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Remarks on the non-commutative Khintchine inequalities for $0 < p < 2$

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Abstract

We show that the validity of the non-commutative Khintchine inequality for some q with $1 < q < 2$ implies its validity (with another constant) for all $1 \leqslant p < q$. We prove this for the inequality involving the Rademacher functions, but also for more general "lacunary" sequences, or even non-commutative analogues of the Rademacher functions. For instance, we may apply it to the "$Z(2)$-sequences" previously considered by Harcharras. The result appears to be new in that case. It implies that the space ℓ_1^n contains (as an operator space) a large subspace uniformly isomorphic (as an operator space) to $R_k + C_k$ with $k \sim n^{1/2}$. This naturally raises several interesting questions concerning the best possible such k. Unfortunately we cannot settle the validity of the non-commutative Khintchine inequality for $0 < p < 1$ but we can prove several would be corollaries. For instance, given an infinite scalar matrix $[x_{ij}]$, we give a necessary and sufficient condition for $[\pm x_{ij}]$ to be in the Schatten class S_p for almost all (independent) choices of signs ± 1. We also characterize the bounded Schur multipliers from S_2 to S_p. The latter two characterizations extend to $0 < p < 1$ results already known for $1 \leqslant p \leqslant 2$. In addition, we observe that the hypercontractive inequalities, proved by Carlen and Lieb for the Fermionic case, remain valid for operator space valued functions, and hence the Kahane inequalities are valid in this setting.
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The non-commutative Khintchine inequalities play a very important rôle in the recent developments in non-commutative Functional Analysis, and in particular in Operator Space Theory, see [28,27]. Just like their commutative counterpart for ordinary L_p-spaces, they are a central tool to understand all sorts of questions involving series of random variables, or random vectors, in relation with unconditional or almost unconditional convergence in non-commutative L_p [30]. The commutative version is also crucial in the factorization theory for linear maps between L_p-spaces [21,22] in connection with Grothendieck's theorem. The non-commutative analogues of Grothendieck's theorem reflect the same close connection with the Khintchine inequalities, see e.g. the recent paper [36]. Moreover, in the non-commutative case, further motivation for their study comes from Random Matrix Theory and Free Probability. For instance one finds that the Rademacher functions (i.e. i.i.d. ± 1-valued) independent random variables satisfy the same inequalities as the freely independent ones in non-commutative L_p for $p < \infty$.

For reasons that hopefully will appear below, the case $p < 2$ is more delicate, and actually the case $p < 1$ is still open. When $p < 2$, let us say for convenience that a sequence (f_k) in classical L_p satisfies the classical Khintchine inequality KI_p if there is a constant c_p such that for all finite scalar sequences (a_j) we have

$$\Big(\sum |a_j|^2\Big)^{1/2} \leqslant c_p \Big\| \sum a_j f_j \Big\|_p.$$

Now assume that (f_k) is orthonormal in L_2. Then it is easy to see that if $p < q < 2$, KI_q implies KI_p. Indeed, let $S = \sum a_j f_j$. Let θ be such that $1/q = (1-\theta)/p + \theta/2$. We have

$$\Big(\sum |a_j|^2\Big)^{1/2} \leqslant c_q \|S\|_q \leqslant c_q \|S\|_p^{1-\theta} \|S\|_2^{\theta} = c_q \|S\|_p^{1-\theta} \Big(\sum |a_j|^2\Big)^{\theta/2},$$

and hence after a suitable division we obtain KI_p with $c_p = (c_q)^{1/(1-\theta)}$. The heart of this simple argument is that the span of the sequence (f_k) is the same in L_p and in L_q or in L_2. In sharp contrast, the analogue of this fails for operator spaces. The span of the Rademacher functions in L_p is not isomorphic as operator space to its span in L_q, although they have the same underlying Banach space. This is reflected in the form of the non-commutative version of the Khintchine inequalities first proved by Lust-Piquard in [16] and labelled as (Kh_q) below for the case of non-commutative L_q. Nevertheless, it turns out that the above simple minded extrapolation argument can still be made to work, this is our main result but this requires a more sophisticated version of Hölder's inequality, that (apparently) forces us to restrict ourselves to $p \geqslant 1$.

Let $1 \leqslant q \leqslant 2$. Let (r_k) be the Rademacher functions on $\Omega = [0,1]$. Let (x_k) be a finite sequence in a non-commutative L_q-space. The non-commutative Khintchine inequalities say (when $1 \leqslant q \leqslant 2$) that there is a constant β_q independent of $x = (x_k)$ such that

$$|||x|||_q \leqslant \beta_q \Big(\int \Big\| \sum r_k(t) x_k \Big\|_q^q \, dt \Big)^{1/q} \tag{0.1}$$

where

$$|||x|||_q \overset{\text{def}}{=} \inf_{x_k = a_k + b_k} \Big\{ \Big\| \Big(\sum a_k^* a_k\Big)^{1/2} \Big\|_q + \Big\| \Big(\sum b_k b_k^*\Big)^{1/2} \Big\|_q \Big\}. \tag{0.2}$$

This was first proved in [16] for $1 < q < 2$ and in [18] for $q = 1$ (the converse inequality is easy and holds with constant 1). One of the two proofs in [18] derives this from the non-commutative (little) Grothendieck inequality proved in [25].

In this paper, we follow an approach very similar to the original one in [25] to show that the validity of (0.1) for some q with $1 < q < 2$ implies its validity (with another constant) for all $1 \leqslant p < q$; we also make crucial use of more recent ideas from [13]. For that deduction the only assumption needed on (r_k) is its orthonormality in $L_2([0,1])$. Thus our approach yields (0.1) also for more general "lacunary" sequences than the Rademacher functions. For instance, we may apply it to the "$Z(2)$-sequences" considered in [9] (see also [10,2]). The result appears to be new in that case. Our argument can be viewed as an operator space analogue of the classical fact, in Rudin's style [31], that if a sequence of characters Λ spans a Hilbert space in $L_q(G)$ (G compact Abelian group, e.g. $G = \mathbb{T}$) for some $q < 2$ then it also does for all $p < q$. It implies that the space ℓ_1^n contains (as an operator space) a large subspace uniformly isomorphic (as an operator space) to $R_k + C_k$ with $k \sim n^{\frac{1}{2}}$. Another corollary (see Theorem 4.2) is that there is a constant c such that, for any n, the usual "basis" of S_1^n contains a c-unconditional subset of size $\geqslant n^{3/2}$. This opens the door to various questions concerning the best possible size of such subspaces and subsets. See the end of Section 1 for some speculation on this.

Unfortunately we cannot prove our result (at the time of this writing) for $0 < p < 1$, for lack of a proof of Step 3 below. Thus we leave open the validity of (0.1) for $0 < q < 1$. Nevertheless we will be able to prove several partial results in that direction. In particular (see Sections 3 and 5), if $0 < p \leqslant 2$, given arbitrary scalar coefficients $[x_{ij}]$, we give a necessary and sufficient condition for the random matrix

$$[\pm x_{ij}]$$

to be in the Schatten class S_p for almost all choices of signs. This happens iff $[x_{ij}]$ admits a decomposition of the form $x_{ij} = a_{ij} + b_{ij}$ with

$$\sum_i \Big(\sum_j |a_{ij}|^2\Big)^{p/2} < \infty \quad \text{and} \quad \sum_j \Big(\sum_i |b_{ij}|^2\Big)^{p/2} < \infty.$$

We also show that $[x_{ij}]$ defines a bounded Schur multiplier from S_2 to S_p iff it admits a decomposition of the form $x_{ij} = \psi_{ij} + \chi_{ij}$ with

$$\sum_i \sup_j |\psi_{ij}|^{2p/(2-p)} < \infty \quad \text{and} \quad \sum_j \sup_i |\chi_{ij}|^{2p/(2-p)} < \infty.$$

In those two results, only the case $0 < p < 1$ is new. In passing we remind the reader that when $0 < p < 1$, L_p-spaces (commutative or not), in particular the Schatten class S_p, are not normed spaces. They are only p-normed, i.e. for any pair x, y in the space we have $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$.

In the final section, we turn to the Kahane inequalities. Recall that the latter are a vector valued version of the Khintchine inequalities valid for functions with values in an *arbitrary* Banach space. It is natural to wonder whether there are non-commutative analogues when one uses the *operator space valued* non-commutative L_p-spaces introduced in [27]. We observe that the hypercontractive inequalities, proved by Carlen and Lieb [7] for the Fermionic case, remain valid for operator space valued functions, and hence the Kahane inequalities are valid in this Fermionic setting. The point of this simple remark is that Kahane's inequality now appears as the Bosonic case. The same remark is valid in any setting for which hypercontractivity has been established. This applies in particular to Biane's free hypercontractive inequalities [5].

1. The case $1 \leqslant p < 2$

Actually, $L_2([0,1])$ or $L_2(G)$ can be replaced here by any non-commutative L_2-space $L_2(\varphi)$ associated to a semi-finite generalized (i.e. "non-commutative") measure space, and (r_k) is then replaced by an orthonormal sequence (ξ_k) in $L_2(\varphi)$. Then the right-hand side of (0.1) is replaced by $\|\sum \xi_k \otimes x_k\|_{L_q(\varphi\times\tau)}$. More precisely, by a (semi-finite) generalized measure space (N,φ) we mean a von Neumann algebra N equipped with a faithful, normal, semi-finite trace φ. Without loss of generality, we may always reduce consideration to the σ-finite case. Throughout this paper, we will use freely the basics of non-commutative integration as described in [23] or [33, Chapter IX].

Let us fix another generalized measure space (M,τ). The inequality we are interested in now takes the following form:

$$(K_q)\qquad \begin{cases} \exists\beta_q \text{ such that for any finite sequence} \\ x=(x_k) \text{ in } L_q(\tau) \text{ we have} \\ |||x|||_q \leqslant \beta_q \left\| \sum \xi_k \otimes x_k \right\|_{L_q(\varphi\times\tau)} \\ \text{where } |||\cdot|||_q \text{ is defined as in (0.2).} \end{cases}$$

In the Rademacher case, i.e. when $(\xi_k)=(r_k)$, we denote (Kh_q) instead of (K_q), and we refer to these as the non-commutative Khintchine inequalities.

We can now state our main result for the case $q \geqslant 1$.

Theorem 1.1. *Let $1<q<2$. Recall that (ξ_k) is assumed orthonormal in $L_2(\varphi)$. Then $(K_q) \Rightarrow (K_p)$ for all $1 \leqslant p < q$.*

Here is a sketch of the argument. We denote

$$S = \sum \xi_k \otimes x_k.$$

Let $\mathcal{D}$ be the collection of all "densities," i.e. all f in $L_1(\tau)_+$ with $\tau(f)=1$. Fix p with $0<p \leqslant q$. Then we denote for $x=(x_k)$

$$C_q(x) = \inf\left\{ \left\| \sum \xi_k \otimes y_k \right\|_q \right\}$$

where $\|\cdot\|_q$ is the norm in $L_q(\varphi\otimes\tau)$ and the infimum runs over all sequences $y=(y_k)$ in $L_q(\tau)$ for which there is f in $\mathcal{D}$ such that

$$x_k = \left(f^{\frac{1}{p}-\frac{1}{q}} y_k + y_k f^{\frac{1}{p}-\frac{1}{q}} \right)/2.$$

Note that $C_p(x) = \|S\|_p$.

Remark 1.2. Assume $x_k = x_k^*$ for all k. Then

$$|||x|||_q = \inf\left\{ \left\| \left(\sum \alpha_k^* \alpha_k \right)^{1/2} \right\|_q \right\} \tag{1.1}$$

where the infimum runs over all decompositions

$$x_k = \mathrm{Re}(\alpha_k) = \left(\alpha_k + \alpha_k^*\right)/2.$$

Indeed, $x_k = a_k + b_k$ implies $x_k = \mathrm{Re}(a_k + b_k^*)$. Let $\alpha_k = a_k + b_k^*$. We have (assuming $q \geqslant 1$)

$$\left\| \left(\sum \alpha_k^* \alpha_k \right)^{1/2} \right\|_q \leqslant \left\| \left(\sum a_k^* a_k \right)^{1/2} \right\|_q + \left\| \left(\sum b_k b_k^* \right)^{1/2} \right\|_q.$$

Therefore $\inf \|(\sum \alpha_k^* \alpha_k)^{1/2}\|_q \leqslant |||x|||_q$. Since the converse inequality is obvious, this proves (1.1).

The proof of Theorem 1.1 is based on a variant of "Maurey's extrapolation principle" (see [21, 22]). This combines three steps: (here $C', C'', C''', \ldots$ are constants independent of $x = (x_k)$ and we wish to emphasize that here p remains fixed while the index q in $C_q(x)$ is such that $p < q \leqslant 2$).

Step 1. Assuming (K_q) we have

$$|||x|||_p \leqslant C' C_q(x).$$

Step 2.

$$C_2(x) \leqslant C'' |||x|||_p.$$

Actually we will prove also the converse inequality (up to a constant).

Step 3.

$$C_q(x) \leqslant C''' C_p(x)^{1-\theta} C_2(x)^{\theta}$$

where $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$. (Recall $p < q < 2$ so that $0 < \theta < 1$.)

The three steps put all together yield

$$|||x|||_p \leqslant C' C''' C_p(x)^{1-\theta} \left(C'' |||x|||_p\right)^{\theta}$$

and hence

$$|||x|||_p \leqslant C'''' C_p(x) = C'''' \|S\|_p.$$

Proof of Step 1. This is easy: We simply apply (K_q) to $y=(y_k)$. More precisely, fix $\varepsilon>0$. Let $y=(y_k)$ and f in $\mathcal{D}$ such that $x_k=(f^{\frac{1}{p}-\frac{1}{q}}y_k+y_kf^{\frac{1}{p}-\frac{1}{q}})/2$ and

$$\Big\|\sum \xi_k\otimes y_k\Big\|_q<C_q(x)(1+\varepsilon).$$

By (K_q) we have $|||y|||_q<\beta_qC_q(x)(1+\varepsilon)$. Let a_k,b_k be such that $y_k=a_k+b_k$ with

$$\Big\|\Big(\sum a_k^*a_k\Big)^{1/q}\Big\|+\Big\|\Big(\sum b_kb_k^*\Big)^{1/2}\Big\|_q\leqslant\beta_qC_q(x)(1+\varepsilon)$$

we have

$$2x_k=f^{\frac{1}{p}-\frac{1}{q}}(a_k+b_k)+(a_k+b_k)f^{\frac{1}{p}-\frac{1}{q}}.$$

But it is easy to check that for some $g,h\in\mathcal{D}$ there are α_k,β_k such that

$$a_k=\alpha_kg^{\frac{1}{q}-\frac{1}{2}},\qquad b_k=h^{\frac{1}{q}-\frac{1}{2}}\beta_k$$

with

$$\Big(\sum\|\alpha_k\|_2^2\Big)^{1/2}\leqslant\Big\|\Big(\sum a_k^*a_k\Big)^{1/2}\Big\|_q$$

and

$$\Big(\sum\|\beta_k\|_2^2\Big)^{1/2}\leqslant\Big\|\Big(\sum b_kb_k^*\Big)^{1/2}\Big\|_q.$$

Thus we find

$$2x_k=f^{\frac{1}{p}-\frac{1}{q}}\alpha_kg^{\frac{1}{q}-\frac{1}{2}}+f^{\frac{1}{p}-\frac{1}{q}}h^{\frac{1}{q}-\frac{1}{2}}\beta_k+\alpha_kg^{\frac{1}{q}-\frac{1}{2}}f^{\frac{1}{p}-\frac{1}{q}}+h^{\frac{1}{q}-\frac{1}{2}}\beta_kf^{\frac{1}{p}-\frac{1}{q}}.$$

Let $\frac{1}{r}=\frac{1}{p}-\frac{1}{2}$. Note that by Hölder's inequality (since $\frac{1}{p}-\frac{1}{2}=(\frac{1}{p}-\frac{1}{q})+(\frac{1}{q}-\frac{1}{2})$)

$$\big\|f^{\frac{1}{p}-\frac{1}{q}}h^{\frac{1}{q}-\frac{1}{2}}\big\|_r\leqslant1\quad\text{and}\quad\big\|g^{\frac{1}{q}-\frac{1}{2}}f^{\frac{1}{p}-\frac{1}{q}}\big\|_r\leqslant1.$$

Let $x_k'=f^{\frac{1}{p}-\frac{1}{q}}h^{\frac{1}{q}-\frac{1}{2}}\beta_k+\alpha_kg^{\frac{1}{q}-\frac{1}{2}}f^{\frac{1}{p}-\frac{1}{q}}$. Then, again by Hölder, we have

$$|||x'|||_p\leqslant\Big(\sum\|\alpha_k\|_2^2\Big)^{1/2}+\Big(\sum\|\beta_k\|_2^2\Big)^{1/2}\leqslant\beta_qC_q(x)(1+\varepsilon).$$

Similarly, let

$$x_k''=f^{\frac{1}{p}-\frac{1}{q}}\alpha_kg^{\frac{1}{q}-\frac{1}{2}}+h^{\frac{1}{q}-\frac{1}{2}}\beta_kf^{\frac{1}{p}-\frac{1}{q}}.$$

We claim that

$$|||x''|||_p\leqslant\beta_qC_q(x)(1+\varepsilon).$$

Thus we obtain, since $2x = x' + x''$

$$2|||x|||_p \leqslant |||x'|||_p + |||x''|||_p \leqslant 2\beta_q C_q(x)(1+\varepsilon),$$

and hence Step 1 holds with $C' = \beta_q$.

We now check the above claim. Define θ by $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$ and let

$$x_k(z) = f^{\frac{1}{p} - \frac{1}{q(z)}} \alpha_k g^{\frac{1}{q(z)} - \frac{1}{2}}$$

where $\frac{1}{q(z)} \overset{\text{def}}{=} \frac{1-z}{p} + \frac{z}{2}$. We will use the probability measure μ_θ on the boundary of the complex strip $\mathcal{S} = \{0 < \Re(z) < 1\}$ that is the Jensen (i.e. harmonic) measure for the point θ. This gives mass θ (resp. $1-\theta$) to the vertical line $\{\Re(z) = 1\}$ (resp. $\{\Re(z) = 0\}$). By perturbation, we may assume that f and g are suitably bounded below so that $x_k(.)$ is a "nice" $L_p(\tau)$-valued analytic function on $\mathcal{S}$, i.e. bounded and continuous on $\bar{\mathcal{S}}$. Then, since $q(\theta) = q$, we have by Cauchy's formula

$$f^{\frac{1}{p} - \frac{1}{q}} \alpha_k g^{\frac{1}{q} - \frac{1}{2}} = x_k(\theta) = \int_{\Re(z) \in \{0,1\}} x_k(z)\, d\mu_\theta(z),$$

but $\forall t \in \mathbb{R}$ $x_k(it) = U(it)\alpha_k V(it) g^{\frac{1}{p} - \frac{1}{2}}$ and $x_k(1+it) = f^{\frac{1}{p} - \frac{1}{2}} U(1+it)\alpha_k V(1+it)$, where $U(it) = U(1+it) = f^{it(\frac{1}{p} - \frac{1}{2})}$ and $V(it) = V(1+it) = g^{it(\frac{1}{2} - \frac{1}{p})}$ are unitary. This yields

$$f^{\frac{1}{p} - \frac{1}{q}} \alpha_k g^{\frac{1}{q} - \frac{1}{2}} = (1-\theta)\alpha_k^{(0)} g^{\frac{1}{p} - \frac{1}{2}} + \theta f^{\frac{1}{p} - \frac{1}{2}} \alpha_k^{(1)}$$

where $\alpha^{(1)}$ (resp. $\alpha^{(0)}$) are the corresponding averages over $\{\Re(z) = 1\}$ (resp. $\{\Re(z) = 0\}$) satisfying $(\sum \|\alpha_k^{(0)}\|_2^2)^{1/2} \leqslant (\sum \|\alpha_k\|_2^2)^{1/2}$ and $(\sum \|\alpha_k^{(1)}\|_2^2)^{1/2} \leqslant (\sum \|\alpha_k\|_2^2)^{1/2}$, and hence we find $|||(f^{\frac{1}{p} - \frac{1}{q}} \alpha_k g^{\frac{1}{q} - \frac{1}{2}})|||_p \leqslant (\sum \|\alpha_k\|_2^2)^{1/2}$. Similarly, we find $|||(h^{\frac{1}{q} - \frac{1}{2}} \beta_k f^{\frac{1}{p} - \frac{1}{q}})|||_p \leqslant (\sum \|\beta_k\|_2^2)^{1/2}$. Thus we obtain as claimed

$$|||x''|||_p \leqslant \Big(\sum \|\alpha_k\|_2^2\Big)^{1/2} + \Big(\sum \|\beta_k\|_2^2\Big)^{1/2} \leqslant \beta_q C_q(x)(1+\varepsilon). \qquad \square$$

Proof of Step 2. Assume $|||x|||_p < 1$, i.e. $x_k = a_k + b_k$ with

$$\Big\|\Big(\sum a_k^* a_k\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum b_k b_k^*\Big)^{1/2}\Big\|_p < 1.$$

By semi-finiteness of τ, we may assume there exists $f_0 > 0$ in $\mathcal{D}$. (In the finite case we can simply take $f_0 = 1$.) Let $f' = (\varepsilon(f_0)^{2/p} + \sum a_k^* a_k + \sum b_k b_k^*)^{1/2}$. We can choose $\varepsilon > 0$ small enough so that

$$\|f'\|_p^p < 2$$

(using the fact that $L_{p/2}(\tau)$ is $p/2$-normed). We can then write

$$x_k = a_k' f' + f' b_k'$$

where $a'_k = a_k(f')^{-1}$ and $b'_k = (f')^{-1}b_k$. Let $f = (f')^p(\tau(f'^p))^{-1}$. Note that $f \in \mathcal{D}$ and we have

$$x_k = \alpha_k f^{\frac{1}{p}-\frac{1}{2}} + f^{\frac{1}{p}-\frac{1}{2}}\beta_k \tag{1.2}$$

where

$$\alpha_k = \|f'\|_p a'_k f^{\frac{1}{2}}, \qquad \beta_k = \|f'\|_p f^{\frac{1}{2}} b'_k.$$

Note that $\sum a'^*_k a'_k = (f')^{-1}\sum a^*_k a_k (f')^{-1} \leqslant 1$ and similarly $\sum b'_k b'^*_k \leqslant 1$. Therefore

$$\Big(\sum \|\alpha_k\|_2^2\Big)^{1/2} = \|f'\|_p \Big\|\Big(\sum a'^*_k a'_k\Big)^{1/2} f^{\frac{1}{2}}\Big\|_2 \leqslant \|f'\|_p \leqslant 2^{\frac{1}{p}},$$

and similarly

$$\Big(\sum \|\beta_k\|_2^2\Big)^{1/2} \leqslant 2^{\frac{1}{p}}.$$

We will now modify this to obtain $\alpha_k = \beta_k$. More precisely we claim there are y_k in $L_2(\tau)$ such that $x_k = (f^{\frac{1}{p}-\frac{1}{2}} y_k + y_k f^{\frac{1}{p}-\frac{1}{2}})/2$ and

$$\Big(\sum \|y_k\|_2^2\Big)^{1/2} \leqslant 2\Big(\Big(\sum \|\alpha_k\|_2^2\Big)^{1/2} + \Big(\sum \|\beta_k\|_2^2\Big)^{1/2}\Big). \tag{1.3}$$

Let $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. To prove this claim, let E be the dense subspace of $L_2(\tau) \oplus \cdots \oplus L_2(\tau)$ formed of families $h = (h_k)$ such that $f^{\frac{1}{r}} h_k + h_k f^{\frac{1}{r}} \in L_2(\tau)$ for all k. Then for all h in E we have $\sum \langle x_k, h_k\rangle = \sum \langle \alpha_k f^{\frac{1}{r}} + f^{\frac{1}{r}}\beta_k, h_k\rangle$ and hence

$$\Big|\sum \langle x_k, h_k\rangle\Big| \leqslant \Big(\sum \|\alpha_k\|_2^2\Big)^{1/2}\Big(\sum \big\|h_k f^{\frac{1}{r}}\big\|_2^2\Big)^{1/2} + \Big(\sum \|\beta_k\|_2^2\Big)^{1/2}\Big(\sum \big\|f^{\frac{1}{r}} h_k\big\|_2^2\Big)^{1/2}.$$

By an elementary calculation one verifies easily that $\|f^{\frac{1}{r}} h_k\|_2^2 \leqslant \|f^{\frac{1}{r}} h_k + h_k f^{\frac{1}{r}}\|_2^2$ and similarly $\|h_k f^{\frac{1}{r}}\|_2^2 \leqslant \|f^{\frac{1}{r}} h_k + h_k f^{\frac{1}{r}}\|_2^2$. Therefore we find

$$\Big|\sum \langle x_k, h_k\rangle\Big| \leqslant \Big(\Big(\sum \|\alpha_k\|_2^2\Big)^{1/2} + \Big(\sum \|\beta_k\|_2^2\Big)^{1/2}\Big)\Big(\sum \big\|f^{\frac{1}{r}} h_k + h_k f^{\frac{1}{r}}\big\|_2^2\Big)^{1/2}.$$

From this our claim that there are (y_k) in $L_2(\tau)$ satisfying (1.3) follows immediately by duality. Then (1.3) implies

$$\Big\|\sum \xi_k \otimes y_k\Big\|_2 = \Big(\sum \|y_k\|_2^2\Big)^{1/2} \leqslant 4 \cdot 2^{\frac{1}{p}},$$

and we obtain $C_2(x) \leqslant 2^{2+\frac{1}{p}} |||x|||_p$, completing Step 2. $\square$

Remark 1.3. Conversely we have

$$|||x|||_p \leqslant C_2(x). \tag{1.4}$$

Indeed, if $C_2(x) < 1$ then $x_k = (f^{\frac{1}{p}-\frac{1}{2}} y_k + y_k f^{\frac{1}{p}-\frac{1}{2}})/2$ with

$$\left\| \sum \xi_k \otimes y_k \right\|_2 = \left(\sum \|y_k\|_2^2 \right)^{1/2} < 1.$$

Let now

$$a_k = \left(y_k f^{\frac{1}{p}-\frac{1}{2}}\right)/2 \quad \text{and} \quad b_k = \left(f^{\frac{1}{p}-\frac{1}{2}} y_k\right)/2.$$

We have (recall the notation $|T| = \sqrt{T^*T}$)

$$2\left(\sum a_k^* a_k\right)^{1/2} = \left| \left(\sum y_k^* y_k\right)^{1/2} f^{\frac{1}{p}-\frac{1}{2}} \right|$$

and hence (setting $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$) by Hölder

$$2\left\| \left(\sum a_k^* a_k\right)^{1/2} \right\|_p \leqslant \left\| \left(\sum y_k^* y_k\right)^{1/2} \right\|_2 \left\| f^{\frac{1}{p}-\frac{1}{2}} \right\|_r < 1.$$

Similarly $\|(\sum b_k b_k^*)^{1/2}\|_p < 1/2$. Thus we obtain $|||x|||_p < 1$. By homogeneity this proves (1.4).

Proof of Step 3. Fix $\varepsilon > 0$. Let y_k be such that $x_k = (f^{\frac{1}{r}} y_k + y_k f^{\frac{1}{r}})/2$ with

$$\left\| \sum \xi_k \otimes y_k \right\|_2 < C_2(x)(1 + \varepsilon).$$

Let us assume that (M, τ) is M_n equipped with usual trace. We will use the orthonormal basis for which f is diagonal with coefficients denoted by (f_i). We have then

$$(y_k)_{ij} = 2\left(f_i^{\frac{1}{r}} + f_j^{\frac{1}{r}}\right)^{-1} (x_k)_{ij}.$$

We define $y_k(\theta)$ by setting

$$y_k(\theta)_{ij} = 2\left(f_i^{\frac{\theta}{r}} + f_j^{\frac{\theta}{r}}\right)^{-1} (x_k)_{ij}.$$

Note that $y_k(0) = x_k$ while $y_k(1) = y_k$. Let $T(\theta) = \sum \xi_k \otimes y_k(\theta)$.

We claim that if $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$ and $1 \leqslant p < q \leqslant 2$

$$\left\| T(\theta) \right\|_q \leqslant c \left\| T(0) \right\|_p^{1-\theta} \left\| T(1) \right\|_2^{\theta} \tag{1.5}$$

for some constant c depending only on p and q. As observed in [13], when $p > 1$, this is easy to prove using the boundedness of the triangular projection on S_p. The case $p = 1$ is a consequence of Theorem 1.1a in [13] (the latter uses [26, Theorem 4.5]). See Appendix A for a detailed justification.

Therefore we obtain

$$C_q(x) \leqslant \left\|T(\theta)\right\|_q \leqslant 2^{1-\theta} c C_p(x)^{1-\theta}\big(C_2(x)(1+\varepsilon)\big)^{\theta},$$

i.e. we obtain Step 3 in the matricial case. Note that the argument works assuming merely that the density f has finite spectrum. □

Proof of Theorem 1.1. Combining the 3 steps, we have already indicated the proof in the case $M = M_n$ or assuming merely that the density f has finite spectrum. We will now prove the general semi-finite case. We return to Step 2. We claim that for any $\delta > 0$ we can find (x'_k) such that $\|\!|(x_k) - (x'_k)\|\!|_p < \delta \|\!|x\|\!|_p$ and such that

$$C_2(x') \leqslant 2 \cdot 2^{\frac{1}{p}}(1+\delta)\|\!|x'\|\!|_p,$$

where the definition of $C_2(x')$ is now restricted to densities with finite spectrum.

Indeed, one may assume by homogeneity that $0 < \|\!|x\|\!|_p < 1$. Let r be defined by $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Let $\delta' = (\delta/n)\|\!|x\|\!|_p$. Then let $f, y_k, \ldots$ be as in the above proof of Step 2 and let $g \in \mathcal{D}$ be an element with finite spectrum such that $\|f^{\frac{1}{r}} - g^{\frac{1}{r}}\|_r < (2 \cdot 2^{\frac{1}{p}})^{-1}\delta'$. Note that g exists by the semi-finiteness of τ. Then let

$$x'_k = \big(g^{\frac{1}{r}} y_k + y_k g^{\frac{1}{r}}\big)/2.$$

Note that (by Hölder) $\|x'_k - x_k\|_p < \delta'$ and hence (assuming $p \geqslant 1$) $\|\!|x - x'\|\!|_p < \delta \|\!|x\|\!|_p$.

We now observe that the proof of Step 3 applies if we replace (x, f) by (x', g). Thus if we apply the three steps to x' we obtain for some constant C_4

$$\|\!|x'\|\!|_p \leqslant C_4 C_p(x') = C_4 \left\|\sum \xi_k \otimes x'_k\right\|_p.$$

But since (x'_k) is an arbitrary close perturbation of (x_k) in L_p-norm, we conclude that (K_p) holds. □

Remark 1.4. In Theorem 1.1, the assumption that (ξ_k) is orthonormal in $L_2(\varphi)$ (that is only used in Step 2) can be replaced by the following one: for any finite sequence $y = (y_k)$ in $L_2(M, \tau)$ we have

$$\left\|\sum \xi_k \otimes y_k\right\|_{L_2(\varphi \times \tau)} \leqslant \Big(\sum \|y_k\|_2^2\Big)^{1/2}. \tag{1.6}$$

The proof (of Step 2) for that case is identical.

Assume for simplicity that (M, τ) is M_n equipped with its usual trace.

Let $S = \sum \xi_k \otimes x_k$, $x_k \in M_n$. Equivalently $S = [S_{ij}]$ with $S_{ij} \in L_2(\varphi)$. Consider $f \in \mathcal{D}$. The proof of Step 3 becomes straightforward if (1.5) holds. In the case $p \geqslant 1$, we invoked [13] to claim that (1.5) is indeed true, but we do not know whether it still holds when $0 < p < 1$.

Nevertheless, there is a situation when (1.5) is easy to check, when the following condition (γ', γ'') holds:

Condition $(\boldsymbol{\gamma'}, \boldsymbol{\gamma''})$. Let γ', γ'' be positive numbers. We say that $S = \sum \xi_k \otimes x_k$ satisfies the condition (γ', γ'') if we can find f in $\mathcal{D}$ and y_k such that $x_k = (f^{\frac{1}{r}} y_k + y_k f^{\frac{1}{r}})/2$ and such that $T = \sum \xi_k \otimes y_k$ satisfies *simultaneously* the following two bounds:

$$\|T\|_2 \leqslant \gamma' C_2(x), \tag{1.7}$$

$$\big\|\big(1 \otimes f^{\frac{1}{r}}\big) T\big\|_p \leqslant \gamma'' \|S\|_p. \tag{1.8}$$

If we set $F = 1 \otimes f$, we can rewrite (1.8) as

$$\big\|F^{\frac{1}{r}} T\big\|_p \leqslant \gamma'' \big\|F^{\frac{1}{r}} T + T F^{\frac{1}{r}}\big\|_p / 2, \tag{1.9}$$

and hence by the triangle inequality (or its analogue for $p < 1$), since $S = (F^{\frac{1}{r}} T + T F^{\frac{1}{r}})/2$, we have automatically for a suitable γ''' (depending only on γ'' and p)

$$\big\|T F^{\frac{1}{r}}\big\|_p \leqslant \gamma''' \|S\|_p. \tag{1.10}$$

Remark 1.5. The reason why condition (γ', γ'') resolves our problem is that the one-sided version of (1.5) is quite easy: we have

$$\big\|F^{-\frac{\theta}{r}} S\big\|_q \leqslant \|S\|_p^{1-\theta} \big\|F^{-\frac{1}{r}} S\big\|_2^{\theta}. \tag{1.11}$$

Indeed, if we let $T = F^{-\frac{1}{r}} S$ then (1.11) becomes

$$\big\|F^{\frac{1-\theta}{r}} T\big\|_q \leqslant \big\|F^{\frac{1}{r}} T\big\|_p^{1-\theta} \|T\|_2^{\theta} \tag{1.12}$$

and the latter holds by Lemma 1.8 below.

Theorem 1.6. *Let (ξ_k) be a sequence in $L_2(\varphi)$ orthonormal or merely satisfying* (1.6). *Let $0 < p < q < 2$. Then, if we assume the condition (γ', γ'') (as above but for any S), the implication $(K_q) \Rightarrow (K_p)$ holds, where the resulting constant β_p depends on p, q, β_q and also on γ', γ''.*

For simplicity we will prove this again assuming that (M, τ) is M_n equipped with its usual trace. See the above proof of Theorem 1.1 for indications on how to check the general case.

Remark 1.7. If (K_p) holds, then there are constants (γ', γ'') depending only on p such that the condition (γ', γ'') holds. Indeed by the above proof of Step 2 we have

$$2x_k = f^{\frac{1}{r}} y_k + y_k f^{\frac{1}{r}}$$

with

$$\Big(\sum \|y_k\|_2^2\Big)^{1/2} \leqslant C'' |||x|||_p$$

and hence by Hölder and (1.6)

$$\big\| F^{\frac{1}{r}} T \big\|_p \leqslant \|T\|_2 \leqslant \Big(\sum \|y_k\|_2^2 \Big)^{1/2} \leqslant C'' |||x|||_p.$$

Now if (K_p) holds we have

$$|||x|||_p \leqslant \beta_p \|S\|_p = \beta_p/2 \big\| F^{\frac{1}{r}} T + T F^{\frac{1}{r}} \big\|_p$$

therefore we find $\|F^{\frac{1}{r}} T\|_p \leqslant C'' \beta_p/2 \|F^{\frac{1}{r}} T + T F^{\frac{1}{r}}\|_p$ i.e. (1.9) holds.

The following two lemmas will be used.

Lemma 1.8. *Let (M, τ) be a generalized measure space. Consider $F \geqslant 0$ in $L_1(\tau)$. Assume $0 < p < q < 2$. Let $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and let θ be such that $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$. Then for any V in $L_2(\tau)$ we have*

$$\big\| F^{\frac{1-\theta}{r}} V \big\|_q \leqslant \big\| F^{\frac{1}{r}} V \big\|_p^{1-\theta} \|V\|_2^\theta$$

and

$$\big\| V F^{\frac{1-\theta}{r}} \big\|_q \leqslant \big\| V F^{\frac{1}{r}} \big\|_p^{1-\theta} \|V\|_2^\theta.$$

Proof. It suffices to show

$$\big\| V F^{(1-\theta)(\frac{1}{p} - \frac{1}{2})} \big\|_q \leqslant \big\| V F^{\frac{1}{p} - \frac{1}{2}} \big\|_p^{1-\theta} \|V\|_2^\theta \tag{1.13}$$

since we obtain the other inequality by replacing V by V^*. Since the complex interpolation of non-commutative L_p-spaces is valid in the whole range $0 < p < \infty$ [35], this can be deduced from the 3 line lemma. Alternatively, this also follows from Hölder's inequality, together with [15]. Indeed,

$$\big\| V F^{(1-\theta)(\frac{1}{p} - \frac{1}{2})} \big\|_q = \big\| |V| F^{(1-\theta)(\frac{1}{p} - \frac{1}{2})} \big\|_q = \big\| |V|^\theta |V|^{1-\theta} F^{(1-\theta)(\frac{1}{p} - \frac{1}{2})} \big\|_q$$

and hence by Hölder (recall $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$)

$$\leqslant \|V\|_2^\theta \big\| |V|^{1-\theta} F^{(\frac{1}{p} - \frac{1}{2})(1-\theta)} \big\|_{\frac{p}{1-\theta}}.$$

But by [15] (see also [1]) we have

$$\big\| |V|^{1-\theta} F^{(\frac{1}{p} - \frac{1}{2})(1-\theta)} \big\|_{\frac{p}{1-\theta}} \leqslant \big\| |V| F^{\frac{1}{p} - \frac{1}{2}} \big\|_p^{1-\theta}$$

and hence we obtain (1.13). □

Lemma 1.9. *(See [13].) Let Q_j $(j=1,\dots,n)$ be mutually orthogonal projections in M and let λ_j $(j=1,\dots,n)$ be non-negative numbers.*

(i) *For any $1 \leqslant q \leqslant \infty$ and any x in $L_q(\tau)$*

$$\left\| \sum_{i,j=1}^{n} \frac{\lambda_i \vee \lambda_j}{\lambda_i + \lambda_j} Q_i x Q_j \right\|_{L_q(\tau)} \leqslant \frac{3}{2} \|x\|_{L_q(\tau)}$$

and

$$\left\| \sum_{i,j=1}^{n} \frac{\lambda_i \wedge \lambda_j}{\lambda_i + \lambda_j} Q_i x Q_j \right\|_{L_q(\tau)} \leqslant \frac{1}{2} \|x\|_{L_q(\tau)}.$$

(ii) *For any $1 < q < \infty$ there is a constant $t(q)$, depending only on q, such that for any x in $L_q(\tau)$*

$$\left\| \sum_{i,j=1}^{n} \frac{\lambda_i}{\lambda_i + \lambda_j} Q_i x Q_j \right\|_{L_q(\tau)} \leqslant t(q) \|x\|_{L_q(\tau)}$$

and

$$\left\| \sum_{i,j=1}^{n} \frac{\lambda_j}{\lambda_i + \lambda_j} Q_i x Q_j \right\|_{L_q(\tau)} \leqslant t(q) \|x\|_{L_q(\tau)}.$$

(iii) *For any s with $1 < q < s \leqslant \infty$, any density $f \in \mathcal{D}$ and any $x \in L_s(\tau)$, we have*

$$\max\left\{ \left\| f^{\frac{1}{q}-\frac{1}{s}} x \right\|_q, \left\| x f^{\frac{1}{q}-\frac{1}{s}} \right\|_q \right\} \leqslant t(q) \left\| f^{\frac{1}{q}-\frac{1}{s}} x + x f^{\frac{1}{q}-\frac{1}{s}} \right\|_{L_q(\tau)}.$$

Proof. This was used in [13] (see also [11,12] for related facts). For the convenience of the reader we sketch the argument. We may easily reduce to the case $\sum Q_j = 1$.

(i) expresses the fact that

$$\frac{\lambda_i \vee \lambda_j}{\lambda_i + \lambda_j} \quad \text{and} \quad \frac{\lambda_i \wedge \lambda_j}{\lambda_i + \lambda_j}$$

are (completely) contractive Schur multipliers on $L_q(M_n)$ for any $1 \leqslant q \leqslant \infty$ (see [13]).

(ii) Using a permutation of the (Q_j) we may assume $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. By the boundedness of the triangular projection when $1 < q < \infty$ (see the seminal paper [20] and [30, §8] for more references to the literature), it suffices to check (ii) when x is either upper or lower triangular with respect to the decomposition (Q_j). More precisely it suffices to check this when either $x = x^+$ or $x = x^-$ where $x^+ = \sum_{i \leqslant j} Q_i x Q_j$ and $x^- = \sum_{i > j} Q_i x Q_j$. But since $\lambda_i \vee \lambda_j = \lambda_j$ and $\lambda_i \wedge \lambda_j = \lambda_i$ if $i \leqslant j$, the case when x is upper triangular (i.e. $x = x^+$) follows from the first part. The lower triangular case (i.e. $x = x^-$) is similar.

(iii) By density we may reduce this to the case when f has finite spectrum, so that $f = \sum \lambda_j Q_j$. Then (iii) essentially reduces to (ii). □

Proof of Theorem 1.6. We choose $q > 1$ with $0 < p < q < 2$. We use the same notation as for Theorem 1.1. By the observations made before Theorem 1.6, it suffices to verify (1.5). By Lemma 1.8 applied with $V = T$ and $V = T^*$ we have

$$\big\|F^{\frac{1-\theta}{r}}T\big\|_q \leqslant \big\|F^{\frac{1}{r}}T\big\|_p^{1-\theta}\|T\|_2^{\theta},$$
$$\big\|TF^{\frac{1-\theta}{r}}\big\|_q \leqslant \big\|TF^{\frac{1}{r}}\big\|_p^{1-\theta}\|T\|_2^{\theta}.$$

Let $\lambda_i = f_i^{\frac{\theta}{r}}$. By Lemma 1.9 we have

$$\big\|\big[\big(f_i^{\frac{\theta}{r}} + f_j^{\frac{\theta}{r}}\big)^{-1} f_i^{\frac{1}{r}} T_{ij}\big]\big\|_q \leqslant t(q)\big\|F^{\frac{1-\theta}{r}}T\big\|_q \tag{1.14}$$

and similarly

$$\big\|\big[\big(f_i^{\frac{\theta}{r}} + f_j^{\frac{\theta}{r}}\big)^{-1} T_{ij} f_j^{\frac{1}{r}}\big]\big\|_q \leqslant t(q)\big\|TF^{\frac{1-\theta}{r}}\big\|_q. \tag{1.15}$$

Note that we have

$$T(\theta) = \big(f_i^{\frac{\theta}{r}} + f_j^{\frac{\theta}{r}}\big)^{-1}\big(f_i^{\frac{1}{r}} + f_j^{\frac{1}{r}}\big)T_{ij}.$$

Therefore by the triangle inequality and (1.14), (1.15)

$$\big\|T(\theta)\big\|_q \leqslant t(q)\big(\big\|F^{\frac{1}{r}}T\big\|_p^{1-\theta} + \big\|TF^{\frac{1}{r}}\big\|_p^{1-\theta}\big)\|T\|_2^{\theta}$$

and hence by condition (γ', γ'')

$$\begin{aligned}\big\|T(\theta)\big\|_q &\leqslant t(q)\big((\gamma'')^{1-\theta} + (\gamma''')^{1-\theta}\big)\big\|\big(F^{\frac{1}{r}}T + TF^{\frac{1}{r}}\big)/2\big\|_p^{1-\theta}\|T\|_2^{\theta}\\ &\leqslant t(q)\big((\gamma'')^{1-\theta} + (\gamma''')^{1-\theta}\big)\|S\|_p^{1-\theta}\|T\|_2^{\theta}.\end{aligned}$$

Therefore we obtain (1.5). By condition (γ', γ'') we have $\|T(1)\|_2 = \|T\|_2 \leqslant \gamma' C_2(x)$, and also $C_q(x) \leqslant \|T(\theta)\|_q$ so we conclude that Step 3 holds. □

Remark 1.10. Theorem 1.1 implies as a special case the following fact possibly of independent interest: if for some $0 < q < 2$ we have

$$(C_q)\quad \begin{cases} \exists C\ \forall a_k \in L_p(\tau), \\ \Big\|\Big(\sum a_k^* a_k\Big)^{1/2}\Big\|_{L_q(\tau)} \leqslant C\Big\|\sum \xi_k \otimes a_k\Big\|_{L_q(\varphi\times\tau)}, \end{cases}$$

then (C_p) holds (for a different constant C) for all p with $0 < p < q$. In this case Step 3 is easy to verify (only right multiplication appears in this case).

Remark 1.11. If $0 < p \leqslant 2$, the converse inequality to (K_p) is valid assuming that $\varphi(1) = 1$ and $\xi_k \in L_2(N, \varphi)$ is orthonormal or satisfies (1.6).

Indeed, for any $t \geqslant 0$ in $N \otimes M$ since $p/2 \leqslant 1$ and $\varphi(1) = 1$, by the operator concavity of $t \mapsto t^{p/2}$ (see [4, pp. 115–120]), we have

$$\|t\|_{p/2} \leqslant \left\|\mathbb{E}^M(t)\right\|_{p/2}$$

and hence, if $S = \sum \xi_k \otimes x_k$, we have

$$\|S\|_p = \left\|S^*S\right\|_{p/2}^{1/2} \leqslant \left\|\mathbb{E}^M\left(S^*S\right)\right\|_{p/2}^{1/2} \leqslant \left\|\left(\sum x_k^* x_k\right)^{1/2}\right\|_p,$$

and similarly

$$\|S\|_p \leqslant \left\|\left(\sum x_k x_k^*\right)^{1/2}\right\|_p.$$

From this we easily deduce

$$\|S\|_p \leqslant c(p) |||x|||_p$$

where $c(p) = 1$ if $1 \leqslant p \leqslant 2$ and $c(p) = 2^{\frac{1}{p}-1}$ if $0 < p \leqslant 1$. The preceding remark shows that the assumption that φ is finite cannot be removed.

Remark 1.12. To extend Theorem 1.1 to the case $0 < p < 1$ the difficulty lies in Step 3, or in proving a certain form of Hölder inequality such as (1.5). Note that a much *weaker* estimate allows to conclude:

It suffices to show that there is a function $\varepsilon \to \delta(\varepsilon)$ tending to zero with $\varepsilon > 0$ such that when $f \in \mathcal{D}$ we have ($\alpha = \frac{1}{p} - \frac{1}{2} = \frac{1}{r}$) ($1 < q < 2$):

$$\left[\|x\|_2 \leqslant 1, \left\|f^\alpha x + x f^\alpha\right\|_p \leqslant \varepsilon\right] \quad \Rightarrow \quad \left\|f^{\alpha(1-\theta)} x + x f^{\alpha(1-\theta)}\right\|_q \leqslant \delta(\varepsilon).$$

This might hold even if Step 3 poses a problem.

In the case $2 \leqslant q < \infty$, the formulation of (K_q) must be changed. When $2 < q < \infty$, and $x = (x_k)$ is a finite sequence in $L_q(\tau)$, we set

$$|||x|||_q \stackrel{\text{def}}{=} \max\left\{\left\|\left(\sum x_k^* x_k\right)^{1/2}\right\|_q, \left\|\left(\sum x_k x_k^*\right)^{1/2}\right\|_q\right\}.$$

We will then say (when $2 < q < \infty$) that (ξ_k) satisfies (K_q) if there is a constant β_q such that for any such $x = (x_k)$ we have

$$\left\|\sum \xi_k \otimes x_k\right\|_q \leqslant \beta_q |||x|||_q.$$

By [16], this holds when (ξ_k) are the Rademacher functions on [0, 1].

In operator space theory, all L_q-spaces, in particular the Schatten class S_q, can be equipped with a "natural" operator space structure (see [28, §9.5 and 9.8]). Let C_q (resp. R_q) denote the closed span of $\{e_{i1} \mid i \geqslant 1\}$ (resp. $\{e_{1j} \mid j \geqslant 1\}$) in S_q. We denote by $C_{q,n}$ and $R_{q,n}$ the corresponding n-dimensional subspaces. By definition, the "sum space" $R_q + C_q$ is the quotient space $(R_q \oplus C_q)/N$ where $N = \{(x, -{}^t x) \mid x \in R_q\}$. We define similarly $R_{q,n} + C_{q,n}$. The intersection $R_q \cap C_q$ is defined as the subspace $\{x, {}^t x\}$ in $R_q \oplus C_q$. Here the direct sums are meant (say) in the operator space sense, i.e. in the ℓ_∞-sense. Let us denote by $\mathrm{Rad}(n, q)$ (resp. $\mathrm{Rad}(q)$) the linear span of the first n (resp. the sequence of all the) Rademacher functions in $L_q([0, 1])$. The operator space structure induced on the space $\mathrm{Rad}(q)$ is entirely described by the non-commutative Khintchine inequalities (see [28, §9.8]): the space $\mathrm{Rad}(q)$ is completely isomorphic to $R_q + C_q$ when $1 \leqslant q \leqslant 2$ and to $R_q \cap C_q$ when $2 \leqslant q < \infty$. The case $0 < q < 1$ is open.

Note that $\mathrm{Rad}(q, n)$ is an n-dimensional subspace of $\ell_q^{2^n}$, so (Kh_q) implies that $R_{q,n} + C_{q,n}$ uniformly embeds into $\ell_q^{2^n}$ for all $1 \leqslant q < 2$. The next result improves significantly the dimension of the embedding.

First recall that two operator spaces E, F are called completely c-isomorphic if there is an isomorphism $u : E \to F$ such that $\|u\|_{cb}\|u^{-1}\|_{cb} \leqslant c$.

Theorem 1.13. *Let $1 \leqslant q < 2$. For any n, there is a subspace of ℓ_q^n with dimension $k = [n^{1/2}]$ that is completely c-isomorphic to $R_{q,k} + C_{q,k}$ where c is a constant depending only on q.*

Proof. By [9], we know that there is a subset of $[1, e^{it}, \dots, e^{int}]$ with cardinality $k = [n^{1/2}]$ such that the corresponding set $\{\xi_1, \dots, \xi_k\}$ satisfies (K_q) for all q such that $2 \leqslant q \leqslant 4$ and hence by duality for all q such that $4/3 \leqslant q \leqslant 2$. By Theorem 1.1, the same set satisfies (K_p) for all $1 \leqslant p \leqslant 2$ (with a constant β_p independent of n). □

Problem. What is the correct estimate of k in Theorem 1.13? In particular is it true for $k \sim [n^\alpha]$ with α any number in $(0, 1)$ instead of $\alpha = \frac{1}{2}$? Is it true for $k \sim [\delta n]$ $(0 < \delta < 1)$?

The case $q = 1$ is particularly interesting. It is natural to expect a positive answer with k proportional to n by analogy with the Banach space case (see [8]). One could even dream of an operator space version of the Kashin decomposition (cf. [32])! Another bold conjecture would be the operator space generalization of Schechtman's and Bourgain, Lindenstrauss and Milman's results, as refined by Talagrand in [34]:

Problem. Let E be an n-dimensional operator subspace of S_1. Assume that for some $p > 1$ E embeds c-completely isomorphically into S_p. Is it true that E can then be embedded c'-completely isomorphically (the constant c' being a function of p and c) into S_1^{2n}?

2. Conditional expectation variant

Again we consider (ξ_k) in $L_2(N, \varphi)$ and coefficients (x_k) in $L_2(M, \tau)$, but, in addition, we give ourselves a von Neumann subalgebra $\mathcal{M} \subset M$ such that $\varphi_{|\mathcal{M}}$ is semi-finite and we denote by $E : M \to \mathcal{M}$ the conditional expectation with respect to $\mathcal{M}$. Recall that E extends to a contractive projection (still denoted abusively by E) from $L_q(M, \tau)$ onto $L_q(\mathcal{M}, \tau)$ for all $1 \leqslant q \leqslant \infty$.

Consider $x = (x_k)$ with $x_k \in L_q(\tau)$. In this section, we define

$$|||x|||_{q,\mathcal{M}} = \inf_{x_k = a_k + b_k} \left\{ \left\| \left(E \sum a_k^* a_k \right)^{1/2} \right\|_q + \left\| \left(E \sum a_k a_k^* \right)^{1/2} \right\|_q \right\}.$$

We then define again

$$C_{q,\mathcal{M}}(x) = \inf \left\{ \left\| \sum \xi_k \otimes y_k \right\|_q \right\}$$

where the infimum runs over all y_k in $L_q(\tau)$ such that there is f in $\mathcal{D} \cap L_1(\mathcal{M})$ such that $x_k = (f^{\frac{1}{r}} y_k + y_k f^{\frac{1}{r}})/2$.

Then the proof described in Section 1 extends with no change to this situation and shows that if there is $\beta_q(\mathcal{M})$ such that for all finite sequences $x = (x_k)$ in $L_q(\tau)$ we have

$$|||x|||_{q,\mathcal{M}} \leqslant \beta_q(\mathcal{M}) \left\| \sum \xi_k \otimes x_k \right\|_q \tag{2.1}$$

then for any p with $1 \leqslant p < q$ there is a constant $\beta_p(\mathcal{M})$ such that (2.1) holds for any $x = (x_k)$ in $L_p(\tau)$ when q is replaced by p. The main new case we have in mind is the case when $L_q(M, \tau) = S_q$ (Schatten class) and $\mathcal{M}$ is the subalgebra of *diagonal* operators (so that $L_q(\mathcal{M}) \simeq \ell_q$) on ℓ_2.

Thus we state for future reference:

Theorem 2.1. *Both Theorems* 1.1 *and* 1.6 *remain valid when* $|||\cdot|||_q$ *is replaced by* $|||\cdot|||_{q,\mathcal{M}}$.

Proof. The verification of this assertion is straightforward. Note that the conditional expectation E satisfies $E(axb) = aE(x)b$ whenever a, b are in $\mathcal{M}$ and x in $L_q(M, \tau)$. This is used to verify Step 2. The proof of the other steps require no significant change. □

Let $L_q(M, \tau) = S_q$ (Schatten q-class) and let $\mathcal{M} \subset B(\ell_2)$ be the subalgebra of diagonal operators with conditional expectation denoted by E. Consider a family $x = (x_{ij})$ with $x_{ij} \in S_q$. We will denote by $(x_{ij})_{k\ell}$ the entries of each matrix $x_{ij} \in S_q$, and we set

$$\hat{x}_{ij} = (x_{ij})_{ij} e_{ij} \quad \text{and} \quad \hat{x} = (\hat{x}_{ij}).$$

Let us denote

$$\|x\|_{R_q} \stackrel{\text{def}}{=} \left\| \left(\sum_{ij} x_{ij} x_{ij}^* \right)^{1/2} \right\|_q \quad \text{and} \quad \|x\|_{C_q} \stackrel{\text{def}}{=} \left\| \left(\sum_{ij} x_{ij}^* x_{ij} \right)^{1/2} \right\|_q.$$

Lemma 2.2. *For any* $q \geqslant 1$ *we have*

$$\|\hat{x}\|_{R_q} \leqslant \|x\|_{R_q} \quad \textit{and} \quad \|\hat{x}\|_{C_q} \leqslant \|x\|_{C_q}, \tag{2.2}$$

and consequently

$$|||\hat{x}|||_q \leqslant |||x|||_q. \tag{2.3}$$

Proof. Indeed, this follows from the convexity of the norms involved and the identity

$$\hat{x}_{ij} = \int \overline{z'_i}\,\overline{z''_j} D(z') x_{ij} D(z'')\, dm(z')\, dm(z'')$$

where $z' = (z'_i)$, $z'' = (z''_j)$ denote elements of $\mathbb{T}^{\mathbb{N}}$ equipped with its normalized Haar measure m. $\square$

Now consider $\lambda_{ij} \in \mathbb{C}$. We define

$$[\lambda]_p = \inf\left\{\left(\sum_i\left(\sum_j |a_{ij}|^2\right)^{p/2} + \sum_j\left(\sum_i |b_{ij}|^2\right)^{p/2}\right)^{1/p}\right\} \tag{2.4}$$

where the inf runs over all possible decompositions $\lambda_{ij} = a_{ij} + b_{ij}$.

Lemma 2.3. *Let $\hat{x}_{ij} = \lambda_{ij} e_{ij}$ (i.e. $\lambda_{ij} = (x_{ij})_{ij}$). We have*

$$\|\hat{x}\|_{C_q} = \left(\sum_j\left(\sum_i |\lambda_{ij}|^2\right)^{q/2}\right)^{1/q} \quad \textit{and} \quad \|\hat{x}\|_{R_q} = \left(\sum_i\left(\sum_j |\lambda_{ij}|^2\right)^{q/2}\right)^{1/q}.$$

Moreover, for any $0 < q < \infty$

$$|||\hat{x}|||_{q,\mathcal{M}} = [\lambda]_q \leqslant |||x|||_{q,\mathcal{M}}$$

where $\mathcal{M}$ is the subalgebra of diagonal *operators on ℓ_2.*

Proof. The first assertion is an immediate calculation. We now show that for any $0 < q < \infty$

$$[\lambda]_q \leqslant |||x|||_{q,\mathcal{M}}. \tag{2.5}$$

Indeed, let us denote

$$\|x\|_{C_q,\mathcal{M}} = \left\|\left(E\sum_{ij} x_{ij}^* x_{ij}\right)^{1/2}\right\|_q \quad \text{and} \quad \|x\|_{R_q,\mathcal{M}} = \left\|\left(E\sum x_{ij} x_{ij}^*\right)^{1/2}\right\|_q.$$

Then a simple verification shows that

$$\|x\|_{C_q,\mathcal{M}} = \left(\sum_k\left(\sum_{ij\ell}\left|(x_{ij})_{\ell k}\right|^2\right)^{q/2}\right)^{1/q} \geqslant \left(\sum_k\left(\sum_\ell\left|(x_{\ell k})_{\ell k}\right|^2\right)^{q/2}\right)^{1/q}$$

and similarly we find

$$\|x\|_{R_q,\mathcal{M}} \geqslant \left(\sum_\ell\left(\sum_k\left|(x_{\ell k})_{\ell k}\right|^2\right)^{q/2}\right)^{1/q}.$$

Then (2.5) follows immediately. In particular $[\lambda]_q \leqslant |||\hat{x}|||_{q,\mathcal{M}}$, and since the converse is immediate we obtain $|||\hat{x}|||_{q,\mathcal{M}} = [\lambda]_q$. $\square$

Remark 2.4. It seems worthwhile to point out that (2.2) is no longer valid when $0 < q < 1$ (even with a constant). Indeed, restricting to the case when $x_{ij} = 0\ \forall i \neq j$, these inequalities imply

$$\Big(\sum_j \big|(x_{jj})_{jj}\big|^q\Big)^{1/q} \leqslant \Big\|\Big(\sum_j x_{jj}^* x_{jj}\Big)^{1/2}\Big\|_q. \tag{2.6}$$

Now let us consider the case

$$x_{jj} = \sum_{k=1}^n e_{jk}.$$

On one hand we have $x_{jj}^* x_{jj} = nP$ where P is the rank one orthogonal projection onto $n^{-1/2}\sum e_k$, so that $(\sum x_{jj}^* x_{jj})^{1/2} = nP$ and hence $\|(\sum x_{jj}^* x_{jj})^{1/2}\|_q = n$. But on the other hand $(x_{jj})_{jj} = 1$ and hence

$$\Big(\sum \big|(x_{jj})_{jj}\big|^q\Big)^{1/q} = n^{1/q}.$$

This shows that (2.6) and hence (2.2) fails for $q < 1$. The same example shows (a fortiori) that (2.3) also fails for $q < 1$.

Remark 2.5. Let $j : L_p(M,\tau) \to L_p(M',\tau')$ be an isometric embedding. Let $y_k = j(x_k)$. Then clearly

$$\int \Big\|\sum r_k(t) y_k\Big\|_p^p dt = \int \Big\|\sum r_k(t) x_k\Big\|_p^p dt.$$

However, when $0 < p < 1$, we do not see how to prove that there is a constant C such that

$$\big|\big|\big|(x_k)\big|\big|\big|_p \leqslant C\big|\big|\big|(y_k)\big|\big|\big|_p,$$

although when $p \geqslant 1$ this holds with $C = 1$ using a conditional expectation.

This may be an indication that (Kh_p) does not hold for $0 < p < 1$, at least in the same form as for $p \geqslant 1$.

3. The case $0 < p < 1$

In (K_q), we may consider the case when $L_q(\tau) = S_q$ (Schatten q-class) and the sequence $x = (x_k)$ is of the form $x_{ij} = \lambda_{ij} e_{ij}$ with $\lambda_{ij} \in \mathbb{C}$. For this special case, the approach used in the preceding section works for all $0 < p < q$. Thus, we obtain

Theorem 3.1. *Let (ε_{ij}) be an i.i.d sequence of $\{+1,-1\}$-valued random variables on a probability space with $\mathbb{P}(\varepsilon_{ij}=\pm 1)=1/2$. Then for any $0<p<1$ there is a constant β_p such that*

$$[\lambda]_p \leqslant \beta_p\left(\int\left\|\sum \varepsilon_{ij}\lambda_{ij}e_{ij}\right\|_{S_p}^p d\mathbb{P}\right)^{1/p}$$

where $[\lambda]_p$ is defined in (2.4).

Remark. Of course, by [16,18], the case $1\leqslant p\leqslant 2$ is already known.

Remark 3.2. Since S_p is p-normed when $0<p<1$, the converse inequality is obvious: we have

$$\|a\|_p \leqslant \left(\sum_i\left\|\sum_j a_{ij}e_{ij}\right\|_p^p\right)^{1/p} = \left(\sum_i\left(\sum_j |a_{ij}|^2\right)^{p/2}\right)^{1/2}$$

and similarly $\|b\|_p \leqslant (\sum_j(\sum_i |b_{ij}|^2)^{p/2})^{1/2}$. Therefore

$$\sup_{\varepsilon_{ij}=\pm 1}\left\|\sum \varepsilon_{ij}\lambda_{ij}e_{ij}\right\|_p \leqslant [\lambda]_p.$$

By well-known general results (cf. [14]) this allows us to formulate

Corollary 3.3. *Let $\lambda_{ij}\in\mathbb{C}$ be arbitrary complex scalars. The following are equivalent.*

(i) *The matrix $[\varepsilon_{ij}\lambda_{ij}]$ belongs to S_p for almost all choices of signs $\varepsilon_{ij}=\pm 1$.*
(ii) *Same as* (i) *for all choices of signs.*
(iii) *There is a decomposition $\lambda_{ij}=a_{ij}+b_{ij}$ with*

$$\sum_i\left(\sum_j |a_{ij}|^2\right)^{p/2} < \infty \quad \textit{and} \quad \sum_j\left(\sum_i |b_{ij}|^2\right)^{p/2} < \infty$$

i.e. in short $[\lambda]_p<\infty$.

Remark 3.4. Note that when $0<p<1$, the spaces S_p or $L_p(\tau)$ are p-normed, i.e. their norm satisfies for any pair of elements x, y

$$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p. \tag{3.1}$$

Remark 3.5. Assume here that $0<p\leqslant 2$. Note that $[\lambda]_p<1$ implies that there is a sequence $f_i>0$ with $\sum f_i\leqslant 1$ such that, if we set $\frac{1}{r}=\frac{1}{p}-\frac{1}{2}$ we have

$$\left(\sum_{ij}\left|\left(f_i^{\frac{1}{r}}+f_j^{\frac{1}{r}}\right)^{-1}\lambda_{ij}\right|^2\right)^{1/2} \leqslant 2. \tag{3.2}$$

Indeed, we have $\lambda_{ij} = a_{ij} + b_{ij}$ with $(\sum_i(\sum_j |a_{ij}|^2)^{p/2})^{1/p} + (\sum_j(\sum_i |b_{ij}|^2)^{p/2})^{1/p} < 1$ we then set $a_i = (\sum_j |a_{ij}|^2)^{p/2}$ and $b_j = (\sum_i |b_{ij}|^2)^{p/2}$ so that $(\sum a_i) + (\sum b_j) < 1$. Note that:

$$\left|\left(a_i^{\frac{1}{r}} + b_j^{\frac{1}{r}}\right)^{-1}(\lambda_{ij})\right| \leqslant a_i^{-\frac{1}{r}}|a_{ij}| + |b_{ij}|b_j^{-1/r}$$

and hence by Hölder

$$\left(\sum_{ij}\left|\left(a_i^{\frac{1}{r}} + b_j^{\frac{1}{r}}\right)^{-1}\lambda_{ij}\right|^2\right)^{1/2} \leqslant \left(\sum\left|a_i^{-1/r}a_i^{1/p}\right|^2\right)^{1/2} + \left(\sum\left|b_j^{-1/r}b_j^{1/p}\right|^2\right)^{1/2}$$

$$= \left(\sum a_i\right)^{1/2} + \left(\sum b_j\right)^{1/2} \leqslant 2.$$

Let $f_i = a_i + b_i$. Then $\sum f_i < 1$, (3.2) holds and, if we perturb f_i slightly, we may assume $f_i > 0$ for all i.

We will use the following well-known elementary fact.

Proposition 3.6. *Let X be a p-normed space $(0 < p \leqslant 1)$, i.e. we assume*

$$\forall x, y \in X \quad \|x + y\|^p \leqslant \|x\|^p + \|y\|^p.$$

Then there is a constant χ_p such that for any finite sequence (x_k) in X and any sequence of real numbers (α_k) we have

$$\left\|\sum \alpha_k r_k x_k\right\|_{L_p(X)} \leqslant \chi_p \sup_k |\alpha_k| \left\|\sum \varepsilon_k x_k\right\|_{L_p(X)}. \tag{3.3}$$

Here (r_k) denote the Rademacher functions on $[0, 1)$ and $L_p(X) = L_p([0, 1]; X)$.

Proof. If $\alpha_k \in \{-1, 1\}$, we have equality in (3.3) with $\chi_p = 1$. If $\alpha_k \in \{-1, 0, 1\}$ we can write $\alpha_k = (\beta_k + \gamma_k)/2$ with $\beta_k \in \{-1, 1\}$, $\gamma_k \in \{-1, 1\}$ and then we obtain (3.2) (using the p-triangle inequality (3.1)) with $\chi_p = 2^{\frac{1}{p}-1}$. For the general case, we can write any α_k in $[-1, 1]$ as a series $\alpha_k = \sum_1^\infty \alpha_k(m)\xi_k(m)$ with $\alpha_k(m) \in \{-1, 0, 1\}$ and $|\xi_k(m)| \leqslant 2^{-m}$. We then obtain (3.3) with

$$\chi_p = 2^{\frac{1}{p}-1}\left(\sum_1^\infty 2^{-mp}\right)^{1/p} = 2^{\frac{1}{p}-1}\left(2^p - 1\right)^{-1/p}. \qquad \square$$

Proof of Theorem 3.1. Let $S = \sum \varepsilon_{ij}\lambda_{ij}e_{ij}$ and let $x_{ij} = e_{ij}\lambda_{ij}$. We already know by [16,18] the case $1 \leqslant p < 2$. We will show that the condition (γ', γ'') holds, and hence that Theorem 3.1 follows from Theorem 1.6.

Again we assume that $M = M_n$. Let $\mathcal{M} \subset M$ be the subalgebra of diagonal matrices with associated conditional expectation denoted by E. Let $x = (x_{ij})$. Then by Lemma 2.3 for any $0 < q < 2$ we have

$$[\lambda]_q = |||x|||_{q,\mathcal{M}}.$$

So the inequality in Theorem 3.1 boils down to $|||x|||_{p,\mathcal{M}} \leqslant \beta_p \| \sum \varepsilon_{ij} x_{ij} \|_p$. We need to observe that when we run the proof of Theorem 1.6 with $|||x|||_{p,\mathcal{M}}$ in place of $|||x|||_p$ we only need to know (K_q) for a family (y_k) such that each y_k lies in the closure in $L_q(\tau)$ of elements in $\mathcal{M} x_k \mathcal{M}$. When $\mathcal{M}$ is the algebra of diagonal operators, that means that y_k is obtained from x_k by a Schur multiplier, so that in any case when (x_k) is the family (x_{ij}) given as above by $x_{ij} = e_{ij}\lambda_{ij}$, then all the families (y_{ij}) are also of the same form i.e. we have $y_{ij} = e_{ij}\mu_{ij}$ for some scalars μ_{ij}, and for the latter we know by [16,18] that the $\mathcal{M}$-version of (K_q) holds for $1 \leqslant q \leqslant 2$.

So we will be able to conclude if we can verify the condition (γ', γ''). We claim that for some constant C

$$\left\| f^{\frac{1}{p}-\frac{1}{2}} T \right\|_p \leqslant C \left\| f^{\frac{1}{p}-\frac{1}{2}} T + T f^{\frac{1}{p}-\frac{1}{2}} \right\|_p \tag{3.4}$$

where f is any positive diagonal matrix and $T = \sum \varepsilon_{ij} y_{ij}$, with y_{ij} of the form $y_{ij} = e_{ij}\mu_{ij}$ as above. Indeed, we have

$$f_i^{\frac{1}{p}-\frac{1}{2}} \leqslant f_i^{\frac{1}{p}-\frac{1}{2}} + f_j^{\frac{1}{p}-\frac{1}{2}}$$

and hence, by Proposition 3.6, (3.4) holds with $C = \chi_p$. Thus, we have condition (γ', γ'') with $\gamma'' = \chi_p$ and by Remark 3.5 we can arrange to have, say, $\gamma' = 4$. Thus, modulo the above observation, we may view Theorem 3.1 as a corollary to Theorem 2.1. □

Remark 3.7. Assume $\lambda_{ij} \in L_p(M, \tau)$ (or simply $\lambda_{ij} \in S_p$) and let $x_{ij} = e_{ij} \otimes \lambda_{ij} \in L_p(B(\ell_2) \otimes M)$. Then, at the time of this writing, we do not know whether Theorem 3.1 remains valid for the series $\sum \varepsilon_{ij} e_{ij} \otimes \lambda_{ij}$, with $[\lambda]_p$ replaced by

$$[[\lambda]]_p = \inf\left\{ \left(\sum_i \left\| \left(\sum_j a_{ij}^* a_{ij} \right)^{1/2} \right\|_p^p \right)^{1/p} + \left(\sum_j \left\| \left(\sum_i b_{ij} b_{ij}^* \right)^{1/2} \right\|_p^p \right)^{1/p} \right\}$$

where the infimum runs over all decomposition, $\lambda_{ij} = a_{ij} + b_{ij}$ in $L_p(\tau)$. By (Kh_p) this clearly holds when $p \geqslant 1$.

4. Remarks on $\sigma(q)$-sets and $\sigma(q)_{cb}$-sets

In [9] (see also [10]) the following notion is introduced:

Definition 4.1. A subset $E \subset \mathbb{N} \times \mathbb{N}$ is called a $\sigma(q)$-set $(0 < q \leqslant \infty)$ if the system $\{e_{ij} \mid (i, j) \in E\}$ is an unconditional basis of its closed linear span in S_q.

Equivalently, there is a constant C such that for any finitely supported family of scalars $\{\lambda_{ij} \mid (i, j) \in E\}$ and any bounded family of scalars (α_{ij}) with $\sup |\alpha_{ij}| \leqslant 1$ we have

$$\left\| \sum_{(i,j)\in E} \alpha_{ij} \lambda_{ij} e_{ij} \right\|_{S_q} \leqslant C \left\| \sum_{(i,j)\in E} \lambda_{ij} e_{ij} \right\|_{S_q}.$$

The smallest such constant C is denoted by $\sigma_q(E)$.

The "operator space" version of this notion is as follows: E is called a $\sigma(q)_{cb}$-set if there is a C such that for any finitely supported family $\{\lambda_{ij} \mid (i,j) \in E\}$ in S_q and any (α_{ij}) as before we have

$$\Big\| \sum_{(i,j)\in E} \alpha_{ij} e_{ij} \otimes \lambda_{ij} \Big\|_{S_q(\ell_2 \otimes \ell_2)} \leqslant C \Big\| \sum_{(i,j)\in E} e_{ij} \otimes \lambda_{ij} \Big\|_{S_q(\ell_2 \otimes \ell_2)}.$$

We then denote by $\sigma_q^{cb}(E)$ the smallest such constant C.

It is not known whether $\sigma(q)$-sets are automatically $\sigma(q)_{cb}$-sets when $q \neq 2$. (The case $q = 2$ is trivial: every subset E is $\sigma(2)_{cb}$.) By the non-commutative Khintchine inequalities [16,18], if $1 \leqslant q < 2$, $E \subset \mathbb{N} \times \mathbb{N}$ is a $\sigma(q)$-set (resp. $\sigma(q)_{cb}$-set) iff there is a constant C' such that for all families $\{\lambda_{ij} \mid (i,j) \in E\}$ with λ_{ij} scalar (resp. $\lambda_{ij} \in S_q$) we have

$$[\lambda]_q \leqslant C' \Big\| \sum_{(i,j)\in E} \lambda_{ij} e_{ij} \Big\|_{S_q}$$

$$\bigg(\text{resp.} \quad [[\lambda]]_q \leqslant C' \Big\| \sum_{(i,j)\in E} e_{ij} \otimes \lambda_{ij} \Big\|_{S_q(\ell_2 \otimes \ell_2)}\bigg).$$

The proof of Theorem 1.1, modified as in Theorem 2.1, yields the following complement to [9]:

Theorem 4.2. *Assume $1 \leqslant p < q < 2$. Any $\sigma(q)$-set (resp. $\sigma(q)_{cb}$-set) $E \subset \mathbb{N} \times \mathbb{N}$ is a $\sigma(p)$-set (resp. $\sigma(p)_{cb}$-set).*

Corollary 4.3. *There is a constant $c \geqslant 1$ such that, for any n, the usual "basis" $\{e_{ij}\}$ of S_1^n contains a c-unconditional subset of size $\geqslant n^{3/2}$.*

Proof. By [9, Theorem 4.8] there is a constant $c \geqslant 1$ such that, for any n, the set $[n] \times [n]$ contains a further ("Hankelian") subset that is a $\sigma(4)_{cb}$-set (and hence by duality also $\sigma(4/3)_{cb}$) with constant $\leqslant c$ and cardinal $\geqslant n^{3/2}$. $\square$

Problem. What is the "right" order of growth in the preceding statement? Can $3/2$ be replaced by any number < 2?

Remark 4.4. As observed in [9], if $2 < p < q$, it is easy to show by interpolation that any $\sigma(q)$-set (resp. $\sigma(q)_{cb}$-set) is a $\sigma(p)$-set (resp. $\sigma(p)_{cb}$-set). Moreover, any such set is a $\sigma(q')$-set (resp. $\sigma(q')_{cb}$-set) where $q'^{-1} = 1 - q^{-1}$. However, the fact that e.g. $\sigma(q') \Rightarrow q(1)$ is new as far as we know.

5. Grothendieck–Maurey factorization for Schur multipliers ($0 < p < 1$)

Consider a bounded linear map $u : H \to L_p(\tau)$ on a Hilbert space H with $0 < p \leqslant 2$. To avoid technicalities, we assume that the range of u lies in a finite dimensional von Neumann subalgebra of M on which τ is finite. When $p \geqslant 1$, it is known that there is f in $L_1(\tau)_+$ with $\tau(f) = 1$ and a bounded linear map $\tilde{u} : H \to L_2(\tau)$ such that

$$\forall x \in H \quad u(x) = f^{\frac{1}{p}-\frac{1}{2}}\tilde{u}(x) + \tilde{u}(x) f^{\frac{1}{p}-\frac{1}{2}}$$

and $\|\tilde{u}\| \leqslant K_p \|u\|$ where K_p is a constant independent of u.

In the case $p = 1$, this fact is easy to deduce from the dual form proved in [25] for maps from M to H; the latter is often designated as the non-commutative "little GT" (here GT stands for Grothendieck's theorem). It is easy to deduce this statement from (Kh_p) (see [18] for more details) in the case $1 \leqslant p < 2$ (note that $p = 2$ is trivial). See [17] for a proof that the best constant K_p remains bounded when p runs over $[1, 2]$. We refer the reader to [17,19,13] for various generalizations.

It seems natural to conjecture that the preceding factorization of u remains valid for any p with $0 < p < 1$. Unfortunately, we leave this open. Nevertheless, in analogy with Section 3, we are able to prove the preceding factorization in the special case of Schur multipliers as follows.

Theorem 5.1. *Let* $0 < p < 1$. *Let* r *be such that* $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. *Consider a Schur multiplier*

$$u_\varphi : [x_{ij}] \to [x_{ij}\varphi_{ij}]$$

where $\varphi_{ij} \in \mathbb{C}$. *The following are equivalent*:

(i) u_φ *is bounded from* S_2 *to* S_p.
(ii) φ *admits a decomposition as* $\varphi = \psi + \chi$ *with* $\sum_i \sup_j |\psi_{ij}|^r < \infty$ *and* $\sum_j \sup_i |\chi_{ij}|^r < \infty$.
(iii) *There is a sequence* $f_i \geqslant 0$ *with* $\sum f_i < \infty$ *such that* $|\varphi_{ij}| \leqslant f_i^{\frac{1}{r}} + f_j^{\frac{1}{r}}$.

Proof. (Sketch) (ii) $\Leftrightarrow$ (iii) is elementary, and (ii) $\Rightarrow$ (i) is easy. The main point is (i) $\Rightarrow$ (ii). To prove this, the scheme is the same as in Section 3. We again use extrapolation starting from the knowledge that Theorem 5.1 holds when $p = q$ for some q with $1 \leqslant q < 2$. Let us fix p with $0 < p < 1$. For any q with $p \leqslant q \leqslant 2$, we denote

$$C'_q(\varphi) = \inf\bigl\{\|u_y : S_2 \to S_q\|\bigr\}$$

where the infimum runs over all $y = (y_{ij})$ for which there is $f_i \geqslant 0$ with $\sum f_i \leqslant 1$ such that $\varphi_{ij} = (f_i^{\frac{1}{p}-\frac{1}{q}} y_{ij} + y_{ij} f_j^{\frac{1}{p}-\frac{1}{q}})/2$. We also denote

$$]\varphi[_p = \inf\bigl\{\|\psi\|_{\ell_r(\ell_\infty)} + \|{}^t\chi\|_{\ell_r(\ell_\infty)}\bigr\}$$

where the infimum runs over all decompositions $\varphi = \psi + \chi$.

Note that $C'_p(\varphi) = \|u_\varphi : S_2 \to S_p\|$. Let $1 \leqslant q < 2$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$. We have then by the same arguments as in Section 1:

Step 1′. $]\varphi[_p \leqslant C' C'_q(\lambda)$.

Step 2′. $C'_2(\varphi) \leqslant C'']\varphi[_p$.

Step 3′. $C'_q(\varphi) \leqslant C''' C'_p(\varphi)^{1-\theta} C'_2(\varphi)^{\theta}$.

Note that obviously $\|u_y : S_2 \to S_2\| = \sup|y_{ij}|$ so that we have again equivalence in Step 2′. To verify Step 3′ we argue exactly as for Theorem 3.1. □

Corollary 5.2. *Let* $0 < p \leqslant 2 \leqslant q \leqslant \infty$. *Let* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. *With this value of* r, *the properties* (i) *and* (ii) *in the preceding theorem are equivalent to*:

(i)′ u_φ *is bounded from* S_q *to* S_p.

Proof. Assume (i)′. Since S_p has cotype 2 [35], u_φ factors through a Hilbert space by [25]. By an elementary averaging argument (see e.g. [29]), the factorization can be achieved using only Schur multipliers. Thus we must have $\varphi = \varphi_1\varphi_2$ with φ_1 (resp. φ_2) bounded from S_2 to S_p (resp. S_q to S_2). If we now apply Theorem 5.1 (resp. the results of [29,36]) to φ_1 (resp. φ_2), and use an arithmetic/geometric type inequality of the form $f^{\frac{1}{p}-\frac{1}{2}} g^{\frac{1}{2}-\frac{1}{q}} \leqslant c(f^{\frac{1}{p}-\frac{1}{q}} + g^{\frac{1}{p}-\frac{1}{q}})$ for all $f, g \geqslant 0$, we obtain (iii). The other implications are easy. □

Problem. Characterize the bounded Schur multipliers from S_q to S_p when $p < q < 2$ or when $2 < p < q \leqslant \infty$.

Some useful information on this problem can be derived from [13]. The difficulty is due to the fact that, except when $q = 1, 2, \infty$, we have no characterization of the bounded Schur multipliers on S_q.

Remark. By general results, actually Theorem 5.1 implies Theorem 3.1. Indeed, the same idea as in [18] can be used to see this. Moreover, as pointed out by Q. Xu, the converse implication is also easy: just observe that, by Theorem 3.1, any Schur multiplier bounded from S_2 to S_p must be a bounded "multiplier" from $\ell_2(\mathbb{N} \times \mathbb{N})$ to $\ell_p(\ell_2) + {}^t\ell_p(\ell_2)$. Then a well-known variant of Maurey's classical factorization yields (ii) or (iii) in Theorem 5.1.

Although the recent paper [13] established several important factorization theorems for maps between non-commutative L_p-spaces, there seems to be some extra difficulty to extend the Maurey factorization theorem when $0 < p < 1$. The next result points to the obstacle. To avoid technicalities we again restrict to the finite dimensional case, so we assume (M, τ) as before but with M finite dimensional. For any $\varepsilon > 0$, we denote

$$\mathcal{D}_\varepsilon = \{f \in \mathcal{D} \mid f \geqslant \varepsilon 1\}.$$

For any x in M, we let

$$T(x)y = xy + yx.$$

Note that if $x > 0$ then $T(x)$ is an isomorphism on M so that $T(x)^{-1}$ makes sense.

Let B be any Banach space. Given a linear map $u : B \to L_p(\tau)$, we denote by $M_p(u)$ the smallest constant C such that for any finite sequence (x_j) in B

$$\left\|\left|(ux_j)\right|\right\|_p \leqslant C\Big(\sum \|x_j\|^2\Big)^{1/2}.$$

We denote by $\mathcal{M}_p(u)$ the smallest constant C such that there is $\varepsilon > 0$ and a probability λ on $\mathcal{D}_\varepsilon$ such that

$$\forall x \in B \quad \int \left\| T\left(f^{\frac{1}{r}}\right)^{-1} ux \right\|_2^2 d\lambda(f) \leqslant C^2 \|x\|^2 \tag{5.1}$$

where (as before) $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$.

We have then

Proposition 5.3. *There is a constant $\beta > 0$ such that for any u as above we have*

$$\frac{1}{\beta}\mathcal{M}_p(u) \leqslant M_p(u) \leqslant \mathcal{M}_p(u).$$

Proof. The main point is to observe that if ε is chosen small enough (compared to $\dim(M)$) we have for any finite sequence $y = (y_j)$ in $L_p(\tau)$

$$\left\|\left|(y_j)\right|\right\|_p \leqslant \inf_{f \in \mathcal{D}_\varepsilon} \left(\sum \left\| T\left(f^{\frac{1}{r}}\right)^{-1} y_j \right\|_2^2 \right)^{1/2} \leqslant \beta \left\|\left|(y_j)\right|\right\|_p \tag{5.2}$$

where β is a fixed constant, independent of the dimension of M.

Then $M_p(u) \leqslant \mathcal{M}_p(u)$ follows immediately. To prove the converse, assume $M_p(u) \leqslant 1$. Then by (5.2) we have

$$\inf_{y \in \mathcal{D}_\varepsilon} \sum \left\| T\left(f^{\frac{1}{r}}\right)^{-1} u(x_j) \right\|_2^2 \leqslant \beta^2 \sum \|x_j\|^2.$$

By a well-known Hahn–Banach type argument (see e.g. [28, Exercise 2.2.1]), there is a net (λ_i) on $\mathcal{D}_\varepsilon$ such that

$$\forall x \in B \quad \lim_i \int \left\| T\left(f^{\frac{1}{r}}\right)^{-1} u(x) \right\|_2^2 d\lambda_i(f) \leqslant \beta^2 \|x\|^2.$$

We may as well assume that the net corresponds to an ultrafilter. Setting $\lambda = \lim \lambda_i$, we obtain (5.1) and hence $\mathcal{M}_p(u) \leqslant \beta$. $\square$

Remark 5.4. Now assume $1 \leqslant p < 2$. Note then that $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ satisfies $-1 \leqslant -\frac{2}{r} = 1 - \frac{2}{p} < 0$. Therefore the function $t \to t^{-\frac{2}{r}}$ is *operator convex* (see e.g. [4, p. 123]). Using this and assuming $\mathcal{M}_p(u) \leqslant 1$, we claim that there is, for some $\varepsilon > 0$, a density F in $\mathcal{D}_\varepsilon$ such that

$$\left\| T\left(F^{1/r}\right)^{-1} ux \right\|_2 \leqslant \beta' \|x\|, \quad \forall x \in H. \tag{5.3}$$

Indeed we first observe that

$$\left\| T\left(f^{\frac{1}{r}}\right)^{-1} y \right\|_2^2 \simeq \left\| T(f)^{-\frac{1}{r}} y \right\|_2^2 = \left\langle T(f)^{-\frac{2}{r}} y, y \right\rangle \tag{5.4}$$

where $\simeq$ means that the (squared) norms are equivalent with equivalence constants depending only on r, and hence if we set

$$F = \int f \, d\lambda(f)$$

we deduce from (5.1) that, for some constant c, we have

$$\langle T(F)^{-2/r} ux, ux \rangle \leqslant c\|x\|^2$$

and hence using (5.4) again we obtain (5.3).

Note that if B is Hilbertian any bounded linear u from B to $L_p(\tau)$ satisfies the factorization of the form (5.3) if $1 \leqslant p \leqslant 2$. This follows immediately by duality from either [17] or [19].

However, what happens for $0 < p < 1$ is unclear: Can we still get rid of λ as in the preceding remark?

6. A non-commutative Kahane inequality

In vector-valued probability theory, the following inequalities due to Kahane (see [14]) play an important role. For any $0 < p < q < \infty$, there is a constant $K(p,q)$ such that for any Banach space X and any finite sequence (x_k) of elements of X we have

$$\left\| \sum r_k x_k \right\|_{L_q(X)} \leqslant K(p,q) \left\| \sum r_k x_k \right\|_{L_p(X)} \tag{6.1}$$

where (r_k) denotes as before the Rademacher functions.

As observed by C. Borell (see [6]) Kahane's result can be deduced from the hypercontractive inequality for the semi-group $T(t)$ defined on $L_2([0,1])$ by $T(t)\prod_{k\in A} r_k = e^{-t|A|}\prod_{k\in A} r_k$ for any finite set $A \subset \mathbb{N}$. The hypercontractivity says that if $1 < p < q < \infty$ and if $e^{-2t} \leqslant (p-1)(q-1)^{-1}$ then $\|T(t): L_p \to L_q\| = 1$. Since $T(t) \geqslant 0$ for all $t \geqslant 0$, this implies that for any Banach space X, we also have

$$\left\| T(t): L_p(X) \to L_q(X) \right\| = 1.$$

In particular, if $S = \sum r_k x_k$ then $T(t)S = e^{-t}S$ and hence we find

$$\|S\|_{L_q(X)} \leqslant (q-1)^{1/2}(p-1)^{-1/2}\|S\|_{L_p(X)},$$

which yields (6.1) for $p > 1$ (and the case $0 < p \leqslant 1$ can be easily deduced from this using Hölder's inequality).

The goal of this section is to remark that this approach is valid mutatis mutandis in the "anti-symmetric" or Fermionic setting considered in [7]. Let (M, τ) be a von Neumann algebra equipped with a faithful normal trace τ such that $\tau(1) = 1$. Let $\{Q_k \mid k \geqslant 0\}$ be a spin system in M. By this we mean that Q_k are self-adjoint unitary operators such that

$$\forall k \neq \ell \quad Q_k Q_\ell = -Q_\ell Q_k.$$

For any finite set $A \subset \mathbb{N}$, ordered so that $A = \{k_1, \dots, k_m\}$ with $k_1 < k_2 < \cdots < k_m$, we set

$$Q_A = Q_{k_1} Q_{k_2} \dots Q_{k_m},$$

with the convention

$$Q_\phi = 1.$$

We will assume that M is generated by $\{Q_k\}$. In that case, M is the so-called hyperfinite factor of type II_1, i.e. the non-commutative analogue of the Lebesgue interval [0, 1].

Let $V(t) : L_2(\tau) \to L_2(\tau)$ be the semi-group defined for all $A \subset \mathbb{N}$ ($|A| < \infty$) by

$$V(t) Q_A = e^{-t|A|} Q_A.$$

Carlen and Lieb [7] observed that the semi-group $V(t)$ is completely positive (see [7, (4.2), p. 36]) and proved that if $e^{-2t} \leqslant (p-1)(q-1)^{-1}$

$$\big\| V(t) : L_p(\tau) \to L_q(\tau) \big\| = 1.$$

We take the occasion of this paper to point out that the Kahane inequality remains valid in this setting provided one works with the "vector-valued non-commutative L_p-spaces" $L_p(\tau; E)$ introduced in [27]. Here E is an operator space, i.e. $E \subset B(H)$ for some Hilbert space H, and $L_p(\tau; E)$ is defined as the completion if $L_p(\tau) \otimes E$ for the norm denoted by $\| \cdot \|_{L_p(\tau; E)}$ defined as follows.

For any f in the algebraic tensor product $L_p(\tau) \otimes E$

$$\|f\|_{L_p(\tau;E)} = \inf\big\{ \|a\|_{L_{2p}(\tau)} \|b\|_{L_{2p}(\tau)} \big\} \tag{6.2}$$

where the infimum runs over all possible factorizations of f of the form

$$f = a \cdot g \cdot b \tag{6.3}$$

with $g \in M \otimes E$ such that

$$\|g\|_{M \otimes_{\min} E} \leqslant 1.$$

In (6.3), the map $(a, g, b) \to a \cdot g \cdot b$ is obtained by linear extension from

$$\big(a, (m \otimes e), b\big) \to amb \otimes e.$$

Our observation boils down to the following.

Lemma 6.1. *If $T : L_p(\tau) \to L_q(\tau)$ ($1 \leqslant p, q \leqslant \infty$) is completely positive and bounded, then for any operator space $E \neq \{0\}$ the operator $T \otimes id_E$ extends to a bounded operator from $L_p(\tau; E)$ to $L_q(\tau; E)$ such that*

$$\big\| T \otimes id_E : L_p(\tau; E) \to L_q(\tau; E) \big\| = \big\| T : L_p(\tau) \to L_q(\tau) \big\|.$$

Proof. By density, it suffices to prove this when M is generated by a finite subset $\{Q_0, Q_1, \dots, Q_n\}$, say with even cardinality (i.e. n odd). Then $M = M_{2^k}$ with $k = (n+1)/2$ and it is well known (see e.g. [24]) that any c.p. map $T : M \to M$ is of the form

$$T(x) = \sum a_j^* x a_j \tag{6.4}$$

for some finite set a_j in M.

Now assume $f \in L_p(\tau) \otimes E$ with $\|f\|_{L_p(\tau;E)} < 1$. We can write $f = a^* \cdot g \cdot b$ with $\|a\|_{2p}, \|b\|_{2p} < 1$ and $\|g\|_{M \otimes_{\min} E} < 1$. Assume $\|T : L_p(\tau) \to L_q(\tau)\| = 1$. Let $\alpha = (\sum a_j^* a^* a a_j)^{1/2}$ and $\beta = (\sum a_j^* b^* b a_j)^{1/2}$. Since $\alpha^2 = T(a^*a)$ and $\beta^2 = T(b^*b)$ we have $\|\alpha\|_{2q} < 1$ and $\|\beta\|_{2q} < 1$. Fix $\varepsilon > 0$. We have

$$aa_j = \alpha_j \left(\varepsilon 1 + \alpha^2\right)^{1/2},$$
$$ba_j = \beta_j \left(\varepsilon 1 + \beta^2\right)^{1/2}$$

where $\alpha_j = aa_j(\varepsilon 1 + \alpha^2)^{-1/2}$ and $\beta_j = ba_j(\varepsilon 1 + \beta^2)^{-1/2}$ satisfy

$$\sum \alpha_j^* \alpha_j = \left(\varepsilon 1 + \alpha^2\right)^{-1/2} \alpha^2 \left(\varepsilon 1 + \alpha^2\right)^{1/2} \leqslant 1$$

and similarly $\sum \beta_j^* \beta_j \leqslant 1$. This implies clearly (by the defining property of an operator space!)

$$\left\| \sum \alpha_j^* \cdot g \cdot \beta_j \right\|_{M \otimes_{\min} E} < 1.$$

We have

$$(T \otimes id_E)(f) = \left(\varepsilon 1 + \alpha^2\right)^{1/2} \hat{g} \left(\varepsilon 1 + \beta^2\right)^{1/2}$$

where $\hat{g} = \sum \alpha_j^* \cdot g \cdot \beta_j$, and hence we conclude by (6.2)

$$\begin{aligned} \left\|(T \otimes id_E)(f)\right\|_{L_q(\tau;E)} &\leqslant \left\|\left(\varepsilon 1 + \alpha^2\right)^{1/2}\right\|_{2q} \left\|\left(\varepsilon 1 + \beta^2\right)^{1/2}\right\|_{2q} \\ &\leqslant \left(\varepsilon + \left\|\alpha^2\right\|_q\right)^{1/2} \left(\varepsilon + \left\|\beta^2\right\|_q\right)^{1/2} \leqslant 1 + \varepsilon, \end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, we obtain the announced result by homogeneity. □

Remark 6.2. Q. Xu pointed out to me that Lemma 6.1 remains valid in the non-hyperfinite case.One can check this using the following fact: consider y in $L_p(\tau) \otimes M_n$, then $y \in B_{L_p(\tau;M_n)}$ iff there are λ, μ in $B_{L_p(\tau)}$ such that $\left(\begin{smallmatrix} \lambda & y \\ y^* & \mu \end{smallmatrix}\right) \geqslant 0$ where $\geqslant 0$ is meant in $L_p(\tau \times \mathrm{tr})$ (see e.g. [28, Exercise 11.5] for the result at the root of this fact). A similar statement is valid with $B(H)$ in place of M_n.

Theorem 6.3. *Let $1 < p < q < \infty$. Assume $e^{-2t} \leqslant (p-1)(q-1)^{-1}$, then for any operator space E*

$$\left\| V(t) : L_p(\tau; E) \to L_q(\tau; E) \right\| \leqslant 1.$$

Consequently, for any $1 \leqslant p < q < \infty$ there is a constant $K'(p,q)$ such that for any E and any finite sequence x_k in E we have

$$\left\| \sum Q_k \otimes x_k \right\|_{L_q(\tau;E)} \leqslant K'(p,q) \left\| \sum Q_k \otimes x_k \right\|_{L_p(\tau;E)}.$$

Proof. The first part follows from the preceding lemma by [7]. Let $f = \sum Q_k \otimes x_k$. In particular, if $1 < p < q < \infty$ we have

$$\|f\|_{L_q(\tau;E)} \leqslant (q-1)^{1/2}(p-1)^{-1/2}\|f\|_{L_p(\tau;E)}. \tag{6.5}$$

Let $0 < \theta < 1$ be defined by

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q}.$$

By [27, p. 40] we have isometrically

$$L_p(\tau;E) = \big(L_1(\tau;E), L_q(\tau;E)\big)_\theta$$

and hence

$$\|f\|_{L_p(\tau;E)} \leqslant \|f\|_{L_1(\tau;E)}^{1-\theta} \|f\|_{L_q(\tau;E)}^{\theta},$$

which when combined with (6.5) yields

$$\|f\|_{L_q(\tau;E)} \leqslant \big((q-1)^{1/2}(p-1)^{-1/2}\big)^{\frac{1}{1-\theta}} \|f\|_{L_1(\tau;E)}. \qquad \square$$

Remark. Obviously Theorem 6.3 is also valid for other hypercontractive semi-groups, as the ones in [5].

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Appendix A

The main technical difficulty in our proof of Step 3 above is (1.5). We will first show how this follows from Theorem 1.1 in [13]. We will then also outline a direct more self-contained argument.

Let (M,τ) be a generalized (possibly non-commutative) measure space, with associated space $L_p(\tau)$. Since it is easy to pass from the finite to the semifinite case, we assume τ finite. Consider a density $f > 0$ in M with $\tau(f) = 1$, with finite spectrum, i.e. we assume that $f = \sum_1^N f_j Q_j$ where $0 < f_1 \leqslant f_2 \leqslant \cdots \leqslant f_N$, $1 = \sum_1^N Q_j$ and Q_j are mutually orthogonal projections in M. We now introduce for any x in $L_p(\tau)$ $(1 \leqslant p \leqslant \infty)$

$$\|x\|_{L_p(f)} = \big\|f^{\frac{1}{p}}x\big\|_{L_p(\tau)} + \big\|xf^{\frac{1}{p}}\big\|_{L_p(\tau)}. \tag{A.1}$$

We will denote by $L_p(f)$ the space $L_p(\tau)$ equipped with the norm $\|\cdot\|_{L_p(f)}$. Then [13, Theorem 1.1] implies in particular that for any $0<\theta<1$ and any $1<p<\infty$ we have

$$L_{p(\theta)}(f) \simeq \big(M, L_p(f)\big)_\theta \tag{A.2}$$

where $p(\theta)^{-1} = \frac{1-\theta}{\infty} + \frac{\theta}{p} = \theta/p$, and where $\simeq$ means that the norms on both sides are equivalent with equivalence constants depending only on p and θ. Note that by the triangle inequality and by Lemma 1.9(ii), we have

$$\big\|f^{\frac{1}{p}}x + xf^{\frac{1}{p}}\big\|_{L_p(\tau)} \leqslant \|x\|_{L_p(f)} \leqslant 2t(p)\big\|f^{\frac{1}{p}}x + xf^{\frac{1}{p}}\big\|_{L_p(\tau)}. \tag{A.3}$$

Let us denote

$$T(f)x = fx + xf.$$

With this notation, the dual norms

$$\|x\|_{L_p(f)^*} = \sup\big\{\big|\tau(xy)\big| \bigm| \|y\|_{L_p(f)} \leqslant 1\big\}$$

satisfy for any x in $L_{p'}(\tau)$ the following dual version to (A.3)

$$\big(2t(p)\big)^{-1}\big\|T\big(f^{\frac{1}{p}}\big)^{-1}x\big\|_{L_{p'}(\tau)} \leqslant \|x\|_{L_p(f)^*} \leqslant \big\|T\big(f^{\frac{1}{p}}\big)^{-1}x\big\|_{L_{p'}(\tau)}. \tag{A.4}$$

Note that with our simplifying assumptions on f, $T(f)$ is an isomorphism on $L_p(\tau)$.

Here and in the sequel we will denote by $c_1, c_2, \ldots$ constants depending only on p and θ.

Recall (see e.g. [3]) that we have isometrically for any $0<\theta<1$

$$\big(M, L_p(f)\big)^*_\theta = \big(L_1(\tau), L_p(f)^*\big)_\theta.$$

Therefore (A.2) implies in particular that for any x in $L_{p'}(\tau)$

$$\|x\|_{L_{p(\theta)}(f)^*} \leqslant c_1\|x\|^{1-\theta}_{L_1(\tau)}\|x\|^{\theta}_{L_p(f)^*}. \tag{A.5}$$

Using (A.4), (A.5) implies

$$\big\|T\big(f^{\frac{\theta}{p}}\big)^{-1}(x)\big\|_{L_{p(\theta)'}(\tau)} \leqslant c_2\|x\|^{1-\theta}_{L_1(\tau)}\big\|T\big(f^{\frac{1}{p}}\big)^{-1}x\big\|^{\theta}_{L_{p'}(\tau)}. \tag{A.6}$$

In Step 3 of the present paper, we used the special case $p=2$. If we denote $q = p(\theta)'$ we have $\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{2}$ so that (A.6) becomes

$$\big\|T\big(f^{\frac{\theta}{2}}\big)^{-1}(x)\big\|_q \leqslant c_2\|x\|_1^{1-\theta}\big\|T\big(f^{\frac{1}{2}}\big)^{-1}x\big\|_2^{\theta}, \tag{A.7}$$

and we obtain (1.5) for $p = 1$. The case $1 < p < 2$ can be derived by the same argument, but this is anyway much easier because of the simultaneous boundedness on L_p and L_2 of the triangular projection.

For the convenience of the reader, we now give a direct argument, based on the same ideas as [13]. We want to show (A.7). Note that it is equivalent to (change x to $T(f^{\frac{1}{2}}y)$): for all y in M

$$\big\|T\big(f^{\frac{\theta}{2}}\big)^{-1}T\big(f^{\frac{1}{2}}\big)y\big\|_q \leqslant c_4\big\|T\big(f^{\frac{1}{2}}\big)y\big\|_1^{1-\theta}\|y\|_2^{\theta}. \tag{A.8}$$

By the triangle inequality and by Lemma 1.9(ii) we have

$$\begin{aligned}\big\|T\big(f^{\frac{\theta}{2}}\big)^{-1}T\big(f^{\frac{1}{2}}\big)y\big\|_q &\leqslant \big\|T\big(f^{\frac{\theta}{2}}\big)^{-1}f^{\frac{\theta}{2}}f^{\frac{1-\theta}{2}}y\big\|_q + \big\|T\big(f^{\frac{\theta}{2}}\big)^{-1}yf^{\frac{1-\theta}{2}}f^{\frac{\theta}{2}}\big\|_q \\ &\leqslant t(q)\big(\big\|f^{\frac{1-\theta}{2}}y\big\|_q + \big\|yf^{\frac{1-\theta}{2}}\big\|_q\big).\end{aligned}$$

Therefore to show (A.7) (or (A.8)) it suffices to show

$$\big\|f^{\frac{1-\theta}{2}}y\big\|_q + \big\|yf^{\frac{1-\theta}{2}}\big\|_q \leqslant c_6\big\|f^{\frac{1}{2}}y + yf^{\frac{1}{2}}\big\|_1^{1-\theta}\|y\|_2^{\theta}. \tag{A.9}$$

Recall that $f = \sum_1^N f_j Q_j$. We denote

$$y^+ = \sum_{i\leqslant j} Q_i y Q_j, \qquad y^- = \sum_{i>j} Q_i y Q_j.$$

Note that y^+ (resp. y^-) is the upper (resp. lower) triangular part of y (with respect to the decomposition $I = \sum Q_j$). We recall that, whenever $1 < q < \infty$, $y \mapsto y^+$ and $y \mapsto y^-$ are bounded linear maps on $L_q(\tau)$ with bounds independent of N, but this fails in case $q = 1$ or $q = \infty$ (see [20] and [30, §8] for references on this).

By the triangle inequality, since $y = y^+ + y^-$, to prove (A.9) it suffices to show $\forall y \in M$

$$\max\big\{\big\|f^{\frac{1-\theta}{2}}y^+\big\|_q, \big\|y^+f^{\frac{1-\theta}{2}}\big\|_q\big\} \leqslant c_7\big\|f^{\frac{1}{2}}y + yf^{\frac{1}{2}}\big\|_1^{1-\theta}\|y\|_2^{\theta} \tag{A.10}$$

and similarly with y^- in place of y^+. Let $L_p^-(\tau) = \{x \in L_p(\tau) \mid x^+ = 0\}$. Let $\Lambda_p = L_p(\tau)/L_p^-(\tau)$. Note that $x^+ + L_p^-(\tau) = x + L_p^-(\tau)$. We will denote abusively by $\|x^+\|_{\Lambda_p}$ the norm in Λ_p of the equivalence class of x^+ modulo $L_p^-(\tau)$. Note that $\|x^+\|_{\Lambda_1} \leqslant \|x\|_1$ for all x in $L_1(\tau)$ and hence $\|f^{\frac{1}{2}}y^+ + y^+f^{\frac{1}{2}}\|_{\Lambda_1} \leqslant \|f^{\frac{1}{2}}y + yf^{\frac{1}{2}}\|_1$ for all y in $L_2(\tau)$. Moreover we have $\|y^+\|_{\Lambda_2} = \|y^+\|_2$. Therefore to show (A.10) it suffices to show

$$\big\|f^{\frac{1-\theta}{2}}y^+\big\|_q \leqslant c_7\big\|f^{\frac{1}{2}}y^+ + y^+f^{\frac{1}{2}}\big\|_{\Lambda_1}^{1-\theta}\big\|y^+\big\|_{\Lambda_2}^{\theta} \tag{A.11}$$

and similarly for $y^+f^{\frac{1-\theta}{2}}$.

We now observe that, by Lemma 1.9(i), the maps

$$T_1 : x \mapsto \sum \frac{\lambda_i \wedge \lambda_j}{\lambda_i + \lambda_j} Q_i x Q_j \quad \text{and} \quad T_2 : x \mapsto \sum \frac{\lambda_i \vee \lambda_j}{\lambda_i + \lambda_j} Q_i x x_j$$

have norm $\leqslant 3/2$ on $L_q(\tau)$ for all $1 \leqslant q \leqslant \infty$, in particular on $L_1(\tau)$. Since these maps preserve $L_1^-(\tau)$, the "same" maps are contractive on Λ_1. Applying this with $\lambda_i = f_i^{\frac{1}{2}}$ and assuming as before that $f_1 \leqslant \cdots \leqslant f_N$, we have $f_i \wedge f_j = f_i$ and $f_i \vee f_j = f_j$ for all $i \leqslant j$ and hence $T_1(f^{\frac{1}{2}} y^+ + y^+ f^{\frac{1}{2}}) = f^{1/2} y^+$ and $T_2(f^{\frac{1}{2}} y^+ + y^+ f^{\frac{1}{2}}) = y^+ f^{\frac{1}{2}}$. This gives us

$$\max\bigl\{\bigl\|f^{\frac{1}{2}} y^+\bigr\|_{\Lambda_1}, \bigl\|y^+ f^{\frac{1}{2}}\bigr\|_{\Lambda_1}\bigr\} \leqslant (3/2)\bigl\|f^{\frac{1}{2}} y^+ + y^+ f^{\frac{1}{2}}\bigr\|_{\Lambda_1}.$$

Thus to show (A.11) it suffices to show

$$\bigl\|f^{\frac{1-\theta}{2}} y^+\bigr\|_q \leqslant c_7 \bigl\|f^{\frac{1}{2}} y^+\bigr\|_{\Lambda_1}^{1-\theta} \bigl\|y^+\bigr\|_{\Lambda_2}^{\theta},$$

and similarly for $y^+ f^{\frac{1-\theta}{2}}$. Now by [26, Theorem 4.5] and by duality we have $(\Lambda_1, \Lambda_2)_\theta \simeq \Lambda_q$ with equivalent norms (and equivalence constants independent of N). Using the analytic function $z \mapsto f^{\frac{z}{2}}$ and a by now routine application of the 3 line lemma (this is essentially the "Stein interpolation principle") this gives us (recall $\|y^+\|_{\Lambda_2} = \|y^+\|_2$)

$$\bigl\|f^{\frac{1-\theta}{2}} y^+\bigr\|_{\Lambda_q} \leqslant c_8 \bigl\|f^{\frac{1}{2}} y^+\bigr\|_{\Lambda_1}^{1-\theta} \bigl\|y^+\bigr\|_2^{\theta}.$$

But now since the "triangular projection" $y \mapsto y^+$ is bounded on $L_q(\tau)$ when $1 < q < \infty$ (and since $(f^{\frac{1-\theta}{2}} y)^+ = f^{\frac{1-\theta}{2}} y^+$) we obtain finally

$$\bigl\|f^{\frac{1-\theta}{2}} y^+\bigr\|_q \leqslant c_9 \bigl\|f^{\frac{1}{2}} y^+\bigr\|_{\Lambda_1}^{1-\theta} \bigl\|y^+\bigr\|_2^{\theta}.$$

By the preceding successive reductions, this completes the proof of (A.7) and hence also of (1.5) for $p = 1$.

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