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Transition to turbulence, small disturbances, and sensitivity analysis I: A motivating problem

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Abstract

For over 100 years, researchers have attempted to predict transition to turbulence in fluid flows by analyzing the spectrum of the linearized Navier–Stokes equations. However, for many simple flows this approach fails to match experimental results. Recently, new scenarios for transition have been proposed that are based on the interaction of the linearized equations of motion with small disturbances to the flow system. These new "mostly linear" theories have increased our understanding of the transition process, but the role of nonlinearity has not been explored in detail. This paper is the first of a two part work in which sensitivity analysis is used to study the effects of small disturbances on transition to turbulence. In this part, we study a highly sensitive one-dimensional Burgers' equation as a motivating problem. Sensitivity analysis is used to predict the large changes in solutions in the presence of a small disturbance. Also, sensitivity analysis is shown to provide more information about the disturbed nonlinear problem than a purely linear analysis of the problem. In the second part of this work, this analysis will be extended to the three-dimensional Navier–Stokes equations to show that small disturbances have great potential to trigger transition to turbulence.

Keywords: Burgers' equation; Navier–Stokes equations; Nonnormality; Transition to turbulence; Small disturbances; Fréchet differentiability; Sensitivity analysis; Sensitivity equations

1. Introduction

One of the longstanding problems in fluid dynamics is to predict when a flow will transition from a laminar to turbulent state. This problem has been investigated experimentally, numerically, and analytically for well over a century. Experiments have shown for various flow configurations that when the Reynolds number reaches a certain critical value, the flow becomes turbulent. However, when classical linear stability analysis is applied to this problem, the method fails to predict the experimentally determined critical Reynolds number for many simple flows [1].

During the past twenty five years nonclassical linear stability techniques have been developed to attack the transition problem (see [2] for a review). These methods produced new insights into the transition process and generated two new "mostly linear" theories on the mechanism of transition. First, researchers discovered that certain small perturbations to a laminar flow can cause an extremely large transient energy growth in the linearized system [3–6]. This

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energy growth was found to be caused by the nonnormality of the linearized operator (see [7, Chapter IV] for a more general study). It has been suggested that when the transient growth becomes "sufficiently large," the flow is "mixed" by the nonlinearity, producing turbulence [8]. This approach can be roughly summarized by saying that flow systems are extremely sensitive to small perturbations in the initial flow. This scenario (or a slight modification thereof) has been proposed by several research groups [3,8–10].

In this transition scenario, it is thought that certain small perturbations to the laminar flow can trigger transition. If the linearized flow operator is stable, then (under suitable hypotheses) small enough perturbations to the laminar flow *cannot* transition [11]. Researchers have hypothesized that as the Reynolds number increases, the perturbations must be extremely small in order to guarantee that the flow will remain near the base flow [8,9]. If this is true, then it is possible that the energy in very small initial flow perturbations can grow and cause transition.

Another new approach to transition uses ideas from robust control theory to study the effects of small forcing on flow systems. The forcing could arise from slight imperfections in a flow experiment, small neglected terms in the mathematical flow model, microscopic wall roughness, etc. Researchers studied the input/output properties of linearized flow systems with certain types of small random (forcing) input and found that the energy in the system could be amplified on the order of the Reynolds number cubed [12,13]. This energy amplification is again due to the nonnormality of the linearized operator. Transition is also thought to be caused by the nonlinearity mixing the energy in the system. This transition scenario can be roughly summarized by saying that flow systems are extremely sensitive to small forcing. In particular, it is possible that the small forcing interacts with the nonlinearity to cause movement or bifurcation of equilibria leading to transition. This "bifurcation under uncertainty" scenario is studied in [14–16]. This phenomenon will appear in this work in our study of a one-dimensional Burgers' equation.

Remark. We emphasize that all of these classical and modern approaches to transition have developed out of the study of the linear Orr–Sommerfeld/Squire equations. These equations are obtained by applying a certain transformation to the linearized Navier–Stokes equations. This transformation is widely used, but the author is not aware of any theory guaranteeing that properties of the linearized Navier–Stokes equations (e.g., the spectrum of the linear operator, transient growth of solutions, etc.) are unchanged after the transformation. Therefore, it is possible that one loses important information in the transformation. However, we do not consider this possibility here and we assume that the Orr–Sommerfeld/Squire operator provides valuable insight into the linearized Navier–Stokes operator.

Both of these modern theories have greatly increased our understanding of the transition process, however they have not yet provided a complete theory explaining the mechanism of transition. In particular, the role of the nonlinearity in transition is not well understood. This is due to the fact that these new theories are primarily concerned with energy growth and the nonlinearity in the Navier–Stokes equations conserves energy in many flow situations. It is the linear term that causes the great increase in energy and therefore much of the work has focused on the linearized system. A complete picture of the actual mechanism of transition would be a great advance in our understanding of turbulence. In particular, this knowledge could lead to improved methods for feedback flow control.

This paper is the first of a two part work in which sensitivity analysis is used to analyze the impact of small disturbances on the transition process in the three-dimensional Navier–Stokes equations. In this paper, we examine a relatively simple model problem in order to (1) emphasize that extremely small disturbances can cause very large changes in solutions of nonlinear partial differential equations; and (2) demonstrate that sensitivity analysis can be used to predict the effects of a small disturbance without solving the full disturbed nonlinear problem. Specifically, we study a highly sensitive one-dimensional Burgers' equation whose solutions are known to change drastically if there is a small disturbance in the boundary conditions. Sensitivity analysis is used to measure the change in the solution with respect to the small disturbances. In particular, the continuous sensitivity equation method is used to differentiate the solution of the undisturbed problem with respect to the disturbance parameter. Numerical results predict the large sensitivity of the solution with respect to the small disturbance. Also, we use high order sensitivities to give an indication of how the nonlinearity interacts with the small disturbance to create the large change in the solutions.

In part two of this work ([17], hereafter referred to as Part II), the analysis presented below is extended to study the role of small disturbances and nonlinearity in transition to turbulence in the Navier–Stokes equations. We use sensitivity analysis to show that the change in a laminar flow with respect to small variations in the initial data or small forcing acting on the system is large when the linearized operator is stable yet nonnormal. Therefore, the solution of the disturbed flow problem can be large (and possibly turbulent) even if the linearized operator is stable and the disturbances are extremely small. This analysis extends the "mostly linear" transition scenarios described above to the nonlinear case. Furthermore, the sensitivity analysis is used to obtain bounds on the magnitudes of the disturbed solutions which could potentially be used to estimate the size of the disturbances that trigger transition.

We note that sensitivity analysis can be applied to study the effects of many types of small disturbances on many different nonlinear problems. The new "mostly linear" approaches to stability analysis discussed above give an indication of the behavior of a nonlinear system under certain general classes of disturbances by studying the linearized system; however, these approaches are not able to provide specific information about the behavior of a perturbed nonlinear system. In contrast, the sensitivity analysis approach presented here is able to anticipate the behavior of a nonlinear system in the presence of specific disturbances. These disturbances are not limited to be of a particular type or class; the only requirement is that the disturbance enter the equation in a differentiable manner. Moreover, this sensitivity analysis approach can also be used to gain insight into the response of a nonlinear system to general classes of disturbances. This is the approach used on the Navier–Stokes equations in Part II of this work.

To begin, we introduce the highly sensitive Burgers' equation in Section 2 and study the transition of solutions in Section 3. In Section 4, we recall theoretical results for sensitivity analysis of semilinear parabolic problems. This theory allows us to prove the differentiability of solutions of Burgers' equation with respect to the disturbance parameter. In Sections 5 and 6, the sensitivities are shown to predict transition. We close with conclusions, applications, and a brief overview of the results contained in Part II.

2. Motivating problem: Burgers' equation

The one-dimensional Burgers' equation has long been used as a simplified model of fluid flow. This is due to the fact that Burgers' equation shares with the Navier–Stokes equations a second-order diffusion term balanced against a quadratic first-order convection term. In this work, we study the one-dimensional Burgers' equation

$$v_t(t,x) + v(t,x)v_x(t,x) = \mu v_{xx}(t,x), \tag{1}$$

with constant nonhomogeneous Dirichlet boundary conditions

$$v(t,-1) = 1, \quad v(t,1) = -1, \tag{2}$$

and initial condition

$$v(0,x) = v_0(x). \tag{3}$$

The subscripts t and x denote partial derivatives with respect to time and space, respectively, and μ is a positive constant which plays the role of the flow viscosity (or the inverse of the Reynolds number). The focus of this work is to use sensitivity analysis to study the change in solutions of this system with respect to a small disturbance, i.e., we will differentiate the solution with respect to the disturbance parameter. In order to do this in a rigorous fashion, in this section we give an abstract formulation of the problem and show global existence and uniqueness of solutions. This abstract framework is used in Sections 5 and 6 to prove the parameter differentiability results.

This Burgers' equation is known to have a unique smooth steady solution [18], i.e., there is only one smooth solution of the boundary value problem

$$U(x)U_x(x) = \mu U_{xx}(x), \quad U(-1) = 1, \quad U(1) = -1, \tag{4}$$

which is given by

$$U(x) = c\tanh(-cx/2\mu). \tag{5}$$

The constant $c \approx 1$ is chosen so that U satisfies the boundary conditions [18]. The function U plays the role of the "laminar flow" for this problem. We investigate the fluctuations u about the equilibrium state U defined by $v(t,x) = U(x) + u(t,x)$. Then u satisfies the fluctuation Burgers' equation

$$u_t + uu_x = \mu u_{xx} - (Uu)_x, \tag{6}$$

$$u(t,-1) = 0 = u(t,1), \tag{7}$$

$$u(0,x) = u_0(x) := v_0(x) - U(x). \tag{8}$$

If the fluctuations u remain small, then the solution $v(t,x)$ of the Burgers' equation (1)–(3) will remain near the base flow U, i.e., solutions will not "transition."

Remark. A fluid flow is normally said to *transition* if it changes from a laminar (nonchaotic) state to a turbulent (chaotic) state. In this work, we generalize this concept to other types of systems and say that a solution transitions from one state to another if the distance between the states is large in some norm. We do not require the latter state to be chaotic; also, the states may be stationary or time-varying. This notion of "transition" allows us to draw parallels between the behavior of a relatively simple model problem (Burgers' equation) and the Navier–Stokes equations.

This fluctuation Burgers' equation can be formulated abstractly as a differential equation over an infinite dimensional Hilbert space of the form

$$\dot{w}(t) = Aw(t) + B\big(w(t), w(t)\big), \quad w(0) = w_0.$$

Let X be the Hilbert space of square integrable functions, $L^2(-1,1)$, with the standard inner product and norm. Define $V = H_0^1(-1,1)$, the Hilbert space of functions $v \in H^1(-1,1)$ satisfying the boundary conditions $v(-1) = 0 = v(1)$, with the inner product $(u,v)_V = \int u_x v_x \, dx$. The linear operator A is given by

$$Au = \mu u_{xx} - (Uu)_x,$$

and is defined for all $u \in D(A) = H^2 \cap V$. We show below that this linear operator generates an analytic C_0-semigroup, denoted e^{At}, and that the fractional power $X^{1/2} = D((-A)^{1/2})$ of the state space is given by V. The bilinear operator B given by

$$B(u,v) = -uv_x$$

maps $V \times V$ into X and satisfies $B(u,v) \leqslant C\|u\|_V \|v\|_V$ for some constant $C > 0$. In this way, the fluctuation Burgers' equation can be written as the abstract differential equation above over X, or, equivalently, it can be written as the nonlinear integral equation

$$w(t) = e^{At} w_0 + \int_0^t e^{A(t-\tau)} B\big(w(\tau), w(\tau)\big)\, d\tau$$

over V. Furthermore, the nonlinearity also conserves energy since

$$\big(B(v,v), v\big)_X = \int_{-1}^{1} \big(v^3/3\big)_x \, dx = \frac{1}{3}\big(v(1)^3 - v(-1)^3\big) = 0$$

for all $v \in V$. In Part II, we give a very similar formulation of the fluctuation Navier–Stokes equations.

Since we are considering the effects of small disturbances on Burgers' equation, we want to consider the disturbed equation

$$\dot{w}(t) = Aw(t) + B\big(w(t), w(t)\big) + f, \quad w(0) = w_0, \tag{9}$$

where $w_0 \in V$, $f \in X$, and the operators A and B are defined above. One can use standard methods to show that for any smooth U, A generates an analytic semigroup, and this equation has a unique solution that exists for all time.

Theorem 2.1. *Let U be any function in $C^1([-1,1])$.*

(1) *The linear operator $-A$ is sectorial, A generates an analytic semigroup, and $X^{1/2} = D((-A)^{1/2}) = V$.*
(2) *For any $T > 0$, $w_0 \in V$, and $f \in X$, there exists a unique solution on $[0,T]$ to the disturbed fluctuation Burgers' equation* (9).

Proof. The operator $-A$ can be used to define a continuous sesquilinear form $a : V \times V \to \mathbb{C}$ through integrating by parts: for $u, v \in V$,

$$a(u,v) := (-Au, v)_H = \int_{-1}^{1} \mu u_x(x) v_x(x) - U(x)u(x)v_x(x)\,dx.$$

Since $L^\infty(-1,1)$ is continuously embedded in V and $U \in C^1([-1,1])$, it is easily shown that there exists positive constants C, c, and λ such that for all $u, v \in V$,

$$\|a(u,v)\| \leqslant C\|u\|_V\|v\|_V, \quad \operatorname{Re} a(v,v) + \lambda\|v\|_H^2 \geqslant c\|v\|_V^2.$$

It is well known that this implies that $X^{1/2} = D((-A)^{1/2}) = V$, $-A$ is sectorial, and A generates an analytic semigroup [19,20].

To show the local existence of a unique solution to the fluctuation Burgers' equation (9), we use the existence theory for semilinear parabolic equations [19–21]. Since A generates an analytic semigroup, we need only show that $F(w) = B(w,w)$ is a locally Lipschitz continuous mapping from X^α to X for some $\alpha \in (0,1)$. This is easily done in the case $\alpha = 1/2$ where $X^{1/2} = V$ (for details, see the example in Section 3.3 in [21]). Therefore, there exists $T^* > 0$ so that there exists a unique solution to the problem on $[0, T^*]$.

The unique solution $w(t)$ will exist on any interval $[0,T]$ if we show that $\|w(t)\|_V = \|w(t)\|_{1/2}$ remains bounded for $0 \leqslant t \leqslant T$ [21, Theorem 3.3.4]. We give a brief sketch of how this can be done. In [22], it is shown that Burgers' equation falls into a general class of flow equations considered in Section III.3.1 of Temam's book [23]. Energy estimates for the solution of the fluctuation Burgers' equation can be derived in a similar fashion as estimates for the two-dimensional Navier–Stokes equations with nonhomogeneous Dirichlet boundary conditions (see Section III.3.2 in [23]). In particular, this procedure can be used to show that the solution w of the fluctuation Burgers' equation is contained in $L^\infty(0,T;V)$ for any $T > 0$, and therefore the solution exists on all of $[0,T]$. □

3. Transition in Burgers' equation

To study transition in the Burgers' equation (1)–(3), we want to know whether the fluctuations about the "laminar flow," U, remain small. If we take the classical approach to transition, we would examine the spectrum of the linear operator A. In this case, the spectrum is known to consist entirely of negative real eigenvalues that are bounded away from the imaginary axis [18,24]; this is true regardless of the constant μ. Therefore, the linear operator is stable and solutions of the fluctuation equation (9) with no forcing (i.e., $f = 0$) and small enough initial data in V will converge to zero (see [16,19,21]); i.e., there is no transition for any μ, or, analogously, any Reynolds number. In this way, this Burgers' problem is similar to Couette or Hagen–Poiseuille (pipe) flow where the Orr–Sommerfeld/Squire operators are stable for all Reynolds numbers (see [25,26] and [27,28], respectively).

Now we employ the two modern approaches to transition outlined in Section 1. First, we examine the sensitivity of the problem to the initial data, i.e., we want to know if certain "relatively small" initial data can produce solutions that transition to another state. For this problem, numerical simulations (not presented here) suggest that solutions of the fluctuation Burgers' equation (6)–(8) converge to zero for any initial data. In [24], this is proved for any initial data that is continuous and satisfies the zero boundary conditions. Thus, transition will not occur for this problem even for certain very large perturbations of the base flow U.

Next, we examine if the problem is sensitive to small forcing. In particular, we examine the effect of a small disturbance in the boundary conditions on the solution of the Burgers' fluctuation problem. As mentioned in Section 1, it is well known that solutions to Burgers' equation can be "supersensitive" with respect to small disturbances in the equation and boundary conditions [29–33]. For the most part, this phenomenon has been studied using asymptotic analysis. However, we examine this problem from a different point of view using sensitivity analysis and make connections with the transition problem. For this problem, we will see that the small disturbance effectively moves the base flow which causes solutions to transition to another state. For more details on this transition scenario, see [14–16].

Remark. Due to the relative simplicity of the one-dimensional Burgers' equation, solutions will not transition to a turbulent state. However, solutions of the fluctuation Burgers' equation (6)–(8) will "transition" to another state that is not near the zero state. This is equivalent to solutions of Burgers' equation (1)–(3) diverging away from the "laminar flow." When we say solutions to Burgers' equation transition, we refer to this behavior and not to any turbulent phenomenon.

For the computations in this section, the group finite element method [34,35] is used for the spatial discretization of the fluctuation Burgers' equation. This method provides a major computational advantage over the standard finite element method for many nonlinear partial differential equations. Although the author is unaware of any convergence theory for this method applied to Burgers' equation, computational studies suggest that the accuracy of the group formulation equals, or exceeds, the standard method for both Burgers' equation and other flow equations [34–36]. Matlab's `ode23s` solver is used to approximate the solution of the resulting approximating system of ordinary differential equations.

Consider the fluctuation Burgers' equation with a perturbed boundary condition

$$u_t + uu_x = \mu u_{xx} - (Uu)_x, \tag{10}$$

$$u(t,-1) = 0, \quad u(t,1) = \varepsilon, \tag{11}$$

$$u(0,x) = u_0(x) := v_0(x) - U(x), \tag{12}$$

where ε is a small positive constant. We study the effect of the small boundary disturbance ε on the solution u. To simplify the analysis, we make a change of variables to homogenize the boundary conditions. Let $\psi(x;\varepsilon) = \varepsilon(x+1)/2$ and $y(t,x;\varepsilon) = u(t,x;\varepsilon) - \psi(x;\varepsilon)$. Then y satisfies

$$y_t + yy_x = \mu y_{xx} - (ky)_x + f, \tag{13}$$

$$y(t,-1) = 0, \quad y(t,1) = 0, \tag{14}$$

$$y(0,x) = y_0(x) := u_0(x) - \psi(x), \tag{15}$$

where $k(x;\varepsilon) = U(x) + \psi(x;\varepsilon)$ and $f(x;\varepsilon) = -(U(x)\psi(x;\varepsilon))_x - \psi(x;\varepsilon)\psi_x(x;\varepsilon)$. An application of Theorem 2.1 shows that both of these problems have a unique solution.

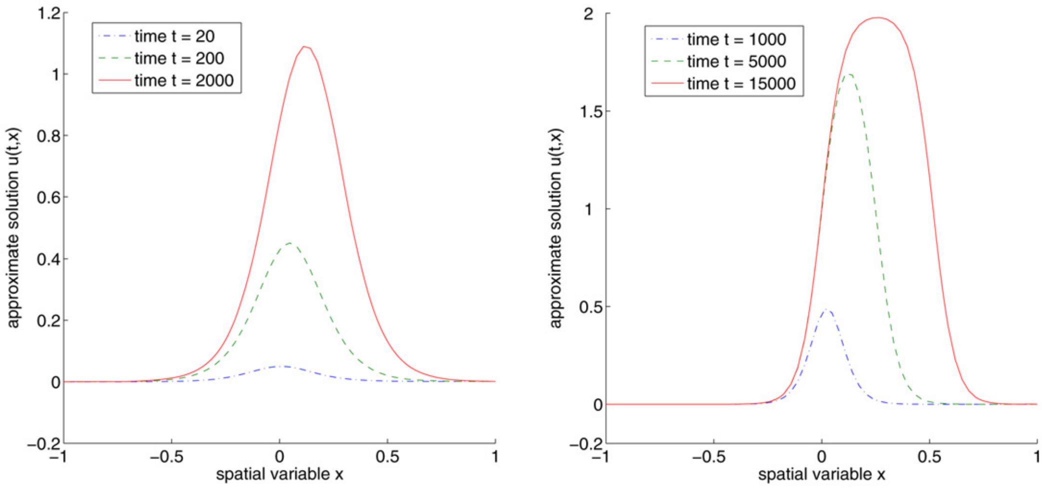
Proposition 3.1. *For any $T > 0$, $\varepsilon \in \mathbb{R}$, and $y_0 \in H_0^1$, the transformed equation* (13)–(15) *and the disturbed Burgers' fluctuation equation* (10)–(12) *each have a unique solution on* $[0,T]$.

Proof. As before, set $X = L^2(-1,1)$, $V = H_0^1(-1,1)$, and define the operators $Ay = \mu y_{xx} - (ky)_x$ with $D(A) = H^2 \cap H_0^1$, and $B(y,z) = -yz_x$. The transformed equation can then be written

$$\dot{y}(t) = Ay(t) + B\big(y(t), y(t)\big) + f, \quad y(0) = y_0,$$

where $f \in X$ and $y_0 \in V$. This problem falls into the general form of the disturbed fluctuation Burgers' equation considered in Theorem 2.1; therefore, there exists a unique solution to this problem on any time interval $[0,T]$. Inverting the change of variables shows that there is a unique solution u on any time interval $[0,T]$ to the fluctuation Burgers' equation (10)–(12). □

Assume $u_0(x) = 0$ for all x so that initially there is no fluctuation to the base flow U. Without the disturbance (i.e., $\varepsilon = 0$), the solution to the fluctuation Burgers' equation (10)–(12) remains zero for all time. This implies that the "flow" does not fluctuate around the laminar flow U, and therefore transition does not occur in this case. However, when $\varepsilon \neq 0$, the zero function no longer satisfies the disturbed boundary condition and solutions with initial data $u_0(x) = 0$ cannot remain at the zero state. Figure 1(a) shows time snapshots of the approximate solution with $\varepsilon = 10^{-3}$ and $\mu = 0.1$. The small disturbance causes the approximate solution to "transition" to an order one steady state. Decreasing μ increases the sensitivity. Figure 1(b) shows the smaller disturbance of $\varepsilon = 10^{-4}$ cause the solution to transition. Similar behavior has been observed for a wide variety of initial data. The small disturbance causes the zero equilibrium state to *move* and this causes the "transition" [16]. If we let $\mu = 1/R$, then these examples (and other computations not presented here) suggest that a boundary disturbance ε on the order of R^{-3} causes the solution to transition to an order one steady state. This is analogous to the results mentioned earlier for energy amplification in the Orr–Sommerfeld/Squire system.



(a) $\mu = 0.1$, $u_0(x) = 0$, and $\varepsilon = 10^{-3}$ (b) $\mu = 0.05$, $u_0(x) = 0$, and $\varepsilon = 10^{-4}$

Fig. 1. Approximate solution of the disturbed fluctuation equation (10)–(12) with U defined in Eq. (5).

4. Sensitivity analysis for semilinear parabolic equations

In the remainder of this paper, we focus on using sensitivity analysis to predict the effects of the small boundary disturbance on the solution of the fluctuation Burgers' equation (10)–(12). Sensitivity analysis is the process of measuring how solutions of a differential equation change with respect to parameters of interest. The change in the solution can be quantified by computing the derivative of the solution with respect to the parameters. In our case, we are interested in computing the derivatives of the zero solution of the fluctuation Burgers' equations with respect to the disturbance parameter ε.

Since the sensitivity is a derivative, one could use finite differences to approximate the sensitivity. However, this approach can lead to difficulties in practice. First, one has the problem of finding a good step size for the finite difference calculation. Also, if the evaluation of the state is computationally expensive as is the case in many applications (such as in flow problems), this method can be very inefficient.

Another approach is to approximate the sensitivity by solving an auxiliary sensitivity equation. There are two approaches to deriving a sensitivity equation: a discretize-then-differentiate method and a continuous sensitivity equation method. In the former method, one first discretizes the problem and then differentiates the discrete system with respect to the parameters of interest to obtain the linear sensitivity equations. Once the state equation has been solved, the state information is used to solve the sensitivity equations. Since this method only involves solving a linear system, it is much more computationally efficient than finite difference computations. The method can also be automated to differentiate through existing code [37]. The main drawback of this approach is that there is no guarantee that the computed sensitivity is near the true sensitivity.

In this work, we use the latter approach, the continuous sensitivity equation method (CSEM), to approximate sensitivities. Here, the infinite dimensional state equation is implicitly differentiated with respect to the parameters leading to linear equations for the sensitivities. Since the state and sensitivity equations are both infinite dimensional, different numerical schemes or levels of refinement can be used for each equation. In particular, one can take advantage of the linear nature of the sensitivity equations and solve them at a relatively low computational cost. This is especially advantageous if the original equation is nonlinear and requires a large amount of computational effort to solve. For details on the CSEM and applications to flow problems, see [38–40] and references therein.

An important feature of the continuous sensitivity equation method is that in many cases the sensitivities can be shown to exist and satisfy the appropriate sensitivity equations. For ordinary differential equations, this type of result is classical [41]. Parameter differentiability theory for various infinite dimensional evolution equations has been obtained more recently [42–48]. These theoretical results are important since they can provide insight for choosing appropriate numerical methods to approximate the sensitivities. In this work, we utilize the differentiability theory for semilinear parabolic problems found in Henry's text [21, Theorem 3.4.4 and Corollary 3.4.5].

Theorem 4.1 *(Henry). Suppose X and Y are Banach spaces, $-A$ is sectorial on X, $\alpha \in (0,1)$, U is open in X^α, and Q is open in Y. Suppose also that $F : U \times Q \to X$ is k times continuously Fréchet differentiable or analytic over $U \times Q$. For $x_0 \in U$ and $q \in Q$, let $x = x(t; x_0, q)$ be the solution of the parameter dependent semilinear parabolic problem*

$$\dot{x}(t) = Ax(t) + F\big(x(t); q\big), \quad x(0) = x_0, \tag{16}$$

on the interval $0 < t < T(x_0, q)$. Then on the interval $0 < t < T(x_0, q)$, $x(t; x_0, q)$ is k times continuously Fréchet differentiable or analytic with respect to x_0 and q as a mapping from $X^\alpha \times Q$ into X^α.

Henry proves this result using a parameter dependent version of the contraction mapping theorem due to Hale [49,50]. This theorem has been extended to include a time dependent nonlinear term and a parameter and/or time dependent linear operator [21,51]; however we do not require these cases here.

Since the solution $x(t; x_0, q)$ of the differential equation (16) satisfies the integral equation

$$x(t; x_0, q) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} F\big(x(\tau; x_0, q); q\big)\, d\tau, \tag{17}$$

an immediate consequence of this theorem is that equations can be derived for the derivatives of the solution with respect to the initial data, x_0, and parameter, q. Define the first-order sensitivity operators $S_1(t) = D_{x_0}x(t; x_0, q)$ and $S_2(t) = D_q x(t; x_0, q)$, and sensitivities $s_1(t) = S_1(t)x_0$ and $s_2(t) = S_2(t)q$.

Corollary 4.1 *(Henry). Under the assumptions of Theorem* 4.1*, the sensitivities $s_1(t)$ and $s_2(t)$ satisfy the integral equations*

$$s_1(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\big[D_x F\big(x(\tau; q); q\big)\big]s_1(\tau)\, d\tau,$$

$$s_2(t) = \int_0^t e^{A(t-\tau)}\big(\big[D_x F\big(x(\tau; q); q\big)\big]s_2(\tau) + \big[D_q F\big(x(\tau; q); q\big)\big]q\big)\, d\tau,$$

and are mild solutions of the linear initial value problems

$$\dot{s}_1(t) = As_1(t) + \big[D_x F\big(x(t; q); q\big)\big]s_1(t), \quad s_1(0) = x_0,$$
$$\dot{s}_2(t) = As_2(t) + \big[D_x F\big(x(t; q); q\big)\big]s_2(t) + \big[D_q F\big(x(t; q); q\big)\big]q, \quad s_2(0) = 0.$$

Differentiating the integral equation (17) *with respect to x_0 and q gives integral equations for higher order sensitivities.*

Each higher order sensitivity satisfies an integral equation that directly corresponds to a linear differential equation. In general, the sensitivities are only mild solutions of these differential equations; i.e., they may not be classical solutions [21]. Note that the differential sensitivity equations can be obtained by formally differentiating the original differential equation (16) with respect to the parameter (either x_0 or q), interchanging the order of differentiation, and using the chain rule.

5. Using sensitivity analysis to predict transition

In the next sections, we use the abstract framework presented above to rigorously derive equations for the sensitivities with respect to the disturbance parameter ε of the zero solution of the fluctuation Burgers' equation. We show that the computed sensitivities predict the large change in the zero solution with respect to the small disturbance. Also, higher order sensitivities are used to study the interaction of the nonlinearity and the small disturbance.

Let $u(t, x; \varepsilon)$ be the solution of the disturbed fluctuation Burgers' equation (10)–(12). Note that we have made explicit the dependence of the solution on the disturbance parameter ε. Recall that the transformed equation can be written abstractly (see Proposition 3.1) as

$$\dot{y}(t) = A(\varepsilon)y(t) + B\big(y(t), y(t)\big) + f(\varepsilon), \quad y(0;\varepsilon) = y_0(\varepsilon). \tag{18}$$

The following simple lemma shows that the nonlinear term is analytic in y.

Lemma 5.1. *Let U and Y be Banach spaces and suppose $B : U \times U \to Y$ is a continuous bilinear form on U, i.e., B is linear in each argument and there exists a positive constant C such that $\|B(u,v)\|_Y \leqslant C\|u\|_U\|v\|_U$ for all $u, v \in U$. The function $F : U \to Y$ defined by $F(u) = B(u,u)$ is analytic at every point $u \in U$. The first two Fréchet derivatives of F are given by*

$$\big[D_u F(u)\big]v = B(u,v) + B(v,u), \qquad \big[D_u^2 F(u)\big](v,w) = B(v,w) + B(v,w)$$

for all $u, v, w \in U$, and all higher order derivatives are zero. The function F and its derivatives are continuous in $u \in U$.

Proof. Since B is bilinear, $B(u,u) - B(v,v) = B(u, u-v) - B(v-u, v)$ for all $u, v \in U$. All of the results now follow directly from the definitions and the continuity of the bilinear form. □

Since the initial condition of the transformed equation depends on the parameter ε, we require a slight extension of Theorem 4.1.

Lemma 5.2. *Assume the hypotheses of Theorem 4.1 and let $x(t;q)$ be the solution of the parameter dependent problem*

$$\dot{x}(t) = Ax(t) + F\big(x(t);q\big), \quad x(0;q) = x_0(q),$$

on $0 < t < T(q)$. If the initial data $x_0 : Q \subset Y \to X^\alpha$ is k times continuously differentiable or analytic in a neighborhood of q_0, then for $0 < t < T(q)$ the solution $x(t;q)$ is k times differentiable or analytic as a mapping from Q into X^α. Sensitivity equations for the derivatives are given in a similar fashion to Corollary 4.1.

Proof. The proof follows by the same methods used to prove Theorem 4.1 (see [21]). □

With this framework in place, we can now prove that the solution $u(t,x;\varepsilon)$ of the disturbed fluctuation Burgers' equation (10)–(12) is analytic with respect to ε.

Theorem 5.1. *Let $T > 0$ and suppose $u_0(\varepsilon) \in H^1(-1,1)$ with $u_0(-1) = 0$ and $u_0(1) = \varepsilon$. If $u_0(\varepsilon)$ is analytic in ε, then the solution $u(t,x;\varepsilon)$ of the disturbed fluctuation Burgers' equation (10)–(12) on $[0,T]$ is analytic with respect to ε.*

Proof. This result follows if the solution y of the transformed equation (13)–(15) is analytic with respect to ε. Recall the abstract formulation of the transformed equation given in (18). To apply the sensitivity theory presented in Lemma 5.2 (and Theorem 4.1), we rewrite this problem so that the parameter ε does not appear in the linear operator A. Note $A(\varepsilon)$ can be split as $A = A_0 + A_1(\varepsilon)$, where $A_0 y = \mu y_{xx} - (Uy)_x$ and $A_1(\varepsilon)y = -(\psi(\varepsilon)y)_x$. Therefore, y satisfies

$$\dot{y}(t) = A_0 y(t) + F\big(y(t);\varepsilon\big), \quad y(0;\varepsilon) = y_0(\varepsilon), \tag{19}$$

where $F(y;\varepsilon) = A_1(\varepsilon)y + B(y,y) + f(\varepsilon)$. Theorem 2.1 shows that $-A_0$ is sectorial and $F(y;\varepsilon)$ is analytic in y by Lemma 5.1. Due to the assumptions on $u_0(\varepsilon)$, $y_0(\varepsilon)$ is in V and is analytic in ε. Also, it is easy to check that $F(y;\varepsilon)$ is analytic in ε. Therefore, the requirements of Theorem 4.1 and Lemma 5.2 are satisfied and the solution $y(t;\varepsilon)$ is analytic in ε. □

Define the sensitivity $S(t,x;\varepsilon)$ to be the derivative of the fluctuations u with respect to ε,

$$S(t,x;\varepsilon) = \frac{\partial u}{\partial \varepsilon}(t,x;\varepsilon). \tag{20}$$

Theorem 5.1 allows us to derive a sensitivity equation for S by formally differentiating the disturbed fluctuation Burgers' equation (10)–(12) with respect to ε, interchanging the order of differentiation, and using the chain rule (again, see Corollary 4.1).

Corollary 5.1. *Under the assumptions of Theorem* 5.1*, the sensitivity $S(t,x;\varepsilon)$ defined above is the unique solution of the linear partial differential equation*

$$S_t(t,x;\varepsilon) = \mu S_{xx}(t,x;\varepsilon) - \big(U(x)S(t,x;\varepsilon)\big)_x - \big(u(t,x;\varepsilon)S(t,x;\varepsilon)\big)_x, \tag{21}$$

$$S(t,-1;\varepsilon) = 0, \quad S(t,1;\varepsilon) = 1, \tag{22}$$

$$S(0,x;\varepsilon) = 0. \tag{23}$$

The solution $S(t,x;\varepsilon) = \partial_\varepsilon u(t,x;\varepsilon)$ of the sensitivity equation should be understood through the change of variables $y(t,x;\varepsilon) = u(t,x;\varepsilon) - \psi(x;\varepsilon)$. Once we have an equation for $\partial_\varepsilon y(t,x;\varepsilon)$, we derive the equation for S by inverting the transformation; this explains the incompatibility of the initial and boundary condition in the S sensitivity equation.

Proof of Corollary 5.1. We again work with the transformed problem. The integral form of the equation is given by

$$y(t;\varepsilon) = e^{A_0 t} y_0(\varepsilon) + \int_0^t e^{A_0(t-\tau)} F\big(y(\tau;\varepsilon);\varepsilon\big)\,d\tau.$$

Differentiating through this equation with respect to ε and using the chain rule gives an integral equation for the derivative $y_\varepsilon(t;\varepsilon)$. It can be checked that the corresponding differential equation gives the above partial differential equation (21)–(23) after inverting the transformation. □

Notice that the sensitivity equation is linear and depends on the solution $u(t,x;\varepsilon)$ of the disturbed fluctuation Burgers' equation.

We are particularly interested in examining how the base flow U changes with respect to ε. If the sensitivity of the base flow is large, then it is possible that a very small disturbance can cause the solution to transition to another state. Since the base flow corresponds to the zero solution of the Burgers' fluctuation equation, we are interested in the case where initial fluctuations in (12) are zero (i.e., $u(0,x) = u_0(x) = 0$) and we want to evaluate the sensitivity S at $\varepsilon = 0$. Let $s(t,x)$ be the sensitivity at $\varepsilon = 0$, i.e.,

$$s(t,x) = S(t,x;\varepsilon)|_{\varepsilon=0}.$$

With zero initial data and $\varepsilon = 0$, the term involving $u(t,x;\varepsilon)$ vanishes in the S sensitivity equation (21)–(23) which gives

$$s_t(t,x) = \mu s_{xx}(t,x) - \big(U(x)s(t,x)\big)_x, \tag{24}$$

$$s(t,-1) = 0, \quad s(t,1) = 1, \quad s(0,x) = 0. \tag{25}$$

Again, if the solution of this equation is large, then the zero solution of the Burgers' fluctuation equation will go through a large change in response to the small boundary disturbance ε.

To approximate the sensitivity numerically we use standard finite elements with N equally spaced nodes for the spatial discretization and Matlab's `ode23s` to solve the resulting approximating ordinary differential equation system. Figure 2 shows the finite element approximation to the sensitivity $s(t,x)$ with $N = 64$ and $\mu = 0.1$ over the time interval $0 \leqslant t \leqslant 4$. The sensitivity is increasing and is already fairly large at $t = 4$. This shows that the solution to the disturbed Burgers' fluctuation equation can undergo a large change. Integrating further in time shows that the computed sensitivity continues to increase and it eventually becomes exceptionally large, especially near $x = 0$ (see the time snapshots in Fig. 3). The size of the sensitivity for $\mu = 0.01$ is much more dramatic. Figure 4 shows that as time increases, the sensitivity slowly increases until reaching a steady state of magnitude approximately 10^{14}. The main support of the sensitivity is over the small interval $[-0.1, 0.1]$. The sensitivity is similar to a delta function centered at $x = 0$. Due to the large gradients in the solution, it is unclear even with 1028 evenly spaced mesh points that the solution has numerically converged. This behavior indicates that the solution of the disturbed fluctuation Burgers' equation with zero initial data can change drastically near $x = 0$.

These results show that the sensitivity predicts that a small boundary disturbance ε will cause the zero solution of the Burgers' fluctuation equation to undergo a drastic change. Of course, we already observed this behavior above

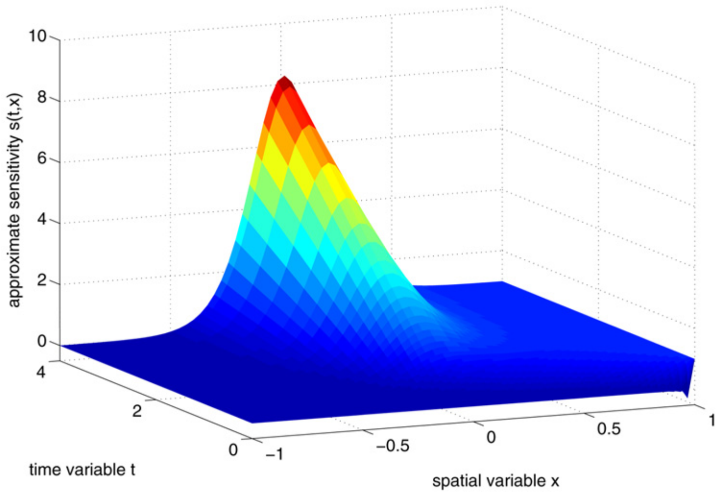


Fig. 2. Approximate sensitivity of the zero solution of the disturbed Burgers' fluctuation equation with respect to ε over the time interval $[0, 4]$. The sensitivity is computed using the sensitivity equation (24)–(25) with $\mu = 0.1$ and $N = 64$.

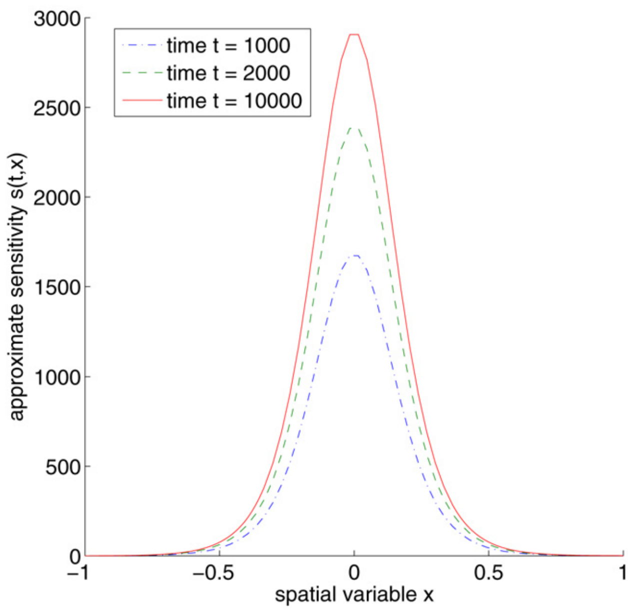


Fig. 3. Approximate solution of the sensitivity equation (24)–(25) with $\mu = 0.1$, $N = 64$ and $u(t, x; 0) = 0$.

through direct numerical simulation of the disturbed problem. It is important to note that sensitivity analysis can be used to predict transition when direct numerical simulation of the disturbed problem is costly or unfeasible. If one suspects that a certain small disturbance is causing transition in a flow system, sensitivity analysis can be used to determine the effect of the disturbance on the flow at a relatively low computational cost.

6. Higher order sensitivities

The two modern approaches to transition discussed in the introduction are primarily based on the study of a linearized flow system. In this section, we use sensitivity analysis to distinguish the behavior of solutions to the linearized and nonlinear fluctuation systems. In particular, higher order sensitivities with respect to the disturbance parameter

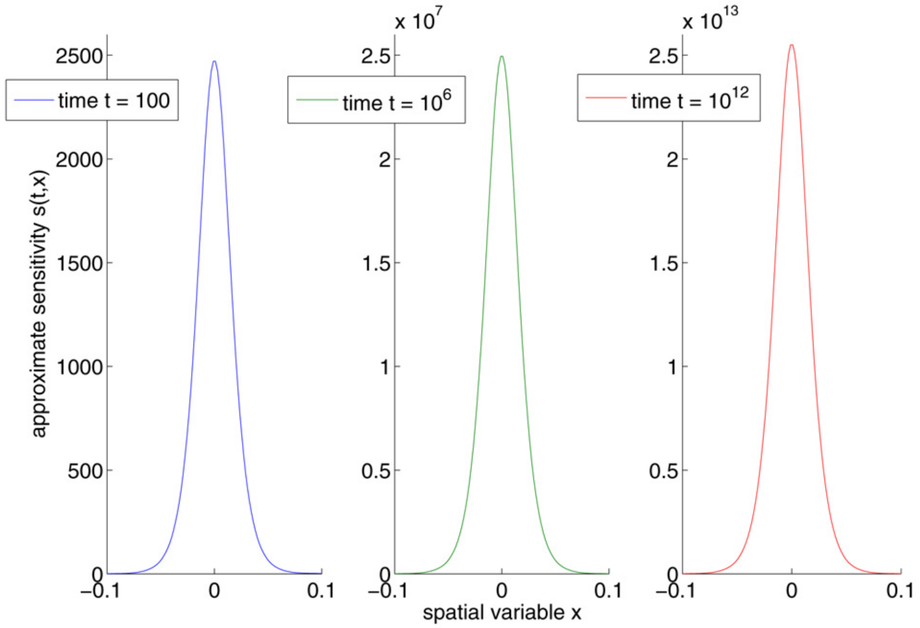


Fig. 4. Approximate solution of the sensitivity equation (24)–(25) with $\mu = 0.01$, $N = 1028$ and $u(t, x; 0) = 0$. The sensitivities are nearly zero for x outside of $[-0.1, 0.1]$. Note the scales on the vertical axes.

are used to obtain estimates of the size of the disturbance that will cause transition. We show that this information is unavailable through the study of the linearized problem.

Consider the disturbed Burgers' fluctuation equation (10)–(12) with zero initial fluctuation, $u(0, x) = u_0(x) = 0$. If we drop the nonlinear term uu_x, we obtain the linearized fluctuation system

$$z_t(t, x; \varepsilon) = \mu z_{xx}(t, x; \varepsilon) - \big(U(x)z(t, x; \varepsilon)\big)_x,$$
$$z(t, -1; \varepsilon) = 0, \quad z(t, 1; \varepsilon) = \varepsilon, \quad z(0, x; \varepsilon) = z_0(x) = 0.$$

We use z to denote the solution of the linearized fluctuation problem and keep u as the solution of the nonlinear problem. Sensitivity analysis is performed on this problem to study how the linearized solution z changes with respect to the small disturbance ε.

Since the equation is linear in z and ε enters the linearized fluctuation equation in a linear manner, the sensitivity $p = \partial_\varepsilon z(t, x; \varepsilon)$ will not depend on the solution z or the disturbance ε. This is seen by differentiating through the linearized sensitivity equation to show that $p = p(t, x)$ is the mild solution of

$$p_t(t, x) = \mu p_{xx}(t, x) - \big(U(x)p(t, x)\big)_x,$$
$$p(t, -1) = 0, \quad p(t, 1) = 1, \quad p(0, x) = 0.$$

Again, this can be made rigorous using the methods described above. There are two important details to note here.

(1) The sensitivity p of the solution of the linearized fluctuation system satisfies the same equation (24)–(25) as the sensitivity s of the zero solution of the nonlinear fluctuation system evaluated at $\varepsilon = 0$.
(2) All of the higher derivatives $\partial_\varepsilon^n z(t, x; \varepsilon)$ must be zero for all t and x.

Here are two implications of these observations.

First, since $p(t, x)$ equals $s(t, x)$, the change in the zero solution of the nonlinear fluctuation system is, in a sense, completely governed by the linearized fluctuation system. This gives an idea of why the recent exploration of linear flow systems has provided so much new insight into the transition problem. In contrast, however, the second observation shows that the linearized fluctuation system is also *lacking* important information about the transition problem.

This can be seen as follows. If we expand the solution of the disturbed linearized fluctuation equation in a Taylor series in the parameter ε, we obtain

$$\begin{aligned} z(t,x;\varepsilon) &= z(t,x;0) + \varepsilon\frac{\partial z}{\partial \varepsilon}(t,x;0) + \frac{\varepsilon^2}{2!}\frac{\partial^2 z}{\partial \varepsilon^2}(t,x;0) + \cdots \\ &= z(t,x;0) + \varepsilon p(t,x), \end{aligned}$$

since all of the higher order derivatives of z with respect to ε are zero. This shows that the solution z of the disturbed linearized fluctuation system is completely determined by the solution of the undisturbed linearized system and the p sensitivity equation. In contrast, the solution of the disturbed nonlinear fluctuation problem depends on much more information.

Return to Eq. (21)–(23) for the sensitivity $S(t,x;\varepsilon)$ of the solution of the nonlinear fluctuation equation. In contrast to the linearized fluctuation system, the higher derivatives with respect to ε all depend on ε (and therefore are nonzero) due to the differentiation with respect to ε of the "quadratic" term $(u(t,x;\varepsilon)S(t,x;\varepsilon))_x$. By differentiating through the sensitivity equation, we can rigorously derive equations for the higher order sensitivities $S_n(t,x;\varepsilon) = \partial_\varepsilon^n u(t,x;\varepsilon)$. Again, we are interested in the sensitivity of the zero solution, so define

$$s_n(t,x) = \frac{\partial^n u}{\partial \varepsilon^n}(t,x;\varepsilon)\Big|_{\varepsilon=0}.$$

Since $u(t,x;\varepsilon)$ is analytic in ε, we can expand $u(t,x;\varepsilon)$ in a Taylor series consisting of the sensitivities $s_n(t,x)$.

Theorem 6.1. *The solution $u(t,x;\varepsilon)$ of the disturbed fluctuation Burgers' equation* (10)–(12) *can be expanded in Taylor series by*

$$\begin{aligned} u(t,x;\varepsilon) &= u(t,x;0) + \varepsilon\frac{\partial u}{\partial \varepsilon}(t,x;0) + \frac{\varepsilon^2}{2!}\frac{\partial^2 u}{\partial \varepsilon^2}(t,x;0) + \frac{\varepsilon^3}{3!}\frac{\partial^3 u}{\partial \varepsilon^3}(t,x;0) + \cdots \\ &= \varepsilon s_1(t,x) + \frac{\varepsilon^2}{2!}s_2(t,x) + \frac{\varepsilon^3}{3!}s_3(t,x) + \cdots \end{aligned}$$

since $u(t,x;0) = 0$. The first-order sensitivity $s_1(t,x)$ is the mild solution of (24)–(25). *Equations for the higher order sensitivities $s_n(t,x)$ can be derived by differentiating equation* (21)–(23) *for the sensitivity $S(t,x;\varepsilon)$ with respect to ε and setting $u(t,x;0) = 0$. For example, the differential sensitivity equations for $s_2(t,x)$ and $s_3(t,x)$ are given by*

$$\frac{\partial}{\partial t}s_2(t,x) = \mu\frac{\partial^2}{\partial x^2}s_2(t,x) - \frac{\partial}{\partial x}\big(U(x)s_2(t,x)\big) - 2s_1(t,x)\frac{\partial}{\partial x}s_1(t,x), \tag{26}$$

$$s_2(t,-1) = 0 = s_2(t,1), \quad s_2(0,x) = 0, \tag{27}$$

and

$$\frac{\partial}{\partial t}s_3(t,x) = \mu\frac{\partial^2}{\partial x^2}s_3(t,x) - \frac{\partial}{\partial x}\big(U(x)s_3(t,x)\big) - 3\frac{\partial}{\partial x}\big(s_1(t,x)s_2(t,x)\big), \tag{28}$$

$$s_3(t,-1) = 0 = s_3(t,1), \quad s_3(0,x) = 0. \tag{29}$$

Proof. The analyticity of $u(t,x;\varepsilon)$ with respect to ε was proved in Theorem 5.1. The sensitivity equations are derived by applying Lemma 5.2 to the transformed fluctuation Burgers' equation (18). □

If the higher order sensitivities are large, then the Taylor expansion shows that solutions to the disturbed fluctuation Burgers' equation (10)–(12) can be large even when ε is small. Furthermore, the size of the higher order sensitivities can be used to estimate how large the disturbance ε must be in order to cause the solution $u(t,x;\varepsilon)$ to reach a certain magnitude. This is information that the linearized fluctuation problem does not provide.

To illustrate this procedure with the nonlinear fluctuation Burgers' equation, we numerically approximate the second and third-order sensitivities $s_2(t,x)$ and $s_3(t,x)$ using the sensitivity equations (26)–(27) and (28)–(29), respectively. To solve these equations we must also approximate $s_1(t,x)$ which we have done above using the first-order sensitivity equation (24)–(25). For the spatial discretization, we use standard finite elements with N equally spaced nodes for all three sensitivity equations. For the time integration, we use Matlab's ODE solver `ode23s` to give approximate values of s_1 at various (discrete) times. The trapezoid rule is then used to approximate s_2 and s_3 at these same set of time values. The approximation methods used here were chosen mainly for simplicity; if desired, one could use different spatial discretizations, different meshes, time integrators, etc. on the three different sensitivity equations.

Figure 5 shows the finite element approximation to the solution $s_2(t,x)$ of the second-order sensitivity equation with $\mu = 0.1$ and $N = 128$. Note that the magnitude of the sensitivity is quite large. Therefore, the second term in the Taylor series expansion can be large even if ε is small. Figure 6 shows the approximation to $s_3(t,x)$. Again, since this sensitivity is so large, the third term in the Taylor expansion may not be negligible even if ε is extremely small.

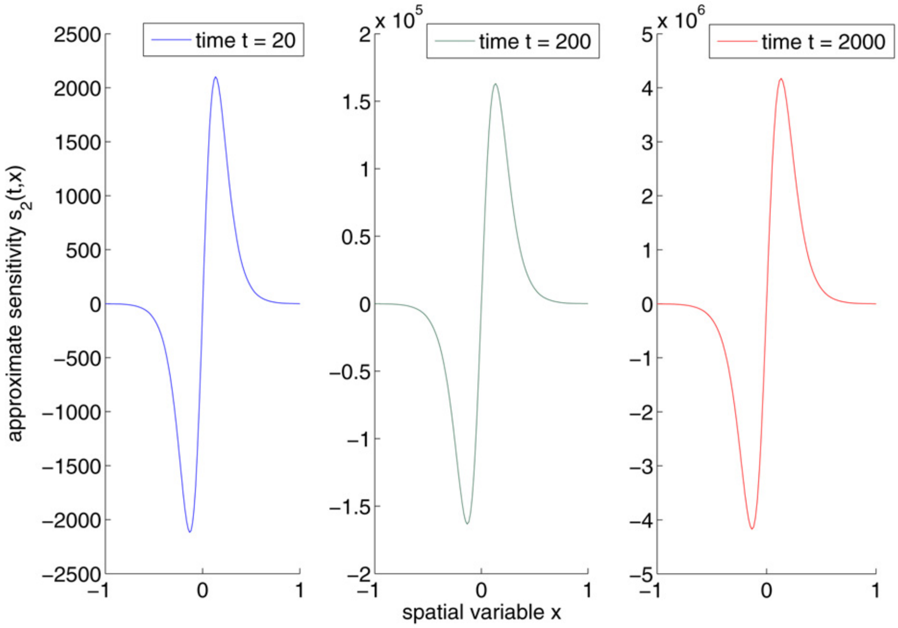


Fig. 5. Approximate solution of the second-order sensitivity equation (26)–(27) with $\mu = 0.1$ and $N = 128$. Note the scales on the vertical axes.

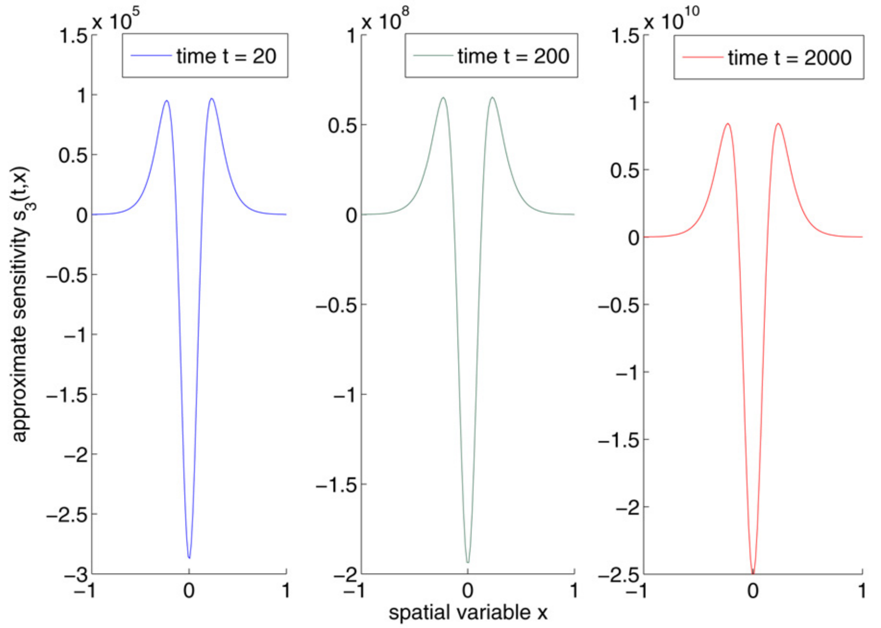


Fig. 6. Approximate solution of the third-order sensitivity equation (28)–(29) with $\mu = 0.1$ and $N = 128$. Note the scales on the vertical axes.

As with the first-order sensitivity $s(t,x) = s_1(t,x)$, numerical experiments (not presented here) show that decreasing μ causes these sensitivities to become much larger. Therefore, if μ is small, ε can be quite small and still cause the solution to transition.

One may use these sensitivities to estimate the size of the disturbance ε that causes the solution to the disturbed problem to become "large." In particular, one can compute each term in the Taylor series expansion to estimate the size of the solution to the disturbed problem. As a simple example, when $\varepsilon = 0.1$ the first three sensitivities reach a magnitude of (roughly) 10^3, 10^6, and 10^{10}, respectively. Using a Taylor expansion and neglecting any remainder term, when t is large we have

$$\left\|u(t,x;\varepsilon)\right\|_{L^\infty} \approx \varepsilon 10^3 + \frac{\varepsilon^2}{2}10^6 + \frac{\varepsilon^3}{6}10^{10}.$$

To keep the magnitude of $u(t,x;\varepsilon)$ less than 10^{-p} for some p, ε must be approximately 10^{-3p}. This method could be extended to other nonlinear problems to provide an efficient way to estimate the size of the solution to the disturbed system.

7. Conclusion

In this first paper in a two part work, we used the continuous sensitivity equation method to study the effects of a small disturbance on solutions of a highly sensitive Burgers' equation. We showed that the computed sensitivities predict the large change in solutions in the presence of the disturbance. Also, higher order sensitivities were used to distinguish the effects of the small disturbance on solutions of the linearized and nonlinear fluctuation problems. In particular, the linearized problem was shown to completely determine the first-order sensitivity but not provide information on any of the higher order sensitivities. These higher order sensitivities can be used in a Taylor series to efficiently estimate the solution of the disturbed problem.

In the second part of this work, we extend the methods introduced here to study the effects of small disturbances on transition to turbulence in the three-dimensional Navier–Stokes equations. We consider the equations for fluctuations about a laminar flow and treat nonzero initial data and forcing as disturbances. Roughly, we show that if the linearized operator A is stable, yet either

(1) the C_0-semigroup e^{At} is "large" over some finite period of time (which occurs when A is nonnormal), or
(2) the spectrum of the linear operator A is "close" to the imaginary axis,

then these small disturbances can cause a large change in the zero solution of the fluctuation equation. Furthermore, we use a Taylor series expansion to give rigorous estimates on the size of the solution of the fluctuation Navier–Stokes equations in the presence of these small disturbances. The estimates for the fluctuations $w(t;w_0,f)$ as a function of a small initial fluctuation w_0 and a small forcing f take the form

$$\left\|w(t;w_0,0)\right\|_\alpha \leqslant e^{-\omega t}\sum_{n=1}^{\infty}\frac{1}{n!}c_n(t;\alpha,\omega)M^{2n-1}\|w_0\|_\alpha^n,$$

$$\left\|w(t;0,f)\right\|_\alpha \leqslant \sum_{n=1}^{\infty}\frac{1}{n!}d_n(t;\alpha,\omega)M^{2n-1}\|f\|_{H_\sigma}^n.$$

Here, the constant M is large if e^{At} is large and the coefficients c_n and d_n are large if A has spectrum near the imaginary axis. The analysis extends the "mostly linear" transition theories discussed in Section 1 by showing that very small disturbances have the potential to trigger transition to turbulence in the full nonlinear Navier–Stokes equations.

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