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Hamilton cycles in graph bundles over a cycle with tree as a fibre☆

Tomaž Pisanski *, Janez Žerovnik

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia

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ABSTRACT

A sufficient and necessary condition for the existence of a Hamilton cycle in a graph bundle with a cycle as a base and a tree as a fibre is obtained.

1. Introduction

In 1982, Batagelj and Pisanski [2] proved that the cartesian product of a tree T and a cycle C_n has a Hamilton cycle if and only if $n \geq \Delta(T)$, where $\Delta(T)$ denotes the maximum valence of T. Let $D(G)$ denote the minimum of $\Delta(T)$ over all spanning trees T of G. They introduced the cyclic Hamiltonicity $cH(G)$ of G as the smallest integer n for which the cartesian product $G\Box C_n$ is Hamiltonian. They also conjectured that $cH(G) \leq D(G) \leq cH(G) + 1$. Later the conjecture was proved in [4]. In this note it is shown that the original result extends in a certain way to graph bundles.

The notion of *graph bundle* was first introduced in 1982 by Pisanski and Vrabec in [8]. Unfortunately, the paper remained in the form of an 80-page preprint in which the basic theory of graph bundles was developed. Over the years about 30 papers devoted to the study of graph bundles emerged.

The Hamiltonian properties of the graph bundle $C_n\Box^a T$ with the base C_n and tree fibre T depend only on the structure of the automorphism a of the tree T. We are able to give a complete generalization of the cartesian product result to the graph bundle case in terms of the structure of the automorphism a. Let T/a be the quotient tree of T with respect to a. We prove that the graph bundle $C_n\Box^a T$ has a Hamilton cycle if and only if $n \geq h(T, a)$ where $h(T, a)$ is the maximum value of $\lceil d(v, a)/o(v, a)\rceil$ over all vertices $v \in V(T)$ and $o(v, a)$ denotes the number of elements in the orbit of v under the automorphism a while $d(v, a)$ is the valence of the vertex corresponding to the orbit of v in the tree T/a.

In this paper we are interested mostly in simple or simplicial graphs with no loops or parallel edges. However, in the construction of such graphs we have to use more general graphs with loops and parallel edges or even pending-edges (or semi-edges) permitted. We introduce a structure that we call a pre-graph; see also [6]. A *pre-graph* G is is quadruple $G = (V, S, i, r)$ where V is the set of vertices, S is the set of *arcs* (also known as semi-edges, darts, sides, etc.), i is the *initial mapping* $i : S \to V$, specifying the origin or the initial vertex for each arc, while r is the *reversal involution*: $r : S \to S, r^2 = 1$.

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* Corresponding author.
E-mail addresses: Tomaz.Pisanski@fmf.uni-lj.si (T. Pisanski), Janez.Zerovnik@imfm.uni-lj.si (J. Žerovnik).

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We may also define the *terminal mapping* $t : S \to V$ as $t(s) := i(r(s))$, specifying the terminal vertex for each arc. An arc s with $r(s) \neq s$ forms an edge $e = \{s, r(s)\}$, which is called *proper* if $|e| = 2$ and is called a *half-edge* if $|e| = 1$. Define $\partial(e) = \{i(s), t(s)\}$. A pre-graph without half-edges is called a *(general) graph*. Note that G is a graph if and only if the involution has no fixed points. A proper edge e with $|\partial(e)| = 1$ is called a *loop* and two edges e, e' are *parallel* if $\partial(e) = \partial(e')$. A graph without loops and parallel edges is called *simple*. The *valence* of a vertex v is defined as $val(v) = |\{s \in S | i(s) = v\}|$.

Now we shall briefly introduce voltage graphs. Voltage graphs are obtained from pre-graphs by assigning group elements to arcs. More precisely, a *permutation voltage graph* X is a 7-tuple $X = (V, S, i, r, \Gamma, M, a)$ where (V, S, i, r) is the underlying pre-graph called the *base pre-graph*, Γ is a permutation group acting on the set M and a is a mapping $a : S \to \Gamma$, called *permutation voltage assignment* satisfying the following axiom: For each $s \in S$ we have $a(r(s)) = a^{-1}(s)$.

Any voltage graph X defines the so-called *permutation derived graph* or *covering graph* Y as follows:

$$V(Y) := V \times M, \qquad S(Y) := S \times M,$$
$$i(s, m) := (i(s), m), \quad \text{for any } (s, m) \in S(Y),$$
$$r(s, m) := (r(s), a(s)[m]), \quad \text{for any } (s, m) \in S(Y).$$

If $s \in S$ is a half-edge, i.e. if $r(s) = s$ its voltage $a(s)$ must satisfy the condition: $a(s) = a^{-1}(s)$. Hence $a = a^{-1}$ or equivalently, it is of order at most two: $a^2 = \text{id}$. If a has any fixed points then the derived structure Y has half-edges and remains a pre-graph.

If $M = V(F)$ is a vertex set of a graph F, then the permutation voltage graph X gives rise to another graph Z as follows:

$$V(Z) := V(X) \times V(F), \qquad S(Z) := S(X) \times V(F) \cup V(X) \times S(F),$$
$$i(s, m) := (i(s), m), \quad \text{for any } (s, m) \in S(X) \times V(F),$$
$$r(s, m) := (r(s), a(s)[m]), \quad \text{for any } (s, m) \in S(X) \times V(F),$$
$$i(v, s) := (v, i(s)), \quad \text{for any } (v, s) \in V(X) \times S(F),$$
$$r(v, s) := (v, r(s)), \quad \text{for any } (v, s) \in V(X) \times S(F),$$

This graph Z is called a *pre-bundle* with base X and fibre F. If Γ is a subgroup of the automorphism group *Aut F* then Z is called a *graph bundle*. The bundle over X with fibre F is denoted by $X\Box^a F$.

Recall that a denotes the voltage assignment on X. There are two special cases: If F is empty graph mK_1, the resulting bundle is a covering graph over X. If a is trivial, then the graph $X\Box F$ is simply the well-known cartesian product of graphs.

Intuitively, on the pre-image of each vertex $v \in V$ in the covering graph we have drawn a copy of the graph F to obtain the graph bundle. The edges of copies of F are called *degenerate* edges. Other edges are called *non-degenerate*.

Note that voltage graphs enable us to develop a combinatorial analog of the theory of covering spaces and bundles in algebraic topology. The reader may find more on the theory of voltage graphs and covering graphs in [3,5,7].

It is well-known [8] that for any spanning tree T of X, the cartesian product $T\Box F$ is a spanning graph of $X\Box^a F$.

2. Trees and their automorphisms

Let T be a tree. Consider a graph bundle $C_n\Box^a T$, i.e. a bundle with base C_n and fibre a tree T. Let us label the vertices of C_n consecutively from 0 to $n-1$. Let the voltages be identity on the path from 0 to $n-1$ and let the voltage on the arc from $n-1$ to 0 be $a \in Aut(T)$.

Before studying such bundles let us recall some basic properties of trees and their automorphisms. Let T be a tree and let L_0 denote its set of leaves, i.e. the vertices of valence 1. For $i > 0$ define L_i to be the set of vertices of T at distance i from leaves. Let L_* denote the set of vertices that are farthest from the leaves. Then L_* consists either of a single vertex or of a pair of adjacent vertices and is called the *center* or the *centroid* of the tree, respectively. The proofs of these lemmas are included for completeness.

Lemma 2.1. *Any automorphism a of the tree T stabilizes each set L_i set-wise.*

Proof. By induction on i. Since each automorphism maps a vertex of valence 1 to a vertex of valence 1, the claim is true for L_0. As it maps adjacent vertices to adjacent vertices, and maps L_i to itself, it must map L_{i+1} to itself. □

Lemma 2.2. *Let a be an automorphism of the tree T and let v be any vertex of T. The orbit of v with respect to a is either:*

1. *a set of independent vertices*
2. *a set of two adjacent vertices*

Furthermore, the last option is possible if and only if the two vertices constitute the centroid of the tree and there are no fixed vertices under a.

Proof. By previous Lemma each orbit is contained in some L_i. Each set except possibly L_* consists of independent vertices, therefore the result follows. If T has a center, then it is a fixed vertex of any automorphism of a tree. If T has a centroid, then it is either fixed point-wise by a or a interchanges the two vertices of the centroid. If the edge belonging to the centroid is removed from T, the tree breaks up into two isomorphic subtrees and a maps one to another. Hence there are no fixed vertices in this case. □

Let $a \in Aut\, T$ be an arbitrary automorphism of T. It induces a cyclic subgroup that partitions the vertex set $V(T)$ into disjoint orbits. For a given vertex $v \in V(T)$ let $O(v, a)$ denote such an orbit whose size is denoted by $o(v, a)$.

Let T/a denote the quotient graph obtained from the tree T by vertex identification of each vertex orbit $O(v, a)$ and with two orbits $O(u, a)$ and $O(v, a)$ being adjacent in T/a if and only if there are two representatives $u \in O(u, a)$ and $v \in O(v, a)$ that are adjacent in T.

Theorem 2.3. *The quotient graph T/a is a tree.*

Proof. Each set L_i is partitioned into equivalence classes and each class from L_i is adjacent to exactly one class from L_{i+1}. Finally, either the class L_* consists of a single vertex or two adjacent fixed vertices. □

Lemma 2.4. *Let v be a vertex in T, such that $O(v, a)$ is a set of k independent vertices. Then the subgraph of $C_n\square^a T$ induced on the vertex set $V(C_n) \times O(v, a)$ is isomorphic to a cycle C_{nk}. However, if $O(v, a)$ consists of two adjacent vertices, then the graph on $V(C_n) \times O(v, a)$ is a Möbius ladder M_n, which, in turn, contains a spanning cycle C_{2n}.*

Proof. The orbit $O(v, a)$ can be written as $O(v, a) = \{v, a(v), a^2(v), \dots, a^{k-1}(v)\}$. The graph induced on $V(C_n) \times O(v, a)$ can be viewed as k copies of the path P_n such that each copy $P_n(i)$ corresponds to the vertex $a^i(v)$ and the last vertex of $P_n(i)$ is joined with the first vertex of $P_n(i+1)$ thus forming a single cycle. If the orbit $O(v, a)$ is not composed of isolated vertices, it is an edge and the resulting cycle C_{2n} has main diagonals, making it a Möbius ladder M_n. □

3. An edge-coloring problem

Let us consider a general question. Let G be a graph and let $o : V(G) \to \{1, 2, \dots\}$ be a mapping. Let $h(G, o)$ denote the least number of colors needed in an edge-coloring of G in such a way that each color appears at vertex v at most $o(v)$ times. Any admissible coloring is called an o-edge-coloring.

Theorem 3.1. *For general graphs G and general function o the computation of $h(G, o)$ is NP-hard.*

Proof. Take $o(v) = 1$ and the problem reduces to the computation of the edge-chromatic number, which in turn, is known to be NP-hard. □

For bipartite graphs there exists a closed form solution.

Theorem 3.2. *For a bipartite graph G the value of $h(G, o)$ is given by*

$$h(G, o) = \max\{\lceil val(v)/o(v)\rceil | v \in V(G)\}.$$

Proof. Essentially we repeat the argument in the Claim of Theorem 3.1. from a recent paper by Noga Alon, [1]. For each vertex v of valence $val(v)$ define the integer $r(v)$ as $r(v) = \lceil val(v)/o(v)\rceil$. Split each vertex v into $o(v)$ vertices $v_1, v_2, \dots v_{o(v)}$ in such a way that each of the new vertices is adjacent either to $r(v)$ or $r(v) - 1$ original vertices and the disjoint neighbor sets cover the original neighbor set of v. The resulting graph G' is bipartite of maximal valence $h(G, o)$. By König's theorem it is $h(G, o)$-edge-colorable. Any optimal edge-coloring of G' induces the corresponding o-edge-coloring of G that satisfies the condition of the theorem. □

The above proof gives also a construction that enables one to label each edge $e = uv$ of the original graph by a triple (e, u_i, v_j) and thus refine the edge-coloring.

4. Results

Before we turn to graph bundles we will give an alternative construction of the Hamilton cycle in the cartesian product of a tree T and a cycle C_n for $n \geq \Delta(T)$. Essentially it is the same as given in [2] and then explained in more algorithmic way in [4]. Think of edges of copies of C_n in $C_n\square T$ to be non-degenerate. They define a spanning subgraph of $C_n\square T$ composed of m copies of C_n, where $m = |V(T)|$. We will use each edge e of T in order to join two cycles into a larger cycle. After all $m - 1$ edges of T are used, the m cycles will be fused into a single, Hamilton cycle. Since T is a bipartite graph it can be edge-colored by $\Delta(T)$ colors. We may choose the colors to be (some) of the edges of a C_n. Each vertex of $C_n\square T$ can be labeled as a pair $(i, t) = i(t)$, where i is an integer mod n and t a vertex from T. Let $C_n(t)$ denote the copy of the cycle at vertex t from T. For any edge $e = st$ of T with t as the endpoint with color $i = c(e)$ we remove edges $(i, i + 1)$ from copies t and s and replace them by the edges $(i(t), i(s))$ and $(i + 1(t), i + 1(s))$. This completely defines the Hamilton cycle.

A similar construction is carried out in more general case of graph bundles. The situation is more complicated, since the number of initial cycles is given by the number of orbits of the automorphism a. The lengths of these cycles are not uniform and attachments are more intricate.

Theorem 4.1. *Let X and F be Hamiltonian graphs. Then any graph bundle $X\square^a F$ with base X and fibre F is Hamiltonian.*

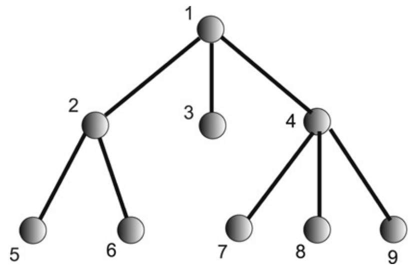

Fig. 1. The tree T_1 from [4].

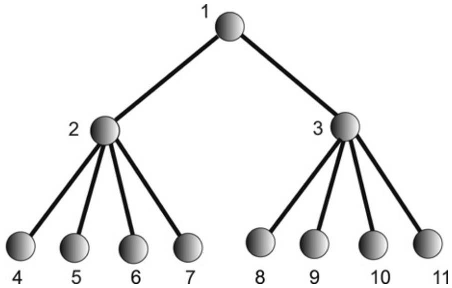

Fig. 2. The second tree T_2.

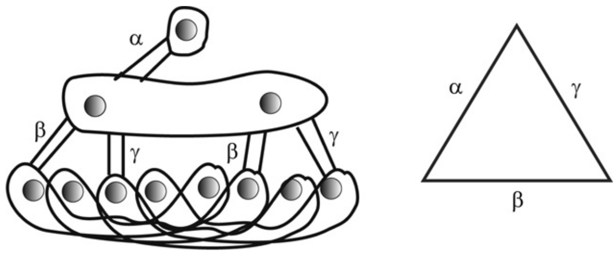

Fig. 3. Fibre edges in $C_3\square^a T_2$ form five cycles: one isomorphic to C_3 and four isomorphic to C_6. The quotient graph T_2/a is isomorphic to $K_{1,5}$. We color the five edges of $K_{1,5}$ with colors α, β, γ so that two edges are colored by β and two by γ.

Proof. Restrict attention to the two spanning cycles $C_n \subseteq X$ and $C_m \subseteq F$. The bundle $C_n\square^a C_m$ contains a spanning subgraph $P_n\square C_m$ which is Hamiltonian for any $m \geq 2$ by the result of Batagelj and Pisanski [2]. □

Here is the main result, a complete analog of the result of Batagelj and Pisanski. Only bundles of the form $X = C_n\square^a T$ are considered. Namely, the bundle $X = T\square^a C_n$ has a tree T as the base and is therefore isomorphic to the trivial bundle, the Cartesian product $T\square C_n$, and the result from [2] applies directly.

Theorem 4.2. *Let T be a tree and a one of its automorphisms. Let T/a denote the quotient tree of T with respect to a. Let $o(v, a)$ denote the number of elements in the orbit of v with respect to a and let $d(v, a)$ denote the valence of the orbit of v with respect to a in T/a. The graph bundle $X = C_n\square^a T$ has a Hamilton cycle if and only if $n \geq h(T/a, o)$, where $h(T/a, o) = \max\{\lceil d(v, a)/o(v, a)\rceil \mid v \in V(T)\}$.*

Proof. The proof has two steps.

($\Leftarrow$). In case $n \geq h(T/a, o)$ we construct a Hamilton cycle in X. Each vertex of T/a represents a cycle in X that passes through each layer, defined by the vertices of T. The existence of such cycles is provided by Lemma 2.4. The only case not covered by this Lemma is the case when the orbit is determined by the centroid of T, but in this case the appropriate spanning cycle has to be taken. The improper edge-coloring of T/a as defined by o determines a collection of edges of type (e, u_i, v_j) colored, say k. This means that the edge e passing from the ith element of orbit of u connects to the jth element of the orbit of v in T. The connection passes from the kth layer and $(k+1)$th layer. From the collection of edges in an orbit of a containing $e \in T/a$ choose one edge $\tilde{e} = u_i v_j \in T$. Now each edge e of T/a together with its color k determines one 4-cycle C_e in the bundle X with two parallel edges shared by two disjoint cycles whose existence was established by Lemma 2.4. Replacing the two edges by the other pair of parallel edges in the 4-cycles we get a cycle covering all the vertices of X covered by the two original cycles. We employ this operation repeatedly for all the edges of T/a. The edge-coloring of T/a forces the 4-cycles C_e to be mutually disjoint.

(⇒). In case $n < h(T/a, o)$ there exists a vertex v in T such that $n < d(v, a)/o(v, a)$. By removing the n copies of all vertices of the orbit of v ($no(v)$ vertices in total) from X the graph will be partitioned into $d(v, a)$ components, therefore, it cannot be Hamiltonian. □

Example 1. Our first example has a tree T_1 from [4] p. 52 (Fig. 1). Since $\Delta(T_1) = 1$ the Hamilton cycle in the presence of the trivial automorphism exists if and only if $n \geq 4$. However, if we take $\alpha = (7, 8)$ the quotient tree has maximal valence 3 and there exists a Hamilton cycle in bundle with $n = 3$. In this example we have $h(T/a, o) = \Delta(T/a)$ and the problem reduces to the computation of maximal valence in the quotient tree.

Example 2. Our second example involves a tree T_2 from Fig. 2. Let us assume that the automorphism a is an involution $a = (2, 3)(4, 8)(5, 9)(6, 10)(7, 11)$. In this case $\Delta(T_2) = \Delta(T_2/a) = 5$, however, since $h(T/a, o) = 3$ we may construct a Hamilton cycle already in $C_3\Box^a T_2$ (Fig. 3).

As an analog of the cyclic Hamiltonicity $cH(G)$ of Batagelj and Pisanski [2] let us define *cyclic bundle Hamiltonicity* $cbH(G)$ of the graph G as the min$\{n|$ such that there exists $a \in Aut\ (G)$ and $C_n\Box^a G$ has a Hamilton cycle$\}$. Clearly, $cbH(G) \leq cH(G)$. However, it would be interesting to formulate and prove a theorem that would hold for cyclic bundle Hamiltonicity and would be similar to the one that was formulated in [2] and proved in [4] for cyclic Hamiltonicity. Graph bundles remain an interesting class of graphs having a number of challenging properties.

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