Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics Letters 20 (2007) 1218–1222

Applied Mathematics Letters

www.elsevier.com/locate/aml

Coefficient bounds for some families of starlike and convex functions of complex order

Osman Altıntaş[a], Hüseyin Irmak[a], Shigeyoshi Owa[b], H.M. Srivastava[c,*]

[a] *Department of Mathematics Education, Başkent University, Bağlıca Campus, TR-06810 Ankara, Turkey*
[b] *Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*
[c] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*

Received 4 January 2007; accepted 31 January 2007

Abstract

In the present work, the authors determine coefficient bounds for functions in certain subclasses of starlike and convex functions of complex order, which are introduced here by means of a family of nonhomogeneous Cauchy–Euler differential equations. Several corollaries and consequences of the main results are also considered.

Keywords: Analytic functions; Coefficient bounds; Starlike functions of complex order; Convex functions of complex order; Nonhomogeneous Cauchy–Euler differential equations; Inequalities

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\gamma)$ if it also satisfies the following inequality:

$$\Re\left[1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] > 0 \quad (z \in \mathbb{U};\ \gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}). \tag{1.2}$$

* Corresponding author. Tel.: +1 604 721 7455; fax: +1 604 721 8962.
E-mail addresses: oaltintas@baskent.edu.tr (O. Altıntaş), hisimya@baskent.edu.tr (H. Irmak), owa@math.kindai.ac.jp (S. Owa), harimsri@math.uvic.ca (H.M. Srivastava).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.01.003

Furthermore, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}(\gamma)$, if it also satisfies the following inequality:

$$\Re\left(1+\frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right)>0 \quad (z\in\mathbb{U};\gamma\in\mathbb{C}^*). \tag{1.3}$$

The function classes $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ were considered earlier by Nasr and Aouf [10–12] and Wiatrowski [15], respectively (see also [8,9,14]).

We also let $\mathcal{SC}(\gamma,\lambda,\beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition:

$$\Re\left[1+\frac{1}{\gamma}\left(\frac{z[\lambda zf'(z)+(1-\lambda)f(z)]'}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)\right]>\beta \quad \left(f(z)\in\mathcal{A};0\leqq\lambda\leqq 1;0\leqq\beta<1;\gamma\in\mathbb{C}^*;z\in\mathbb{U}\right). \tag{1.4}$$

Clearly, we have the following relationships:

$$\mathcal{SC}(\gamma,0,0)\equiv\mathcal{S}^*(\gamma) \quad \text{and} \quad \mathcal{SC}(\gamma,1,0)\equiv\mathcal{C}(\gamma).$$

Recently, the function class satisfying the inequality (1.4) was considered by Altıntaş et al. [4]. For other special cases of the function class $\mathcal{SC}(\gamma,\lambda,\beta)$, we refer the reader to the investigations by (for example) Altıntaş et al. [1–3, 5–7]. The main object of the present investigation is to derive some coefficient bounds for functions in the subclass $\mathcal{B}(\gamma,\lambda,\beta;\mu)$ of $\mathcal{A}$, which consists of functions $f(z)\in\mathcal{A}$ satisfying the following *nonhomogeneous* Cauchy–Euler differential equation:

$$z^2\frac{d^2w}{dz^2}+2(1+\mu)z\frac{dw}{dz}+\mu(1+\mu)w=(1+\mu)(2+\mu)g(z)$$
$$(w:=f(z)\in\mathcal{A};\ g(z)\in\mathcal{SC}(\gamma,\lambda,\beta);\ \mu\in\mathbb{R}\setminus(-\infty,-1]). \tag{1.5}$$

2. Coefficient estimates for the function class $\mathcal{SC}(\gamma,\lambda,\beta)$

For functions in the class $\mathcal{SC}(\gamma,\lambda,\beta)$, we first establish the following result.

Theorem 1. *Let the function $f(z)\in\mathcal{A}$ be defined by (1.1). If the function $f(z)$ is in the class $\mathcal{SC}(\gamma,\lambda,\beta)$, then*

$$|a_n|\leqq\frac{\prod_{j=0}^{n-2}[j+2|\gamma|(1-\beta)]}{(n-1)![1+\lambda(n-1)]} \quad (n\in\mathbb{N}^*:=\mathbb{N}\setminus\{1\}=\{2,3,4,\ldots\}). \tag{2.1}$$

Proof. Let the function $f(z)\in\mathcal{A}$ be given by (1.1) and let the function $\mathcal{F}(z)$ be defined by

$$\mathcal{F}(z):=\lambda zf'(z)+(1-\lambda)f(z) \quad \left(f(z)\in\mathcal{A};0\leqq\lambda\leqq 1;z\in\mathbb{U}\right).$$

Then, from (1.4) and the definition of the function $\mathcal{F}(z)$ above, it is easily seen that

$$\Re\left[1+\frac{1}{\gamma}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}-1\right)\right]>\beta$$

with

$$\mathcal{F}(z)=z+\sum_{k=2}^{\infty}A_kz^k\in\mathcal{A} \quad \left(A_k:=[1+\lambda(k-1)]a_k;k\in\mathbb{N}^*\right).$$

Thus, by setting

$$\frac{1+\frac{1}{\gamma}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}-1\right)-\beta}{1-\beta}=h(z)$$

or, equivalently,

$$z\mathcal{F}'(z)=[1+\gamma(1-\beta)(h(z)-1)]\mathcal{F}(z), \tag{2.2}$$

we get

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{2.3}$$

Since

$$\Re(h(z)) > 0 \quad (0 \leqq \beta < 1; \gamma \in \mathbb{C}^*),$$

we conclude that

$$|c_n| \leqq 2 \quad (n \in \mathbb{N}).$$

We also find from (2.2) and (2.3) that

$$(n-1)A_n = 2\gamma(1-\beta)[1 + A_2 + A_3 + \cdots + A_{n-1}].$$

In particular, for $n = 2, 3, 4$, we have

$$A_2 = 2\gamma(1-\beta) \Rightarrow |A_2| \leqq 2|\gamma|(1-\beta),$$
$$2A_3 = 1 + A_2 \Rightarrow |A_3| \leqq \frac{2|\gamma|(1-\beta)[1+2|\gamma|(1-\beta)]}{2!},$$

and

$$3A_4 = 1 + A_2 + A_3 \Rightarrow |A_4| \leqq \frac{2|\gamma|(1-\beta)[1+2|\gamma|(1-\beta)][2+2|\gamma|(1-\beta)]}{3!},$$

respectively. Using the principle of mathematical induction, we obtain

$$|A_n| \leqq \frac{\prod_{j=0}^{n-2}[j + 2|\gamma|(1-\beta)]}{(n-1)!} \quad (n \in \mathbb{N}^*). \tag{2.4}$$

Moreover, by the relationship between the functions $f(z)$ and $\mathcal{F}(z)$, it is clear that

$$A_n = [1 + \lambda(n-1)]a_n \quad (n \in \mathbb{N}^*), \tag{2.5}$$

just as we indicated above.

The inequality (2.1) now follows from (2.4) and (2.5). This evidently completes the proof of Theorem 1. □

By choosing suitable values of the admissible parameters β, λ, and γ in Theorem 1 above, we deduce the following corollaries.

Corollary 1. *If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{SC}(\gamma, \lambda, 0)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2}(j + 2|\gamma|)}{(n-1)![1+\lambda(n-1)]} \quad (n \in \mathbb{N}^*).$$

Corollary 2 (*cf., e.g., Nasr and Aouf [10]*). *If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(\gamma)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2}(j + 2|\gamma|)}{(n-1)!} \quad (n \in \mathbb{N}^*).$$

Corollary 3 (*cf., e.g., Nasr and Aouf [10]*). *If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}(\gamma)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2}(j + 2|\gamma|)}{n!} \quad (n \in \mathbb{N}^*).$$

Corollary 4. *If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{SC}(1-\alpha, \lambda, \beta)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2} [j + 2(1-\alpha)(1-\beta)]}{(n-1)![1+\lambda(n-1)]} \quad (n \in \mathbb{N}^*).$$

Corollary 5 (*cf.* Robertson [13]). *If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(1-\alpha)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2} [j + 2(1-\alpha)]}{(n-1)!} \quad (n \in \mathbb{N}^*).$$

Corollary 6 (*cf.* Robertson [13]). *If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}(1-\alpha)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2} [j + 2(1-\alpha)]}{n!} \quad (n \in \mathbb{N}^*).$$

3. Coefficient bounds for the function class $\mathcal{B}(\gamma, \lambda, \beta; \mu)$

Our main coefficient bounds for functions in the class $\mathcal{B}(\gamma, \lambda, \beta; \mu)$ are given by Theorem 2 below.

Theorem 2. *Let the function $f(z) \in \mathcal{A}$ be defined by (1.1). If the function $f(z)$ is in the class $\mathcal{B}(\gamma, \lambda, \beta; \mu)$, then*

$$|a_n| \leqq \frac{(1+\mu)(2+\mu) \prod_{j=0}^{n-2} [j + 2|\gamma|(1-\beta)]}{(n-1)!(n+\mu)(n+1+\mu)[1+\lambda(n-1)]} \quad (n \in \mathbb{N}^*). \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{A}$ be given by (1.1). Also let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{SC}(\gamma, \lambda, \beta), \tag{3.2}$$

so that

$$a_n = \frac{(1+\mu)(2+\mu)}{(n+\mu)(n+1+\mu)} b_n \quad \left(n \in \mathbb{N}^*; \mu \in \mathbb{R} \setminus (-\infty, -1]\right). \tag{3.3}$$

Thus, by using Theorem 1, we readily obtain

$$|a_n| \leqq \frac{(1+\mu)(2+\mu) \prod_{j=0}^{n-2} [j + 2|\gamma|(1-\beta)]}{(n-1)!(n+\mu)(n+1+\mu)[1+\lambda(n-1)]} \quad (n \in \mathbb{N}^*),$$

which is precisely the assertion (3.1) of Theorem 2. □

Acknowledgements

The first- and the second-named authors were supported by Başkent University (Ankara, Turkey). They would also like to thank Professor Mehmet Haberal (Rector, Başkent University), who generously supports scientific research in all aspects. The present investigation was supported, in part, by the Japanese Ministry of Education, Science and Culture under a Grant-in-Aid for General Scientific Research (No. 046204) and, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

References

[1] O. Altıntaş, H. Irmak, H.M. Srivastava, Fractional calculus and certain starlike functions with negative coefficients, Comput. Math. Appl. 30 (2) (1995) 9–15.

[2] O. Altıntaş, Ö. Özkan, Starlike, convex and close-to-convex functions of complex order, Hacettepe. Bull. Natur. Sci. Engrg. Ser. B 28 (1991) 37–46.
[3] O. Altıntaş, Ö. Özkan, On the classes of starlike and convex functions of complex order, Hacettepe. Bull. Natur. Sci. Engrg. Ser. B 30 (2001) 63–68.
[4] O. Altıntaş, Ö. Özkan, H.M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, Appl. Math. Lett. 13 (3) (1995) 63–67.
[5] O. Altıntaş, Ö. Özkan, H.M. Srivastava, Majorization by starlike functions of complex order, Complex Variables Theory Appl. 46 (2001) 207–218.
[6] O. Altıntaş, Ö. Özkan, H.M. Srivastava, Neighborhoods of a certain family of multivalent functions with negative coefficients, Comput. Math. Appl. 47 (2004) 1667–1672.
[7] O. Altıntaş, H.M. Srivastava, Some majorization problems associated with p-valently starlike and convex functions of complex order, East Asian Math. J. 17 (2001) 175–218.
[8] P.L. Duren, Univalent Functions, in: A Series of Comprehensive Studies in Mathematics, vol. 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
[9] G. Murugusundaramoorthy, H.M. Srivastava, Neighborhoods of certain classes of analytic functions of complex order, J. Inequal. Pure Appl. Math. 5 (2: Article 24) (2004) 1–8. Electronic.
[10] M.A. Nasr, M.K. Aouf, Radius of convexity for the class of starlike functions of complex order, Bull. Fac. Sci. Assiut Univ. Sect. A 12 (1983) 153–159.
[11] M.A. Nasr, M.K. Aouf, Bounded starlike functions of complex order, Proc. Indian Acad. Sci. Math. Sci. 92 (1983) 97–102.
[12] M.A. Nasr, M.K. Aouf, Starlike function of complex order, J. Natur. Sci. Math. 25 (1985) 1–12.
[13] M.S. Robertson, On the theory of univalent functions, Ann. Math. 37 (1936) 374–408.
[14] H.M. Srivastava, S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1992.
[15] P. Wiatrowski, On the coefficients of some family of holomorphic functions, Zeszyty Nauk. Uniw. Łódz Nauk. Mat.-Przyrod Ser. 2 39 (1970) 75–85.