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A FIBRATION THAT DOES NOT ACCEPT TWO DISJOINT MANY-VALUED SECTIONS

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The fibration $\eta:\prod_{i=0}^{\infty} S^{(2^i)}\to\prod_{i=1}^{\infty} \mathrm{RP}^{(2^i)}$ does not accept two disjoint many-valued sections.

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open map — many-valued section
fibration — 1-soft map

In the Lewis list of problems [2, 3] there is the following:

Problem 166 (Bula). Suppose $f:X\to Y$ is an open map, with each of X and Y compact metric and each $f^{-1}(y)$ infinite. Do there exist disjoint closed subsets F and H of X such that $f(F)=f(H)=Y$?

The same question was formulated to the author by E. V. Shchepin in the late 1970s.

It is well known that the problem has an affirmative answer if $\dim Y<\infty$. Nevertheless it turns out that in general the answer is negative.

Every compact set $F\subset X$ with the property $f(F)=Y$ defines a many-valued upper semicontinuous section $s:Y\to X$ of the map $f:X\to Y$ by the formula $s(y)=F\cap f^{-1}(y)$. The reverse statement is also true: Every many-valued upper semicontinuous section s of f defines a compact set $F_s=\bigcup_{y\in Y} s(y)$ such that $f(F)=Y$. Now we can reformulate the problem:

Does an arbitrary open mapping f between compact metric spaces with infinite fibers have two disjoint many-valued sections?

In [1] the following map was considered

$$\eta:\prod_{i=0}^{\infty} S^{(2^i)}\to\prod_{i=0}^{\infty} \mathrm{RP}^{(2^i)},$$

where $\eta=\prod_{i=0}^{\infty}\nu_{2^i}$ and the map $\nu_k:S^k\to\mathrm{RP}^k$ is a 2-fold covering map of the k-dimensional sphere onto real projective k-space. The fibration η as well as every open mapping between compacta has a section in probability measures. The main result of [1] claims that the fibration η does not accept two disjoint such sections.

Theorem 1. *The fibration η does not accept two disjoint many-valued upper semi-continuous sections.*

$P(X)$ denotes the space of all regular Borel probability measures on X in the weak $*$-topology and $P(f): P(X) \to P(Y)$ denotes the map which is generated by the map $f: X \to Y$. There is a natural embedding $Y \hookrightarrow P(Y)$ of which every point $y \in Y$ carries in the Dirac measure δ_y. Let $f_P: P(f)^{-1}(Y) \to Y$ denote the restriction of the map $P(f)$ onto the set $P(f)^{-1}(Y)$ and let $f: X \to Y$ be an n-fold covering map. The map f_P is a locally trivial fibration with the $(n-1)$-dimensional simplex as a fiber. Thus the map f_P is naturally included in an $(n-1)$-dimensional vector bundle $vf: vX \to Y$. The quotient space of all real-valued functions on the set $f^{-1}(y)$ by the subspace of all constant functions may be regarded as a fiber $(vf)^{-1}(y)$. Then the space $P(f^{-1}(y))$ is equal to the image under the quotient map of the set of all nonnegative functions g with the property $\sum_{z \in f^{-1}(y)} g(z) = 1$. It is easy to check that the map f_P has two disjoint sections if and only if the vector bundle vf has everywhere nonzero sections.

We denote by η_k the product $\prod_{i=0}^{k} \nu_{2^i}$ of covering maps.

Lemma 2. *The map $(\eta_k)_P$ does not accept two disjoint sections for any k.*

Proof. Every two disjoint sections s_1, s_2 of the map $(\eta_k)_P$ generate everywhere nonzero sections of the vector bundle $v\eta_k$ by the formula $s(y) = s_1(y) - s_2(y)$. In this case the top Stiefel-Whitney class is equal to zero [5]. The contradiction with calculation in [1] proves the lemma. □

Proof of Theorem 1. Assume the contrary: There exist two disjoint many-valued upper semicontinuous sections u_1, u_2 of the map η. We denote by π_k^∞ the projection of the product $\prod_{i=0}^{\infty} S^{(2^i)}$ onto its subproduct $\prod_{i=0}^{k} S^{(2^i)}$. One can choose a number k such that $\pi_k^\infty(F_{u_1}) \cap \pi_k^\infty(F_{u_2}) = \emptyset$. *Let* ρ be a metric on the manifold $M = \prod_{i=0}^{k} S^{(2^i)}$ and let W_1, W_2 be open disjoint neighbourhoods of the sets $\pi_k^\infty(F_{u_1})$ and $\pi_k^\infty(F_{u_2})$. We define sections s_1, s_2 of the map $(\eta_k)_P$ in the following way. For an arbitrary $x \in \prod_{i=0}^{k} \mathrm{RP}^{(2^i)}$ we define

$$s_1(x) = \sum_{y \in \eta_k^{-1}(x)} \frac{\rho(y, M \setminus W_1)}{\sum_{z \in \eta_k^{-1}(x)} \rho(z, M \setminus W_1)} \cdot \delta_y,$$

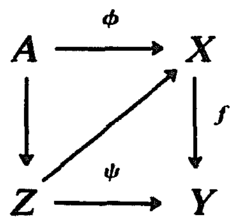
$$s_2(x) = \sum_{y \in \eta_k^{-1}(x)} \frac{\rho(y, M \setminus W_2)}{\sum_{z \in \eta_k^{-1}(x)} \rho(z, M \setminus W_2)} \cdot \delta_y.$$

The measures $s_1(x)$ and $s_2(x)$ are different because they have different supports: $\operatorname{supp} s_1(x) \subseteq W_1$ and $\operatorname{supp} s_2(x) \subseteq W_2$. It is easy to check that the maps s_2, s_1 are continuous.

The contradiction with Lemma 2 proves the theorem. □

Remark. The proof of Theorem 1 implies that any two many-valued upper semicontinuous sections of η coincide in some point.

We recall that a map $f: X \to Y$ is called 1-soft if there exists a solution of the following lifting problem



for an arbitrary closed couple (Z, A) with $\dim Z \leq 1$ and arbitrary continuous maps ϕ, ψ with $f \circ \phi = \psi|_A$.

Michael's theorem [4] claims that a map $f: X \to Y$ is 1-soft if and only if f is monotone-open and the family of fibers $\{f^{-1}(y)\}_{y \in Y}$ is equi-LC^0.

Question. Let $f: X \to Y$ be a 1-soft mapping of compact metric spaces with infinite fibers. Is it true that:

(1) f has two disjoint many-valued upper semicontinuous sections?

(2) f has two disjoint many-valued continuous sections?

(3) f has two many-valued continuous sections which are distinguished in any point $y \in Y$?

(4) The map $f_{\exp}: (\exp f)^{-1}(Y) \to Y$ is a trivial fibration with the Hilbert cube as a fiber?

Here $\exp X$ denotes the space of all nonempty closed subsets of X in the Vietoris topology (another symbol for this space is 2^X). any continuous map between compacta $f: X \to Y$ induces a map $\exp f: \exp X \to \exp Y$. By $f_{\exp}$ we denote the restriction of $\exp f$ onto $(\exp f)^{-1}(Y)$.

References

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