Bistable wavefronts in a diffusive and competitive Lotka–Volterra type system with nonlocal delays

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Abstract

A diffusive Lotka–Volterra type model with nonlocal delays for two competitive species is considered. The existence of a traveling wavefront analogous to a bistable wavefront for a single species is proved by transforming the system with nonlocal delays to a four-dimensional system without delay. Furthermore, in order to prove the asymptotic stability (up to translation) of bistable wavefronts of the system, the existence, regularity and comparison theorem of solutions of the corresponding Cauchy problem are first established for the systems on $\mathbb{R}$ by appealing to the theory of abstract functional differential equations. The asymptotic stability (up to translation) of bistable wavefronts are then proved by spectral methods. In particular, we also prove that the spreading speed is unique by upper and lower solutions technique. From the point of view of ecology, our results indicate that the nonlocal delays appeared in the interaction terms are not sensitive to the invasion of species of spatial isolation.

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1. Introduction

In population dynamics, time delay (which arises from a variety of causes, such as the period of hatching, duration of gestation, see Hale and Verduyn Lunel [21] and Kuang [25]) and spatial...
dispersal or migration from one location to another (see, e.g., Levin [26,27], Silvertown et al. [48]) seem to be inevitable, it refers to the surveys by Wu [59]. Therefore, partial functional differential equations attracted much attention due to its significant sense in mathematical theory and practical fields (see, e.g., Martin and Smith [34,35], Ruan and Wu [45], Travis and Webb [52], Wu [59]). A prototype of such equations take the form

$$\frac{\partial u(x,t)}{\partial t} = D \Delta u(x,t) + f(u_t(x)), \quad (1.1)$$

where \( x \in \Omega \subset \mathbb{R}, t > 0, D \) is a diagonal matrix with positive elements, \( u \in \mathbb{R}^n \), \( u_t(x) \) is an element in \( C([-\tau, 0], \mathbb{R}^n) \) parameterized by \( x \in \Omega \) and is defined by \( u(x, t + s) \) for \( s \in [-\tau, 0] \), \( 0 \leq \tau < \infty \) denotes the maximal delay [59], \( f : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n \) is a continuous function. In fact, (1.1) is not sufficiently precise in population dynamics [59], since the individuals may be at the different location in their history. Motivated by this, Britton [5,6] considered comprehensively the two factors and introduced the so-called spatio-temporal delay or nonlocal delay, that is, the delay term involves a weighted spatio-temporal average over the whole of the infinite spatial domain and the whole of the previous times. He derived the following reaction diffusion equation with nonlocal delay on the domain \( \Omega \subset \mathbb{R}^n \)

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t) \left[ 1 + \alpha u - (1 + \alpha) \int_{-\infty}^{t} \int_{\Omega} g(x - y, t - s) u(y,s) dy ds \right], \quad (1.2)$$

in which the process of natural selection and random spatial migration are clear. In (1.2), \( g \) can be regarded as a weighted function concerning with the random walk and the historical effect of the individuals. Since then, many investigators considered the following reaction diffusion equation (system) with nonlocal delay

$$\frac{\partial u(x,t)}{\partial t} = D \Delta u(x,t) + f(u(x,t), (g \ast u)(x,t)), \quad (1.3)$$

where \((g \ast u)(x,t)\) denotes a convolution with respect to both of spatial and temporal variables. The related results can refer to the surveys of Gourley et al. [19,20] and Ruan [44].

In the study of (1.3), traveling wave solutions, which has the form \( u(x,t) = \Phi(x + ct) \) for some \( c \in \mathbb{R} \) accounting for the wave speed of propagation and \( \Phi \) interpreted as the wave profile, attracted much attention due to its significant sense in mathematical theory and other fields, see, e.g., Ai [1], Al-Omari and Gourley [2,3], Ashwin et al. [4], Faria et al. [12], Gourley and Kuang [17], Li and Wang [29], Li and Wu [30], Wang and Li [56], Wang et al. [57,58] and the references cited therein. Recently, Gourley and Ruan [18] studied the traveling wave solutions of the following competition diffusion system with nonlocal delays

$$\begin{cases} 
\frac{\partial u_1(x,t)}{\partial t} = d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \left[ 1 - a_1 u_1(x,t) - b_1 (g_1 \ast u_2)(x,t) \right], \\
\frac{\partial u_2(x,t)}{\partial t} = d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \left[ 1 - a_2 u_2(x,t) - b_2 (g_2 \ast u_1)(x,t) \right], 
\end{cases} \quad (1.4)$$
where \((g_1 \ast u_2)(x, t)\) and \((g_2 \ast u_1)(x, t)\) are defined by

\[
\begin{align*}
(g_1 \ast u_2)(x, t) &= \int_{-\infty}^{t} \int_{-\infty}^{\infty} G_1(x - y, t - s)k_1(t - s)u_2(y, s) \, dy \, ds, \\
(g_2 \ast u_1)(x, t) &= \int_{-\infty}^{t} \int_{-\infty}^{\infty} G_2(x - y, t - s)k_2(t - s)u_1(y, s) \, dy \, ds,
\end{align*}
\]

in which the kernel function \(k_i\) satisfies

\[
k_i(s) = \frac{1}{\tau_i} e^{-\frac{1}{\tau_i} s}, \quad i = 1, 2,
\]

and for \(i = 1, 2\), \(G_i\) is a weighting function describing the distribution at past times of the individuals of the species \(u_{3-i}\) who are at position \(x\) at time \(t\). The \(u_i\) individuals diffuse at diffusivity \(d_i\); thus \(G_i\) must satisfy

\[
\frac{\partial G_i}{\partial t} = d_{3-i} \frac{\partial^2 G_i}{\partial x^2}, \quad G_i(x, 0) = \delta(x),
\]

where \(\delta(x)\) is the general Dirac function, so that \(G_1, G_2\) are both fundamental solutions of heat equations, i.e.,

\[
G_i(x, t) = \frac{1}{\sqrt{4d_{3-i}\pi t}} e^{-\frac{x^2}{4d_{3-i}t}}, \quad i = 1, 2.
\]

Thus, if we let \(\theta = t - s\) and \(z = x - y\), then it is easy to see that

\[
\begin{align*}
(g_1 \ast u_2)(x, t) &= \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-\frac{1}{\tau_1} \theta} \frac{1}{\sqrt{4d_1\pi \theta}} e^{-\frac{\theta^2}{4d_1\pi \theta}} u_2(x - z, t - \theta) \, d\theta \, dz, \\
(g_2 \ast u_1)(x, t) &= \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-\frac{1}{\tau_2} \theta} \frac{1}{\sqrt{4d_2\pi \theta}} e^{-\frac{\theta^2}{4d_2\pi \theta}} u_1(x - z, t - \theta) \, d\theta \, dz.
\end{align*}
\]

With these assumptions, system (1.4) has a trivial equilibrium \(E_0 = (0, 0)\), two semitrivial spatially homogeneous equilibria \(E_1 = (1/a_1, 0)\) and \(E_2 = (0, 1/a_2)\) and the positive spatially homogeneous equilibrium \(E^*\) defined by

\[
E^* = \left( \frac{a_2 - b_1}{a_1 a_2 - b_1 b_2}, \frac{a_1 - b_2}{a_1 a_2 - b_1 b_2} \right) \triangleq (k_1, k_2)
\]

provided that \(a_1 a_2 \neq b_1 b_2\) and either (i) \(a_2 > b_1\) and \(a_1 > b_2\) or (ii) \(a_2 < b_1\) and \(a_1 < b_2\). By linear chain technique and geometric singular perturbation theory [13], they established the existence of a traveling wave solution \(\Phi(x + ct)\) connecting \(E_1\) with \(E_2\) if

\[
b_2 < a_1 \quad \text{and} \quad a_2 < b_1
\]
and \( \tau_1 \) and \( \tau_2 \) are sufficiently small. In other words, when the positive spatially homogeneous equilibrium \( E^* \) defined by (1.6) is absent in the first quadrant, a transition can occur between \( E_1 \) and \( E_2 \) in the form of a traveling wavefront solution. We note that if (1.7) holds, then, in the two-dimensional \((u_1, u_2)\) phase plane, the diffusionless undelayed system (removal of delay can be effected by setting each \( k_i(t) = \delta(t) \) in (1.4))

\[
\begin{align*}
\frac{du_1(t)}{dt} &= r_1 u_1(t) \left[ 1 - a_1 u_1(t) - b_1 u_2(t) \right], \\
\frac{du_2(t)}{dt} &= r_2 u_2(t) \left[ 1 - a_2 u_2(t) - b_2 u_1(t) \right]
\end{align*}
\]  

(1.8)
is monostable (\( E_2 \) is stable). However, if

\[
a_1 < b_2 \quad \text{and} \quad a_2 < b_1
\]  

(1.9)
or

\[
a_1 > b_2 \quad \text{and} \quad a_2 > b_1
\]  

(1.10)
holds, then the coexistence equilibrium \( E^* \) of (1.8) exists in the first quadrant. Furthermore, (1.8) is bistable (\( E_1 \) and \( E_2 \) are stable) if (1.9) holds and monostable (\( E^* \) is stable) if (1.10) holds. It is well known from [9,16,23,24,51,54] that under the condition (1.9) or (1.10) the reaction diffusion system

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \left[ 1 - a_1 u_1(x,t) - b_1 u_2(x,t) \right], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \left[ 1 - a_2 u_2(x,t) - b_2 u_1(x,t) \right]
\end{align*}
\]  

(1.11)
has traveling wavefronts. In particular, Volpert et al. [55, Theorem 3.2.2] implies that (1.11) has a traveling wavefront from a stable equilibrium \( E_1 \) to another stable equilibrium \( E_2 \), which we call a bistable wavefront. Our attention now is to ask if “nonlocal delays are harmless” in the sense that bistable wavefront for system (1.11) persist when the so-called nonlocal delays are incorporated in the interaction terms of (1.11). Such a persistence issue or harmless nonlocal delays has been addressed by Gourley and Ruan [18] in stability of two semitrivial spatially homogeneous equilibria \( E_1 \) and \( E_2 \) and the positive spatially homogeneous equilibrium \( E^* \) of (1.4) under the bounded region in \( \mathbb{R}^N \) and the homogeneous Neumann boundary conditions. For the sake of brevity, in this paper we only consider the bistable case here. The monostable case will be studied in another paper.

To the best of our knowledge, there are no results on the existence of bistable wavefronts can be applied to system (1.4) with (1.9). In particular, we should mention the recent result of Ou and Wu [42], in which they considered the persistence of wavefronts for both monostable and bistable systems with nonlocal delay. However, their results are not applied to our system since the number of equilibrium of (1.4) with (1.9) does not satisfy their requirement [42]. Moreover, some other known results concerning with traveling wave solutions of reaction diffusion systems with (nonlocal) delay also do not apply to (1.4) with (1.9) because these results are only concerned with monostable systems (e.g., [28,32,57,60,64]). It is known from [8] that the monostable and
bistable systems have many differences. In order to overcome this difficulty, in this paper, by introducing new variables, we shall convert (1.4) into a system without time delay, which is similar to those of Ashwin et al. [4] and Gourley and Ruan [18]. Thus, the abstract results on bistable systems [55] can be applied to establish the existence of bistable wavefronts of (1.4). The related results on predator–prey, cooperative and competitive systems without delay, one refers to [30,34–40] and the references cited therein.

After the existence of bistable wavefronts of (1.4) has established, a natural problem is whether such a bistable wavefront of (1.4) can determine the long term behavior of the corresponding Cauchy type problem. As we know, many known results (see [7,47,49,55,58]) implied that the bistable wavefronts of reaction diffusion equations (1.1) are stable if delays are absent or finite, which naturally determine the long time dynamical behavior of the corresponding initial value problem if the initial value satisfies certain conditions. This motivates us to consider the stability of bistable wavefronts of reaction diffusion systems with nonlocal delays. Note that the squeezing technique [7,49,58] and spectral theory [46,55] are difficult in applying to (1.4) since the delay is infinite. Thus, new technique should be developed to establish the stability of bistable wavefronts of (1.4). In the current paper, we establish the asymptotic stability of bistable wavefronts of (1.4) in the sense of the general super norm by proving the stability of bistable wavefronts of a corresponding system without delay according to the spectral method. Our result implies that even for the nonlocal systems, bistable wavefronts also can determine the long time behavior of the corresponding initial value problems. This is probably the first time the asymptotic stability of traveling wavefronts of reaction diffusion systems with nonlocal delay has been studied.

We also note that the wave speed of traveling wavefront is unique for bistable systems (equations) and delayed equations, see [7,40,49,55,61]. For the delayed systems, this problem remains open. This motivates us to consider the uniqueness of wave speed of bistable wavefronts of (1.4). As might be expected, we also affirm that the wave speed of bistable wavefronts of (1.4) is unique by constructing proper upper and lower solution.

For the sake of convenience, we list the organization of this paper. In Section 2, we consider the existence of bistable wavefronts of (1.4). Section 3 is concerned with the existence and regularity of the initial value problem of (1.4). In Section 4, we are devoted to investigating the asymptotical stability of bistable wavefronts of (1.4) by spectral method. The uniqueness of wave speed is given in Section 5. The paper ends with a further discussion in both mathematical theory and ecological background.

2. Existence of bistable wavefronts

In this section, we first recall some known results on traveling wavefronts of bistable system without time delay. Then we will prove the existence of bistable wavefronts of (1.4).

2.1. Preliminaries

We begin our preliminaries from the definition of traveling wavefronts of the following system

\[
\frac{\partial u(x,t)}{\partial t} = D \Delta u(x,t) + f(u(x,t)),
\]

(2.1)

where \( x \in \mathbb{R}, D = \text{diag}(d_1, \ldots, d_n) \) with \( d_i > 0, i = 1, 2, \ldots, n, t > 0, u \in \mathbb{R}^n \), \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function.
**Definition 2.1.** A traveling wave solution of (2.1) is a special solution $u(x, t) = \Phi(x + ct)$ for some wave speed $c \in \mathbb{R}$ and wave profile $\Phi(t) \in C^2(\mathbb{R}, \mathbb{R}^n)$. In addition, if $\Phi(t)$ is monotone in $t \in \mathbb{R}$, then it is called a traveling wavefront.

It is clear that a traveling wave solution $\Phi(t)$ of (2.1) satisfies

$$c \Phi'(t) = D \Phi''(t) + f(\Phi(t)), \quad t \in \mathbb{R}. \quad (2.2)$$

Recalling the practical sense of traveling wave solutions (e.g., it can refer to the pioneer work of traveling wave solutions by Fisher [14]), we often require that

$$\lim_{t \to -\infty} \Phi(t) = \Phi_-, \quad \lim_{t \to \infty} \Phi(t) = \Phi_+,$$

with $\Phi_- < \Phi_+$ and $f(\Phi_-) = 0$. Similar to the definition of traveling wave solutions of (2.1), a traveling wave solution $\Phi(t) = (\phi_1(t), \phi_2(t))$ of (1.4) satisfies

$$\begin{cases}
    d_1 \phi_1''(t) - c \phi_1'(t) + r_1 \phi_1(t) \left[1 - a_1 \phi_1(t) - b_1 \phi_2(t)\right] = 0, \\
    d_2 \phi_2''(t) - c \phi_2'(t) + r_2 \phi_2(t) \left[1 - a_2 \phi_2(t) - b_2 \phi_1(t)\right] = 0,
\end{cases} \quad (2.4)$$

where $(g_1 \ast \phi_2)(t)$ and $(g_2 \ast \phi_1)(t)$ are defined by

$$\begin{align*}
    (g_1 \ast \phi_2)(t) &= +\infty \int_{-\infty}^{t} \int_{0}^{+\infty} e^{-\frac{\theta}{\tau_1}} \frac{1}{\sqrt{4\pi \tau_2}} e^{-\frac{s^2}{4\tau_2}} \phi_2(t - c\theta - s) \, ds \, d\theta, \\
    (g_2 \ast \phi_1)(t) &= +\infty \int_{-\infty}^{t} \int_{0}^{+\infty} e^{-\frac{\theta}{\tau_2}} \frac{1}{\sqrt{4\pi \tau_1}} e^{-\frac{s^2}{4\tau_1}} \phi_1(t - c\theta - s) \, ds \, d\theta.
\end{align*} \quad (2.5)$$

For the existence of bistable wavefronts of (2.2) and (2.3), there are many results, for example, Huang [22], Mischaikow and Hutson [40], Volpert et al. [55], Wu and Li [61] and the references cited therein. In order to prove the existence of traveling wavefronts of (2.4), we need the following existence theorem, which can be found in [55, Theorem 3.3.2].

**Theorem 2.2.** Let the vector-valued function $f(\Phi)$ vanishes at a finite number of points $\Phi^k$ in the interval $[\Phi_-, \Phi_+]$ with $\Phi_- < \Phi^k < \Phi_+$, $k = 1, 2, \ldots, m$. We assume that all the eigenvalues of the matrices $f'(\Phi^k)$ lie in the left half-plane, and that there exist vector $p_k \geq 0$ such that $p_k f'(\Phi^k) > 0$, $k = 1, 2, \ldots, m$. Then there exists a monotone traveling wavefront, i.e., a constant $c$ and a twice continuously differentiable monotone vector-valued function $\Phi(t)$, satisfying (2.2) and (2.3).
2.2. Existence

In order to prove the existence of bistable wavefronts connecting the semitrivial equilibria $E_1$ and $E_2$, we need to introduce some new variables. Let $(g_1 * u_2)(x, t) = u_3(x, t)$, $(g_2 * u_1)(x, t) = u_4(x, t)$. Then it is straightforward to see that (1.4) is equivalent to

\[
\begin{aligned}
\frac{\partial u_1(x, t)}{\partial t} &= d_1 \Delta u_1(x, t) + r_1 u_1(x, t) \left[ 1 - a_1 u_1(x, t) - b_1 u_3(x, t) \right], \\
\frac{\partial u_2(x, t)}{\partial t} &= d_2 \Delta u_2(x, t) + r_2 u_2(x, t) \left[ 1 - a_2 u_2(x, t) - b_2 u_4(x, t) \right], \\
\frac{\partial u_3(x, t)}{\partial t} &= a_1 u_2(x, t) - a_2 u_1(x, t), \\
\frac{\partial u_4(x, t)}{\partial t} &= d_1 \Delta u_4(x, t) + \frac{1}{\tau_1} u_1(x, t) - \frac{1}{\tau_2} u_4(x, t).
\end{aligned}
\]  

(2.6)

Furthermore, in order to apply Theorem 2.2, let us introduce new state variables

\[
\begin{aligned}
u_1 &= \frac{1}{a_1} - u_1^*, \\
u_4 &= \frac{1}{a_1} - u_4^*
\end{aligned}
\]

and drop the star, then (2.6) is equivalent to

\[
\begin{aligned}
\frac{\partial u_1(x, t)}{\partial t} &= d_1 \Delta u_1(x, t) + r_1 \left[ \frac{1}{a_1} - u_1(x, t) \right] \left[ b_1 u_3(x, t) - a_1 u_1(x, t) \right], \\
\frac{\partial u_2(x, t)}{\partial t} &= d_2 \Delta u_2(x, t) + r_2 u_2(x, t) \left[ 1 - \frac{b_2}{a_2} - a_2 u_2(x, t) + b_2 u_4(x, t) \right], \\
\frac{\partial u_3(x, t)}{\partial t} &= d_2 \Delta u_3(x, t) + \frac{1}{\tau_1} u_2(x, t) - \frac{1}{\tau_1} u_3(x, t), \\
\frac{\partial u_4(x, t)}{\partial t} &= d_1 \Delta u_4(x, t) + \frac{1}{\tau_2} u_1(x, t) - \frac{1}{\tau_2} u_4(x, t).
\end{aligned}
\]  

(2.7)

Thus, the corresponding wave system of (2.7) reduces to

\[
\begin{aligned}
c \phi_1'(t) &= d_1 \phi_1''(t) + r_1 \left[ \frac{1}{a_1} - \phi_1(t) \right] \left[ b_1 \phi_3(t) - a_1 \phi_1(t) \right], \\
c \phi_2'(t) &= d_2 \phi_2''(t) + r_2 \phi_2(t) \left[ 1 - \frac{b_2}{a_2} - a_2 \phi_2(t) + b_2 \phi_4(t) \right], \\
c \phi_3'(t) &= d_2 \phi_3''(t) + \frac{1}{\tau_1} \phi_2(t) - \frac{1}{\tau_1} \phi_3(t), \\
c \phi_4'(t) &= d_1 \phi_4''(t) + \frac{1}{\tau_2} \phi_1(t) - \frac{1}{\tau_2} \phi_4(t)
\end{aligned}
\]  

(2.8)

with the asymptotical boundary conditions

\[
\begin{aligned}
\lim_{t \to -\infty} (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)) &= (0, 0, 0, 0), \\
\lim_{t \to \infty} (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)) &= \left( \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_2}, \frac{1}{a_1} \right).
\end{aligned}
\]  

(2.9)
Let

$$\Phi_0 = 0 \quad \text{and} \quad \Phi_+ = \left( \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_2}, \frac{1}{a_1} \right).$$

Then by applying Theorem 2.2 to system (2.7), we have the following existence result.

**Theorem 2.3.** Assume that (1.9) holds. Then there exists a monotone vector-valued function $\Phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)) \in C^2(\mathbb{R}, \mathbb{R}^4)$ satisfying (2.8) and (2.9).

**Proof.** In order to prove our result, it suffices to show that all the eigenvalues of $f'(0)$ and $f'(\Phi_+)$ have negative real part, whereas there exist vectors $p_1 > 0, p_2 > 0$ such that $p_1 f' (\Phi^1) > 0$ and $p_2 f' (\Phi^2) > 0$, where $\Phi^1$ and $\Phi^2$ are given by

$$\Phi^1 = \left( \frac{1}{a_1} - k_1, k_2, \frac{1}{a_1} - k_1 \right), \quad \Phi^2 = \left( \frac{1}{a_1}, 0, 0, \frac{1}{a_1} \right),$$

respectively. For the matrices $f'(0)$ and $f'(\Phi_+)$, it is easy to see that all of their eigenvalues are negative. Direct calculation shows that $f'(\Phi^1)$ is given by

$$f'(\Phi^1) = \begin{bmatrix}
-r_1 a_1 k_1 & 0 & 0 & \frac{1}{\tau_1} \\
0 & -r_2 a_2 k_1 & \frac{1}{\tau_1} & 0 \\
r_1 b_1 k_1 & 0 & -\frac{1}{\tau_1} & 0 \\
0 & r_2 b_2 k_2 & 0 & -\frac{1}{\tau_2}
\end{bmatrix}.$$  

Furthermore, for any $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) > 0$, we know that

$$(\beta_1, \beta_2, \beta_3, \beta_4) f'(\Phi^1) > 0$$

is equivalent to

$$\beta_3 b_1 > \beta_1 a_1, \quad \beta_4 b_2 > \beta_2 a_2, \quad \beta_2 > \beta_3, \quad \beta_1 > \beta_4. \quad (2.10)$$

Since $a_1 < b_2, a_2 < b_1$ holds, it is easy to find $\beta > 0$ such that (2.10) is true.

Similarly, for $\Phi^2$, it is easy to see that

$$\beta f' (\Phi^2) = \left( r_1 \beta_1, r_2 \beta_2, \frac{\beta_2 - \beta_3}{\tau_1}, \frac{\beta_1 - \beta_4}{\tau_2} \right) > 0$$

if $\beta_2 > \beta_3 > 0$ and $\beta_1 > \beta_4 > 0$ hold. Hence, system (2.7) satisfies all the conditions of Theorem 2.2, namely, (2.7) has a traveling wavefront satisfies (2.8) and (2.9). The proof is complete. \(\square\)

**Corollary 2.4.** Assume that (1.9) holds. Then (1.4) has a monotone traveling wavefront connecting $E_1$ with $E_2$, which is a bistable wavefront.

**Remark 2.5.** In this section, we rewrite (1.4) as (2.6) since bistable wavefronts of (1.4) are twice continuous differentiable. Thus, the regularity of $u_3$ and $u_4$ is clear, it refers to [1,4,18].
**Remark 2.6.** In [18], Gourley and Ruan proved the existence of traveling wave solutions of (1.4) if $\tau_1$ and $\tau_2$ are small. The something interesting is that the same $\tau_1$ and $\tau_2$ in our system (1.4) do not affect the existence of bistable wavefronts. In other words, the delays appeared in the interaction terms of system (1.4) are not sensitive to the existence of bistable wavefronts.

### 3. Initial value problem

In this section, we consider the corresponding initial value problem of (1.4). Let us introduce new variables $u_i^* = \frac{1}{a_i} - u_1$ and drop the star. Then (1.4) reduces to

$$
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= \frac{d_1}{\Delta_1} \Delta u_1(x,t) + r_1 \left[ \frac{1}{a_1} - u_1(x,t) \right] \left[ b_1(g_1 \ast u_2)(x,t) - a_1 u_1(x,t) \right], \\
\frac{\partial u_2(x,t)}{\partial t} &= \frac{d_2}{\Delta_1} \Delta u_2(x,t) + r_2 u_2(x,t) \left[ 1 - \frac{b_2}{a_1} - a_2 u_2(x,t) + b_2(g_2 \ast u_1)(x,t) \right].
\end{align*}
$$

We consider (3.1) with the initial value

$$
u_1(x,s) = \psi_1(x,s), \quad u_2(x,s) = \psi_2(x,s) \in C(\mathbb{R} \times (-\infty, 0], \mathbb{R}),$$

where

$$0 \leq \left( \psi_1(x,s), \psi_2(x,s) \right) \leq \left( \frac{1}{a_1}, \frac{1}{a_2} \right), \quad (x,s) \in \mathbb{R} \times (-\infty, 0].$$

#### 3.1. Existence and uniqueness of mild solutions

Let

$$
\beta_1 = r_1 \left( 1 + \frac{b_1}{a_2} \right), \quad \beta_2 = r_2 \left( 1 + \frac{b_2}{a_1} \right),
$$

and for $0 \leq u_1(x,t) \leq \frac{1}{a_1}, 0 \leq u_2(x,t) \leq \frac{1}{a_2}, (x,t) \in \mathbb{R} \times \mathbb{R}^+$, define $F = (F_1, F_2)$ by

$$
\begin{align*}
F_1(u_1, u_2)(x,t) &= \beta_1 u_1(x,t) + r_1 \left[ \frac{1}{a_1} - u_1(x,t) \right] \left[ b_1(g_1 \ast u_2)(x,t) - a_1 u_1(x,t) \right], \\
F_2(u_1, u_2)(x,t) &= \beta_2 u_2(x,t) + r_2 u_2(x,t) \left[ 1 - \frac{b_2}{a_1} - a_2 u_2(x,t) + b_2(g_2 \ast u_1)(x,t) \right].
\end{align*}
$$

Then $F$ satisfies

$$0 = F(0, 0) \leq F(u_1, u_2)(x,t) \leq F(v_1, v_2)(x,t) \leq F \left( \frac{1}{a_1}, \frac{1}{a_2} \right)$$
for any $0 \leq (u_1(x,s), u_2(x,s)) \leq (v_1(x,s), v_2(x,s)) \leq \left( \frac{1}{a_1}, \frac{1}{a_2} \right)$, $x \in \mathbb{R}, t > 0, s \in \mathbb{R}$. From (3.4) and (3.5), we now rewrite (3.1) as

$$
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \Delta u_1(x,t) - \beta_1 u_1(x,t) + F_1(u_1, u_2)(x,t), \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \Delta u_2(x,t) - \beta_2 u_2(x,t) + F_2(u_1, u_2)(x,t).
\end{align*}
$$

(3.7)

Let $X$ be a Banach space defined by

$$
X = UBC(\mathbb{R}, \mathbb{R}^2) = \left\{ \Phi : \Phi \text{ is a bounded and uniformly continuous vector-valued function from } \mathbb{R} \text{ to } \mathbb{R}^2 \right\}
$$

equipped with the general super norm $\| \cdot \|$ and

$$
X_I = \left\{ u(x) \in X, \ 0 \leq u(x) \leq \left( \frac{1}{a_1}, \frac{1}{a_2} \right) \text{ for } x \in \mathbb{R} \right\}.
$$

Let $T(t) = (T_1(t), T_2(t))$ and define

$$
\begin{align*}
T_1(t) &= \frac{e^{-\beta_1 t}}{\sqrt{4\pi d_1 t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d_1 t}} u_1^0(y) \, dy \triangleq T_1(t)u_1^0(x), \\
T_2(t) &= \frac{e^{-\beta_2 t}}{\sqrt{4\pi d_2 t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d_2 t}} u_2^0(y) \, dy \triangleq T_2(t)u_2^0(x).
\end{align*}
$$

Then $T(t) : X \to X$ is a $C_0$ semigroup. Moreover, we know from [10,11,43,50] that $T(t)$ is an analytic semigroup for $t > 0$ and $T(t)$ is a positive semigroup in the functional sense. By using the analyticity and the positivity of the semigroup $T(t)$, standard upper and lower solutions technique and the theory of abstract functional differential equations [34,45], it is easy to see that the following result holds.

**Theorem 3.1.** Assume that $(\psi_1(\cdot,s), \psi_2(\cdot,s))$ satisfies (3.3). Then (3.7) and (3.2) (also (3.1) and (3.2)) has a unique mild solution $(u_1(x,t), u_2(x,t))$ defined by

$$
\begin{align*}
u_1(x,t) &= T_1(t)\psi_1(x,0) + \int_0^t T_1(t-s)F_1(u_1, u_2)(x,s) \, ds, \\
u_2(x,t) &= T_2(t)\psi_2(x,0) + \int_0^t T_2(t-s)F_2(u_1, u_2)(x,s) \, ds
\end{align*}
$$

(3.8)

for all $x \in \mathbb{R}$, $t > 0$. Moreover, $(u_1(\cdot,t), u_2(\cdot,t)) \in X_I$ for all $t > 0$. 

3.2. Regularity of mild solutions

We now consider the following initial value problem

\[
\begin{align*}
\frac{\partial v_1(x,t)}{\partial t} &= d_1 \Delta v_1(x,t) + r_1 \left[ \frac{1}{a_1} - v_1(x,t) \right] \left[ b_1 v_3(x,t) - a_1 v_1(x,t) \right], \\
\frac{\partial v_2(x,t)}{\partial t} &= d_2 \Delta v_2(x,t) + r_2 v_2(x,t) \left[ 1 - \frac{b_2}{a_1} - a_2 v_2(x,t) + b_2 v_4(x,t) \right], \\
\frac{\partial v_3(x,t)}{\partial t} &= d_2 \Delta v_3(x,t) + \frac{1}{\tau_1} v_2(x,t) - \frac{1}{\tau_1} v_3(x,t), \\
\frac{\partial v_4(x,t)}{\partial t} &= d_1 \Delta v_4(x,t) + \frac{1}{\tau_2} v_1(x,t) - \frac{1}{\tau_2} v_4(x,t), \\
(v_1(x,0), v_2(x,0), v_3(x,0), v_4(x,0)) &= (v_1(x), v_2(x), v_3(x), v_4(x)).
\end{align*}
\]

For \((u_1, u_2) \in X\), let \((T_3(t), T_4(t)) : X \to X\) be

\[
\begin{align*}
T_3(t)u_1(x) &\triangleq e^{-\frac{t}{\tau_1}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d_2 t}} u_1(y) dy, \\
T_4(t)u_2(x) &\triangleq e^{-\frac{t}{\tau_2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d_1 t}} u_2(y) dy.
\end{align*}
\]

Then (3.9) has a unique classical solution \(v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t)\) defined by

\[
\begin{align*}
v_1(x,t) &= T_1(t)v_1(x) + \int_0^t T_1(t-s) \left\{ \beta_1 v_1(x,s) \\
&+ r_1 \left[ \frac{1}{a_1} - v_1(x,s) \right] \left[ b_1 v_3(x,s) - a_1 v_1(x,s) \right] \right\} ds, \\
v_2(x,t) &= T_2(t)v_2(x) + \int_0^t T_2(t-s) \left\{ \beta_2 v_2(x,s) \\
&+ r_2 v_2(x,s) \left[ 1 - \frac{b_2}{a_1} - a_2 v_2(x,s) + b_2 v_4(x,s) \right] \right\} ds, \\
v_3(x,t) &= T_3(t)v_3(x) + \frac{1}{\tau_1} \int_0^t T_3(t-s)v_2(x,s) ds, \\
v_4(x,t) &= T_4(t)v_4(x) + \frac{1}{\tau_2} \int_0^t T_4(t-s)v_1(x,s) ds.
\end{align*}
\]

Let the initial values of (3.9) be given by

\[
v_i(x,0) = \psi_i(x,0), \quad i = 1, 2, 3, 4,
\]

(3.10)
where $\psi_1(x,0), \psi_2(x,0)$ are defined by (3.2) and $\psi_3(x,0), \psi_4(x,0)$ are defined by

$$
\psi_3(x,0) = \frac{1}{\tau_1} \int_0^\infty T_3(\theta)\psi_2(x,-\theta) \, d\theta, \quad \psi_4(x,0) = \frac{1}{\tau_2} \int_0^\infty T_4(\theta)\psi_1(x,-\theta) \, d\theta.
$$

Then the property of semigroup implies that

$$
v_3(x,t) = e^{-\frac{t}{\tau_1}} \sqrt{\frac{v_2}{4\pi d_2 t}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4d_2 t}} \psi_3(x-y,0) \, dy
$$

$$
\quad + \int_0^t e^{-\frac{t-s}{\tau_1}} \frac{1}{\tau_1 \sqrt{4\pi d_2 (t-s)}} \int_{-\infty}^\infty e^{-\frac{y^2}{4d_2 (t-s)}} \psi_2(x-y,s) \, dy \, ds
$$

$$
\quad = \frac{1}{\tau_1} \int_0^\infty T_3(t)\psi_2(x,t-\theta) \, d\theta
$$

and

$$
v_4(x,t) = \frac{1}{\tau_2} \int_0^\infty T_4(t)\psi_1(x,t-\theta) \, d\theta.
$$

Substitute $v_3, v_4$ into $v_1, v_2$, then $v_1(x,t)$ and $v_2(x,t)$ are independent of $v_3$ and $v_4$.

The following lemma will show that $(u_1, u_2) = (v_1, v_2)$ for all $x \in \mathbb{R}$ and $t > 0$.

**Lemma 3.2.** If the initial values of (3.9) are given by (3.10), then $(u_1(x,t), u_2(x,t)) = (v_1(x,t), v_2(x,t))$ holds for all $x \in \mathbb{R}$ and $t > 0$.

**Proof.** Substituting $v_3$ into $v_1$, it follows that for all $t > 0$,

$$
v_1(x,t) = T_1(t)v_1(x,0) + \int_0^t T_1(t-s) \left\{ \beta_1 v_1(x,s) + r_1 \left[ \frac{1}{a_1} - v_1(x,s) \right] \right\} \, ds,
$$

and Theorem 3.1 implies that

$$
u_1(x,t) = T_1(t)u_1(x,0) + \int_0^t T_1(t-s) \left[ \beta_1 u_1(x,s) + r_1 \left[ \frac{1}{a_1} - u_1(x,s) \right] \right] \, ds.
$$
Note that the norm of $T(t)$ is not greater than 1 in $X$, then

$$\begin{align*}
\sup_{x \in \mathbb{R}} |v_1(x, t) - u_1(x, t)| & \leq \sup_{x \in \mathbb{R}} |v_1(x, 0) - u_1(x, 0)| \\
& \quad + L \int_0^t \left[ \sup_{s \leq \theta} \sup_{x \in \mathbb{R}} |v_1(x, s) - u_1(x, s)| + \sup_{s \leq \theta} \sup_{x \in \mathbb{R}} |v_2(x, s) - u_2(x, s)| \right] d\theta,
\end{align*}$$

(3.11)

where

$L = \beta_1 + \beta_2 + r_1 \left[ 1 + \frac{b_1}{a_1} + \frac{b_1}{a_2} \right] + r_2 \left[ 1 + \frac{b_2}{a_2} + \frac{b_2}{a_1} \right].$

Similarly, the following estimates hold for $t > 0$

$$\begin{align*}
\sup_{x \in \mathbb{R}} |v_2(x, t) - u_2(x, t)| & \leq \sup_{x \in \mathbb{R}} |v_2(x, 0) - u_2(x, 0)| \\
& \quad + L \int_0^t \left[ \sup_{s \leq \theta} \sup_{x \in \mathbb{R}} |v_1(x, s) - u_1(x, s)| + \sup_{s \leq \theta} \sup_{x \in \mathbb{R}} |v_2(x, s) - u_2(x, s)| \right] d\theta.
\end{align*}$$

(3.12)

Furthermore, (3.11) and (3.12) imply that

$$\begin{align*}
\sup_{x \in \mathbb{R}} |v_2(x, t) - u_2(x, t)| & + \sup_{x \in \mathbb{R}} |v_1(x, t) - u_1(x, t)| \\
& \leq \sup_{x \in \mathbb{R}} |v_2(x, 0) - u_2(x, 0)| + \sup_{x \in \mathbb{R}} |v_1(x, 0) - u_1(x, 0)| \\
& \quad + 2L \int_0^t \left[ \sup_{s \leq \theta} \sup_{x \in \mathbb{R}} |v_1(x, s) - u_1(x, s)| + \sup_{s \leq \theta} \sup_{x \in \mathbb{R}} |v_2(x, s) - u_2(x, s)| \right] d\theta \\
& \leq \sup_{x \in \mathbb{R}} |v_2(x, 0) - u_2(x, 0)| + \sup_{x \in \mathbb{R}} |v_1(x, 0) - u_1(x, 0)| \\
& \quad + 4L \int_0^t \left[ \sup_{s \leq \theta} \sup_{x \in \mathbb{R}} |v_1(x, s) - u_1(x, s)| + \sup_{x \in \mathbb{R}} |v_2(x, s) - u_2(x, s)| \right] d\theta.
\end{align*}$$

(3.13)

Define $w(t)$ by

$$w(t) = \sup_{x \in \mathbb{R}} |v_1(x, t) - u_1(x, t)| + \sup_{x \in \mathbb{R}} |v_2(x, t) - u_2(x, t)|.$$
Then (3.13) implies
\[ w(t) \leq w(0) + 4L \int_0^t \left[ \sup_{s \leq \theta} w(s) \right] d\theta \quad \text{for } t > 0. \]

Furthermore, we have for all \( t > 0, \)
\[ w(t) + \sup_{s \leq 0} w(s) \leq w(0) + \sup_{s \leq 0} w(s) + 4L \int_0^t \left[ \sup_{s \leq \theta} w(s) \right] d\theta \]
\[ \leq w(0) + \sup_{s \leq 0} w(s) + 4L \int_0^t \left( \sup_{0 \leq s \leq \theta} w(s) + \sup_{s \leq 0} w(s) \right) d\theta. \]

For any given \( t > 0, \) let \( m(t) = \sup_{0 \leq s \leq t} \left[ w(s) + \sup_{z \leq 0} w(z) \right]. \) Since
\[ \int_0^t \left( \sup_{0 \leq s \leq \theta} w(s) + \sup_{s \leq 0} w(s) \right) d\theta \]
is nondecreasing with respect to \( t > 0, \) then
\[ m(t) \leq m(0) + 4L \int_0^t m(s) ds. \]

Thus, the Gronwall’s inequality implies that \( m(t) \equiv 0 \) if \( m(0) = 0. \) That is to say, if all the assumptions of Lemma 3.2 are satisfied, then
\[ \left| v_1(x, t) - u_1(x, t) \right| + \left| v_2(x, t) - u_2(x, t) \right| = 0, \quad x \in \mathbb{R}, \ t \geq 0. \]

The proof is complete. \( \square \)

From Lemma 3.2 we know that the smoothness of \( v_i(x, t) \) implies that \( u_i(x, t) \) \((i = 1, 2)\) is a classical solution of (3.1). Moreover, we can define \( u_3 \) and \( u_4 \) as follows
\[
\begin{align*}
    u_3(x, t) &= \frac{1}{\tau_1} \int_0^\infty T_3(t) u_2(x, t - \theta) d\theta, \\
    u_4(x, t) &= \frac{1}{\tau_2} \int_0^\infty T_4(t) u_1(x, t - \theta) d\theta.
\end{align*}
\] (3.14)

It is clear that \( u_3 \) and \( u_4 \) are twice differential with respect to \( x \) and differentiable with respect to \( t > 0. \) Thus, the smoothness of \( u_1(x, t) \) and \( u_2(x, t) \) imply that the following result holds.
Theorem 3.3. The mild solution $u_1(x, t), u_2(x, t)$ defined in Theorem 3.1 is the classical solution of (3.1) and (3.2) for $x \in \mathbb{R}, t > 0$. Moreover, if $u_3(x, t), u_4(x, t)$ are defined by (3.14), then $u_i(x, t)$ ($i = 1, 2, 3, 4$) satisfies (2.7) in the sense of the classical solution.

Remark 3.4. Theorem 3.3 implies that we can obtain the information of solutions of (3.1) and (3.2) by investigating of the corresponding undelayed system (2.7). The key idea will be used in the following two sections. Namely, we will consider the following initial value problem

$$\begin{align*}
\frac{\partial u_1(x, t)}{\partial t} &= d_1 \Delta u_1(x, t) + r_1 \left[ \frac{1}{\alpha_1} - u_1(x, t) \right] \left[ b_1 u_3(x, t) - a_1 u_1(x, t) \right], \\
\frac{\partial u_2(x, t)}{\partial t} &= d_2 \Delta u_2(x, t) + r_2 u_2(x, t) \left[ 1 - \frac{b_2}{\alpha_1} - a_2 u_2(x, t) + b_2 u_4(x, t) \right], \\
\frac{\partial u_3(x, t)}{\partial t} &= d_3 \Delta u_3(x, t) + \frac{1}{\tau_1} u_2(x, t) - \frac{1}{\tau_1} u_3(x, t), \\
\frac{\partial u_4(x, t)}{\partial t} &= d_4 \Delta u_4(x, t) + \frac{1}{\tau_2} u_1(x, t) - \frac{1}{\tau_2} u_4(x, t), \quad \text{for } x \in \mathbb{R}, \ t > 0, \\
u_i(x, 0) &= \psi_i(x) \quad \text{for } x \in \mathbb{R}, \ i = 1, 2, 3, 4.
\end{align*}$$

(3.15)

3.3. Comparison principle

Let $\Psi_1(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))$ and $\Psi_2(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)) \in C(\mathbb{R}, \mathbb{R}^4)$ with $0 \leq \varphi_2(x) \leq \psi_1(x) \leq \Phi_+, \ x \in \mathbb{R}$. Then it is easy to see that the following comparison principle holds for (3.15).

Lemma 3.5. Assume that

$$u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t), u_4(x, t))$$

and

$$v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t), v_4(x, t))$$

are solutions of (3.15) with initial values $\Psi_1$ and $\Psi_2$, respectively. Then

$$0 \leq v(x, t) \leq u(x, t) \leq \Phi_+, \ x \in \mathbb{R}, \ t > 0.$$

Lemma 3.6. For $\Psi_1, \Psi_2, u, v$ defined in Lemma 3.5,

$$u_i(x, t) - v_i(x, t) \geq \Theta_1(J, t - t_0) \int_z^{z+1} \left[ u_i(y, t_0) - v_i(y, t_0) \right] dy \geq 0$$

(3.16)

for $i = 1, 2, 3, 4, J > 0, x, z \in \mathbb{R}$ with $|x - z| \leq J$, and $t > t_0 \geq 0$, where

$$\begin{align*}
\Theta_1(J, t - t_0) &= \frac{e^{-\beta_1 t}}{\sqrt{4\pi d_1(t - t_0)}} e^{-\frac{(J+1)^2}{4d_1(t - t_0)}}, \\
\Theta_2(J, t - t_0) &= \frac{e^{-\beta_2 t}}{\sqrt{4\pi d_2(t - t_0)}} e^{-\frac{(J+1)^2}{4d_2(t - t_0)}}, \\
\Theta_3(J, t - t_0) &= \frac{e^{-\frac{1}{\tau_1} t}}{\sqrt{4\pi d_2(t - t_0)}} e^{-\frac{(J+1)^2}{4d_2(t - t_0)}}, \\
\Theta_4(J, t - t_0) &= \frac{e^{-\frac{1}{\tau_2} t}}{\sqrt{4\pi d_2(t - t_0)}} e^{-\frac{(J+1)^2}{4d_2(t - t_0)}}.
\end{align*}$$

(3.17)
The proof is similar to that of Smith and Zhao [49, Theorem 2.2], so we omit it here.

**Remark 3.7.** Lemma 3.6 implies that each nontrivial traveling wavefront of (2.7) is strictly monotone. And so, bistable wavefronts of (1.4) connecting \( E_1 \) with \( E_2 \) are also strictly monotone.

### 3.4. Upper and lower solutions method

**Definition 3.8.** Assume that \((u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))\) is \(C^2\) in \(x \in \mathbb{R}\) and \(C^1\) in \(t > 0\) and \(0 \leq (u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t)) \leq \Phi^+\). Then \((u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))\) is called an upper solution of (3.15) if it satisfies

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &\geq d_1 \Delta u_1(x,t) + r_1 \left[ a_1 - b_1 \bar{u}_3(x,t) - a_1 u_1(x,t) \right], \\
\frac{\partial u_2(x,t)}{\partial t} &\geq d_2 \Delta u_2(x,t) + r_2 u_2(x,t) \left[ 1 - \frac{b_2}{a_1} - a_2 \bar{u}_2(x,t) + b_2 \bar{u}_4(x,t) \right], \\
\frac{\partial u_3(x,t)}{\partial t} &\geq d_2 \Delta u_3(x,t) + r_2 \bar{u}_2(x,t) - \frac{1}{\tau_1} \bar{u}_3(x,t), \\
\frac{\partial u_4(x,t)}{\partial t} &\geq d_1 \Delta u_4(x,t) + r_1 \bar{u}_1(x,t) - \frac{1}{\tau_2} \bar{u}_4(x,t), \\
(u_1(x,0), u_2(x,0), u_3(x,0), u_4(x,0)) &\geq (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)).
\end{align*}
\]

(3.17)

A lower solution of (3.15) can be defined by reversing the inequalities above.

**Remark 3.9.** It is well known that if \((u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))\) and \((v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t))\) are two upper solutions of (3.15), then

\[
\bar{w}(x,t) = \left( \min \left\{ u_1(x,t), v_1(x,t) \right\}, \min \left\{ u_2(x,t), v_2(x,t) \right\}, \min \left\{ u_3(x,t), v_3(x,t) \right\}, \min \left\{ u_4(x,t), v_4(x,t) \right\} \right)
\]

is a generalized upper solution of (3.15). Similarly, we can define a generalized lower solution \(w(x,t)\) of (3.15). We can refer to Ye and Li [62].

By Definition 3.8 and Remark 3.9, the following result is obvious.

**Lemma 3.10.** Assume that \((\bar{u}_1(x,t), \bar{u}_2(x,t), \bar{u}_3(x,t), \bar{u}_4(x,t))\) is a (generalized) upper solution of (3.15) and \((u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))\) is a (generalized) lower solution of (3.15). We further assume that

\[
(\bar{u}_1(x,0), \bar{u}_2(x,0), \bar{u}_3(x,0), \bar{u}_4(x,0)) \geq (u_1(x,0), u_2(x,0), u_3(x,0), u_4(x,0)).
\]

Then the following are true

(i) \((\bar{u}_1(x,t), \bar{u}_2(x,t), \bar{u}_3(x,t), \bar{u}_4(x,t)) \geq (u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))\),
(ii) (3.15) has a unique solution \((u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))\) satisfying
\[
(u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t)) \leq (\tilde{u}_1(x,t), \tilde{u}_2(x,t), \tilde{u}_3(x,t), \tilde{u}_4(x,t)).
\]

**Remark 3.11.** Due to Lemma 3.10, we will call generalized upper (lower) solution as upper (lower) solution for simplicity.

### 4. Asymptotical stability of traveling fronts

In this section, we shall consider the asymptotic stability of traveling wavefronts of (1.4) by the spectral theory, although this question is full of difficulties mentioned in Section 1. By using the regularity result given in Section 3 and similar technique to that of Section 2, we shall consider the system (2.7).

Define the spaces \(C_0(\mathbb{R}, \mathbb{R}^4)\) and \(C_0^2(\mathbb{R}, \mathbb{R}^4)\) by
\[
C_0(\mathbb{R}, \mathbb{R}^4) = \{ u : u \in C(\mathbb{R}, \mathbb{R}^4), \lim_{x \to \pm \infty} |u(x)| = 0 \},
\]
\[
C_0^2(\mathbb{R}, \mathbb{R}^4) = \{ u, u', u'' \in C_0(\mathbb{R}, \mathbb{R}^4) \}.
\]

It is clear that \(C_0(\mathbb{R}, \mathbb{R}^4)\) and \(C_0^2(\mathbb{R}, \mathbb{R}^4)\) are Banach spaces if we equip them with the following norms
\[
\|u\|_{C_0} = \sup_{\xi \in \mathbb{R}} |u(\xi)|, \quad u \in C_0(\mathbb{R}, \mathbb{R}^4),
\]
\[
\|u\|_{C_0^2} = \max \{ \|u\|_{C_0}, \|u'\|_{C_0}, \|u''\|_{C_0} \}, \quad u \in C_0^2(\mathbb{R}, \mathbb{R}^4),
\]
respectively, where \(|\cdot|\) denotes the general super norm in \(\mathbb{R}^4\).

**Definition 4.1.** A traveling wavefront \(\Phi(x + ct)\) of (2.7) is said to be **asymptotically stable with phase shift** according to the norm \(\|\cdot\|_{C_0}\) if there exists a small positive constant \(\varepsilon\) such that the initial value \(\Psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))\) of (3.15) satisfies
\[
0 \leq \Psi(x) \leq \Phi_+, \quad \Phi(x) - \Psi(x) \in C_0 \quad \text{and} \quad \|\Phi(x + \kappa) - \Psi(x)\|_{C_0} < \varepsilon
\]
for some \(\kappa \in \mathbb{R}\), then there exist \(M > 0, \beta > 0\) and \(h \in \mathbb{R}\) such that
\[
\|u(x,t) - \Phi(x + ct + h)\|_{C_0} \leq Me^{-\beta t}, \quad t > 0,
\]
where \(u(x,t)\) is the solution of (3.15), \(h\) is dependent on the initial value \(\Psi(x)\), \(M > 0, \beta > 0\) are independent of \(t, h\) and \(\Psi(x)\).

It is well known that the stability of traveling wavefronts usually is concerned with the spectral distribution of a linear operator obtained by linearization on a traveling wavefront [46,55]. Therefore, we define the operator \(L = (L_1, L_2, L_3, L_4) : C_0^2(\mathbb{R}, \mathbb{R}^4) \to C_0(\mathbb{R}, \mathbb{R}^4)\) by
\[ L_1: \quad d_1 \psi_1''(t) - c \psi_1'(t) + r_1 \left[ -1 + 2 \phi_1(t) - b_1 \phi_3(t) \right] \psi_1(t) + r_1 b_1 \left[ \frac{1}{a_1} - \phi_1(t) \right] \psi_3(t), \]

\[ L_2: \quad d_2 \psi_2''(t) - c \psi_2'(t) + r_2 \left[ 1 - \frac{b_2}{a_1} - 2 a_2 \phi_2(t) + b_2 \phi_4(t) \right] \psi_2(t) + r_2 b_2 \phi_2(t) \psi_4(t), \]

\[ L_3: \quad d_2 \psi_3''(t) - c \psi_3'(t) + \frac{1}{\tau_1} \psi_2(t) - \frac{1}{\tau_1} \psi_3(t), \]

\[ L_4: \quad d_1 \psi_4''(t) - c \psi_4'(t) + \frac{1}{\tau_2} \psi_1(t) - \frac{1}{\tau_2} \psi_4(t), \]

where \( \Psi(x) = (\psi_1, \psi_2, \psi_3, \psi_4) \in C_0^2(\mathbb{R}, \mathbb{R}^4) \) and \( \Phi(x) = (\phi_1, \phi_2, \phi_3, \phi_4) \) is the traveling wavefronts of (2.7).

By Volpert et al. [55, p. 227, Theorem 2.1], the following result is true.

**Lemma 4.2.** The traveling wavefronts of (2.7) is asymptotic stable with phase shift according to the norm \( \| \cdot \|_{C_0} \) if the following are true:

(i) The derivative \( \Phi'(t) \) belongs to \( C_0 \) and the operator \( L \) has a simple zero eigenvalue, and no other eigenvalues with a nonnegative real part.

(ii) For arbitrary real \( \xi \) all eigenvalues of the matrix

\[ -\xi^2 D + f(0) \quad \text{and} \quad -\xi^2 D + f(\Phi_+) \]

have negative real part.

The following result is obvious from the definition of traveling wavefronts.

**Lemma 4.3.** \( \Phi' \in C_0^2(\mathbb{R}, \mathbb{R}^4) \) and \( L \Phi' = 0 \).

Now, consider the matrix

\[
E = \begin{bmatrix}
  r_1 (-1 + 2 \phi_1 - b_1 \phi_3) & 0 & r_1 b_1 \left( \frac{1}{a_1} - \phi_1 \right) & 0 \\
  0 & r_2 (1 - \frac{b_2}{a_1} - 2 a_2 \phi_2 + b_2 \phi_4) & 0 & r_2 b_2 \phi_2 \\
  0 & \frac{1}{\tau_1} & -\frac{1}{\tau_1} & 0 \\
  \frac{1}{\tau_2} & 0 & 0 & -\frac{1}{\tau_2}
\end{bmatrix}.
\]

It is clear that the matrix \( E \) is an irreducible matrix in the functional sense since the strict monotonicity of traveling wavefronts ensures that \( \frac{1}{a_1} - \phi_1(t) > 0 \) and \( b_2 \phi_4(t) > 0 \) hold for any \( t \in \mathbb{R} \), see this by Remark 3.7.

In the remainder of this section, \( \mathbb{C} \) will denote the set of complex numbers, \( \text{Re} \lambda \) and \( \text{Im} \lambda \) denote the real part and image part of \( \lambda \in \mathbb{C} \).

By applying [55, p. 212, Theorem 5.1] to the operator \( L \), we obtain the following result.

**Lemma 4.4.** For \( \lambda \in \mathbb{C} \) and the asymptotic boundary problem

\[ Lu = \lambda u, \quad u \in C_0, \quad (4.1) \]

the following results are true:
(i) (4.1) has no other solution different from zero for $\text{Re} \lambda \geq 0$ and $\lambda \neq 0$;
(ii) each solution $\Psi(x)$ of (4.1) for $\lambda = 0$ has the form $\Psi(x) = k \Phi'(x)$, where $k$ is a constant.

**Lemma 4.5.** The traveling wavefront $\Phi(x + ct)$ of (2.7) is asymptotically stable with phase shift according to the norm $\| \cdot \|_{C_0}$.

**Proof.** Consider the following two matrices

$$E^+ = \begin{bmatrix}
-d_1 \lambda^2 + r_1 & 0 & 0 & \frac{1}{\tau_2} \\
0 & -d_2 \lambda^2 + r_2(1 - \frac{b_2}{a_1}) & 0 & \frac{1}{\tau_1} \\
\frac{b_1}{a_1} & 0 & -d_2 \lambda^2 - \frac{1}{\tau_1} & 0 \\
0 & 0 & 0 & -d_1 \lambda^2 - \frac{1}{\tau_2}
\end{bmatrix},
$$

$$E^- = \begin{bmatrix}
-d_1 \lambda^2 - r_1 & 0 & 0 & \frac{1}{\tau_2} \\
0 & -d_2 \lambda^2 - r_2(1 - \frac{b_2}{a_1}) & 0 & \frac{1}{\tau_1} \\
\frac{b_1}{a_1} & 0 & -d_2 \lambda^2 + \frac{1}{\tau_1} & 0 \\
0 & 0 & 0 & -d_1 \lambda^2 + \frac{1}{\tau_2}
\end{bmatrix},
$$

of which the eigenvalues are

$$-d_1 \lambda^2 - r_1, \quad -d_2 \lambda^2 a_1 - r_2 a_1 + r_2 b_2 \frac{1}{a_1}, \quad \frac{d_2 \lambda^2 \tau_1 + 1}{\tau_1}, \quad \frac{d_1 \lambda^2 \tau_2 + 1}{\tau_2}$$

and

$$-d_1 \lambda^2 a_2 - r_1 a_2 + r_1 b_1 \frac{1}{a_2}, \quad -d_2 \lambda^2 - r_2, \quad \frac{d_2 \lambda^2 \tau_1 + 1}{\tau_1}, \quad \frac{d_1 \lambda^2 \tau_2 + 1}{\tau_2},$$

respectively. It is clear that all the eigenvalues are negative for any $\lambda \in \mathbb{R}$. Combining this with Lemmas 4.4 and 4.5, the traveling wavefront $\Phi(t)$ is asymptotically stable with phase shift according to the norm $\| \cdot \|_{C_0}$ by Lemma 4.2. The proof is complete. □

**Theorem 4.6.** For system (3.1) with (3.2), assume that

(i) $0 \leq (\psi_1(x, s), \psi_2(x, s)) \leq (\frac{1}{a_1}, \frac{1}{a_2})$, $x \in \mathbb{R}$, $s \leq 0$;

(ii) $\lim_{x \to -\infty}(\psi_1(x, 0), \psi_2(x, 0)) = \lim_{x \to -\infty}((g_2 \ast \psi_1)(x, 0), (g_1 \ast \psi_2)(x, 0)) = (0, 0), \quad \lim_{x \to -\infty}(\psi_1(x, 0), \psi_2(x, 0)) = \lim_{x \to -\infty}((g_2 \ast \psi_1)(x, 0), (g_1 \ast \psi_2)(x, 0)) = (\frac{1}{a_1}, \frac{1}{a_2})$;

(iii) there exists $\kappa \in \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} \int_0^\infty \left| \int_{-\infty}^\infty \frac{1}{\tau_1} \frac{1}{4d_2 \pi \theta} e^{-\frac{x^2}{4d_2 \pi \theta}} \left| \psi_2(x, y, -\theta) - \psi_2(x - c \theta + \kappa, y) \right| dy d\theta \right|.$$
is sufficiently small.

Then there exist constants $M > 0$, $b > 0$ and $h \in \mathbb{R}$ such that
\[
\sup_{x \in \mathbb{R}} \left[ |\phi_1(x + ct + h) - u_1(x, t)| + |\phi_2(x + ct + h) - u_2(x, t)| \right] \leq Me^{-bt},
\]
where $M$, $b$ are independent of the initial value and $t$.

5. Uniqueness of wave speeds

In this section, we will study the uniqueness of wave speed by the upper and lower solution technique, which is motivated by the method of Mischaikow and Hutson [40]. But their Theorem 7.3 cannot apply to (2.7) directly since the nonlinear terms of (2.7) do not satisfy their condition (H2). For more details related to this method, one refers to [7,49,61]. Therefore, we need to develop some new techniques to deal with our system (2.7).

In this section, we will always assume that (3.15) satisfies the following conditions:

(H1) $0 \leq (\psi_1, \psi_2, \psi_3, \psi_4) \leq \Phi_+;$

(H2) $\limsup_{x \to -\infty} \psi_i(x)$ is sufficiently small for $i = 1, 2, 3, 4$;

(H3) $\limsup_{x \to \infty} (\frac{1}{a_1} - \psi_i(x))$, $i = 1, 4$, and $\limsup_{x \to \infty} (\frac{1}{a_2} - \psi_i(x))$, $i = 2, 3$, are small.

For $i = 1, 2, 3, 4$, let $\delta_i(t)$ be slowly increasing $C^2$ functions defined on $\mathbb{R}$, namely, the first and twice derivation of them are sufficiently small. Furthermore, assume that $\delta_i(t) = \delta_i^\pm$ for sufficient large $|t|$ and small $\delta_i^\pm > 0$ satisfying
\[
a_2 \delta_2^+ > b_2 \delta_4^+, \quad a_1 \delta_1^- > b_1 \delta_3^- , \quad \delta_3^\pm > \delta_2^\pm, \quad \delta_4^\pm > \delta_1^\pm.
\]

Lemma 5.1. Assume that $(\phi_1(x + ct), \phi_2(x + ct), \phi_3(x + ct), \phi_4(x + ct))$ be a traveling wavefront of (2.7). Define the continuous functions
\[
\tilde{u}_1(x, t) = \min \left\{ \phi_1(x + ct + \xi^+ - \sigma e^{-\beta t}) + \delta_1(x + ct)e^{-\beta t}, \frac{1}{a_1} \right\},
\]
\[
\tilde{u}_2(x, t) = \min \left\{ \phi_2(x + ct + \xi^+ - \sigma e^{-\beta t}) + \delta_2(x + ct)e^{-\beta t}, \frac{1}{a_2} \right\},
\]
\[
\tilde{u}_3(x, t) = \min \left\{ \phi_3(x + ct + \xi^+ - \sigma e^{-\beta t}) + \delta_3(x + ct)e^{-\beta t}, \frac{1}{a_2} \right\},
\]
\[
\tilde{u}_4(x, t) = \min \left\{ \phi_4(x + ct + \xi^+ - \sigma e^{-\beta t}) + \delta_4(x + ct)e^{-\beta t}, \frac{1}{a_1} \right\}.
\]
Then there exist constants $\sigma > 0, \beta > 0, \xi^+ \geq 0$ such that $(\bar{u}_1(x,t), \bar{u}_2(x,t), \bar{u}_3(x,t), \bar{u}_4(x,t))$ is an upper solution of (3.15) for $x \in \mathbb{R}, t > 0$ if (H1)–(H3) hold.

**Proof.** The initial value condition can be proved easily by choosing sufficiently large $\xi^+ > 0$. We now prove the differential inequalities in the definition of upper solution. Let $x + ct + \xi^+ - \sigma e^{-\beta t} = \varsigma$. Then

$$
\frac{\partial \bar{u}_i(x,t)}{\partial t} = c\phi'_i(\varsigma) + \sigma \beta e^{-\beta t} \phi'_i(\varsigma) + c\delta'_i(x + ct)e^{-\beta t} - \beta \delta_i(x + ct)e^{-\beta t}.
$$

$$
\Delta \bar{u}_i(x,t) = \phi''_i(\varsigma) + \delta''_i(x + ct)e^{-\beta t}.
$$

If $\bar{u}_1(x,t) = \frac{1}{a_1}$ holds, then it is clear that $\bar{u}_1(x,t)$ is an upper solution of (2.7). If $\bar{u}_1(x,t) < \frac{1}{a_1}$, then the definitions of traveling wavefronts and the upper lower solutions imply that we only need to prove that

$$
\sigma \beta \phi'_i(\varsigma) + c\delta'_i(x + ct) - \beta \delta_i(x + ct) \geq d_1\delta''_i(x + ct) - r_1\delta_1(x + ct)[b_1\bar{u}_3(x,t) - a_1\bar{u}_1(x,t)]
$$

$$
+ r_1\left[\frac{1}{a_1} - \phi_1(\varsigma)\right][-a_1\delta_1(x + ct) + b_1\delta_3(x + ct)]. \tag{5.1}
$$

Let $M > 0$ be large enough such that $\phi_1(-M), \phi_3(-M), \frac{1}{a_1} - \phi_1(M)$ and $\frac{1}{a_1} - \phi_3(M)$ are sufficiently small and $\delta_i(t)$ are constant for $|t| > M$ and $i = 1, 2, 3, 4$. If $|\varsigma| \leq M$, then Remark 3.7 implies that (5.1) holds if $\sigma > 0$ is large enough.

For $\varsigma \leq -M$, it suffices to prove that

$$
-2\beta \delta_1^- > -r_1\delta_1^- [b_1\delta_3^- - a_1\delta_1^-] + r_1\left[\frac{1}{a_1} - \phi_1(\varsigma)\right][-a_1\delta_1^- + b_1\delta_3^-]. \tag{5.2}
$$

Note that $b_1\delta_3^- - a_1\delta_1^- < 0$ holds and $\delta_1^- > 0$ is small, e.g., $\delta_1^- < \frac{1}{2a_1}$, then (5.2) is true if $\beta > 0$ is small enough.

If $\varsigma \geq M$, then

$$
\frac{1}{a_1} - \phi_1(\varsigma) \to 0 , \quad b_1\bar{u}_3(x,t) - a_1\bar{u}_1(x,t) \to \frac{b_1}{a_2} - 1 > 0
$$

as $\varsigma \to \infty$, which implies that (5.1) holds if $\beta > 0$ is small enough. Thus, $\bar{u}_1(x,t)$ is an upper solution of (2.7).

Similarly, we can prove that $\bar{u}_2(x,t)$ is an upper solution if $a_2\delta_2^+ > b_2\delta_4^+$ and $\sigma > 0$ is large enough, $\beta > 0$ is small enough.

For $i = 3$, the result is clear if $\bar{u}_3(x,t) = \frac{1}{a_2}$. Otherwise, it suffices to prove that

$$
\sigma \beta \phi'_3(\varsigma) + c\delta'_3(x + ct) - \beta \delta_3(x + ct) \geq d_2\delta''_3(x + ct) - \frac{1}{\tau_1}\delta_3(x + ct) + \frac{1}{\tau_1}\delta_2(x + ct),
$$
which is true if \( \delta^{\pm}_1 > \delta^{\pm}_2 \) and \( \sigma > 0 \) is sufficiently large, \( \beta > 0 \) is small enough. Similarly, we can prove that \( \bar{u}_4(x,t) \) is the upper solution if \( \delta^{\pm}_4 > \delta^{\pm}_1 \), \( \sigma > 0 \) is large enough and \( \beta > 0 \) is sufficiently small. The proof is complete. \( \square \)

**Lemma 5.2.** Assume that \((\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct), \phi_4(x+ct))\) be a traveling wavefront of (2.7). Define the continuous functions

\[
\begin{align*}
    u_1(x,t) &= \max\{\phi_1(x+ct + \xi^- + \sigma e^{-\beta t}) - \delta_1(x+ct)e^{-\beta t}, 0\}, \\
    u_2(x,t) &= \max\{\phi_2(x+ct + \xi^- + \sigma e^{-\beta t}) - \delta_2(x+ct)e^{-\beta t}, 0\}, \\
    u_3(x,t) &= \max\{\phi_3(x+ct + \xi^- + \sigma e^{-\beta t}) - \delta_3(x+ct)e^{-\beta t}, 0\}, \\
    u_4(x,t) &= \max\{\phi_4(x+ct + \xi^- + \sigma e^{-\beta t}) - \delta_4(x+ct)e^{-\beta t}, 0\}.
\end{align*}
\]

Then there exist constants \( \sigma > 0, \beta > 0, \xi^- \leq 0 \) such that \((u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))\) is a lower solution of (3.15) for \( x \in \mathbb{R}, t > 0 \) if (H1)–(H3) hold.

The proof of Lemma 5.2 is similar to that of Lemma 5.1, so we omit it here.

**Theorem 5.3.** Assume that \( \Phi_1(x+c_1t) \) is a traveling wavefront of (2.7) connecting 0 with \( \Phi_+ \). Then \( c = c_1 \).

**Proof.** It is clear that the traveling wavefront \( \Phi_1 \) satisfies (H1)–(H3). Let \( t = 0 \). Then we always can choose \( \xi^+ \) and \( \xi^- \) in Lemmas 5.1–5.2 such that

\[
    \underline{u}(x,0) \leq \Phi_1(x) \leq \bar{u}(x,0)
\]

for all \( x \in \mathbb{R} \). Then the comparison principle and asymptotic behavior of the upper and lower solutions imply what we wanted. The proof is complete. \( \square \)

**Theorem 5.4.** For the traveling wavefronts connecting \( E_1 \) and \( E_2 \) of (1.4), the wave speed is unique.

**Remark 5.5.** From the proof of Theorem 5.3, we also can find that the wave speed is unique for any bistable waves \( \Phi^* \) of (2.7) (not require the monotonicity) which connects 0 with \( \Phi_+ \) and satisfies \( 0 \leq \Phi^* \leq \Phi_+ \). And the similar result holds for (1.4).

**Remark 5.6.** From the upper and lower solutions in this section, we also can find the solution \( u(x,t) \) of (1.4) is convergent as \( |x + ct| \to \infty \) if the initial value satisfies (H1)–(H3).

6. Further discussion

In this section, we further consider the traveling wavefronts of the reaction diffusion system with nonlocal delay and bistable nonlinearities. We organize this section by two cases, one is the mathematical theory, another is the ecological sense.
6.1. Mathematical theory

Let us consider the following system with two equations

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \Delta u_1(x,t) + f_1(u_1(x,t), (g_1 * u_2)(x,t)), \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \Delta u_2(x,t) + f_2(u_2(x,t), (g_2 * u_1)(x,t)),
\end{align*}
\]  

(6.1)

where \( f = (f_1, f_2) \) is bistable in the sense of

\[
\begin{align*}
\frac{du_1(t)}{dt} &= f_1(u_1(t), u_2(t)), \\
\frac{du_2(t)}{dt} &= f_2(u_1(t), u_2(t)).
\end{align*}
\]  

(6.2)

Moreover, we assume that

\[ f_1(\phi^+_1, \phi^+_2) = f_2(\phi^+_1, \phi^+_2) = 0, \]

where \( \phi^-_1 < \phi^+_1, \phi^-_2 < \phi^+_2 \) and all the eigenvalues of \( f'(\phi^+_1, \phi^+_2) \) have negative real part. Furthermore, there are only finite point \( (\psi^+_i, \psi^-_i) \in [\phi^-_1, \phi^+_1] \times [\phi^-_2, \phi^+_2], i = 1, 2, \ldots, m \), such that

\[ f_1(\psi^+_i, \psi^-_i) = f_2(\psi^+_i, \psi^-_i) = 0, \quad i = 1, 2, \ldots, m. \]

Moreover, at least one eigenvalue of \( f'(\psi^+_i, \psi^-_i) \) for \( i = 1, 2, \ldots, m \), has positive real part and there exists \( p_i \geq 0, i = 1, 2, \ldots, m \), such that \( p_i f'(\psi^+_i, \psi^-_i) > 0 \) holds for all \( i = 1, 2, \ldots, m \).

Assume that \( g_1, g_2 \) are defined by (1.5), then we can transform (6.1) into a higher dimensional reaction diffusion system without delay. Thus, we can consider (6.1) by using similar technique to that of (1.4). Furthermore, from our discussion, it is easy to see that our results can be generalized to the higher dimensional case.

We should like to mention some results on the stability of traveling wavefronts of bistable equation. Chen [7] and Zinner [63] employed the so-called squeezing technique to consider abstract integral equations and the discrete Nagumo equation, which was also used by Ma and Zou [33], Smith and Zhao [49] for delayed equation. Recently, Wu and Li [61] developed the technique to a reaction diffusion system with two equations. We expect to develop the squeezing technique to higher dimension systems. If so, we can establish some asymptotic stability results on traveling wavefronts, which are different from that of the current paper. We shall consider this problem in our further research.

In addition, we can generalize the kernel functions \( g_1, g_2 \) to other forms, for example,

\[
\begin{align*}
(\alpha_1 * u_2)(x,t) &= \int_0^\infty \int_{-\infty}^\infty \frac{\theta}{\tau_1^2} e^{-\frac{1}{\tau_1^2} \theta} \frac{1}{\sqrt{4d_2 \pi \theta}} e^{-\frac{\theta^2}{4d_2 \theta}} u_2(x-y, t-\theta) \, dy \, d\theta, \\
(\alpha_2 * u_1)(x,t) &= \int_0^\infty \int_{-\infty}^\infty \frac{\theta}{\tau_2^2} e^{-\frac{1}{\tau_2^2} \theta} \frac{1}{\sqrt{4d_1 \pi \theta}} e^{-\frac{\theta^2}{4d_1 \theta}} u_1(x-y, t-\theta) \, dy \, d\theta,
\end{align*}
\]  

(6.3)
where $\alpha_1, \alpha_2$ is the so-called strong kernel ($g_1, g_2$ is the so-called weak kernel). Recently, Lin and Yuan [31] considered the following system

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \Delta u_1(x,t) + r_1 u_1(x,t) [1 - a_1 u_1(x,t) - b_1(\alpha_1 * u_2)(x,t)], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \Delta u_2(x,t) + r_2 u_2(x,t) [1 - a_2 u_2(x,t) - b_2(\alpha_2 * u_1)(x,t)],
\end{align*}
\]

(6.4)

and established the traveling wave solutions (which cannot ensure the monotonicity) connecting $E_1$ and $E_2$ if $k_1k_2 \leq 0$ and the method is similar to that of Gourley and Ruan [18]. It is interesting to consider the traveling wavefronts of (6.4) if (1.9) holds.

Now consider the following system without delay

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \Delta u_1(x,t) + r_1 u_1(x,t) [1 - a_1 u_1(x,t) - b_1 u_2(x,t)], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \Delta u_2(x,t) + r_2 u_2(x,t) [1 - a_2 u_2(x,t) - b_2 u_1(x,t)].
\end{align*}
\]

(6.5)

If (1.9) holds, then there are some results on the existence of traveling wavefronts of (6.5), see Volpert et al. [55]. Comparing the existence result of (6.5) with that of (1.1), we find that the non-local delays appeared in interaction terms are not sensitive to the existence of bistable traveling fronts of (1.1).

6.2. Ecological background

In this paper, we established the existence and stability of traveling wavefronts of (1.4), which implies that there exists a zone which will lead to the separation of two species in spatial domain if the initial value of (2.7) satisfies (H1)–(H3) in Section 5. More precisely, for any given $0 < \varepsilon < \min\{\frac{1}{a_1}, \frac{1}{a_2}\}$, there is only a finite space interval such that $u_1(x,t) > \varepsilon$, $u_2(x,t) > \varepsilon$ if $t > 0$ is large enough (see this by Remark 5.6). This is similar to the spatial isolation phenomena reported by [41,53]. However, traveling wavefronts of (1.4) also indicate that the invasion of one species of $u_1, u_2$ if the wave speed $c \neq 0$, which implies that the unsymmetry of spatial isolation and a traveling wavefront is an invasion wave. If the wave speed $c = 0$, then the spatial isolation is symmetrical. Thus, the sign of wave speed $c$ will determine which species has stronger competitive capacity than the others. From this view, the wave speed $c$ should be a function of $a_1, a_2, b_1, b_2$. How to determine the sign of $c$ remains open as a challenge problem and is interesting in ecological science. The similar discussion on reaction diffusion systems with bistable nonlinearities can refer to Mischaikow and Hutson [40].

Furthermore, motivated by the corresponding results of the following Huxley equation

\[
\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + (u^2(x,t) - 1)(u(x,t) - a),
\]

(6.6)

with $a \in (-1, 1)$. It is well known that (6.6) can be used to describe the phase transform process in physics and has a unique traveling wavefront $\varphi(\xi)$ under the sense of phase shift, and its wave speed is $\sqrt{2a}$.

By the stability of traveling wavefronts (see, e.g., Chen [7] and Fusco et al. [15]), we know that the equilibrium 1 and the equilibrium $-1$ is symmetry if $a = 0$. If $a > 0$, then the equilibrium
−1 is dominant, whereas the equilibrium 1 is in a dominant state if \( a < 0 \). The more details refers to Fusco et al. [15]. Motivated by the result concerning with (6.6), we conjecture that the sign of \( c \) (partly) depends on the distances between two of \( E_1, E_2, E^* \) and 0 although this has not been verified.

References


