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Exceptional times for the dynamical discrete web

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Abstract

The dynamical discrete web (DyDW), introduced in the recent work of Howitt and Warren, is a system of coalescing simple symmetric one-dimensional random walks which evolve in an extra continuous dynamical time parameter τ. The evolution is by independent updating of the underlying Bernoulli variables indexed by discrete space–time that define the discrete web at any fixed τ. In this paper, we study the existence of exceptional (random) values of τ where the paths of the web do not behave like usual random walks and the Hausdorff dimension of the set of such exceptional τ. Our results are motivated by those about exceptional times for dynamical percolation in high dimension by Häggstrom, Peres and Steif, and in dimension two by Schramm and Steif. The exceptional behavior of the walks in the DyDW is rather different from the situation for the dynamical random walks of Benjamini, Häggstrom, Peres and Steif. For example, we prove that the walk from the origin S_0^τ violates the law of the iterated logarithm (LIL) on a set of τ of Hausdorff dimension one. We also discuss how these and other results should extend to the dynamical Brownian web, the natural scaling limit of the DyDW.

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1. Introduction

In this paper, we present a number of results concerning a dynamical version of coalescing random walks, which was recently introduced in [1]. Our results concern sets of dynamical times of Hausdorff dimension less than or equal to one (and of zero Lebesgue measure) where the system of coalescing walks behaves exceptionally. The results are analogous to and were motivated by the model of dynamical percolation and its exceptional times [2,3]. In this section, we define the basic model treated in this paper, which we call the dynamical discrete web (DyDW), recall some facts about dynamical percolation, and then briefly describe our main results. The justification for calling this model a discrete web is that there is a natural scaling limit, the dynamical Brownian web (DyBW), which was proposed by Howitt and Warren in [1] and completely constructed in [4]. As we shall explain (see Section 6), the exceptional times results for the DyDW should extend to the continuum DyBW.

Before defining the specific models treated in this paper, we comment on the study of exceptional values of dynamical time, or analogous parameters, in a variety of models. In [2], one motivation given for studying exceptional times in dynamical percolation is to understand the (lack of) stability of "critical infinite clusters"; here the corresponding objects are simple symmetric random walks that exceptionally are slightly subdiffusive. Beyond that, since both the DyDW and some dynamical extensions have been used as models of hydrological drainage and erosion (see, e.g., [5,6]), dynamical times where exceptional behavior occurs can potentially be relevant for modeling time-sporadic phenomena in particular physical settings that correspond to individual ω's in the underlying probability space of the model. An intriguing speculative possibility of that type has been raised in the rather different physical context of spin glasses (see Sec. 4.2 of [7]), where ω corresponds to a realization of the microscopic disorder of a particular sample of the material being modeled, and dynamical time is replaced by the temperature parameter.

Returning to the dynamical time setting, we note that exceptional times for other dynamical versions of random walks in various spatial dimensions have been studied in [8–10] and elsewhere, but, as we shall see, these are quite different from the dynamical random walks of the DyDW.

1.1. The discrete web

The discrete web (DW) is a collection of coalescing one-dimensional simple random walks starting from every point in the discrete space–time domain $\mathbb{Z}^2_{even} = \{(x,t) \in \mathbb{Z}^2 : x+t \text{ is even}\}$. The Bernoulli-percolation-like structure is highlighted by defining $\xi_{x,t}$ for $(x,t) \in \mathbb{Z}^2_{even}$ to be the increment of the random walk at location x at time t. These Bernoulli variables are symmetric and independent and the paths of all the coalescing random walks can be reconstructed by assigning to each point (x,t) an arrow from (x,t) to $\{x+\xi_{x,t}, t+1\}$ and considering all the paths starting from arbitrary points in $\mathbb{Z}^2_{even}$ that follow the arrow configuration $\aleph$.

1.2. The dynamical discrete web

In the DyDW, there is, in addition to the random walk discrete time parameter, an additional (continuous) dynamical time parameter τ. The system starts at $\tau = 0$ as an ordinary DW and then evolves in τ by randomly switching the direction of each arrow at a fixed rate independently of all other arrows. We will generally do the switching by having at each $(x,t) \in \mathbb{Z}^2_{even}$ a Poisson clock ring at rate one and then reset the direction of the arrow at random; thus the rate of switching will be 1/2. This amounts to extending the percolation substructure ξ_z^0 to time varying functions ξ_z^τ defining a (right-continuous) dynamical arrow configuration $\tau \rightsquigarrow \aleph(\tau)$ and $W(\tau)$, the dynamical discrete web at time τ, is defined as the web constructed from $\aleph(\tau)$.

If one follows the arrows starting from the (space–time) origin (0, 0), the dynamical path S_0^τ begins at $\tau = 0$ as a simple symmetric random walk and then evolves dynamically in τ. At any fixed time τ, S_0^τ has the same law as at time $\tau = 0$. As a consequence, if μ is the probability distribution of a simple symmetric random walk starting from the origin and A is any event with $\mu(A) = 1$, we have for any deterministic τ that $\mathbb{P}(S_0^\tau \in A) = 1$. By a straightforward application of Fubini's theorem, this implies that

$$\mathbb{P}(S_0^\tau \in A \text{ for Lebesgue a.e. } \tau) = 1. \tag{1.1}$$

Following [8], for any event such that (1.1) holds, a natural question is whether (1.1) can be strengthened to

$$\mathbb{P}(S^\tau \in A \text{ for all } \tau \geq 0) = 1, \tag{1.2}$$

i.e., *do there exist some exceptional times τ at which S_0^τ violates some almost sure properties of the standard random walk?* or stated differently, is the random walk sensitive to the dynamics introduced on the DW?

1.3. Analogies with dynamical percolation

Similar questions have been investigated in percolation. Static (site) percolation models are defined also in terms of independent Bernoulli variables ξ_z^0, indexed by points z in some d-dimensional lattice, which in general are asymmetric with parameter p. There is a critical value p_c when the system has a transition from having an infinite cluster (connected component) with probability zero to having one with probability one. It is expected that at $p = p_c$ there are no infinite clusters and this is proved for $d = 2$ and for high d (see, e.g., [11]). In dynamical percolation, one extends ξ_z^0 to time varying functions ξ_z^τ, as in the case of coalescing walks, except that the transition rates for the jump processes ξ_z^τ are chosen to have the critical asymmetric $(p_c, 1 - p_c)$ distribution to be invariant. The question raised in [2] was whether there are exceptional times when an infinite cluster (say, one containing the origin) occurs, even though this does not occur at deterministic times. This was answered negatively in [2] for large d and, more remarkably, was answered positively by Schramm and Steif for $d = 2$ in [3], where they further obtained upper and lower bounds on the Hausdorff dimension (as a subset of the dynamical time axis) of these exceptional times. In [12], the exact Hausdorff dimension was obtained.

1.4. Main results

We apply in this paper the approaches used for dynamical percolation to the dynamical discrete web. Although we restrict attention to one-dimensional random walks whose paths are in two-dimensional space–time and hence analogous to $d = 2$ dynamical percolation, we use both the high d and $d = 2$ methods of [2,3].

–Tameness. A natural initial question is whether there might be exceptional dynamical times τ for which the walk from the origin $S_0^\tau(t)$ is transient (say to $+\infty$). Our first main result (see

Theorem 2.1), modeled after the high-d dynamical percolation results of [2], is that there are no such exceptional times.

–Existence of exceptional times. For a simple symmetric random walk S, it is well known that $\liminf_{t\uparrow\infty} S(t)/\sqrt{t} = -\infty$ a.s. (and, of course, $\limsup_{t\uparrow\infty} S(t)/\sqrt{t} = +\infty$ a.s.). In the following, we will say that a path is subdiffusive if it violates this a.s. property of the standard random walk.

Definition 1.1 (*K^+-subdiffusivity*). Let $K \in (0, \infty)$. A path π starting at $x = 0$ at time $t = 0$ is said to be K^+-subdiffusive iff there exists $j \geq 0$ such that

$$\forall t > 0, \quad \pi(t) \geq -j - K\sqrt{t}. \tag{1.3}$$

We say that π is subdiffusive iff there exists $K \in (0, \infty)$ such that $\liminf_{t\uparrow\infty} \pi(t)/\sqrt{t} \geq -K$ or $\limsup_{t\uparrow\infty} \pi(t)/\sqrt{t} \leq K$.

In Proposition 4.1, we prove that for K large enough, there is a strictly positive probability for having a dynamical time $\tau \in [0, 1]$ at which S_0^τ is K^+-subdiffusive. Propositions 5.3 and 5.5 give lower and upper bounds on the (deterministic) Hausdorff dimension of these exceptional times in $[0, \infty)$. Interestingly, the Hausdorff dimensions depend non-trivially on the constant K so that the dimension tends to zero (respectively, one) as $K \to 0$ (respectively, $K \to \infty$). In particular, as a direct consequence of Proposition 5.3, we obtain the following result.

Theorem 1.2. *The set of times $\tau \in [0, \infty)$ at which S_0^τ is subdiffusive has Hausdorff dimension one. Hence, the set of exceptional times for the law of the iterated logarithm (LIL) also has Hausdorff dimension one.*

Since a set of exceptional times has zero Lebesgue measure (see (1.1)), we see that the set of exceptional times for the LIL (or for subdiffusivity) is in a sense as large as it can be. This is strikingly in contrast with the dynamical one-dimensional random walk of [8] where there are no exceptional times for which the LIL fails (in [8], we recall that the analogue of S_0^τ is simply defined as

$$\bar{S}_0^\tau(n) = \sum_{i=1}^{n} X_i^\tau, \tag{1.4}$$

where $\{X_i^\tau\}_i$ are independent $\{-1, +1\}$-valued Markov jump processes with rate 1 and uniform initial distribution). To explain why the walks of [8] can behave so differently from those of the discrete web, we note that a single switch in the dynamical random walk of (1.4) affects only one increment of the walk while single switches in the discrete web can change the path of the walker by a "macroscopic" amount. Indeed, the difference between the path S_0^τ before and after a single switch is given by the difference between two independent simple random walks starting two spatial units apart. This corresponds to the excursion of a (non-simple) random walk from zero whose mean duration is infinite. It follows that a simple random walk is more sensitive to the extra noise induced by the dynamics on the discrete web than to the one induced by the dynamics considered in [8]. Rephrasing [3] in our context, since our dynamical random walk "changes faster" than the one in [8], it has "more chances" to exhibit exceptional behavior. Mathematically, "changing fast" corresponds to having small correlations over short time intervals and the main ingredient for proving our exceptional times results will be the correlation estimate (3.16) of Proposition 3.1.

By an obvious symmetry argument, there are also exceptional dynamical times τ for which $S^\tau(t) \le j + K\sqrt{t}$ for all t. One may ask whether there are exceptional τ for which $|S^\tau(t)| \le j + K\sqrt{t}$ for all t. Proposition 5.8 shows, at least for small K, that there are no such exceptional times. The case of large K is unresolved.

1.5. Scaling limits

In Section 6, we discuss the continuum analogue of the dynamical random walk, the dynamical Brownian motion constructed in [4]. We briefly recall there the main ideas of the construction along with some elementary properties of that object. Then, we outline the main ideas that are needed to extend the results for exceptional times from the discrete level to the continuum.

2. Tameness

In this section, we prove the following theorem.

Theorem 2.1. $\mathbb{P}\left(S_0^\tau \text{ is recurrent for all } \tau \ge 0\right) = 1.$

Recall the definition of the dynamical percolation model given in the introduction. For Bernoulli percolation on a homogeneous graph with critical probability p_c, let $\theta(p)$ be the probability that the origin belongs to an infinite cluster. In Section 3 of [2], it is proved that if for some $C < \infty$

$$\theta(p) \le C(p - p_c) \quad \text{for } p \ge p_c, \tag{2.5}$$

then in the corresponding dynamical percolation model, there is almost surely no dynamical time τ at which percolation occurs. In our setting, an entirely parallel argument can be used to show tameness of the dynamical discrete web with respect to recurrence.

Following [2], we start by giving a very general tameness criterion. Let $\mathbb{P}_p$ be the probability measure for the static web when the probability for having a right arrow at a given site of $\mathbb{Z}^2_{even}$ is p. Let S_0 be the simple random walk starting from the origin and let A be a measurable set of paths such that $\mathbb{P}_{1/2}(S_0 \in A) = 0$. In the following, $\mathbb{P}_p(S_0 \in A)$ plays the role that $\theta(p)$ did for dynamical percolation. Our first lemma is the analogue to Lemma 3.1 in [2].

Lemma 2.2. *Let A be such that $\{S_0 \in A\}$ is an increasing event w.r.t. the basic Bernoulli $\{\xi_z\}$ variables and such that $\mathbb{P}_{1/2}(S_0 \in A) = 0$. Let N_A be the cardinality of the set $\{\tau \in [0,1] : S_0^\tau \in A\}$. Suppose there exists $c < \infty$ such that*

$$\mathbb{P}_p(S_0 \in A) \le c\left(p - \frac{1}{2}\right) \quad \text{for all } p \ge \frac{1}{2}; \tag{2.6}$$

then $\mathbb{E}(N_A) < \infty$.

Proof. Let $m > 1$. We first estimate $\mathbb{E}(N_m)$ where N_m is the number of $i \in \{1, 2, \ldots, m\}$ such that there exists $\tau \in [\frac{i-1}{m}, \frac{i}{m}]$ for which $S_0^\tau \in A$. For a given $i \le m$, define

$$\bar{\xi} = \bar{\xi}(i) = \left\{ \sup_{\tau \in \left[\frac{i-1}{m}, \frac{i}{m}\right]} \xi_z^\tau \right\}_{z \in \mathbb{Z}^2_{even}}. \tag{2.7}$$

This naturally induces a new arrow configuration $\bar{\aleph}$ for which the probability to find a right arrow at any given site is given by

$$\bar{p} = 1 - \frac{1}{2}\exp\left(-\frac{1}{2m}\right) \leq \frac{1}{2} + \frac{1}{4m}. \tag{2.8}$$

For such a configuration, the path $\bar{S}_0$ starting from the origin is a drifting random walk coupled with S_0^τ in such a way that

$$\forall \tau \in \left[\frac{i-1}{m}, \frac{i}{m}\right], \quad S_0^\tau \leq \bar{S}_0, \tag{2.9}$$

which implies

$$\mathbb{P}\left(\exists \tau \in \left[\frac{i-1}{m}, \frac{i}{m}\right] \text{ with } S_0^\tau \in A\right) \leq \mathbb{P}(\bar{S}_0 \in A) \tag{2.10}$$

$$\leq c\left(\bar{p} - \frac{1}{2}\right) \leq \frac{c}{4m}. \tag{2.11}$$

Hence, $E(N_m) \leq m\frac{c}{4m} = c/4$ for all $m > 1$. Since $N_A = \liminf_{m\uparrow\infty} N_m$, Fatou's lemma completes the proof. □

We now turn to the proof of Theorem 2.1. We extend in the usual way a random walk π, initially defined for integer times, to be a continuous (piecewise linear) function $\pi(t)$ for $t \in [0, \infty)$ by setting $\pi(t) = \pi(k) + (t-k)(\pi(k+1) - \pi(k))$ for $k \leq t \leq k+1$. For any $n \geq 0$, let A_n be the set of (piecewise linear) simple random walks π starting from the origin and such that for all $t \geq 0$, $\pi(t) > -n$. It is well known that

$$\mathbb{P}_p(S_0 \in A) = 1 - \left(1 - \frac{(2p-1)}{p}\right)^n, \quad \text{for } p \in \left[\frac{1}{2}, 1\right]. \tag{2.12}$$

Clearly, A_n satisfies the hypotheses of Lemma 2.2, implying that $\mathbb{E}(N_{A_n}) < \infty$.

In Lemma 3.4 of [2], it is proved that for any homogeneous graph with $\theta(p_c) = 0$, the number N of times $\tau \in [0, 1]$ such that in dynamical percolation the origin belongs to an infinite cluster is a.s. either 0 or ∞. By exactly the same reasoning, one can show that N_{A_n} is either 0 or ∞; we give the proof below for completeness. Since $\mathbb{E}(N_{A_n}) < \infty$ for every n, we then have that $N_{A_n} = 0$ for every n and this together with the corresponding result for transience to $-\infty$ will complete the proof of Theorem 2.1.

It remains to show that a.s. N_{A_n} is either 0 or ∞. Arguing by contradiction, we assume that for some $0 < k < \infty$, $P(N_{A_n} = k) > 0$. Then, by an application of the martingale convergence theorem, there will be some large finite subset Λ of $\mathbb{Z}^2_{even}$ and some realization ω of $\xi_\Lambda = \{\xi_z^\tau : z \in \Lambda, \tau \in [0, 1]\}$ so that the regular conditional distribution $\tilde{\mathbb{P}}$ of $\mathbb{P}$ given ω has $\tilde{\mathbb{P}}(N_{A_n} = k) \geq 0.9$. Let $\mathcal{A}_s$ denote the event that $S_0^\tau \in A_n$ for at least one $\tau \in [0, s)$. Then $\tilde{\mathbb{P}}(\mathcal{A}_s)$ is non-decreasing and left-continuous in s with $\tilde{\mathbb{P}}(\mathcal{A}_0) = 0$ and $\tilde{\mathbb{P}}(\mathcal{A}_1) \geq 0.9$; it is also right-continuous because at the countably many τ_j's where for some z, ξ_z^τ changes, $\tilde{\mathbb{P}}(S_0^{\tau_j} \in A_n) = 0$. Thus for some $\gamma \in (0, 1)$, $\tilde{\mathbb{P}}(\mathcal{A}_\gamma) = 0.5$. Let $\mathcal{B}_\gamma$ denote the event that $S_0^\tau \in A_n$ for at least k different τ's in $[\gamma, 1]$. Then since $\{N_{A_n} = k\} \subset \mathcal{A}_\gamma \cup \mathcal{B}_\gamma$, it follows that $\tilde{\mathbb{P}}(\mathcal{B}_\gamma) \geq 0.4$. Letting ξ^γ denote $\{\xi_z^\gamma : z \in \mathbb{Z}^2_{even}\}$, we have that the conditional probabilities $\tilde{\mathbb{P}}(\mathcal{A}_\gamma | \xi^\gamma)$ and $\tilde{\mathbb{P}}(\mathcal{B}_\gamma | \xi^\gamma)$

are increasing functions of ξ^γ so that by the Markov property and the Harris–FKG inequalities,

$$\tilde{\mathbb{P}}(\mathcal{A}_\gamma \cap \mathcal{B}_\gamma) = \int \tilde{\mathbb{P}}(\mathcal{A}_\gamma|\xi^\gamma)\tilde{\mathbb{P}}(\mathcal{B}_\gamma|\xi^\gamma)\mathrm{d}\tilde{\mathbb{P}} \geq \tilde{\mathbb{P}}(\mathcal{A}_\gamma)\cdot\tilde{\mathbb{P}}(\mathcal{B}_\gamma) \geq 0.2. \tag{2.13}$$

Since $\{N_{A_n} = k\}$ and $\mathcal{A}_\gamma \cap \mathcal{B}_\gamma$ are disjoint, it follows that $\tilde{\mathbb{P}}(N_{A_n} = k) \leq 0.8$ which contradicts our earlier assumption and completes the proof.

Remark 2.3. Another property of the static discrete web with respect to which the dynamical one is tame is the almost sure coalescence of all of its paths. Indeed, the difference between two independent random walks is again a (non-simple) random walk and the proof of Theorem 2.1 can easily be adapted to show that at every dynamical time τ two walkers always meet and coalesce after some finite time t.

3. Sensitivity to the dynamics

In the following, $(C([0,1]), |.|_\infty)$ denotes the space of continuous functions on [0, 1] equipped with the sup norm. In order to prepare for our results about exceptional times, we need to prove that the arrow configuration in the DyDW decorrelates fast enough to allow exceptional behavior for the dynamical random walk. This will be done by proving that on a large (diffusive) scale and for $\tau \neq \tau'$, the paths S_0^τ and $S_0^{\tau'}$ evolve almost independently. More precisely, if for any (small) $\delta > 0$ and any $\pi \in C([0,1])$ we set $\tilde{\pi}(t) \equiv \pi(t/\delta^2)\,\delta$, we will prove that for a certain open set $O \in C([0,1])$, we have the following decorrelation inequality:

$$\mathbb{P}(\tilde{S}_0^\tau \in O, \tilde{S}_0^{\tau'} \in O) \leq \mathbb{P}(\tilde{S} \in O)^2 + K\left(\frac{\delta}{|\tau - \tau'|}\right)^a, \tag{3.14}$$

where S is a simple symmetric random walk and K does not depend on δ, τ and τ'. In other words, the inequality (3.14) estimates the sensitivity of the event O to the dynamics.

We now turn to our specific choice for O. Recall that we aim to prove that at some exceptional τ's the path S_0^τ is K^+-subdiffusive, which requires that the walk starting from the origin is abnormally tilted to the right. Hence, it is natural to study the noise sensitivity of the event

$$O = \{\forall t \in [0,1], \pi(t) > -1 \text{ and } \pi(1) > 1\} \tag{3.15}$$

which occurs for paths slightly tilted to the right. Studying noise sensitivity for this event is analogous to the corresponding question concerning left–right crossing of a square in dynamical percolation as studied in [3]. The previous discussion motivates the following proposition.

Proposition 3.1. *For $O = \{\forall t \in [0,1],\ \pi(t) > -1$ and $\pi(1) > 1\}$, there exist $K, a \in (0,\infty)$ (independent of δ, τ and τ') such that*

$$\mathbb{P}(\tilde{S}_0^\tau \in O, \tilde{S}_0^{\tau'} \in O) \leq \mathbb{P}(\tilde{S} \in O)^2 + K\left(\frac{\delta}{|\tau - \tau'|}\right)^a, \tag{3.16}$$

where S is a simple symmetric random walk.

In order to prove the proposition, we start by highlighting the fact that along the t-axis, the pair $(S_0^\tau, S_0^{\tau'})$ alternates between times at which the two paths are equal (they "stick together") and times at which they move independently. Recall that if S_0^τ and $S_0^{\tau'}$ coincide at time t *and* if the clock at $(S_0^\tau(t), t)$ does not ring on $[\tau, \tau')$, the increments of S_0^τ and $S_0^{\tau'}$ at time t are equal (i.e., S_0^τ and $S_0^{\tau'}$ stick together). Otherwise, the two increments are independent. This suggests the following time decomposition of the pair $(S_0^\tau, S_0^{\tau'})$. Define inductively $\{T_k\}_{k\geq 0}$ with $T_0 = 0$ and for any $k \geq 0$,

$$T_{2k+1} = \inf\{n \in \mathbb{N}, n \geq T_{2k} : \text{the clock at } (S_0^\tau(n), n) \text{ rings in } [\tau, \tau')\},$$

$$T_{2k+2} = \inf\{n \in \mathbb{N}, n > T_{2k+1} : S_0^\tau(n) = S_0^{\tau'}(n)\},$$

$$\Delta T_k = T_{2k+1} - T_{2k}, \quad \text{where we have } \mathbb{P}(\Delta T_k \geq j) = \mathrm{e}^{-|\tau-\tau'|j}.$$

On the interval of integer time $[T_{2k}, T_{2k+1}]$, the paths $S_0^\tau, S_0^{\tau'}$ coincide and at time T_{2k+1} they move independently until meeting at time T_{2k+2}. Hence, if we skip the intervals $\{[T_{2k}, T_{2k+1})\}_{k\geq 0}$, $(S_0^\tau, S_0^{\tau'})$ behave as two independent random walks $(S_d^\tau, S_d^{\tau'})$, while if we skip $\{[T_{2k+1}, T_{2k+2})\}_{k\geq 0}$, the two walks coincide with a single random walk S_s. Furthermore, since S_s is constructed from the arrow configuration at sites different from the ones used to construct $(S_d^\tau, S_d^{\tau'})$, it is independent of $(S_d^\tau, S_d^{\tau'})$.

Now, skipping the intervals $\{[T_{2k}, T_{2k+1})\}_{k\geq 0}$ corresponds to making the random time change $t \to C(t)$ where C is the right-continuous inverse of

$$t + \sum_{k\leq l(t)} \Delta T_k \quad \text{with } l(t) = \#\{i \in \mathbb{N}, i \leq t : S_d^\tau(i) = S_d^{\tau'}(i)\}. \tag{3.17}$$

Skipping $\{[T_{2k+1}, T_{2k+2})\}_{k\geq 0}$ corresponds to making the time change $t \to t - C(t)$. This analysis yields the following lemma.

Lemma 3.2. *There exist three independent simple symmetric random walks $(S_s, S_d^\tau, S_d^{\tau'})$ and an independent sequence of independent non-negative integer valued random variables $\{\Delta T_k\}_{k\geq 0}$ with $\mathbb{P}(\Delta T_k \geq j) = \mathrm{e}^{-(\tau'-\tau)j}$ such that*

$$S_0^\tau(t) = S_d^\tau(C(t)) + S_s(t - C(t)), \tag{3.18}$$

$$S_0^{\tau'}(t) = S_d^{\tau'}(C(t)) + S_s(t - C(t)), \tag{3.19}$$

where C is the right-continuous inverse of (3.17).

In the following, the pair $(S_0^\tau, S_0^{\tau'})$ will be referred to as a *sticky pair of random walks*. We note that the previous lemma has a continuous analogue called a sticky pair of Brownian motions—see Section 6 for more details.

Heuristically, in order to prove Proposition 3.1, we need to show that at large (diffusive) scales Eqs. (3.18) and (3.19) become

$$S_0^\tau(t) \approx S_d^\tau(t), \tag{3.20}$$

$$S_0^{\tau'}(t) \approx S_d^{\tau'}(t), \tag{3.21}$$

or equivalently that $C(t) \approx t$ (see Lemma 3.4). The following three lemmas prepare the justification of this informal approximation. Let $\delta > 0$. We recall that for a path S, $\tilde{S}(\cdot) \equiv S(\cdot/\delta^2)\,\delta$. In the following, we set $\Delta \equiv \delta/|\tau - \tau'|$ and for $O \subset C([0, 1])$ and any $r \geq 0$, we define

$$O + r \equiv \{\pi \in C([0, 1]) \text{ s.t. } \exists\, \bar{\pi} \in O \text{ s.t. } |\pi - \bar{\pi}|_\infty \leq r\}.$$

Lemma 3.3. *Let S be a simple symmetric random walk. For the O defined in (3.15) and any $\alpha < \frac{1}{2}$,*

$$\mathbb{P}(\tilde{S} \in [O + \Delta^{\alpha}] \setminus O) \leq c' \Delta^{\alpha} \tag{3.22}$$

where $c' \in (0, \infty)$ is independent of Δ and δ.

Proof.

$$\mathbb{P}(\tilde{S} \in [O + \Delta^{\alpha}] \setminus O) \leq \mathbb{P}\left(\inf_{t\in[0,1]} \tilde{S}(t) \in (-1 - \Delta^{\alpha}, -1]\right) + \mathbb{P}(\tilde{S}(1) \in (1 - \Delta^{\alpha}, 1]). \tag{3.23}$$

We will first prove that the second term on the right-hand side of this inequality is of order Δ^{α} and then that the first term has the same bound.

In [13], it is proved that a sequence of rescaled standard random walks $\{S(\cdot/\delta^2)\delta\}_{\delta>0}$ and a Brownian motion B can be constructed on the same probability space in such a way that for any $\alpha < \frac{1}{2}$, the quantity $\mathbb{P}(|B - S(\cdot\ /\delta^2)\delta|_{\infty} > \delta^{\alpha})$ goes to 0 faster than any power of δ. On this probability space,

$$\mathbb{P}(\tilde{S}(1) \in (1 - \Delta^{\alpha}, 1]) \leq \mathbb{P}(B(1) \in [1 - 2\Delta^{\alpha}, 1 + \Delta^{\alpha}]) + P(|\tilde{S} - B|_{\infty} \geq \Delta^{\alpha}). \tag{3.24}$$

Let $\alpha < \frac{1}{2}$. Since $\Delta^{\alpha} > \delta^{\alpha}$ (because $|\tau - \tau'| \leq 1$), the last term on the right-hand side of (3.24) is bounded by $O(\delta)$, and consequently by $O(\Delta)$. Since $B(1)$ has a Gaussian density, the first term on the right-hand side of this inequality is clearly bounded by $c\Delta^{\alpha}$ giving us the correct bound for the second term on the right-hand side of (3.23). To get bounds of the same order for the first term from (3.23), we proceed along the same lines, approximating the random walk by a Brownian motion and using the fact that the well-known explicit density function for its maximum is bounded. This completes the proof of the lemma. □

Lemma 3.4. *Define $\bar{C}(t) = C(t/\delta^2)\delta^2$ where C is defined in (3.17) (note that the random clock C and the paths are rescaled in a different manner).*

For any $1 > \beta > 0$

$$\mathbb{P}\left(\sup_{t\in[0,1]} (t - \bar{C}(t)) \geq \Delta^{\beta}\right) \leq \tilde{c}\Delta^{1-\beta}, \tag{3.25}$$

where $\tilde{c} \in (0, \infty)$ is independent of Δ and δ.

Proof. Recall from Lemma 3.2 that

$$C^{-1}(t) = t + L(t), \quad \text{where } L(t) = \sum_{k\leq l(t)} \Delta T_k, \tag{3.26}$$

$\{\Delta T_k\}$ are independent geometric random variables, l is the discrete local time at the origin of $S_d^{\tau} - S_d^{\tau'}$ and C^{-1} is the right-continuous inverse of C. In the following, we set $\bar{L}(t) \equiv L(t/\delta^2)\,\delta^2$. We first prove that

$$\mathbb{P}(\bar{L}(1) \geq \Delta^{\beta}) \leq \tilde{c}\Delta^{1-\beta}. \tag{3.27}$$

By the Markov inequality,

$$\mathbb{P}(\bar{L}(1) \geq \Delta^{\beta}) \leq \mathbb{E}\left(l(1/\delta^2)\delta\right)(\mathbb{E}(\Delta T_1)\delta)\frac{1}{\Delta^{\beta}} \tag{3.28}$$

$$\text{with } \mathbb{E}(\Delta T_1) = \sum_{k=1}^{\infty} \mathrm{e}^{-|\tau-\tau'|\,k} = \frac{\exp(-|\tau-\tau'|)}{1-\exp(-|\tau-\tau'|)}. \tag{3.29}$$

Now

$$\mathbb{E}(l(1/\delta^2)) = \sum_{k \leq 1/\delta^2} \mathbb{P}(S_d^{\tau}(k) - S_d^{\tau'}(k) = 0)$$

and it is a standard fact that the probability in the summation is $O(1/\sqrt{k})$ as $k \to \infty$; thus $\mathbb{E}(l(1/\delta^2)\delta)$ is uniformly bounded in δ as $\delta \to 0$. Furthermore, since $E(\Delta T_1) = O(|\tau - \tau'|^{-1})$, we have $\delta E(\Delta T_1) = O(\Delta)$ and thus (3.27) follows.

Next, on the event $\{\bar{L}(1) \leq \Delta^{\beta}\}$, (3.26) implies that for any $t \in [0, 1]$:

$$(\bar{C})^{-1}(t) \leq t + \Delta^{\beta}. \tag{3.30}$$

Since $\bar{C}(t) \leq t$ and $\bar{C}$ is an increasing function of t, it follows that on $\{\bar{L}(1) \leq \Delta^{\beta}\}$, for all $t \in [0, 1]$, we have

$$t - \bar{C}(t) \leq \Delta^{\beta}. \tag{3.31}$$

The lemma thus follows from (3.27).

Lemma 3.5. *For any continuous function f, define $\omega_f(\epsilon) = \sup_{s,t\in[0,1],|s-t|<\epsilon} |f(t) - f(s)|$ to be the modulus of continuity of f on* [0, 1].

Let $\alpha, \beta \in (0, \infty)$ be such that $\beta/2 > \alpha$. For any $r \geq 0$, there exists c (independent of Δ and δ) such that

$$\mathbb{P}\left(\omega_{\tilde{S}}(\Delta^{\beta}) \geq \frac{\Delta^{\alpha}}{2}\right) \leq c\Delta^{r}. \tag{3.32}$$

Proof. Let $m, n \geq 0$ and define

$$\tilde{M} \equiv \int_0^1 \int_0^1 \frac{|\tilde{S}(t) - \tilde{S}(s)|^n}{|t-s|^m} \,\mathrm{d}t\mathrm{d}s. \tag{3.33}$$

By the Garsia, Rodemich and Rumsey inequality [14], we have for $m > 2$ and all $s, t \in [0, 1]$

$$|\tilde{S}(t) - \tilde{S}(s)| \leq \frac{8m}{m-2}(4\tilde{M})^{\frac{1}{n}}|t-s|^{\frac{m-2}{n}}. \tag{3.34}$$

It is well known that $\mathbb{E}(|\tilde{S}(t) - \tilde{S}(s)|^n) \leq c'|t-s|^{\frac{n}{2}}$, where c' is uniform in δ. Hence, (3.33) implies that if $\frac{n}{2} - m > -1$, then $\mathbb{E}(\tilde{M}) \leq c < \infty$ so that for every $r \geq 0$,

$$\mathbb{P}(\tilde{M} > \Delta^{-r}) \leq c\Delta^{r}. \tag{3.35}$$

On the other hand, on $\{\tilde{M} \leq \Delta^{-r}\}$, (3.34) yields

$$\omega_{\tilde{S}}(\Delta^{\beta}) \leq \frac{8m}{m-2}(4\Delta^{-r})^{\frac{1}{n}}|\Delta^{\beta}|^{\frac{m-2}{n}} \tag{3.36}$$

$$\leq c(n,m)\,\Delta^{\frac{1}{n}(\beta(m-2)-r)}. \tag{3.37}$$

Since $\beta/2 > \alpha$, one can always take (for fixed α, β, r) n, m large enough such that both $\frac{n}{2} - m > -1$ and $\frac{1}{n}(\beta(m-2) - r) > \alpha$. For such a choice, and taking Δ small enough so that $c(n,m)\Delta^{\frac{1}{n}(\beta(m-2)-r)} \leq \Delta^\alpha/2$, we obtain then on $\{\tilde{M} \leq \Delta^{-r}\}$

$$\omega_{\tilde{S}}(\Delta^\beta) \leq \Delta^\alpha/2. \tag{3.38}$$

Hence, for small enough values of Δ, the claim of the lemma follows from (3.35). The claim is obviously satisfied for larger values of Δ. $\square$

We are now ready to prove Proposition 3.1. Recall the definitions of S_d^τ and $S_d^{\tau'}$ in Lemma 3.2. For any $\alpha > 0$, we have

$$\begin{aligned}\mathbb{P}(\tilde{S}_0^\tau \in O, \tilde{S}_0^{\tau'} \in O) &\leq \mathbb{P}(\tilde{S}_d^\tau \in O + \Delta^\alpha, \tilde{S}_d^{\tau'} \in O + \Delta^\alpha)\\ &\quad + 2\mathbb{P}(\tilde{S}_0^\tau \in O, \tilde{S}_d^\tau \in (O + \Delta^\alpha)^c),\end{aligned} \tag{3.39}$$

where $(O + \Delta^\alpha)^c$ is the complementary set of $O + \Delta^\alpha$. Note that we used the equidistribution of $(\tilde{S}_d^\tau, \tilde{S}^\tau)$ and $(\tilde{S}_d^{\tau'}, \tilde{S}^{\tau'})$. We start by dealing with the first term on the right-hand side of the inequality. Since $\tilde{S}_d^\tau, \tilde{S}_d^{\tau'}$ are independent and distributed like a rescaled simple symmetric random walk $\tilde{S}$, we have

$$\begin{aligned}\mathbb{P}(\tilde{S}_d^\tau \in O + \Delta^\alpha, \tilde{S}_d^{\tau'} \in O + \Delta^\alpha) &= \mathbb{P}(\tilde{S}_d^\tau \in O + \Delta^\alpha)\, \mathbb{P}(\tilde{S}_d^{\tau'} \in O + \Delta^\alpha)\\ &\leq \mathbb{P}(\tilde{S} \in O)^2 + 2\mathbb{P}(\tilde{S} \in [O + \Delta^\alpha] \setminus O).\end{aligned} \tag{3.40}$$

The latter inequality and Lemma 3.3 imply that

$$\mathbb{P}(\tilde{S}_d^\tau \in O + \Delta^\alpha, \tilde{S}_d^{\tau'} \in O + \Delta^\alpha) \leq \mathbb{P}(\tilde{S} \in O)^2 + 2c'\Delta^\alpha, \tag{3.41}$$

for any $\alpha < 1/2$. By (3.39) and (3.41), Proposition 3.1 follows if there are $c'', a' \in (0, \infty)$ such that

$$\mathbb{P}[\tilde{S}_0^\tau \in O, \tilde{S}_d^\tau \in (O + \Delta^\alpha)^c] \leq c''\Delta^{a'}. \tag{3.42}$$

This inequality can be justified as follows. Let $0 < \beta < 1$. By Lemma 3.2

$$\begin{aligned}\tilde{S}_0^\tau(t) &= \tilde{S}_d^\tau(\bar{C}(t)) + \tilde{S}_s(t - \bar{C}(t)) &(3.43)\\ &= \tilde{S}_d^\tau(t) + [\tilde{S}_d^\tau(\bar{C}(t)) - \tilde{S}_d^\tau(t)] + \tilde{S}_s(t - \bar{C}(t)). &(3.44)\end{aligned}$$

The last equality implies that for any $0 < \beta < 1$ with $\alpha < \beta/2$,

$$\begin{aligned}\mathbb{P}[\tilde{S}_0^\tau \in O, \tilde{S}_d^\tau \in (O + \Delta^\alpha)^c] &\leq \mathbb{P}(|\tilde{S}_0^\tau - \tilde{S}_d^\tau|_\infty \geq \Delta^\alpha)\\ &\leq \mathbb{P}\left(|\tilde{S}_s(t - \bar{C}(t))|_\infty \geq \frac{\Delta^\alpha}{2}\right)\\ &\quad + \mathbb{P}\left(|\tilde{S}_d^\tau(t) - \tilde{S}_d^\tau(\bar{C}(t))|_\infty \geq \frac{\Delta^\alpha}{2}\right)\\ &\leq 2\mathbb{P}\left(\omega_{\tilde{S}}(\Delta^\beta) \geq \frac{\Delta^\alpha}{2}\right) + 2\mathbb{P}(|t - \bar{C}(t)|_\infty \geq \Delta^\beta)\\ &\leq 2c\Delta^r + 2\tilde{c}\Delta^{1-\beta}\end{aligned}$$

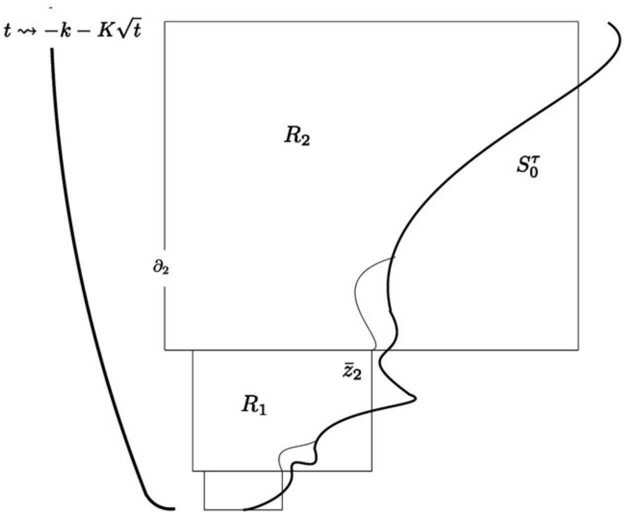

Fig. 1. Construction of the first three boxes (R_0, R_1, R_2) with t the vertical and x the horizontal coordinate. The thin curves represent segments of the paths starting from $\bar{z}_i$, for $i = 1, 2, 3$, for which the events A_0^τ, A_1^τ and A_2^τ occur.

where $r > 0$ and the last inequality is given by Lemmas 3.4 and 3.5. So far we have only needed $\alpha \in (0, 1/2)$ and thus we can indeed choose $\beta \in (0, 1)$ and then $\alpha < \beta/2$ so that Proposition 3.1 follows.

4. Existence of exceptional times

In this section we prove the following result.

Proposition 4.1. *For K large enough*

$$\mathbb{P}(\exists \tau \in [0, 1],\ s.t.\ S_0^\tau \text{ is } K^+\text{-subdiffusive}) > 0. \tag{4.45}$$

(For a definition of K^+-subdiffusivity, see Definition 1.1.)

Let $\gamma > 2$ and $d_k = 2(\lfloor \frac{\gamma^k}{2} \rfloor + 1)$, where $\lfloor x \rfloor$ is the integer part of x. We construct inductively a sequence of boxes R_k with diffusive space–time scale in the following manner (see Fig. 1).

- R_0 is the rectangle with vertices $(-d_0, 0)$, $(+d_0, 0)$, $(-d_0, d_0^2)$ and $(+d_0, d_0^2)$.
- Let $\bar{z}_k = (x_k, t_k)$ be the middle point of the lower edge of R_k (e.g., $\bar{z}_0 = (0, 0)$). R_{k+1} is the rectangle of height d_{k+1}^2 and width $2d_{k+1}$ such that $\bar{z}_{k+1}$ coincides with the upper right vertex of R_k (see Fig. 1).

Note that for our particular choice of d_k, $\bar{z}_k$ always belongs to $\mathbb{Z}^2_{even}$ for $k \geq 0$ and a simple computation leads to the following lemma. Note further that x_k, t_k, d_k and hence the rectangles R_k and their left boundaries ∂_k all depend on the parameter γ.

Lemma 4.2. *Let $\partial^\gamma = \partial^\gamma(t)$ denote the right-continuous function obtained by joining together the left boundaries ∂_k of R_k. For any $K > 0$, let $\gamma(K)$ be the solution in $(2, \infty)$ of $K = (\gamma - 2)\sqrt{\frac{\gamma+1}{\gamma-1}}$. Then,*

$$\forall t \geq 0, \quad \partial^{\gamma(K)}(t) \geq -3 - K\sqrt{t}. \tag{4.46}$$

Proof. On $[t_n, t_{n+1})$, we have $\partial^\gamma(t) = \partial^\gamma(t_n) = x_n - d_n = (d_0 + d_1 + \cdots + d_{n-1} - d_n)$. If γ is such that

$$\partial^\gamma(t_n) \geq -(3 + K\sqrt{t_n}) \quad \text{for } n = 0, 1, 2, \ldots, \tag{4.47}$$

then we will have $\partial^\gamma(t) \geq -(3 + K\sqrt{t})$ for all $t \geq 0$ as desired.

The inequality (4.47) can be rewritten as

$$d_n \leq 3 + d_0 + \cdots + d_{n-1} + K[{d_0}^2 + \cdots + d_{n-1}^2]^{1/2}. \tag{4.48}$$

Using the bound $d_n \leq 2 + \gamma^n$ on the left-hand side of (4.48) and the bounds $d_j \geq \gamma^j$ on the right-hand side, it follows that in order to verify (4.48) it suffices to have, for $n = 0, 1, 2, \ldots$,

$$\gamma^n \leq 1 + \frac{\gamma^n - 1}{\gamma - 1} + K\sqrt{\frac{\gamma^{2n} - 1}{\gamma^2 - 1}}. \tag{4.49}$$

Using the elementary bound $\sqrt{\gamma^{2n} - 1} \geq \gamma^n(1 - \gamma^{-2n})$ (for $\gamma \geq 1$), we see that in order to verify (4.49), it suffices to have, for $n = 0, 1, 2, \ldots$,

$$\gamma^n \left(\frac{\gamma - 2}{\gamma - 1} - \frac{K}{\sqrt{\gamma^2 - 1}} \right) \leq 1 - \frac{1}{\gamma - 1} - \frac{K}{\sqrt{\gamma^2 - 1}} \gamma^{-n}, \tag{4.50}$$

which is satisfied by any γ for which $K = (\gamma - 2)\sqrt{\frac{\gamma+1}{\gamma-1}}$. The lemma follows from the fact that $\gamma \to (\gamma - 2)\sqrt{\frac{\gamma+1}{\gamma-1}}$ is a continuous increasing function mapping $(2, \infty)$ onto $(0, \infty)$.

By Lemma 4.2, S_0^τ is K^+-subdiffusive if $S_0^\tau(t) \geq \partial^{\gamma(K)}(t)$. Let $S_{\bar{z}_k}^\tau$ be the path in $W(\tau)$ starting from $\bar{z}_k = (x_k, t_k)$ and define the event

$$A_k^\tau = A_k^\tau(K) = \{\forall t \in [t_k, t_{k+1}]\ S_{\bar{z}_k}^\tau(t) > \partial_k(t), S_{\bar{z}_k}^\tau(t_{k+1}) > x_{k+1}\}. \tag{4.51}$$

(Here ∂_k depends implicitly on $\gamma(K)$.) Since paths in $W(\tau)$ do not cross, if $\cap_{k \leq n} A_k^\tau$ occurs, S_0^τ is forced to remain to the right of ∂_k on $[t_k, t_{k+1}]$ for every $k \leq n$ (see Fig. 1). This implies that if we have

$$\mathbb{P}(\exists \tau \in [0, 1], \bigcap_{k \geq 0} A_k^\tau(K) \text{ occurs}) > 0, \tag{4.52}$$

then

$$\mathbb{P}(\exists \tau \in [0, 1], S_0^\tau \text{ is } K^+\text{-subdiffusive}) > 0. \tag{4.53}$$

In the rest of the section we proceed to verifying (4.52).

In the following, K is temporarily fixed and to ease the notation we write A_k^τ for $A_k^\tau(K)$ and γ for $\gamma(K)$. In order to verify (4.52), we start by proving the following lemma using Proposition 3.1.

Lemma 4.3. *There exists $c \in (0, \infty)$ such that for $\tau, \tau' \in [0, 1]$*

$$\forall n \geq 0, \quad \prod_{k=0}^{n} \frac{\mathbb{P}(A_k^\tau \cap A_k^{\tau'})}{\mathbb{P}(A_k)^2} \leq c \frac{1}{|\tau - \tau'|^b}, \tag{4.54}$$

where $A_k \equiv A_k^0$ and $b = \log(\sup_k[\mathbb{P}(A_k)^{-1}]) / \log\gamma > 0$.

Proof. Let $(S^\tau, S^{\tau'})$ be the paths starting at $(0,0)$, defined as the translated version of the pair $(S^\tau_{\bar z_k}, S^{\tau'}_{\bar z_k}) \in (W(\tau), W(\tau'))$ starting at $\bar z_k$. By translation invariance, $(S^\tau, S^{\tau'})$ is a sticky pair of random walks starting at $(0,0)$ whose distribution is described in Lemma 3.2 and by definition

$$A^\tau_k = \left\{ S^\tau(d_k^2) > d_k,\ \inf_{[0,d_k^2]} S^\tau(t) > -d_k \right\}. \tag{4.55}$$

Recalling the definition of d_k in terms of γ given at the beginning of this section and Proposition 3.1 for $\delta = d_k^{-1}$, we have that there exists $c, a \in (0,\infty)$ such that

$$\mathbb{P}(A^\tau_k, A^{\tau'}_k) \le \mathbb{P}(A_k)^2 + c\left(\frac{1}{\gamma^k|\tau-\tau'|}\right)^a. \tag{4.56}$$

Defining $N_0 = [\frac{-\log(|\tau-\tau'|)}{\log\ \gamma}] + 1$ so that $(\gamma^{N_0}|\tau-\tau'|) \ge 1$, we have for $n > N_0$

$$\begin{aligned}\prod_{k=N_0+1}^{n}\left(\frac{\mathbb{P}(A^\tau_k \cap A^{\tau'}_k)}{\mathbb{P}(A_k)^2}\right) &\le \prod_{k=N_0+1}^{\infty}\left(1 + \frac{c/\mathbb{P}(A_k)^2}{|\tau-\tau'|^a\gamma^{aN_0}\ \gamma^{a(k-N_0)}}\right)\\ &\le \prod_{k=1}^{\infty}\left(1 + \frac{c}{\inf_n \mathbb{P}(A_n)^2}\frac{1}{\gamma^{ak}}\right).\end{aligned} \tag{4.57}$$

The right-hand side of (4.57) is independent of $|\tau-\tau'|$ and is finite. Indeed, we have $0 < \inf_n P(A_n)$ since the boxes R_k have diffusively scaled sizes and therefore $\mathbb{P}(A_k) \to \mathbb{P}(A)$ as $k\to\infty$, where A is the event that a Brownian motion $B(t)$ starting at 0 at time 0 has $B(1) > 1$ and $\inf_{t\in[0,1]} B(t) > -1$.

On the other hand, for $n \le N_0$

$$\begin{aligned}\prod_{k=0}^{n}\frac{\mathbb{P}(A^\tau_k \cap A^{\tau'}_k)}{\mathbb{P}(A_k)^2} &\le \left(\sup_k \frac{1}{\mathbb{P}(A_k)}\right)^{N_0+1}\\ &\le c''\exp\left(\frac{\log[\sup_k(\mathbb{P}(A_k)^{-1})]}{\log\gamma}\log\left(\frac{1}{|\tau-\tau'|}\right)\right)\\ &= c''/|\tau-\tau'|^b,\end{aligned} \tag{4.58}$$

where $c'' = \sup_k(\mathbb{P}(A_k)^{-1})$ and $b = \log[\sup_k(\mathbb{P}(A_k)^{-1})]/\log\gamma$ are in $(0,\infty)$. This and (4.57) imply (4.54). $\square$

Arguing as in [3], the Cauchy–Schwarz inequality and the previous lemma imply that for every $n \ge 0$, we have

$$\mathbb{P}\left(\int_0^1 \prod_{k=0}^{n} 1_{A^\tau_k}\mathrm{d}\tau > 0\right) \ge \frac{\left(\mathbb{E}\left[\int_0^1 \prod_{k=0}^{n} 1_{A^\tau_k}\mathrm{d}\tau\right]\right)^2}{\mathbb{E}\left[\left(\int_0^1 \prod_{k=0}^{n} 1_{A^\tau_k}\mathrm{d}\tau\right)^2\right]} \tag{4.59}$$

$$= \left(\left[\int_0^1\int_0^1 \prod_{k=0}^{n} \frac{\mathbb{P}(A^\tau_k \bigcap A^{\tau'}_k)}{\mathbb{P}(A_k)^2}\mathrm{d}\tau\mathrm{d}\tau'\right]\right)^{-1} \tag{4.60}$$

$$\geq c^{-1}\left(\left[\int_0^1\int_0^1 \frac{1}{|\tau-\tau'|^b}\mathrm{d}\tau\mathrm{d}\tau'\right]\right)^{-1} \tag{4.61}$$

where the equality is a consequence of the stationarity of $\tau \to W(\tau)$ and the independence between the arrow configurations in different boxes R_k. Recall that γ has an implicit dependence on K and that γ increases from 0 to ∞ as K increases on $(0,\infty)$ (see Lemma 4.2). Hence, for K large enough such that $\gamma = \gamma(K) > \sup_k \mathbb{P}(A_k)^{-1}$, we have

$$b = \log\left(\sup_k[\mathbb{P}(A_k)^{-1}]\right)/\log\gamma(K) < 1$$

and $(\tau,\tau') \to |\tau-\tau'|^{-b} \in L^1([0,1]\times[0,1], \mathrm{d}\tau\,\mathrm{d}\tau')$. (4.61) then implies that

$$\inf_n \mathbb{P}\left(\int_0^1 \prod_{k=0}^n 1_{A_k^\tau}\mathrm{d}\tau > 0\right) \geq p > 0. \tag{4.62}$$

Let E_n be the set of times τ in [0, 1] such that $\bigcap_{k=0}^n A_k^\tau$ occurs. (4.62) implies that $\mathbb{P}(\bigcap_{n=0}^\infty\{E_n \neq \emptyset\}) \geq p > 0$. Since $\{E_n\}$ is obviously decreasing in n, if the E_n were closed subsets of [0, 1] it would follow that $\mathbb{P}((\bigcap_{n=0}^\infty E_n) \neq \emptyset) \geq p > 0$.

Unfortunately, the set of times at which one arrow is (or any finitely many are) oriented to the right (resp., to the left) is not in general a closed subset of [0, 1] since we have a right-continuous process, and thus E_n is not in general a closed set. This extra technicality is handled as in Lemma 5.1 in [3], as follows. On the one hand, there are only countably many switching times for all ξ_z^τ's (recall that ξ_z^τ represents the arrow direction at location z). On the other hand, at any switching time τ, $\bigcap_{n\geq 0} A_n^\tau$ does not occur by independence of the ξ_z^τ's. Since there are countably many switching times, this implies that almost surely, the closures $\bar{E}_n$ of E_n satisfy

$$\bigcap_{n=1}^\infty \bar{E}_n = \bigcap_{n=1}^\infty E_n. \tag{4.63}$$

This completes the verification of (4.52) and thus the proof of Proposition 4.1.

5. Hausdorff dimension of exceptional times

In this section, we derive some lower and upper bounds for the Hausdorff dimension of the set of exceptional dynamical times $\tau \in [0,\infty)$ at which S_0^τ becomes subdiffusive.

Definition 5.1. We say that τ is a K-exceptional time if the path S_0^τ in $W(\tau)$ does not cross the moving boundary $t \rightsquigarrow -K\sqrt{t}$. $\mathcal{T}(K)$ is then defined as the set of all K-exceptional times $\tau \in [0,\infty)$.

Clearly, the set consisting of all the K-exceptional times in $[0,\infty)$ is a non-decreasing function of K. The next proposition asserts that for fixed K, the Hausdorff dimension $\dim_H$ of the set of exceptional times is unchanged if $-K\sqrt{t}$ is replaced by $-j-K\sqrt{t}$ for any $j \geq 0$. We note that as in dynamical percolation (see Sec. 6 of [2]), $\dim_H(\mathcal{T}(K))$ is a.s. a constant by the ergodicity in τ of the dynamical discrete web.

Proposition 5.2. *The Hausdorff dimension dim_H of the set $T_j = T_j(K)$ of exceptional times $\tau \in (0,\infty)$ such that S_0^τ does not cross the moving boundary $t \rightsquigarrow -j-K\sqrt{t}$ does not depend on $j \geq 0$ (for fixed K).*

Proof. By the monotonicity in j, it is enough to prove that $\dim_H(T_j) \leq \dim_H(T_0)$ for any positive integer j.

First, $T_0 \supset T'_j \bigcap\{\tau \in [0,1] : \xi^\tau_{(m,m)} = +1 \text{ for } m < j\}$ where T'_j is the set of $\tau \in [0,\infty)$ such that $S^\tau_{(j,j)}(n) \geq -K\sqrt{n}$ for $n \geq j$. Furthermore, $T'_j \supset \bar{T}_j$, where $\bar{T}_j$ is the set of $\tau \in [0,\infty)$ such that $S^\tau_{(j,j)}(n) - j \geq -j - K\sqrt{n-j}$ for $n \geq j$. Note that $\bar{T}_j$ is just the translation (from $(0,0)$ to (j,j)) of T_j. Hence, $\dim_H(T_0)$ is at least the dimension of $\{\tau \in [0,1] : \forall\, m < j, \xi^\tau_{(m,m)} = +1\} \bigcap \bar{T}_j$.

By ergodicity in τ, the a.s. constant $\dim_H(\bar{T}_j)$ is the essential supremum of the random variable $\dim_H(\bar{T}_j \bigcap [0,1])$. On the other hand, since $\bar{T}_j \bigcap [0,1]$ and $\{\tau \in [0,1] : \forall\, m < j, \xi^\tau_{(m,m)} = +1\}$ are independent and the probability to have $\{\forall \tau \in [0,1], \forall m < j, \xi^\tau_{(m,m)} = +1\}$ is strictly positive, it follows that $\dim_H(\{\tau \in [0,1] : \forall\, m < j,\ \xi^\tau_{(m,m)} = +1\} \bigcap \bar{T}_j)$ has the same essential sup as $\dim_H(\bar{T}_j \bigcap [0,1])$. Hence $\dim_H(T_j) = \dim_H(\bar{T}_j) \leq \dim_H(T_0)$ and the conclusion follows. □

5.1. Lower bound

Set $\gamma_0 \equiv \sup_{k,K} 1/\mathbb{P}(A_k(K))$, where $A_k(K)$ is defined by (4.51) with $\tau = 0$. (Note that $\gamma_0 > 2$.) We recall that $\gamma(K)$ is the solution in $(2,\infty)$ of $K = K(\gamma) = (\gamma - 2)\sqrt{\frac{\gamma+1}{\gamma-1}}$ for $K > 0$. In this section, we prove the following proposition using Lemma 4.3 and then arguments identical to certain of those in [3].

Proposition 5.3.

$$\dim_H(\mathcal{T}(K)) \geq 1 - \frac{\log \gamma_0}{\log \gamma(K)} \quad \text{for } K > K(\gamma_0). \tag{5.64}$$

Thus, $\lim_{K\uparrow\infty} \dim_H(\mathcal{T}(K)) = 1$.

Let $K > K(\gamma_0)$. Note that since $K \to \gamma(K)$ is increasing, $\gamma(K) > \gamma_0$. In the following and as in Section 4, we drop the dependence on K in the notation. Consider the random measure σ_n, such that for any Borel set E in $[0,1]$

$$\sigma_n(E) = \int_E \prod_{k=0}^{n} \frac{1_{A^\tau_k}}{\mathbb{P}(A_k)} \mathrm{d}\tau.$$

We note that σ_n is supported by $\bar{E}_n$, the closure of E_n with

$$E_n = \{\tau \in [0,1] : \underset{k\leq n}{\cap} A^\tau_k \text{ occurs}\}. \tag{5.65}$$

For any positive measure σ, define the α-energy of σ as

$$\mathcal{E}_\alpha(\sigma) = \int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^\alpha} \mathrm{d}\sigma(\tau)\mathrm{d}\sigma(\tau'). \tag{5.66}$$

Following [3], we will need the following extension of Frostman's lemma.

Lemma 5.4 (*[3]*). *Let $D_1 \supset D_2 \supset \cdots$ be a decreasing sequence of compact subsets of* $[0,1]$, *and let $\mu_1, \mu_2, \ldots$ be a sequence of positive measures with μ_n supported on D_n. Suppose that there exist $C \in (0,\infty)$ and $\alpha \in (0,1)$ such that for infinitely many values of n,*

$$\mu_n([0,1]) \geq 1/C, \qquad \mathcal{E}_\alpha(\mu_n) \leq C. \tag{5.67}$$

Then the Hausdorff dimension of $\cap D_n$ is at least α.

Using the ergodicity of the dynamical web in the variable τ, we will prove Proposition 5.3 by showing that for $\alpha < 1 - \frac{\log(\gamma_0)}{\log(\gamma(K))}$, $\{\sigma_n\}$ satisfies the hypotheses of this lemma with strictly positive probability. By Lemma 4.3, we have for all n that

$$\mathbb{E}[\sigma_n([0,1])^2] = \int_0^1 \int_0^1 \prod_{k=0}^n \frac{\mathbb{P}(A_k^\tau \cap A_k^{\tau'})}{\mathbb{P}(A_k)^2} \mathrm{d}\tau \mathrm{d}\tau' \leq c \left(\left[\int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^b} \mathrm{d}\tau \mathrm{d}\tau' \right] \right),$$

where

$$b = \log \left[\sup_k (\mathbb{P}(A_k)^{-1}) \right] \Big/ \log \gamma \leq \frac{\log(\gamma_0)}{\log(\gamma)} < 1. \tag{5.68}$$

By the Cauchy–Schwarz inequality

$$\begin{aligned} \mathbb{E}\left[\sigma_n([0,1])^2\right]^{\frac{1}{2}} \mathbb{P}\left[\sigma_n([0,1]) > \frac{1}{2}\right]^{\frac{1}{2}} &\geq \mathbb{E}\left[\sigma_n([0,1]) \cdot 1_{\sigma_n([0,1]) > 1/2}\right] \\ &\geq \mathbb{E}\left[\sigma_n([0,1])\right] - \frac{1}{2} = \frac{1}{2}, \end{aligned}$$

which implies that $\mathbb{P}[\sigma_n([0,1] > \frac{1}{2})] > c_1$ for some $c_1 > 0$ not depending on n.

By Fubini's theorem and Lemma 4.3,

$$\begin{aligned} \mathbb{E}(\mathcal{E}_\alpha(\sigma_n)) &= \int_0^1 \int_0^1 |\tau - \tau'|^{-\alpha} \prod_{k=0}^n \frac{\mathbb{P}(A_k^\tau \cap A_k^{\tau'})}{\mathbb{P}(A_k)^2} \mathrm{d}\tau \mathrm{d}\tau' \\ &\leq c \int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^{b+\alpha}} \mathrm{d}\tau \mathrm{d}\tau'. \end{aligned} \tag{5.69}$$

Taking α such that

$$\alpha < 1 - \frac{\log(\gamma_0)}{\log(\gamma)}, \tag{5.70}$$

we have from (5.68) that $b + \alpha < 1$ and therefore

$$\sup_{n \geq 0} \mathbb{E}(\mathcal{E}_\alpha(\sigma_n)) \leq c_2 < \infty. \tag{5.71}$$

By the Markov inequality, for all n and all T,

$$\mathbb{P}(\mathcal{E}_\alpha(\sigma_n) \geq c_2 T) \leq 1/T. \tag{5.72}$$

Choose T such that $1/T < c_1/2$. Letting

$$U_n^\alpha = \left\{ \sigma_n([0,1]) > \frac{1}{2} \right\} \cap \{\mathcal{E}_\alpha(\sigma_n) \leq c_2 T\}, \tag{5.73}$$

by the choice of T, we have that

$$\mathbb{P}(U_n^\alpha) \geq c_1/2. \tag{5.74}$$

By Fatou's lemma,

$$\mathbb{P}\left(\limsup_{n\uparrow\infty} U_n^\alpha\right) \geq c_1/2. \tag{5.75}$$

By Lemma 5.4, it follows that for α satisfying (5.70), $\cap_{n\geq 0} \bar{E}_n$ has Hausdorff dimension at least α with positive probability. Since $\cap_{n\geq 0} \bar{E}_n = \cap_{n\geq 0} E_n$ (see (4.63)), the same statement holds for $\cap_{n\geq 0} E_n$ and we are done.

5.2. Upper bound

We will prove the following proposition.

Proposition 5.5. $\dim_H(\mathcal{T}(K)) \leq 1 - p(K)$ *where* $p(K) \in (0, 1)$ *is the solution of the equation*

$$f(p, K) \equiv \frac{\sin(\pi p/2)\Gamma(1 + p/2)}{\pi} \sum_{n=1}^{\infty} \frac{(\sqrt{2}K)^n}{n!} \Gamma((n - p)/2) = 1. \tag{5.76}$$

Furthermore, $K \rightsquigarrow p(K)$ *is a continuous decreasing function on* $(0, \infty)$ *with*

$$\lim_{K\uparrow\infty} p(K) = 0 \quad \text{and more significantly} \ \lim_{K\downarrow 0} p(K) = 1. \tag{5.77}$$

To prove Proposition 5.5 we need the following lemma proved in the Appendix.

Lemma 5.6. *Let* $0 < l < 1$. *Let* S_ϵ *be the simple asymmetric random walk with*

$$\mathbb{P}(S_\epsilon(n+1) - S_\epsilon(n) = +1) = \frac{1}{2} + \frac{1}{2}(1 - \mathrm{e}^{-\epsilon}). \tag{5.78}$$

Then there exists $c(l)$ *such that*

$$\mathbb{P}(\forall n,\ S_\epsilon(n) \geq -1 - K\sqrt{n}) \leq c(l)\epsilon^{p(K/l)} \tag{5.79}$$

where $p(K)$ *is the real solution in* $(0, 1)$ *of* (5.76) *(which satisfies* (5.77)*).*

Let us partition $[0, 1]$ into intervals of equal length 2ϵ, and select the intervals containing a K-exceptional time. The union of those is a cover of $\mathcal{T}(K)$ and we now estimate the number $n(\epsilon)$ of intervals in the cover.

Let U_ϵ be the event that there is a time τ in $[0, 2\epsilon]$ such that $\tau \in \mathcal{T}(K)$. From the full dynamical arrow configuration for all $\tau \in [0, 2\epsilon]$, we construct a static arrow configuration as follows. We declare the static arrow at (i, j) to be right oriented if and only if the dynamical arrow is right oriented (i.e., $\xi_{i,j}^\tau = +1$) at some $\tau \in [0, 2\epsilon]$ (a similar construction was used in Section 2). In this configuration, the path S_ϵ starting from the origin and following the arrows is a slightly right-drifting random walk with $\mathbb{P}(S_\epsilon(n+1) - S_\epsilon(n) = +1) = \frac{1}{2} + \frac{1}{2}(1 - \mathrm{e}^{-\epsilon})$. Clearly,

$$\mathbb{P}(U_\epsilon) \leq \mathbb{P}(\forall n, S_\epsilon(n) \geq -1 - K\sqrt{n}). \tag{5.80}$$

Lemma 5.6 implies that for any $l < 1$

$$\mathbb{P}(U_\epsilon) \ \leq c(l)\epsilon^{p(\frac{K}{l})}. \tag{5.81}$$

Hence

$$\mathbb{E}(n(\epsilon)) = O(\epsilon^{p(\frac{K}{l})-1}) \tag{5.82}$$

so that

$$\limsup_{\epsilon\to 0} \mathbb{E}\left(\frac{n(\epsilon)}{\epsilon^{p(K/l)-1}}\right) < \infty. \tag{5.83}$$

By Fatou's lemma, $\liminf_{\epsilon\downarrow 0} n(\epsilon)\,\epsilon^{1-p(K/l)}$ is almost surely bounded, which implies that $dim_H \mathcal{T}(K)$ (which is equal to $dim_H(T_1(K))$ by Proposition 5.2) is bounded above by $1 - p(\frac{K}{l})$ for any $l < 1$. Since $p(K)$ is continuous in K, Proposition 5.5 follows.

Remark 5.7. We conjecture that $1 - p(K)$ is the exact Hausdorff dimension of $\mathcal{T}(K)$.

Finally, Lemma 5.6 also yields the following tameness result.

Proposition 5.8. *Let $K_1, K_2 > 0$ be small enough so that $p(K_1) + p(K_2) > 1$, where $p(K)$ is defined in Proposition 5.5. For any $j \geq 0$,*

$$\mathbb{P}(\exists\tau \in [0, 1] \text{ s.t. } \forall t \geq 0,\ -j - K_1\sqrt{t} \leq S_0^\tau(t) \leq +j + K_2\sqrt{t}) = 0. \tag{5.84}$$

Proof. Define U_ϵ^+ (resp., U_ϵ^-) to be the event that for some $\tau \in [0, 2\epsilon]$ and all $t \geq 0$, $S^\tau(t) \leq +k + K_2\sqrt{t}$ (resp., $S^\tau(t) \geq -k - K_1\sqrt{t}$). U_ϵ^+ (resp., U_ϵ^-) is a decreasing (resp., increasing) event with respect to the basic $\xi_{(i,j)}^\tau$ processes. Hence, using the FKG inequality, we have

$$\mathbb{P}(U_\epsilon^+ \cap U_\epsilon^-) \leq \mathbb{P}(U_\epsilon^+) \cdot \mathbb{P}(U_\epsilon^-).$$

Reasoning as in Proposition 5.5, for any $l < 1$, we have

$$\mathbb{P}(U_\epsilon^-) \leq \mathbb{P}(\forall t \geq 0, S_\epsilon(t) \geq -j - K_1\sqrt{t}) \tag{5.85}$$

$$\leq c_1\epsilon^{p(\frac{K_1}{l})}, \tag{5.86}$$

where S_ϵ is defined as in the proof of Proposition 5.5. The second inequality is given by (5.81) immediately for $j \leq 1$ and with a little bit of extra effort for all j. Symmetrically,

$$\mathbb{P}(U_\epsilon^+) \leq c_2\epsilon^{p(\frac{K_2}{l})}, \tag{5.87}$$

which implies that

$$\mathbb{P}(U_\epsilon^+ \cap U_\epsilon^-) \leq c_1c_2\epsilon^{p(K_1/l)+p(K_2/l)}. \tag{5.88}$$

Take l close enough to 1 so that $p(K_1/l) + p(K_2/l) > 1$ and define N as the cardinality of $\{\tau \in [0, 1] \text{ s.t. } \forall t \geq 0,\ -j - K_1\sqrt{t} \leq S_0^\tau(t) \leq +j + K_2\sqrt{t}\}$. Reasoning as in Lemma 2.2, we have

$$\mathbb{E}(N) = \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon}\mathbb{P}(U_\epsilon^+ \cap U_\epsilon^-) = 0, \tag{5.89}$$

which completes the proof of the proposition. □

Remark 5.9. We conjecture that when $p(K_1) + p(K_2) < 1$, the set of exceptional times for which the path S_0^τ is bounded on both sides, as in (5.84), by

$$-j - K_1\sqrt{t} \quad \text{and} \quad j + K_2\sqrt{t}$$

is non-empty with Hausdorff dimension $1 - p(K_1) - p(K_2)$. Since the techniques used to prove our one-sided result do not seem to work for the double-sided case, we would need a new approach to prove this conjecture.

6. Scaling limit

In this section, we discuss the existence of a dynamical Brownian motion (constructed, using the Brownian web, in [4]) and the occurrence of exceptional times for this object.

6.1. Brownian web and (1, 2) points

Under diffusive scaling, individual random walk paths converge to Brownian motions. In [15], it was proved (extending the results of [16,17]) that the entire collection of discrete paths in the DW converges (in an appropriate sense) to the continuum Brownian web (BW), which can be loosely described as the collection of graphs of coalescing one-dimensional Brownian motions starting from every possible location in $\mathbb{R}^2$ (space–time).

Formally, the Brownian web (BW) is a random collection of paths with specified starting points in space–time. The paths are continuous graphs in a space–time metric space $(\bar{\mathbb{R}}^2, \rho)$ which is a compactification of $\mathbb{R}^2$. (Π, d) denotes the space whose elements are paths with specific starting points. The metric d is defined as the maximum of the sup norm of the distance between two paths and the distance between their respective starting points. (Roughly, the distance between two paths is small when they start from close (space–time) points and remain close afterwards). The Brownian web takes values in a metric space $(\mathcal{H}, d_{\mathcal{H}})$, whose elements are compact collections of paths in (Π, d) with $d_{\mathcal{H}}$ the induced Hausdorff metric. Thus the Brownian web is an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable, where $\mathcal{F}_{\mathcal{H}}$ is the Borel σ-field associated to the metric $d_{\mathcal{H}}$. The next theorem and the following discussion, taken from [15], give some of the key properties of the BW.

Theorem 6.1. *There is an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\mathcal{W}$ whose distribution is uniquely determined by the following three properties:*

(o) *from any deterministic point (x, t) in $\mathbb{R}^2$, there is almost surely a unique path $B_{(x,t)}$ starting from (x, t);*

(i) *for any deterministic, dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure in $(\mathcal{H}, d_{\mathcal{H}})$ of $\{B_{(x,t)} : (x, t) \in \mathcal{D}\}$;*

(ii) *for any deterministic n and $(x_1, t_1), \ldots, (x_n, t_n)$, the joint distribution of $B_{(x_1,t_1)}, \ldots, B_{(x_n,t_n)}$ is that of coalescing Brownian motions (with zero drift and unit diffusion constant).*

This characterization provides a practical construction of the Brownian web. For $\mathcal{D}$ as above, construct coalescing Brownian motion paths starting from $\mathcal{D}$. This defines a *skeleton* for the Brownian web. $\mathcal{W}$ is simply defined as the closure of this precompact set of paths.

We note that generic (e.g., deterministic) space–time points have almost surely only $m_{\text{out}} = 1$ outgoing (to later times) paths from that point and $m_{\text{in}} = 0$ incoming paths passing through that point (from earlier times). These features of the BW are not hard to prove; for example, the

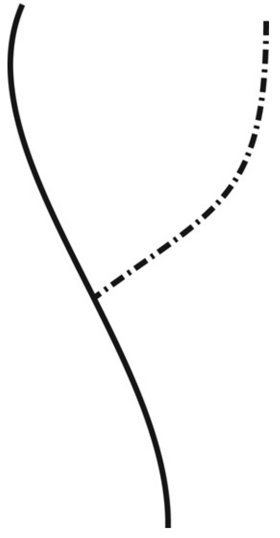

Fig. 2. A schematic diagram of a left $(m_{in}, m_{out}) = (1, 2)$ point. In this example the incoming path connects to the leftmost outgoing path, and the right outgoing path is a newly born path.

reason why $m_{\text{in}} = 0$ at a deterministic (x, t) is that $m_{\text{in}} > 0$ would require an incoming path from $\mathbb{R} \times \{t - \varepsilon\}$ to touch (x, t), but as was already shown by Arratia [16], those paths coalesce by time t into a discrete set of random locations on $\mathbb{R} \times \{t\}$ with zero probability of including a deterministic point. An interesting property of the BW is related to the existence of special points with other values of $(m_{\text{in}}, m_{\text{out}})$. In the following, a dominant role is played by the (1, 2) points as we shall explain. Back in the lattice, (1, 2) points correspond to locations where a path starts at a "microscopic" distance from an old path (that started from an earlier time; we note that in the count of paths, incoming paths that coalesce at some earlier time are identified) and coalesces with it only after some "macroscopic" amount of time. We remark that, with a little work, it can be shown that the paths can be chosen so that the microscopic distance from the old path is actually a single unit lattice spacing. For a (1, 2) point, the single incident path continues along exactly one of the two outward paths. The (1, 2) point is either left-handed or right-handed according to whether the incoming path connects to the left or right outgoing path. See Fig. 2 for a schematic diagram of the "left-handed" case. Both types occur and it is known [15] that each of the two types, as a subset of $\mathbb{R}^2$, has Hausdorff dimension 1.

6.2. *The dynamical Brownian web and exceptional times*

It is natural that there should also exist scaling limits of the DyDW (including those of the random walk from the origin evolving in τ, i.e., a dynamical Brownian motion). Indeed, this was proposed by Howitt and Warren [1] who also studied (two dynamical time distributional) properties of any such limit. In [4], we provided a complete construction that we now briefly describe.

A priori, a direct construction in the continuum appears difficult since the DyDW is entirely based on a modification of the discrete arrow structure of the DW, while in the BW it was unclear a priori whether there even is any arrow structure to modify. Two of the main themes of [4] are thus: (i) "Where is the arrow structure of the BW?" and (ii) "How is it modified to yield the

DyBW (including a dynamical Brownian motion from the origin)?". The answer to the first question is that the arrow structure of the BW comes from the (1, 2) points. Indeed, one can change the direction of the "continuum" arrow at a given (1, 2) point z by simply connecting the incoming path to the newly born path starting from z rather than to the original continuing path. (Back in the lattice, this amounts to changing the direction of an arrow whose switch induces a "macroscopic" effect in the web.) The answer to question (ii) is based on the construction of a Poissonian marking of the (1, 2) points (see [4] for details) that indicates which (1, 2) points get switched and at what value of τ does the switch occur. We note that the main difficulty in the construction of the DyBW lies in the fact that between two dynamical times $\tau < \tau'$, one needs to switch the direction of a set of (1, 2) points dense in $\mathbb{R}^2$ in order to deduce the web at time τ' from the one at time τ.

We proceed to discussing the existence of exceptional times for B_0^τ, the dynamical Brownian motion starting from the origin at dynamical time τ. (We remark that our tameness results, Theorem 2.1 and Remark 2.3, should be extendable to the continuum DyBW, but the arguments will involve some extra Brownian web technology.) Recall that the key ingredient for proving our existence results for the dynamical discrete web is contained in Proposition 3.1 where we estimate how fast the dynamical discrete web decorrelates. The proof of that proposition mostly relies on the observation that $(S_0^\tau, S_0^{\tau'})$ form a sticky pair of random walks. More precisely, we showed in Lemma 3.2 that along the t-axis the pair alternates between periods during which the two paths evolve as a single path (they stick) and periods during which they move independently.

In [4], we proved that $\tau \rightsquigarrow B_0^\tau$ has a similar structure (as suggested in [1]), in that for two distinct dynamical times τ, τ', the paths $B_0^\tau, B_0^{\tau'}$ form a $1/(2|\tau - \tau'|)$-sticky pair of Brownian motions. Such a pair can be simply expressed in terms of three independent standard Brownian motions $(B_d^\tau, B_d^{\tau'}, B_s)$ in the following way.

$$\begin{aligned} B_0^\tau(t) &= B_d^\tau(C(t)) + B_s(t - C(t)), \\ B_0^{\tau'}(t) &= B_d^{\tau'}(C(t)) + B_s(t - C(t)), \end{aligned} \tag{6.90}$$

where C is the continuous inverse of the function

$$C^{-1}(s) = s + \frac{1}{\sqrt{2}|\tau - \tau'|}\, l_0(s) \tag{6.91}$$

and l_0 is the local time at the origin of the process $\left(B_d^\tau - B_d^{\tau'}\right)/\sqrt{2}$. We note that the paths $B_0^\tau, B_0^{\tau'}$ always spend a strictly positive Lebesgue measure of time together, hence the name sticky Brownian motions. Finally, the time the two paths spend together is directly related to the parameter $1/(2|\tau - \tau'|)$ commonly referred to as the "amount of stick" of the pair.

If we denote $\tilde{\pi}(\cdot) = \pi(\cdot/\delta^2)\delta$, the scaling invariance for the Brownian motion combined with (6.90) implies that $(\tilde{B}_0^\tau, \tilde{B}_0^{\tau'})$ is identical in law to a $\delta/(\sqrt{2}|\tau - \tau'|)$-sticky pair of Brownian motions. In other words, the amount of stick of the pair $(\tilde{B}_0^\tau, \tilde{B}_0^{\tau'})$ vanishes as $\delta \to 0$ and from (6.90) and (6.91) we see that for small δ,

$$B_0^\tau(t) \approx B_d^\tau(t) \quad \text{and} \quad B_0^{\tau'}(t) \approx B_d^\tau(t), \tag{6.92}$$

i.e., the two paths become "almost independent". This can be made more precise by establishing (along the same lines as the proof of Proposition 3.1) that for

$$O = \{\forall t \in [0, 1],\ \pi(t) > -1 \text{ and } \pi(1) > 1\}$$

and $\delta > 0$, there exist $K, a \in (0, \infty)$ (independent of δ, τ and τ') such that

$$\mathbb{P}(\tilde{B}_0^{\tau} \in O, \tilde{B}_0^{\tau'} \in O) \le \mathbb{P}(\tilde{B} \in O)^2 + K\left(\frac{\delta}{|\tau - \tau'|}\right)^a \tag{6.93}$$

$$= \mathbb{P}(B \in O)^2 + K\left(\frac{\delta}{|\tau - \tau'|}\right)^a, \tag{6.94}$$

where B is a standard Brownian motion.

Since all the results of Sections 4 and 5.1 for our dynamical random walk are based on the discrete analogue of this result, Propositions 4.1 and 5.3 should be extendable to the continuum in the following manner. Define

$$A = \left\{ \inf_{t\in[0,1]} B(t) > -1, B(1) > 1 \right\},$$

$\gamma_0 = 1/\mathbb{P}(A)$, and let $\bar{\mathcal{T}}(K)$ be the set of τ's belonging to $[0, \infty)$ such that

$$\forall t \ge 0, \quad B_0^{\tau}(t) \ge -1 - K\sqrt{t}. \tag{6.95}$$

Then $\bar{\mathcal{T}}(K)$ should be non-empty with

$$\dim_H(\bar{\mathcal{T}}(K)) \ge 1 - \frac{\log \gamma_0}{\log \gamma(K)} \quad \text{for } K > K(\gamma_0), \tag{6.96}$$

so that $\lim_{K\uparrow\infty} \dim_H(\bar{\mathcal{T}}(K)) = 1$.

We conclude by noting that our upper bound results on the Hausdorff dimension, Propositions 5.5 and 5.8, should also be extendable to the continuum DyBW, but, like the tameness results, that extension would require some extra Brownian web technology beyond what is described in this paper.

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Appendix. Some estimates on random walks (Proof of Lemma 5.6)

We start with the two following lemmas.

Lemma A.1 (*[18]*). *Let $j, K \in (0, \infty)$ and let B be a standard Brownian motion. Then there exists $q \in (0, \infty)$ such that*

$$\lim_{t\to\infty} t^{p(K)/2}\mathbb{P}(\forall s \in [0, t],\ B(s) \ge -j - K\sqrt{s}) = q, \tag{A.97}$$

where $p(K)$ is the solution in $(0, 1)$ of the equation

$$f(p, K) \equiv \frac{\sin(\pi p/2)\Gamma(1 + p/2)}{\pi} \sum_{n=1}^{\infty} \frac{(\sqrt{2}K)^n}{n!}\Gamma((n - p)/2) = 1. \tag{A.98}$$

Furthermore, $p(K)$ is a continuous decreasing function on $(0,\infty)$ with

$$\lim_{K\uparrow\infty} p(K) = 0 \quad \textit{and} \quad \lim_{K\downarrow 0} p(K) = 1. \tag{A.99}$$

Lemma A.2. *Let $K \in (0,\infty)$, $l \in (0,1)$ and let S be a simple symmetric random walk. Then there exists $\bar{c}(K,l) \in (0,\infty)$ such that for every n*

$$n^{p(K/l)/2}\, \mathbb{P}(\forall k \le n, S(k) \ge -1 - K\sqrt{k}) \le \bar{c}(K,l). \tag{A.100}$$

Proof. By Lemma A.1, it suffices to prove that for every $l < 1$, there exists $c(K,l)$ such that

$$\begin{aligned}&\mathbb{P}(\forall k \le n, S(k) \ge -1 - K\sqrt{k}) \le c(K,l)\mathbb{P}(\forall t \in [0, l^2 n],\\ &\qquad B(t) \ge -2 - K\sqrt{t}/l).\end{aligned} \tag{A.101}$$

We now prove the latter inequality. Consider S, the discrete time random walk embedded in the Brownian motion B. Namely, we define inductively a sequence of stopping times t_i with $t_0 = 0$ and

$$t_{i+1} = \inf\{t > t_i : |B(t) - B(t_i)| \ge 1\} \tag{A.102}$$

and then we define $S(i) = B(t_i)$. Note that S and $\{t_i\}$ are independent and therefore

$$\begin{aligned}&\mathbb{P}(\forall k \le n, S(k) \ge -1 - K\sqrt{k})\\ &\quad = \mathbb{P}\left(\forall k \le n,\ S(k) \ge -1 - K\sqrt{k}, l^2 k \le\ t_k \le \frac{k}{l^2}\right) \Big/ \mathbb{P}\left(\forall k \le n, l^2 k \le t_k \le \frac{k}{l^2}\right)\\ &\quad = \mathbb{P}\left(\forall k \le n, B(t_k) \ge -1 - K\sqrt{k},\ l^2 k \le\ t_k \le \frac{k}{l^2}\right) \Big/ \mathbb{P}\left(\forall k \le n, l^2 k \le\ t_k \le \frac{k}{l^2}\right)\\ &\quad \le \mathbb{P}\left(\forall k \le n, B(t_k) \ge -1 - K\sqrt{t_k}/l, l^2 k \le t_k\right) \Big/ \mathbb{P}\left(\forall k \le n,\ l^2 k \le\ t_k \le \frac{k}{l^2}\right).\end{aligned}$$

If for every $k \le n$, we have $B(t_k) \ge -1 - K\sqrt{t_k}/l$ and moreover $l^2 k \le t_k$, then on $[0, l^2 n]$, every time B takes an integer value, B is to the right of $t \rightsquigarrow -1 - K\sqrt{t}/l$. Hence, B remains to the right of $t \rightsquigarrow -2 - K\sqrt{t}/l$ on $[0, l^2 n]$ which implies that

$$\begin{aligned}&\mathbb{P}(k \le n,\ S(k) \ge -1 - K\sqrt{k})\\ &\quad \le \mathbb{P}(\forall t \le l^2 n,\ B(t) \ge -2 - K\sqrt{t}/l) \,/\, \mathbb{P}\left(\forall k \le n,\ l^2 k \le\ t_k \le \frac{k}{l^2}\right).\end{aligned}$$

Finally, t_k is a sum of k i.i.d. random variables with mean 1 (whose common distribution includes 1 in its support). Therefore, $\mathbb{P}(\forall k \in \mathbb{N},\ l^2 k \le\ t_k \le \frac{k}{l^2}) > 0$ and (A.101) follows. This completes the proof of the lemma. □

We are now ready to prove Lemma 5.6. Recall that S_ϵ is a simple random walk with

$$\mathbb{P}(S_\epsilon(k+1) - S_\epsilon(k) = 1) = \frac{1}{2} + \frac{1}{2}(1 - \mathrm{e}^{-\epsilon}).$$

Since $\frac{1}{2} + \frac{1}{2}(1 - e^{-\epsilon}) \le \frac{1}{2}(1+\epsilon)$, it is enough to show the conclusions of the lemma for the simple walk $\bar{S}_\epsilon$ where $p_\epsilon^\pm = \mathbb{P}(\bar{S}_\epsilon(k+1) - \bar{S}_\epsilon(k) = \pm 1) = \frac{1}{2}(1 \pm \epsilon)$.

Let $T_\epsilon = \inf\{n > 0 : \bar{S}_\epsilon(n) < -1 - K\sqrt{n}\}$. We have

$$\mathbb{P}(T_\epsilon = n) = \mathbb{P}(T_0 = n)\ f_\epsilon(n) \tag{A.103}$$

$$\text{with}\, f_\epsilon(n) \equiv (2p_\epsilon^-)^{\frac{1}{2}(n+\lfloor 1+K\sqrt{n}\rfloor+1)}\ (2p_\epsilon^+)^{\frac{1}{2}(n-\lfloor 1+K\sqrt{n}\rfloor-1)}, \tag{A.104}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\le x$ and f_ϵ is the Radon–Nikodym derivative of the distribution of the drifting random walk $\bar{S}_\epsilon$ with respect to that of the non-drifting walk S_0 evaluated on any path from the origin that first is to the left of $-1 - K\sqrt{t}$ at (integer) time n. Since a simple symmetric random walk S eventually hits the moving boundary $t \rightsquigarrow -1 - K\sqrt{t}$, we have

$$\mathbb{P}(T_\epsilon = \infty) = 1 - \sum_{n\ge 1} f_\epsilon(n)\mathbb{P}(T_0 = n) = \sum_{n\ge 1}(1 - f_\epsilon(n))\mathbb{P}(T_0 = n). \tag{A.105}$$

Then, proceeding to a summation by parts we have

$$\mathbb{P}(T_\epsilon = \infty) = \sum_{n\ge 1}(\mathbb{P}(T_0 \ge n) - \mathbb{P}(T_0 \ge n+1))\ [1 - f_\epsilon(n)] \tag{A.106}$$

$$= \sum_{n\ge 1}\mathbb{P}(T_0 \ge n+1)(f_\epsilon(n) - f_\epsilon(n+1)) + (1 - f_\epsilon(1)) \tag{A.107}$$

$$= Q_\epsilon + (1 - f_\epsilon(1)), \tag{A.108}$$

$$\text{with } Q_\epsilon \equiv \sum_{n\ge 1}\mathbb{P}(T_0 \ge n+1) f_\epsilon(n)\left(1 - \frac{f_\epsilon(n+1)}{f_\epsilon(n)}\right). \tag{A.109}$$

We proceed to estimating Q_ϵ. First, for $\epsilon \in (0, 1)$, we have

$$\ln(1+\epsilon) \le \epsilon, \qquad \ln(1-\epsilon) \le -\epsilon,$$

implying that

$$\begin{aligned} f_\epsilon(n) &= \exp\left\{\frac{1}{2}\ln(1-\epsilon)(n + \lfloor 1 + K\sqrt{n}\rfloor + 1) + \frac{1}{2}\ln(1+\epsilon)(n - \lfloor 1 + K\sqrt{n}\rfloor - 1)\right\} \\ &\le \exp\left\{-\frac{\epsilon}{2}(n + \lfloor 1 + K\sqrt{n}\rfloor + 1) + \frac{\epsilon}{2}(n - \lfloor 1 + K\sqrt{n}\rfloor - 1)\right\} \\ &\le \exp\{-\epsilon(\lfloor 1 + K\sqrt{n}\rfloor + 1)\} \\ &\le \exp\{-\epsilon K\sqrt{n}\}. \end{aligned}$$

Next, if we set $\Delta_n \equiv \lfloor 1 + K\sqrt{n+1}\rfloor - \lfloor 1 + K\sqrt{n}\rfloor = \lfloor K\sqrt{n+1}\rfloor - \lfloor K\sqrt{n}\rfloor$, we have

$$\begin{aligned} 1 - \frac{f_\epsilon(n+1)}{f_\epsilon(n)} &= 1 - \exp\left\{\ln(1-\epsilon)\left(\frac{1}{2} + \frac{1}{2}\Delta_n\right) + \ln(1+\epsilon)\left(\frac{1}{2} - \frac{1}{2}\Delta_n\right)\right\} \\ &= 1 - \exp\left(-\epsilon\Delta_n - \frac{\epsilon^2}{2} + o(\epsilon)\Delta_n + o(\epsilon^2)\right) \\ &= \epsilon\Delta_n + \frac{\epsilon^2}{2} + o(\epsilon)\Delta_n + o(\epsilon^2). \end{aligned}$$

By Lemma A.2, for every $l < 1$ there exists $\bar{c}(K, l)$ such that

$$Q_\epsilon \le \sum_{n\ge 1}\mathbb{P}(T_0 \ge n+1)\exp\{-\epsilon K\sqrt{n}\}\left(\epsilon\Delta_n + \frac{\epsilon^2}{2} + o(\epsilon)\Delta_n + o(\epsilon^2)\right) \tag{A.110}$$

$$\leq \bar{c}(K,l)\sum_{n\geq 1}\exp\{-\epsilon K\sqrt{n}\}\left(\epsilon\frac{\Delta_n}{n^{p/2}}+\frac{\epsilon^2}{2n^{p/2}}+o(\epsilon)\frac{\Delta_n}{n^{p/2}}+\frac{o(\epsilon^2)}{n^{p/2}}\right), \tag{A.111}$$

where $p \equiv p(K/l)$ is as in Lemma A.2. Since $\Delta_n = \lfloor K\sqrt{n+1}\rfloor - \lfloor K\sqrt{n}\rfloor$, and $(\sqrt{n+1}-\sqrt{n})\sqrt{n} \to 1/2$ as $n \to \infty$, it is natural to expect that

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^2\sum_{n\geq 1}\exp\{-K\sqrt{\epsilon^2 n}\}\frac{\Delta_n}{\epsilon}\frac{1}{(\sqrt{\epsilon^2 n})^p} &= \lim_{\epsilon\downarrow 0}\epsilon^2\sum_{n\geq 1}\exp\{-K\sqrt{\epsilon^2 n}\}\frac{K}{2\sqrt{n\epsilon^2}}\frac{1}{(\sqrt{\epsilon^2 n})^p}\\ &= \frac{1}{2}\int_0^\infty \exp\{-K\sqrt{t}\}\frac{K}{t^{(p+1)/2}}\mathrm{d}t,\end{aligned} \tag{A.112}$$

where the second equality is due to the Riemann sum on the right-hand side of the first equality. To justify the first equality, one may note that Δ_n is (for large n) either 0 or 1 and then define $N_\ell(n)$ (resp., $N_u(n)$) to be the largest $m \leq n$ (resp., smallest $m > n$) such that $\Delta_m \neq 0$. It is straightforward to show first that $N_u(n) - N_\ell(n)/\sqrt{n} \to 2/K$ as $n \to \infty$ and then to obtain the first equality of (A.112) as a consequence. It is also the case that

$$\lim_{\epsilon\downarrow 0}\epsilon^2\sum_{n\geq 0}\exp\{-K\sqrt{\epsilon^2 n}\}\frac{1}{(\epsilon^2 n)^{p/2}} = \int_0^\infty \exp\{-K\sqrt{t}\}\frac{1}{t^{p/2}}\mathrm{d}t. \tag{A.113}$$

Since $0 < p < 1$, both integrals in (A.112) and (A.113) are finite. Thus, (A.111) yields $Q_\epsilon = O(\epsilon^p) = O(\epsilon^{p(K/l)})$.

Finally, it is easy to prove that $f_\epsilon(1) - 1 = O(\epsilon)$. Since $p(K/l) < 1$, Lemma 5.6 follows from (A.108).

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