset of E having $\frac{1}{2}n$ elements is a base of $M-e$, i.e., $M-e=U_{r,2r-1}$, where $r=\frac{1}{2}n$. Accordingly, M is not a minimal Whitney matroid.

Sufficiency. Suppose that every element of E belongs to a base of M which is not a cobase of M. Let $e \in E$, and B is a base of M which is not a cobase of M such that $e \in B$. The dual version of the proof of the necessity of (b1) of Theorem 7 shows that B is a cocircuit of M. Since $E-B$ is a cobase of M, $(E-S)\cup e$ contains a cocircuit C^*, and $B \cap C^* = e$. By Theorem 5, M is a minimal Whitney matroid, completing the proof.

The cycle matroid of K_4, the complete graph of 4 vertices, is an example of a minimal W-matroid with a W-matroid dual.

3. BINARY W-MATROIDS

In [3, Corollary 1], it is proved that M is a graphical simple W-matroid if and only if it is the cycle matroid of a complete graph. In this section, we consider the more general case that M is a binary simple W-matroid. First, we have

LEMMA 9. *The symmetric difference of two distinct cocircuits of a binary W-matroid M is itself a cocircuit of M.*

Proof. Let C_1^* and C_2^* be distinct cocircuits of M. By the property of a binary matroid, $C_1^* \triangle C_2^*$ is a union of disjoint cocircuits of M. Observing that there exist no disjoint cocircuits in a W-matroid, we conclude that $C_1^* \triangle C_2^*$ is a cocricuit of M.

Let V be the set of nonzero vectors of dimension r, $r \geqslant 2$, over the field $GF(2)$, and let $M(V)$ be the matroid induced by V. Since the cycle matroid of the complete graph of $r+1$ vertices is a restriction of $M(V)$, by Lemma 4(a), we have

LEMMA 10. *$M(V)$ is a W-matroid.*

The matroid $M(V)$ has the following properties:

LEMMA 11. *Let B be a bse of V, and let $S \subseteq B$. Then*

(a) *when $|S| \geqslant 2$, there exists a unique element e in $V - S$ such that $S \cup e$ is a circuit of $M(V)$; and*

(b) *when $|S| \geqslant 1$, there is a unique cocircuit C^* of $M(V)$ such that $C^* \cap B = S$.*

Proof. (a) It follows by the observation that $S \cup e$ is a circuit of $M(V)$ if and only if $e = \sum_{x \in S} x$.

(b) For each element x of B, there is a unique cocircuit C_x^* in $(V - B) \cup x$. Let $C^*(S)$ be the symmetric difference of the family of cocircuits $\{C_x^*; x \in S\}$. By Lemma 9, $C^*(S)$ is a cocircuit of $M(V)$. It is obvious that $C^*(S) \cap B = S$.

If there are two cocircuits C_1^* and C_2^* such that $C_1^* \cap B = C_2^* \cap B = S$, and $C_1^* \neq C_2^*$, then $C_1^* \triangle C_2^*$ is a cocircuit of $M(V)$ which does not intersect B. It is impossible.

Let $D \subseteq V$ be an independent set of $M(V)$. The subset of V

$$OC(D) = \{e; e \in D, \text{ or } D \cup e \text{ contains a circuit of even length}\}$$

is called the odd closure of D.

LEMMA 12. *Suppose that $A \subseteq V$. Then there exist two distinct cocircuits C_1^* and C_2^* such that $C_1^* \cap C_2^* = A$ if and only if A is the odd closure of an independent set of $r - 1$ elements.*

Proof. Necessity. Let C_1^* and C_2^* be cocircuits of $M(V)$ such that $C_1^* \cap C_2^* = A$. Since $V - C_1^*$ and $V - C_2^*$ are hyperplanes of $M(V)$, by a result in linear algebra, the rank of $(V - C_1^*) \cap (V - C_2^*)$ is $r - 2$. Hence we can take a base of $M(V)$ as follows:

$$B = \{e_1, e_2, ..., e_r\},$$

where $e_1 \in C_1^* - C_2^*$, $e_2 \in C_2^* - C_1^*$, and e_i belongs to neither C_1^* nor C_2^* for $i = 3, 4, ..., r$. Consequently, denoting an element

$$e = e_{i_1} + e_{i_2} + \cdots + e_{i_t}, \qquad 1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant r$$

by $e_{i_1, i_2, \ldots, i_t}$, we have

$$A = \{e_{1,2,i_3,\ldots,i_t}; 3 \leqslant i_3 < \cdots < i_t \leqslant r, t \geqslant 2\}.$$

Now it is clear that $\{e_{1,2,\ldots,k}; k = 2, 3, ..., r\}$ is a maximal independent set in A having $r - 1$ elements, and A is the odd closure of this set.

Sufficiency. If A is the odd closure of an independent set $B' = \{e_1, e_2, \ldots, e_{r-1}\}$. Let $B = \{e_1, e_2, \ldots, e_{r-1}, e_r\}$ be a base of $M(V)$. By Lemma 11(b), there exist cocircuits C_1^* and C_2^* such that $C_1^* \cap B = B'$ and $C_2^* \cap B = B$.

Denote a cocircuit C^* of $M(V)$ such that $C^* \cap B = e_i$ by $C^*(i)$. We have

$$C_1^* = C^*(1) \triangle C^*(2) \triangle \cdots \triangle C^*(r-1)$$

and

$$C_2^* = C^*(1) \triangle C^*(2) \triangle \cdots \triangle C^*(r).$$

An element $e_{i_1, i_2, \ldots, i_t} = e_{i_1} + e_{i_2} + \cdots + e_{i_t}$, $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant r$, belongs to $C_1^* \cap C_2^*$ if and only if both $|\{i_1, i_2, \ldots, i_t\} \cap \{1, 2, \ldots, r-1\}|$ and $|\{i_1, i_2, \ldots, i_t\} \cap \{1, 2, \ldots, r\}|$ are odd. Thus $i_t \leqslant r-1$ and t is odd, or, equivalently, it is in the odd closure of B', completing the proof.

COROLLARY 13. *Each pair of cocircuits of $M(V)$ intersects in exactly 2^{r-2} elements.*

Proof. It follows readily by enumeration.

By this corollary, we have the following proposition:

THEOREM 14. *Every binary simple matroid having at least $3 \cdot 2^{r-2}$ elements is a W-matroid.*

Proof. This result follows by the observation that every binary matroid is a restriction of $M(V)$.

The bound given in Theorem 14 is best possible in the sense that there exists a binary simple matroid having $3 \cdot 2^{r-2} - 1$ elements which is not a W-matroid. To construct such a matroid, we take an independent set B' of $M(V)$ having $r-1$ elements, and let A be the odd closure of B'. Then, by Lemma 12 and Corollary 13, the restriction of $M(V)$ to $V - A$ is a binary simple matroid which has $3 \cdot 2^{r-2} - 1$ elements and is not a W-matroid.

Another immediate consequence of Lemma 12 is the following:

THEOREM 15. *Suppose that $T \subseteq V$. Then the restriction of $M(V)$ on T is a W-matroid if and only if, for any independent set B' of $M(V)$ having $r-1$ elements, either there exists $e \in T$ belonging to B' or there exists $e' \in T$ such that the unique circuit in $B' \cup e'$ is of even length.*

Observing that every binary simple matroid is a restriction of $M(V)$, Theorem 15 characterizes the binary simple W-matroids.

Theorem 15 can be stated in another form. Two independent sets

$$B' = \{x_1, x_2, \ldots, x_{r-1}\}$$

and

$$B'' = \{y_1, y_2, \ldots, y_{r-1}\}$$

are said to be equivalent if every element of B'' is in the odd closure of B'. It is easy to verify that this relation is in fact an equivalence relation. Let $\mathscr{B}$ be the collection of independent sets having $r-1$ elements. The equivalence relation defined above decides the equivalence classes of $\mathscr{B}$. Denote the set of equivalence classes of $\mathscr{B}$ uner this relation by $\{N_1, N_2, \ldots, N_s\}$, and M_i is the union of the members, each taken as a subset of V, of N_i, $i = 1, 2, \ldots, s$. Then Theorem 15 asserts that, for $T \subseteq V$, the restriction of $M(V)$ to T is a W-matroid if and only if $T \cap M_i \neq \varnothing$, for each $i = 1, 2, \ldots, s$. Thus, using the algorithm in [1, p. 423], we can construct all the binary simple W-matroids from the family $(M_1, M_2, \ldots, M_s)$.

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