Galois realizability of non-split group extensions of $C_2$ by $(C_2)^r \times (C_4)^s \times (D_4)^t$

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Abstract

The focus of this paper is Galois embedding problems associated with extensions of $C_2$ by groups of the form $(C_2)^r \times (C_4)^s \times (D_4)^t$ over arbitrary fields of characteristic not 2. As an application of our general solution, we obtain obstructions to the realizability of 13 groups of order 32 for which obstructions were not previously known.

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1. Introduction

Let $K$ be a field of characteristic not 2 and let $C_n$ and $D_n$ denote the cyclic group of order $n$ and the dihedral group of order $2n$, respectively. In this paper we study Galois embedding problems with kernel $C_2$ and quotients of the form $(C_2)^r \times (C_4)^s \times (D_4)^t$, with $r, s, t \geq 0$. We apply our results to prove necessary and sufficient conditions for the Galois realizability of 13 groups of order 32 for which realizability conditions were not previously known.

We begin with a precise definition of our use of the term “embedding problem.”

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It is straightforward to show that any solution to a non-split embedding problem with kernel generated (up to isomorphism) by \( \{ C_2 \} \) extends \( \{ C_\pm \} \). We next illustrate our main theorem (Theorem 1) which provides a method for determining solutions to a variety of embedding problems with kernel \( C_2 \) and quotient \( (C_2)^r \times (C_4)^s \times (D_4)^t \). In Section 3, we apply Theorem 1 to 13 specific groups with quotient \( (C_2)^r \times (C_4)^s \times (D_4)^t \) in order to solve the Galois realizability problem for those groups. In other words, for the given group \( E \), we determine the necessary and sufficient conditions on a field \( K \) to have a Galois extension \( M/K \) with Gal \( (M/K) \cong E \). This leaves only 10 groups of order 32 without known realizability conditions.

2. Non-split embedding problems with kernel \( C_2 \) and quotient \( (C_2)^r \times (C_4)^s \times (D_4)^t \)

In this section, we state and prove Theorem 1, which provides a method for determining solutions to a variety of embedding problems with kernel \( C_2 \) and quotient \( (C_2)^r \times (C_4)^s \times (D_4)^t \). Note that a group extension \( E \) of \( C_2 \) by \( G \) is split if and only if \( E \cong C_2 \times G \).

Theorem 1 is based on a result of Ledet [7], which decomposes the obstruction to an embedding into a product of obstructions to related embedding problems. Our result differs from Ledet’s in that we consider a specific quotient, \( (C_2)^r \times (C_4)^s \times (D_4)^t \).

Identify \( C_n \) with \( \langle \zeta_n \rangle \), where \( \zeta_n \) is a primitive \( n \)th root of unity. Let \( L/K \) be a \( (C_2)^r \times (C_4)^s \times (D_4)^t \)-extension, so that \( L \) is the composite of a \( (C_2)^r \)-extension, a \( (C_4)^s \)-extension, and a \( (D_4)^t \)-extension. We begin by recalling the characteristics of these extensions, as described in [7], among others.

Any \( (C_2)^r \)-extension of \( K \) can be written in the form \( K(\sqrt{a_1}, \ldots, \sqrt{a_r}) \), where the \( a_i \) are quadratically independent elements of \( K \). The Galois group of the extension is generated (up to isomorphism) by \( \{ \sigma_{ij} \}_{i=1}^r \subseteq \text{Gal}(L/K) \), where \( \sigma_i(\sqrt{a_j}) = (-1)^{\delta_{ij}} \sqrt{a_j} \) for all \( i, j \leq r \).

\( K \) has a \( C_4 \)-extension if and only if there exists a non-square element \( a \in \bar{K} \) and an element \( \varepsilon \in \bar{K} \) such that \( a = \varepsilon^2 + 1 \). Any \( (C_4)^s \)-extension can be written as \( K(\sqrt{\alpha_1 + \sqrt{\alpha_2 + \sqrt{\alpha_3 + \cdots + \sqrt{\alpha_r + 1}}}}, \ldots, \sqrt{\alpha_1 + \sqrt{\alpha_2 + \sqrt{\alpha_3 + \cdots + \sqrt{\alpha_r + 1}}}}) \), where the \( a_i \) are quadratically independent elements of \( \bar{K} \). Any \( (D_4)^t \)-extension
independent elements of $\hat{K}$, $q_i \in \hat{K}$ and for each $i, r < i \leq s$, there exists $e_i \in \hat{K}$ such that $a_i = e_i^2 + 1$. The Galois group of the extension is generated by $\{\sigma_i\}_{i=r+1}^{t+s+t} \subseteq \text{Gal}(L/K)$, where

$$\sigma_i \left( \sqrt{q_i(a_i + \sqrt{a_i})} \right) = \frac{q_i e_i \sqrt{a_i}}{\sqrt{q_i(a_i + \sqrt{a_i})}}$$

and

$$\sigma_i \left( \sqrt{q_j(a_j + \sqrt{a_j})} \right) = \sqrt{q_j(a_j + \sqrt{a_j})}$$

for all $i \neq j$.

A straightforward computation shows that $\sigma_i(\sqrt{a_j}) = (-1)^{\delta_{ij}} \sqrt{a_j}$ for all $i, j$ with $r < i \leq r + s$ and $r < j \leq r + s$.

Finally, $K$ has a $D_4$-extension if and only if there exist quadratically independent elements $a, b \in \hat{K}$ such that $(a, ab) \equiv 1$ in the Brauer group, $\text{Br}(K)$, that is, if and only if $ab$ is a norm from $K(\sqrt{a})$ to $K$ [1, 7.7(i)]. Any $(D_4)^t$-extension can be written as

$$K\left(\sqrt{q_{r+s+1}(a_{r+s+1} + b_{r+s+1} \sqrt{a_{r+s+1}})}, \sqrt{b_{r+s+1}}, \ldots, \sqrt{q_{r+s+t}(a_{r+s+t} + b_{r+s+t} \sqrt{a_{r+s+t}})}, \sqrt{b_{r+s+t}}\right),$$

where all $a_i, b_j \in \hat{K}$ are pairwise quadratically independent, $a_i b_i = a_i^2 - a_i^2, a_i \in K$, and $\beta_i, q_i \in \hat{K}$ for all $i, r + s < i \leq r + s + t$. The Galois group of the extension is generated by $\{\sigma_i, \tau_i\}_{i=r+1}^{t+s+t} \subseteq \text{Gal}(L/K)$, where

- $\sigma_i \left( \sqrt{q_j(a_i + \sqrt{a_i})} \right) = \frac{q_j \sqrt{a_1 b_1}}{\sqrt{q_j(a_i + \beta_i \sqrt{a_i})}}$ for all $i$;
- $\sigma_i \left( \sqrt{q_j(a_j + \beta_j \sqrt{a_j})} \right) = \sqrt{q_j(a_j + \beta_j \sqrt{a_j})}$ for all $i \neq j$;
- $\sigma_i(\sqrt{b_j}) = \sqrt{b_j}$ for all $i$;
- $\tau_i \left( \sqrt{q_j(a_j + \beta_j \sqrt{a_j})} \right) = \sqrt{q_j(a_j + \beta_j \sqrt{a_j})}$ for all $i, j$;
- $\tau_i(\sqrt{b_j}) = (-1)^{\delta_{ij}} \sqrt{b_j}$ for all $i, j$.

It follows that any $(C_2)^r \times (C_4)^t \times (D_4)^t$-extension $L/K$ can be written in the form

$$L = K\left(\sqrt{a_1}, \ldots, \sqrt{a_r}, \sqrt{q_{r+1}(a_{r+1} + \sqrt{a_{r+1}})}, \ldots, \sqrt{q_{r+s}(a_{r+s} + \sqrt{a_{r+s}})}, \sqrt{q_{r+s+1}(a_{r+s+1} + b_{r+s+1} \sqrt{a_{r+s+1}})}, \sqrt{b_{r+s+1}}, \ldots, \sqrt{q_{r+s+t}(a_{r+s+t} + b_{r+s+t} \sqrt{a_{r+s+t}})}, \sqrt{b_{r+s+t}}\right).$$

(2)
where \( a_1, \ldots, a_{r+s+t}, b_{r+s+1}, \ldots, b_{r+s+t} \in \hat{K} \) are quadratically independent, \( a_i = e_i^2 + 1 \) with \( e_i, q_i \in \hat{K} \) for \( r < i \leq r + s \), and \( a_i, \beta_i \in K \) with \( a_i\beta_i = \alpha_i^2 - a_i^2 \) and \( q_i \in \hat{K} \) for \( i > r + s \). Further, the Galois group of \( L/K \) is generated by \([\sigma_i, \tau_j] | 1 \leq i \leq r + s + t, r + s < j \) where the \( \sigma_i \) and \( \tau_i \) that generate each factor are defined as above, with the additional requirement that \( \sigma_i \) and \( \tau_i \) fix the elements corresponding to the \( j \)th factor for all \( i \neq j \).

We now apply [7, Corollary 2.5] to obtain the following theorem, which gives the obstruction to the embedding problem with kernel \( C_2 \) and quotient \( \text{Gal}(L/K) \).

**Theorem 1.** Let \( L/K \) be a \((C_2)^r \times (C_4)^s \times (D_4)^t\)-extension as in (2) and let \( \sigma_1, \ldots, \sigma_{r+s+t}, \tau_{r+s+1}, \ldots, \tau_{r+s+t} \) be the generators of \( \text{Gal}(L/K) \) defined above. Let

\[
1 \xrightarrow{1} C_2 \xrightarrow{E} (C_2)^r \times (C_4)^s \times (D_4)^t \xrightarrow{1} 1
\]

be a non-split extension of groups, and choose pre-images \( g_1, \ldots, g_{r+s+t} \in E \) of \( \sigma_1, \ldots, \sigma_{r+s+t} \) and \( h_{r+s+1}, \ldots, h_{r+s+t} \in E \) of \( \tau_{r+s+1}, \ldots, \tau_{r+s+t} \). Where appropriate, let \(-1\) denote the image of \(-1\) in \( E \). Then the obstruction to the embedding problem given by \( L/K \) and (3) is

\[
\prod_{i=1}^{r+s+t} (a_i, -1)^d_i \times \prod_{i=r+1}^{r+s+t} [(a_i, 2)(-1, q_i)]^{d_i} \times \prod_{i=r+1}^{r+s+t} [(a_i, -2)(-b_i, 2\alpha_i q_i)]^{d_i} \times (b_i, -1)^{\bar{f}_i} (a_i, -1)^{\bar{f}_i} \times \prod_{i<j}^{r+s+t} (a_i, a_j)^{d_{ij}} \times \prod_{i<j}^{r+s+t} (a_i, b_j)^{e_{ij}} \times \prod_{i<j}^{r+s+t} (b_i, b_j)^{f_{ij}},
\]

(4)

where \( g_i^2 = (-1)^{d_i} \) for \( i \leq r; g_i^2 = (-1)^{d_i} \) for \( r < i \leq r + s + t; h_i^2 = (-1)^{\bar{f}_i} \) for \( r + s < i \leq r + s + t; h_i^2 = (-1)^{\bar{f}_i} \) for \( r + s < i \leq r + s + t; h_i^2 = (-1)^{\bar{f}_i} \) for all \( i, j; g_i h_j = (-1)^{\bar{f}_i} h_j g_i \) for all \( 1 \leq i < j, \) with \( r + s < j \leq r + s + t; \) and \( h_i h_j = (-1)^{\bar{f}_i} h_i h_j \) for \( r + s < i < j \leq r + s + t. \) (If \( \alpha_i = 0, \) then \( -b_i \) is a square in \( \hat{K} \) and we set \( -b_i, 2\alpha_i q_i) = 1 \) in \( \text{Br}(K) \).)

**Proof.** It is straightforward to verify that for all \( i, j, \) \( \sigma_i(\sqrt{\alpha_j}) = (-1)^{\bar{f}_j} \sqrt{\alpha_j} \) and \( \tau_j(\sqrt{\beta_j}) = (-1)^{\bar{f}_j} \sqrt{\beta_j} \). Thus the hypotheses of [7, Corollary 2.5] are satisfied. Hence the obstruction to the embedding problem given by \( L/K \) and (3) is, in Ledet’s notation,

\[
\prod_{i,j,k} (L_i, G_i, \text{res}_{G_i} G_j(y)) \cdot \prod_{i,j,k} (a_{i,h,j,k})^{d(i,h,j,k)}.
\]

The first product corresponds to products of the obstructions to the embedding problem restricted to each of the quotients \( C_2, C_4, \) and \( D_4. \) (These obstructions can be found, for
instance, in [7, Examples 1.3 and 3.1, Proposition 4.2].) This gives us the three factors in the first two lines of (4). The second product translates directly into the last line of (4).

3. Applications of embedding criteria to Galois realizability

For fields of characteristic 2, a classic result of Witt [11] solves realizability problems for finite 2-groups in general. A compilation of conditions for the realizability of groups of order 2, 4, 8, or 16 over fields of characteristic not 2, as well as descriptions of extensions with those Galois groups, can be found in [4,7].

Fourteen of the 51 groups of order 32 are direct products of the form \( C_2 \times G \) and so have realizability conditions that are easily derived from those already known for the groups of order 16. Similarly, conditions for the direct products \( C_4 \times C_8 \), \( C_4 \times D_4 \), and \( C_4 \times Q_8 \) are easily derived from the conditions for the realizability of each factor. Conditions for eleven more groups of order 32 can be found in (or easily derived from results in) [2,8,9]. We give results for 13 additional groups of order 32, leaving only 10 without known realizability conditions.

There is no universally accepted notation for the groups of order 32. Numbering schemes for the groups can be found in [3,5,10]. Our convention is to use the ordered pair \((m,n)\) to identify the group of order 32 corresponding to \(32/m\) in [10] and to \([32,n]\) in GAP’s small groups catalog [3] and to \(32/m\) in [10]. In our proofs, we simplify the notation and refer to the group as \(G_{n}\).

Presentations for the 13 groups of order 32 for which we provide solutions are given in Table 1. In the presentations, \([a, b]\) denotes the commutator \(aba^{-1}b^{-1}\) and \(Z\) denotes the center of the given group.

Table 1

<table>
<thead>
<tr>
<th>Group</th>
<th>Generators</th>
<th>Relators</th>
</tr>
</thead>
<tbody>
<tr>
<td>(16,24) x, y, z</td>
<td>(x^4, y^4, z^2, [x, y], [x, z], zy^{-1}zx^2y)</td>
<td></td>
</tr>
<tr>
<td>(17,38) x, y, z</td>
<td>(x^8, y^2, z^2, [x, y], [x, z], zyzx^4y)</td>
<td></td>
</tr>
<tr>
<td>(19,4) x, y</td>
<td>(x^8, y^4, y^{-1}x^{-1}yxy^5)</td>
<td></td>
</tr>
<tr>
<td>(26,42) x, y, z</td>
<td>(x^8, y^2, z^2, [x, y], (xz)^2, (yz)^2x^4)</td>
<td></td>
</tr>
<tr>
<td>(33,27) w, w, x, y, z</td>
<td>(u^2, w^2, y^2, y^{-1}z^2, [x, y], (xy)^2w, (yz)^2w, [v, w] \leq Z)</td>
<td></td>
</tr>
<tr>
<td>(34,34) x, y, z</td>
<td>(x^4, y^4, z^2, [x, y], (xz)^2, (yz)^2)</td>
<td></td>
</tr>
<tr>
<td>(35,35) x, y, z</td>
<td>(x^4, y^4, [x, y], z^2, xzxz^{-1}, yzyz^{-1})</td>
<td></td>
</tr>
<tr>
<td>(36,28) w, w, x, y, z</td>
<td>(w^2, x^2, y^2, z^2, [x, y], (zx)^2w, (yz)^2, w \in Z)</td>
<td></td>
</tr>
<tr>
<td>(37,29) w, w, x, y, z</td>
<td>(w^2, x^2, y^4, z^2, x^{-1}zwx, yzyz^{-1}, [x, y], w \in Z)</td>
<td></td>
</tr>
<tr>
<td>(38,30) w, w, x, y, z</td>
<td>(w^4, x^2, y^2, z^2, [w, y], zw^{-1}zwx, (yz)^2w^2, x \in Z)</td>
<td></td>
</tr>
<tr>
<td>(39,31) x, y, z</td>
<td>(x^4, y^4, z^2, [x, y], (xz)^2, (yz)^2z^2)</td>
<td></td>
</tr>
<tr>
<td>(44,43) x, y, z</td>
<td>(x^8, y^2, z^2, [y, z], (xy)^2, xzxz^{-1})</td>
<td></td>
</tr>
<tr>
<td>(45,44) x, y, z</td>
<td>(x^8, y^2, [y, z], x^2z, xyx^2y, xzxz^{-1})</td>
<td></td>
</tr>
</tbody>
</table>
The realizability conditions for these 13 groups are given in Table 2. The second column of the table gives a minimum for the size of $\bar{K}/\bar{K}^2$ and the third column lists elements that must be trivial in the Brauer group of $K$, written in terms of the generators given in Table 1.

Propositions 3–5 provide precise statements of the realizability results for three of these groups and are proved in detail. As explained in the text, the proofs for the remaining groups are similar. For each group, the quotient map of the short exact sequence needed in the corresponding proof is given in Table 3.

For our proofs, we rely heavily on the well-known correspondence between certain Galois realizability problems and embedding problems. This correspondence, which we state for clarity below, allows us to use the embedding criteria given in Theorem 1 to obtain the 13 previously unknown realizability results for groups of order 32. Recall that $K$ is a field of characteristic other than 2.

**Theorem 2.** Let $E$ be a finite 2-group and let $N \triangleleft E$ be a proper subgroup of order 2. Let $\kappa : C_2 \rightarrow E$ be the map induced by the isomorphism $C_2 \cong N$. Let $\pi : E \rightarrow E/N$ be the canonical projection.

If $E \not\cong N \times E/N$, then there exists a Galois extension $M/K$ with Galois group $E$ if and only if there exists an $E/N$-extension $L/K$ such that given the non-split exact sequence

$$1 \rightarrow C_2 \rightarrow E \rightarrow E/N \rightarrow 1, \quad (5)$$

there exists a Galois extension $M/K$ and an isomorphism $\varphi : \text{Gal}(M/K) \rightarrow E$ such that $\pi \varphi$ is the natural surjection of Galois groups.
Let $\langle \sigma \rangle$ be a group over $\langle M/K, \varphi \rangle$ solvable. As mentioned above, if a solution $L/K$ can be written as $C_a \times b \times c$, where $a, b, c$ are independent elements, then $L/K$ is realizable as a Galois group over a field $K$. If and only if for some quadratically independent elements $\alpha$ there exist elements $\sigma, \tau$ such that $\alpha = \sigma \cdot \tau$ if and only if there exist elements $\beta, \gamma$ such that $\beta = \gamma$. The group $C_2 \times D_4$ is realizable as a Galois group over a field $K$ if and only if there exists a $C_2 \times D_4$-extension $L/K$ if and only if for some quadratically independent elements $a_1, a_2, b_2 \in K$, there exist elements $a_2 \in K, b_2 \in K$ such that $a_2b_2 = a_2^2 - a_2b_2^2$. Any $C_2 \times D_4$-extension $L/K$ can be written as

$$L = K \left( \sqrt{a_1}, \sqrt{q_2(a_2 + \beta_2 \sqrt{a_2}), \sqrt{b_2}} \right),$$

where $a_1, a_2, b_2, \alpha_2, \text{and} \beta_2$ are as above and $q \in \hat{K}$.

**Proposition 3.** The group $G_{42}$ with presentation

$$\{x, y, z \mid x^8 = y^2 = z^2 = [x, y] = 1, \ xz = zx^{-1}, \ yz = zy^{x^4}y^{-1} \}$$

is realizable as a Galois group over a field $K$ if and only if there exist quadratically independent elements $a, b, c \in \hat{K}$ and an element $u \in K$ such that $(a, c)(b, 2)(-c, u) = (b, -c) = 1$.

**Proof.** Let $L/K$ be a $C_2 \times D_4$-extension as in (6). Note that the set $\{(-1, 1), (1, \sigma), (1, \tau)\}$ generates $C_2 \times D_4$, where $\sigma^2 = \tau^2 = (\sigma \tau)^2 = 1$. These elements satisfy the relations

<table>
<thead>
<tr>
<th>Group</th>
<th>Quotient</th>
<th>$\epsilon(-1)$</th>
<th>Quotient maps</th>
</tr>
</thead>
<tbody>
<tr>
<td>(16,24)</td>
<td>$(C_2)^2 \times C_4$</td>
<td>$x^2$</td>
<td>$x \mapsto (-1, -1, 1), \ z \mapsto (1, -1, 1), \ y \mapsto (1, 1, i)$</td>
</tr>
<tr>
<td>(17,38)</td>
<td>$(C_2)^2 \times C_4$</td>
<td>$x^4$</td>
<td>$y \mapsto (-1, 1, 1), \ z \mapsto (1, 1, 1), \ x \mapsto (1, 1, i)$</td>
</tr>
<tr>
<td>(39,31)</td>
<td>$C_2 \times D_4$</td>
<td>$x^2$</td>
<td>$x \mapsto (i, 1), \ y \mapsto (1, i)$</td>
</tr>
<tr>
<td>(36,28)</td>
<td>$C_2 \times D_4$</td>
<td>$w$</td>
<td>$y \mapsto (-1, 1), \ xz \mapsto (1, \sigma), \ z \mapsto (1, \tau)$</td>
</tr>
<tr>
<td>(34,34)</td>
<td>$C_2 \times D_4$</td>
<td>$y^2$</td>
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<tr>
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</tr>
</tbody>
</table>

Table 3

Short exact sequences for realizability proofs
Suppose there exists a given in Theorem 1, so let \( \sigma_1 = (-1, 1), \sigma_2 = (1, \sigma), \) and \( \tau_2 = (1, \tau). \) Consider the exact sequence

\[
1 \longrightarrow C_2 \xrightarrow{1 \mapsto x^4} G_{42} \xrightarrow{\gamma \mapsto \sigma_1} C_2 \times D_4 \xrightarrow{\theta = \tau \times \tau_2} 1.
\]

In the notation of Theorem 1, let \( g_1 = y, g_2 = x, \) and \( h_2 = z \) in \( G_{42}. \) Then \( g_1^2 = 1 \) and \( g_2^2 = x^4, \) so \( d_1 = 0 \) and \( d_2 = 1. \) Since \( h_2^2 = 1, e_2 = 0. \) Now \( g_1 g_2 = yx = xy = g_2 g_1, \) so \( d_{12} = 0. \) Since \( x^4 \) is in the center of \( G_{42}, \) \( g_1 h_2 = yz = z x^4 y = x^4 h_2 g_1, \) and \( e_{12} = 1. \) The relation \( xz^{-1} = x \) implies that \( z = xzx, \) whence \( zx = x^{-1} z = x^7 z. \) It follows that \( h_2 g_2 = z = x^7 z = x^4 h_2 g_2, \) so \( f_2 = 1. \) Therefore the obstruction to the embedding problem given by \( L/K \) and (7) is \( (a_2, -2)(-b_2, 2a_2 q_2)(a_2, -1)(a_1, b_2) = (a_1, b_2)(a_2, 2)(-b_2, 2a_2 q_2). \)

If \( G_{42} \) is realizable over \( K, \) then certainly \( L/K \) exists and the above embedding problem is solvable. Hence we get sufficient conditions for the proposition. Conversely, given quadratically independent elements \( a_1, a_2, \) and \( b_2 \) in \( K \) and \( u \in K, \) with \( (a_1, b_2)(a_2, 2)(-b_2, u) = (a_2, -b_2) = 1, \) the last equality guarantees the existence of elements \( a_2 \in K \) and \( \beta_2 \in \bar{K} \) such that \( a_2 b_2 = a_2^2 - a_2 \beta_2^3 \) and a set of \( C_2 \times D_4 - \) extensions, parametrized by \( q_2 \) as in (6). Choosing \( q_2 \in K \) such that \( u = 2a_2 q_2 \) yields a \( C_2 \times D_4 - \) extension with \( (a_1, b_2)(a_2, 2)(-b_2, 2a_2 q_2) = 1 \) and therefore \( G_{42} \) is realizable over \( K. \)

Of the groups of order 32 without previously known realizability conditions that are not non-split extensions of \( C_2 \) by \( C_2 \times D_4, \) two are non-split extensions of \( C_2 \) by \( (C_2)^2 \times C_4. \) Recall that the group \( (C_2)^2 \times C_4 \) is realizable over \( K \) if and only if there exist quadratically independent elements \( a, b, c \in \bar{K} \) such that \( (c, -1) \) is trivial in \( Br(K). \) Any \( (C_2)^2 \times C_4 - \) extension \( L/K \) can be written in the form

\[
L = K\left(\sqrt{a_1}, \sqrt{a_2}, \sqrt{g_3(a_3 + \sqrt{a_3})}\right)
\]

for some \( g_3 \in \bar{K}, \) where \( a_1, a_2, a_3 \in \bar{K} \) are quadratically independent and \( a_3 = u^2 + 1. \)

**Proposition 4.** The group \( G_{24} \) with presentation

\[
\langle x, y, z \mid x^4 = y^4 = z^2 = [x, y] = [x, z] = 1, \ yz = zx^2 y \rangle
\]

is realizable as a Galois group over a field \( K \) if and only if there exist quadratically independent elements \( a, b, c \in \bar{K} \) such that \( (a, -1)(b, c) = (c, -1) = 1. \)

**Proof.** Suppose there exists a \( (C_2)^2 \times C_4 - \) extension \( L/K, \) as in (8). We shall use Theorem 1 to find conditions for embedding \( L \) into a \( G_{24} - \) extension of \( K. \) Note that \( \{(-1, 1, 1), (1, -1, 1), (1, 1, i)\} \) is a generating set for \( (C_2)^2 \times C_4 \) that satisfies the relations given in Theorem 1, so we may let \( \sigma_1 = (-1, 1, 1), \sigma_2 = (1, -1, 1), \) and \( \sigma_3 = (1, 1, i). \)
Consider the exact sequence

$$
1 \rightarrow C_2 \overset{1\rightarrow x^2}{\rightarrow} G_{24} \overset{x \mapsto \sigma_1, y \mapsto \sigma_2}{\rightarrow} (C_2)^2 \times C_4 \rightarrow 1.
$$

(9)

Following the notation of Theorem 1, let $g_1 = x$, $g_2 = z$, and $g_3 = y$ in $G_{24}$. Then $g_1^2 = x^2$, $g_2^2 = 1$, and $g_3^2 = 1$, so $d_1 = 1$, $d_2 = 0$, and $d_3 = 0$. Now $g_1g_2 = xz = zx = g_2g_1$, so $d_{12} = 0$. Also $g_1g_3 = xy = yx = g_3g_1$, so $d_{13} = 0$. Finally, $g_2g_3 = zy = x^2yz = x^2g_2g_2$, so $d_{23} = 1$. Thus the obstruction to the embedding problem given by $L/K$ and (9) is $(a_1, -1)(a_2, a_3)$. The sequence is non-split, so by Theorem 1, the obstruction to the embedding problem given by $L/K$ and (9) is $(a_1, -1)(a_2, a_3)$.

Since $G_{24}$ is realizable if and only if $L/K$ exists and the above embedding problem is solvable, this completes the proof of the proposition. □

Finally, the group $G_4$ (see presentation below) is a non-split extension of $C_2$ by $(C_4)^2$. Recall that $(C_4)^2$ itself is realizable as a Galois group over $K$ if and only if there exist quadratically independent elements $a, b \in \bar{K}$ such that $(a, -1)$ and $(b, -1)$ are trivial in $Br(K)$, that is, if and only if there exist quadratically independent elements $a_1, a_2 \in \bar{K}$ such that for some $\varepsilon_1, \varepsilon_2 \in \bar{K}$, $a_1 = \varepsilon_1^2 + 1$ and $a_2 = \varepsilon_2^2 + 1$.

As mentioned in Section 2, any $(C_4)^2$-extension $L/K$ can be written in the form

$$
L = K\left(\sqrt[1]{q_1(a_1 + \sqrt{a_1})}, \sqrt[2]{q_2(a_2 + \sqrt{a_2})}\right)
$$

for some $q_1, q_2 \in \bar{K}$, where $a_1, a_2$ are as above.

Proposition 5. The group $G_4$ with presentation

$$
\langle x, y \mid x^8 = y^4 = 1, \ xy = yx^8 \rangle
$$

is realizable as a Galois group over a field $K$ if and only if there exist quadratically independent elements $a, b \in \bar{K}$ and an element $q \in \bar{K}$ such that $(a, -1) = (b, -1) = (a, 2b)(-1, q) = 1$.

Proof. Let $L$ be a $(C_4)^2$-extension of $K$ as in (10). Note that $\{i, 1, (1, i)\}$ is a generating set for $(C_4)^2$ that satisfies the relations given in Theorem 1, so let $\sigma_1 = (i, 1)$ and $\sigma_2 = (1, i)$. Consider the exact sequence

$$
1 \rightarrow C_2 \overset{1\rightarrow x^4}{\rightarrow} G_4 \overset{x \mapsto \sigma_1, y \mapsto \sigma_2}{\rightarrow} (C_4)^2 \rightarrow 1.
$$

(11)

In the notation of Theorem 1, let $g_1 = x$ and $g_2 = y$ in $G_4$. Then $g_1^4 = x^4$ and $g_2^4 = 1$, so $d_1 = 1$ and $d_2 = 0$. Since $(x^4)$ is a normal subgroup of order 2, $x^4$ is in the center of $G_4$, so $g_1g_2 = xy = yx^5 = x^4yx = x^4g_2g_1$. Thus $d_{12} = 1$. Thus, since the sequence is non-split,
Theorem 1 implies that the obstruction to the embedding problem given by $L/K$ and (11) is 
\[(a_1, 2)(-1, q_1)(a_1, a_2) = (a_1, 2a_2)(-1, q_1).\]

Since $G_4$ is realizable if and only if $L/K$ exists and the above embedding problem is solvable, this completes the proof of the proposition. \[\square\]

References