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Meromorphic functions sharing a set with applications to difference equations

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ABSTRACT

This paper is devoted to proving some uniqueness type results for an entire function $f(z)$ that shares a common set with its shift $f(z+c)$ or its difference operator $\Delta_c f$. We also give some applications to solutions of non-linear difference equations related to a conjecture proposed by C.C. Yang.

1. Introduction

In this paper, a meromorphic function will always be non-constant and meromorphic in the complex plane $\mathbb{C}$, unless specifically stated otherwise. In what follows, we assume that the reader is familiar with the elementary Nevanlinna theory, see [9,14,20]. In particular, for a meromorphic function f, $S(f)$ denotes the family of all meromorphic functions ω such that $T(r,\omega)=S(r,f)=o(T(r,f))$, where $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure. For convenience, we agree that $S(f)$ includes all constant functions and $\hat{S}(f):=S(f)\cup\{\infty\}$.

For a meromorphic function f and a set $S\subset\mathbb{C}$, we define

$$E_f(S)=\bigcup_{a\in S}\big\{z \mid f(z)-a=0,\ \text{counting multiplicities}\big\},$$

$$\overline{E}_f(S)=\bigcup_{a\in S}\big\{z \mid f(z)-a=0,\ \text{ignoring multiplicities}\big\}.$$

We say that f and g share a set S CM, resp. IM, provided that $E_f(S)=E_g(S)$, resp. $\overline{E}_f(S)=\overline{E}_g(S)$. As a special case, let $S=\{a\}$, where $a\in\hat{\mathbb{C}}$. If $E_f(\{a\})=E_g(\{a\})$, resp. $\overline{E}_f(\{a\})=\overline{E}_g(\{a\})$, we say that f and g share the value a CM, resp. IM.

The classical results in the uniqueness theory of meromorphic functions are the 5 IM and 4 CM theorems due to Nevanlinna [16], see also [9,20]. In 1979, Gundersen [4] proved that 4 IM $\neq$ 4 CM and 3 CM + 1 IM = 4 CM. The conclusion 2 CM + 2 IM = 4 CM also given by Gundersen [5], while the case 1 CM + 3 IM still remains an open problem.

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A special topic widely studied in the uniqueness theory is the case when $f(z)$ shares values with its derivatives or differential polynomials. We recall a result of this type from the preceding literature:

Theorem A. *(See [13, Theorem 3].) Let f be a non-constant entire function and a_1, a_2 be two distinct complex numbers. If f and f' share the set $\{a_1, a_2\}$ CM, then f takes one of the following conclusions*:

(i) $f = f'$,
(ii) $f + f' = a_1 + a_2$,
(iii) $f = c_1 e^{cz} + c_2 e^{-cz}$, *with $a_1 + a_2 = 0$, where c, c_1, c_2 are non-zero constants which satisfy $c^2 \neq 1$ and $c_1 c_2 = \frac{1}{4} a_1^2 (1 - \frac{1}{c^2})$.*

It is well known that there exists a set S containing seven elements such that if f and g are two non-constant entire functions and $E_f(S) = E_g(S)$, then $f = g$, see [20, Theorem 10.58]. In a special case, Fang and Zalcman [2, Theorem 1] obtained the following:

Theorem B. *There exists a finite set S containing three elements such that if f is a non-constant entire function and $E_f(S) = E_{f'}(S)$, then $f = f'$.*

There exist some uniqueness results related to the case when two functions share common sets. We recall one of them here:

Theorem C. *(See [3].) Let $S_1 = \{1, -1\}$, $S_2 = \{0\}$. If $f(z)$ and $g(z)$ are non-constant entire functions of finite order such that f and g share the sets S_1 and S_2 CM, then $f = \pm g$ or $f \cdot g = 1$.*

Similarly as to the above situations, one may also consider shared value problems for $f(z)$ with its shifts $f(z+c)$ and their difference polynomials. To this end, we recall a key result [11, Theorem 2], which may be understood as 2 CM + 1 IM theorem for differences:

Theorem D. *Let f be a transcendental meromorphic function of finite order, let $c \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2, a_3 \in \hat{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM, and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

In this paper, we investigate the cases when $f(z)$ shares a common set with $f(z+c)$ or $\Delta_c f := f(z+c) - f(z)$. In particular, we offer difference counterparts to Theorems B and C. We also improve a result in [11] related to Theorem D. Perhaps we could remark here that if we choose $g(z) = f(z+c)$ in Theorem C, then $f(z) = \pm f(z+c)$. Indeed, if $f(z) \cdot f(z+c) = 1$, then $f(z)^2 = f(z)/f(z+c)$, and so $T(r,f) = m(r,f) = S(r,f)$ by Lemma 3.2 below.

This paper is organized as follows. In Section 2, we state that if an entire function $f(z)$ shares a common set with its shift $f(z+c)$ or difference operator $\Delta_c f$, then either $f(z)$ satisfies a certain difference equation or $f(z)$ is of a certain special form. This is a difference counterpart to Theorem A. We also give some results related to Theorems B and C in Section 2. The proofs of these results will be given in Section 3. Section 4 is then devoted to giving an improvement for a result in [11]. In Section 5, we give some applications to non-linear difference equations.

2. Main results

Our first result below may be understood as a difference counterpart to Theorem A, where $f(z)$ shares a common set with its first derivative $f'(z)$. Here $f(z)$ shares a common set with its shift $f(z+c)$.

Theorem 2.1. *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C} \setminus \{0\}$, and let $a(z) \in S(f)$ be a non-vanishing periodic entire function with period c. If $f(z)$ and $f(z+c)$ share the set $\{a(z), -a(z)\}$ CM, then $f(z)$ must take one of the following conclusions*:

(i) $f(z) \equiv f(z+c)$,
(ii) $f(z) + f(z+c) \equiv 0$,
(iii) $f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, *where $\frac{h_1(z+c)}{h_1(z)} = -e^{\gamma}$, $\frac{h_2(z+c)}{h_2(z)} = e^{\gamma}$, $h_1(z)h_2(z) = a(z)^2(1 - e^{-2\gamma})$ and γ is a polynomial.*

Remark 2.2. Suppose $f(z)$ and $f(z+c)$ share the set $\{a(z), b(z)\}$ CM in Theorem 2.1, where $a(z), b(z) \in S(f)$ are non-vanishing periodic entire functions with period c. Defining $g(z) := f(z) - \frac{a(z)+b(z)}{2}$, we see that $g(z)$ and $g(z+c)$ share the set $\{\frac{a(z)-b(z)}{2}, \frac{b(z)-a(z)}{2}\}$ CM. Therefore, we get either $f(z+c) \equiv f(z)$ or $f(z+c) + f(z) \equiv a(z) + b(z)$ or the last case in Theorem 2.1 with $\frac{a(z)-b(z)}{2}$ replacing $a(z)$.

Corollary 2.3. *Under the assumptions of Theorem 2.1, if $f(z)$ and $f(z+c)$ share the sets $\{a(z), -a(z)\}$, $\{0\}$ CM, then $f(z) = \pm f(z+c)$ for all $z \in \mathbb{C}$.*

If $f(z+c)$ is replaced with $\Delta_c f$ in Theorem 2.1, we get the following result:

Theorem 2.4. *Let f be a transcendental entire function of finite order, and let a be a non-zero finite constant. If f and $\Delta_c f$ share the set $\{a, -a\}$ CM, then $f(z+c) \equiv 2f(z)$.*

Remark 2.5. It would be natural to ask what happens if $\{a, -a\}$ is replaced with $\{a(z), b(z)\}$ in Theorem 2.4, where $a(z), b(z) \in S(f)$ are non-vanishing periodic entire functions with period c? This remains open at present.

Theorem 2.6. *There exists a set S with two elements such that if f is a transcendental entire function of finite order with at most finitely many zeros and $E_{f(z)}(S) = E_{f(z+c)}(S)$, then $f(z+c) = \pm f(z)$ for all $z \in \mathbb{C}$.*

Remark 2.7. If the set S has one element only, then Theorem 2.6 is not true. This can be seen by taking $f(z) = e^{z^2}$. Then 0 is a Picard exceptional value for $f(z)$ and $f(z+c)$, while $f(z+c) \not\equiv Af(z)$, where A is any given constant. The assumption on finitely many zeros cannot be deleted, which can be seen by taking $f(z) = \sin z$. Then $f(z)$ and $f(z+\frac{\pi}{2})$ share the set $\{\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\}$ CM, while $f(z+c) \not\equiv \pm f(z)$.

3. Proofs of results

Before proceeding to the actual proofs, we recall a few lemmas that take an important role in the reasoning. The first of these lemmas is a difference analogue of the logarithmic derivative lemma, given by Halburd and Korhonen [6, Corollary 2.2] and Chiang and Feng [1, Corollary 2.6], independently. Presentations in these references are slightly different. The original statement [6, Corollary 2.2] reads as follows:

Lemma 3.1. *Let $f(z)$ be a non-constant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$, and $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)^{1+\varepsilon}}{r^{\delta}}\right) \tag{3.1}$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Making use of [8, Lemma 2.1], we have $T(r+|c|, f) = (1+o(1))T(r, f)$ for all r outside of a possible exceptional set with finite logarithmic measure, provided that f is of finite order. This implies [7, Theorem 2.1], which can be stated as follows.

Lemma 3.2. *Let $f(z)$ be a non-constant meromorphic function of finite order, $c \in \mathbb{C}$, $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{\delta}}\right) = S(r, f), \tag{3.2}$$

where $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set E with finite logarithmic measure.

The following result is an application of Lemma 3.2 to the function $f(z) - a(z)$, see [7, Lemma 2.3].

Lemma 3.3. *Let f be a meromorphic function of finite order, and let $c \in \mathbb{C}$, $n \in \mathbb{N}$. Then for any small periodic function $a(z) \in S(f)$ with period c,*

$$m\left(r, \frac{\Delta_c^n f}{f(z) - a(z)}\right) = S(r, f).$$

The next lemma follows by using a similar reasoning as in the proof of [12, Theorem 1], with apparent modifications. More precisely, we need to replace the differential polynomial of f with the operator $\Delta_c f$ and to use Lemma 3.2 or Lemma 3.3 instead of the logarithmic derivative lemma, if needed. For the convenience of the reader, we give a sketch of the proof here.

Lemma 3.4. *Let $f(z)$ be an entire function of finite order, and let a be a non-zero constant. If f and $\Delta_c f$ share the set $\{a, -a\}$ CM, then*

$$(\Delta_c f - a)(\Delta_c f + a) = (f - a)(f + a)e^{2\gamma}, \tag{3.3}$$

where γ is a polynomial such that $T(r, e^{2\gamma}) = S(r, f)$.

Proof. Let $g := \Delta_c f$. Since f is an entire function of finite order, we have $T(r,g) \leqslant T(r,f) + S(r,f)$ by Lemma 3.2. Since f and g share the set $\{a, -a\}$ CM, we obtain $T(r,f) \leqslant 2T(r,g) + S(r,g)$ by applying the second main theorem. Therefore, $S(r) := S(r,f) = S(r,g)$. Differentiating (3.3), we obtain

$$2gg' = \big(2\gamma'(f-a)(f+a) + 2ff'\big)e^{2\gamma}. \tag{3.4}$$

Defining

$$\psi = \frac{(e^{2\gamma}f')^2 - (g')^2}{(g-a)(g+a)}, \tag{3.5}$$

we get $T(r,\psi) = m(r,\psi) = S(r)$ by repeating the reasoning in [12, pp. 418–419], while making use of Lemma 3.3 again, if needed.

We now proceed to proving $T(r, e^{2\gamma}) = S(r)$.

(A) If $\psi = 0$, then $T(r, e^{2\gamma}) = S(r)$ by (3.5) and Lemma 3.2.

(B) If $\psi \neq 0$, then using a similar discussion as in [12, pp. 419], we first obtain $m(r, \frac{1}{g\pm a}) = S(r)$, and all zeros of $(g-a)(g+a)$ are simple as long as they are not zeros of ψ. Thus

$$2T(r,g) = \overline{N}\left(r, \frac{1}{g-a}\right) + \overline{N}\left(r, \frac{1}{g+a}\right) + S(r). \tag{3.6}$$

Taking derivative in both sides of (3.5) and eliminating $e^{2\gamma}$, we get

$$\big(2\psi(2\gamma'f' + f'') - \psi'f'\big)(g-a)(g+a) = \big(2\psi gf' - (4\gamma'f' + 2f'')g' + 2f'g''\big)g'. \tag{3.7}$$

From (3.7), we know that a simple zero of $(g-a)(g+a)$ must be a zero of the function $2\psi gf' - (4\gamma'f' + 2f'')g' + 2f'g''$.

Define now

$$\psi_1 := \frac{2\psi gf' - (4\gamma'f' + 2f'')g' + 2f'g''}{(f-a)(f+a)}. \tag{3.8}$$

Then $T(r,\psi_1) = S(r)$ follows by using Lemma 3.3 and the lemma of logarithmic derivative. If $\psi_1 \neq 0$, then from (3.8) and Lemma 3.2

$$\begin{aligned} 2T(r,f) &\leqslant m\big(r, (f-a)(f+a)\psi_1\big) + S(r) \\ &\leqslant m(r,f) + m(r,g) + S(r) \\ &\leqslant T(r,f) + m\left(r, f(z)\left(\frac{f(z+c)}{f(z)} - 1\right)\right) + S(r) \\ &\leqslant 2T(r,f) + S(r). \end{aligned}$$

It follows that $T(r,f) = T(r,g) + S(r)$. By (3.6) we now conclude that

$$m\left(r, \frac{1}{f \pm a}\right) = S(r). \tag{3.9}$$

From (3.3), we get

$$\begin{aligned} m\big(r, e^{2\gamma}\big) &\leqslant m\left(r, \frac{\Delta_c f - a}{f-a}\right) + m\left(r, \frac{\Delta_c f + a}{f+a}\right) \\ &\leqslant m\left(r, \frac{\Delta_c f}{f-a}\right) + m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{\Delta_c f}{f+a}\right) + m\left(r, \frac{1}{f+a}\right) + S(r). \end{aligned}$$

Combining (3.9) and Lemma 3.3, $T(r, e^{2\gamma}) = S(r)$ follows.

If $\psi_1 = 0$, we may repeat the reasoning in [12, pp. 420–421] to conclude that $T(r,f) = S(r)$, a contradiction. This completes the proof. □

Proof of Theorem 2.1. Recall that the idea of the proof is similar as to the proof of [12, Theorem 1].

Since $f(z)$ is an entire function of finite order and $f(z)$ and $f(z+c)$ share the set $\{a(z), -a(z)\}$ CM, it is immediate to conclude that

$$\big(f(z+c) - a(z)\big)\big(f(z+c) + a(z)\big) = \big(f(z) - a(z)\big)\big(f(z) + a(z)\big)e^{2\gamma}, \tag{3.10}$$

where γ is a polynomial.

Since $a(z)$ is a periodic entire function with period c, we infer by Lemma 3.2 that

$$m\left(r, \frac{f(z+c)-a(z)}{f(z)-a(z)}\right) = S(r, f) \tag{3.11}$$

and

$$m\left(r, \frac{f(z+c)+a(z)}{f(z)+a(z)}\right) = S(r, f). \tag{3.12}$$

From (3.10)–(3.12), we obtain

$$T\big(r, e^{2\gamma}\big) = m\big(r, e^{2\gamma}\big) = S(r, f). \tag{3.13}$$

Case 1. If $e^{2\gamma} = 1$, from (3.10), then we get $f(z) \equiv f(z+c)$ or $f(z) + f(z+c) \equiv 0$.
Case 2. If $e^{2\gamma} \neq 1$, let $h_1(z) := f(z) - e^{-\gamma} f(z+c)$ and $h_2(z) := f(z) + e^{-\gamma} f(z+c)$. Then

$$f(z) = \frac{1}{2}(h_1 + h_2), \qquad f(z+c) = \frac{1}{2}e^{\gamma}(h_2 - h_1). \tag{3.14}$$

From (3.10), we have

$$h_1(z)h_2(z) = a(z)^2\big(1 - e^{-2\gamma}\big), \tag{3.15}$$

which means that

$$N\left(r, \frac{1}{h_i}\right) = S(r, f), \quad i = 1, 2. \tag{3.16}$$

From the expressions of h_1 and h_2, we get $T(r, h_i) \leqslant 2T(r, f) + S(r, f)$, so that $S(r, h_i) = o(T(r, f))$, $i = 1, 2$.
Let $\alpha := \frac{h_1(z+c)}{h_1(z)}$ and $\beta := \frac{h_2(z+c)}{h_2(z)}$. From (3.16), we have

$$T(r, \alpha) = m(r, \alpha) + N\left(r, \frac{1}{h_1}\right) = S(r, f), \qquad T(r, \beta) = m(r, \beta) + N\left(r, \frac{1}{h_2}\right) = S(r, f). \tag{3.17}$$

From (3.14), we get

$$e^{\gamma} h_2(z) - e^{\gamma} h_1(z) = h_1(z+c) + h_2(z+c). \tag{3.18}$$

Dividing (3.18) with $h_1(z)h_2(z)$, we conclude that

$$\big(\alpha + e^{\gamma}\big)h_1 = \big(e^{\gamma} - \beta\big)h_2. \tag{3.19}$$

From (3.15) and (3.19), it follows that

$$\big(\alpha + e^{\gamma}\big)h_1(z)^2 - \big(e^{\gamma} - \beta\big)a(z)^2\big(1 - e^{-2\gamma}\big) = 0. \tag{3.20}$$

Combining (3.13), (3.17) and (3.20), we get $\alpha = -e^{\gamma}$ and $\beta = e^{\gamma}$. Otherwise, we get $T(r, h_1) = S(r, f)$. Combining (3.14) and (3.15), we conclude that $T(r, f) = S(r, f)$, which is impossible. Thus, we have completed the proof of Theorem 2.1. □

Proof of Corollary 2.3. It suffices to consider the case (iii) in Theorem 2.1. We first assume that $f(z_0) = 0$. Since $f(z)$ and $f(z+c)$ share 0 CM, then $h_1(z_0) + h_2(z_0) = 0$ and $h_1(z_0 + c) + h_2(z_0 + c) = 0$. Hence

$$\frac{h_1(z_0 + c)}{h_1(z_0)} \cdot \frac{h_2(z_0)}{h_2(z_0 + c)} = 1.$$

From $\frac{h_1(z+c)}{h_1(z)} = -e^{\gamma}$ and $\frac{h_2(z+c)}{h_2(z)} = e^{\gamma}$, we obtain

$$\frac{h_1(z_0 + c)}{h_1(z_0)} \cdot \frac{h_2(z_0)}{h_2(z_0 + c)} = -1,$$

a contradiction. Hence 0 must be the Picard exceptional value of $f(z)$ and $f(z+c)$, which implies that $h_1(z) + h_2(z) \neq 0$. Since $h_1(z)$ and $h_2(z)$ are finite order entire functions, then we can write $h_1(z) + h_2(z) = e^{P(z)}$, where $P(z)$ is a polynomial. Combining this with $h_1(z)h_2(z) = a(z)^2(1 - e^{-2\gamma})$, we get the following equation

$$\frac{a(z)^2(1 - e^{-2\gamma}) + h_1^2}{h_1} = e^{P(z)} = 2f(z).$$

So we get

$$N\left(r,\frac{1}{h_1^2}\right)=S(r,f)\quad\text{and}\quad N\left(r,\frac{1}{h_1^2+a(z)^2(1-e^{-2\gamma})}\right)=S(r,f).$$

Applying the second main theorem for three small target functions [9, Theorem 2.5] and the standard Valiron–Mohon'ko theorem [15], we get

$$T(r,f)+S(r,f)=T\left(r,h_1^2\right)\leqslant N\left(r,h_1^2\right)+N\left(r,\frac{1}{h_1^2}\right)+N\left(r,\frac{1}{h_1^2+a(z)^2(1-e^{-2\gamma})}\right)+S(r,h_1)=S(r,f),$$

which is a contradiction. So we can remove the case (iii) to get $f(z)=\pm f(z+c)$. □

Proof of Theorem 2.4. From Lemma 3.4, we must have $T(r,e^{2\gamma})=S(r,f)$. If $e^{2\gamma}=1$, thus $f(z+c)\equiv 2f(z)$. If $e^{2\gamma}\neq 1$, using a method similar to the proof of Theorem 2.1, we easily get $\frac{h_1(z+c)}{h_1(z)}=1-e^{\gamma}$, $\frac{h_2(z+c)}{h_2(z)}=1+e^{\gamma}$, $h_1(z)h_2(z)=a^2(1-e^{-2\gamma})$ and γ is a polynomial. Then we get

$$h_1(z+c)h_2(z+c)=h_1(z)h_2(z)\left(1-e^{\gamma(z)}\right)\left(1+e^{\gamma(z)}\right)=a^2\left(1-e^{-2\gamma(z+c)}\right).$$

Thus, by computing, we can get

$$e^{2\gamma(z)}+e^{-2\gamma(z)}-e^{-2\gamma(z+c)}\equiv 1.$$

From the above equation and [20, Theorem 1.56], we get $e^{2\gamma}=1$, which is a contradiction to our assumption. That implies $f(z+c)\equiv 2f(z)$. Thus, we have completed the proof of Theorem 2.4. □

Proof of Theorem 2.6. Assume that $S=\{a,-a\}$, $a\in\mathbb{C}\setminus\{0\}$. From the proof of Theorem 2.1 above, we have $N(r,h_1)+N(r,\frac{1}{h_1})=S(r,h_1)$. Since f is an entire function and has finitely many zeros, then we can write $2f(z)=P(z)e^{Q(z)}=h_1(z)+h_2(z)$, where $P(z)$ and $Q(z)$ are polynomials. Combining this with $h_1(z)h_2(z)=a^2(1-e^{-2\gamma})$, we get the following equation

$$\frac{a^2(1-e^{-2\gamma})+h_1^2}{h_1}=P(z)e^{Q(z)}=2f(z).$$

We observe that $N(r,\frac{1}{h_1^2+a^2(1-e^{-2\gamma})})=S(r,h_1)$. Using the second main theorem for three small target functions [9, Theorem 2.5], we get $T(r,h_1)=S(r,h_1)$, a contradiction. So we can remove the case (iii) of Theorem 2.1. □

4. Improvements of Theorem D

Heittokangas et al. [10,11] investigated the cases when $f(z)$ shares three small periodic functions with its shift or its difference polynomials. As examples, we state the following theorems, in addition to Theorem D above:

Theorem E. *(See [11, Theorem 7].) Let $f(z)$ be a transcendental meromorphic function of finite order, $c\in\mathbb{C}$, and let $a_1,a_2,a_3\in\hat{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 CM, and if*

$$\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f-a_1})+\overline{N}(r,\frac{1}{f-a_2})}{T(r,f)}<\frac{1}{4},\tag{4.1}$$

then $f(z)=f(z+c)$ or $f(z)=f(z+2c)$ for all $z\in\mathbb{C}$.

Theorem F. *(See [11, Theorem 8].) Let $f(z)$ be a transcendental meromorphic function of finite order, $c\in\mathbb{C}$, and let $a_1,a_2,a_3\in\hat{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 IM, and if*

$$\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)=S(r,f),\tag{4.2}$$

then $f(z)=f(z+c)$ or $f(z)=f(z+2c)$ for all $z\in\mathbb{C}$.

It is natural to ask about conditions to imply that f is periodic with period c in the preceding theorems. To this end, we prove

Theorem 4.1. *Let f be a transcendental meromorphic function of finite order, $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \hat{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 IM, and if*

$$\limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a_1}) + N(r, \frac{1}{f-a_2})}{T(r, f)} < \frac{1}{7}, \tag{4.3}$$

then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

To prove Theorem 4.1, we need the following result, given by Sun and Xu [17, Theorem 1]. For convenience of reader, we recall the proof given in [17].

Theorem G. *Let f_1 and f_2 be meromorphic functions such that*

$$\limsup_{r\notin E} \frac{\overline{N}(r, f_j) + \overline{N}(r, \frac{1}{f_j})}{T(r, f_j)} < \frac{1}{7}, \quad j = 1, 2, \tag{4.4}$$

where E is a set with finite linear measure. If f_1 and f_2 share 1 IM, then $f_1 = f_2$ or $f_1 \cdot f_2 = 1$.

Proof. Define

$$\psi := \frac{f_1''}{f_1'} - \frac{f_2''}{f_2'} - \frac{2f_1'}{f_1 - 1} + \frac{2f_2'}{f_2 - 1}.$$

Suppose $\psi = 0$. Integrating twice results in

$$\frac{1}{f_1 - 1} = \frac{A}{f_2 - 1} + B.$$

If now $B \neq 0, -1$, then $\overline{N}(r, 1/(f_1 - (B+1)/B)) = \overline{N}(r, f_2)$. Thus, an immediate contradiction follows by using (4.4) together with the second main theorem. A similar reasoning results in a contradiction, unless either $A = 1$, $B = 0$, hence $f_1 = f_2$, or $A = -1$, $B = -1$ implying that $f_1 \cdot f_2 = 1$.

To complete the proof, it remains to show that the case $\psi \neq 0$ is not possible. If $\psi \neq 0$, we conclude that

$$\begin{aligned} N_{1)}(r) &\leqslant N\left(r, \frac{1}{\psi}\right) \leqslant T(r, \psi) \leqslant N(r, \psi) + S(r, f_1) + S(r, f_2) \\ &\leqslant \sum_{j=1}^{2} \left(\overline{N}(r, f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) + N_0\left(r, \frac{1}{f_j'}\right) + \overline{N}_{(2}\left(r, \frac{1}{f_j - 1}\right) + S(r, f_j) \right), \end{aligned} \tag{4.5}$$

where $N_{1)}(r)$, resp. $N_0(r, \frac{1}{f_j'})$, resp. $\overline{N}_{(2}(r, \frac{1}{f_j - 1})$, denotes the counting function of common simple 1-points of f_1 and f_2, resp. the zeros of f_j' which are not the zeros of f_j or of $f_j - 1$, resp. the zeros of f_j with multiplicity at least 2.

Since f_1 and f_2 share 1 IM, then

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f_2 - 1}\right) = \overline{N}\left(r, \frac{1}{f_1 - 1}\right) &= N_{1)}(r) + \left\{ \overline{N}_{1)}\left(r, \frac{1}{f_1 - 1}\right) - N_{1)}(r) \right\} + \overline{N}_{(2}\left(r, \frac{1}{f_1 - 1}\right) \\ &\leqslant N_{1)}(r) + \overline{N}_{(2}\left(r, \frac{1}{f_2 - 1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f_1 - 1}\right). \end{aligned} \tag{4.6}$$

From (4.6), it is not difficult to conclude that

$$\begin{aligned} \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{f_j - 1}\right) &\leqslant \frac{1}{2} \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{f_j - 1}\right) + N_{1)}(r) + \sum_{j=1}^{2} \overline{N}_{(2}\left(r, \frac{1}{f_j - 1}\right) \\ &\leqslant N_{1)}(r) + \frac{1}{2} \sum_{j=1}^{2} \left\{ \overline{N}\left(r, \frac{1}{f_j - 1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f_j - 1}\right) \right\} + \frac{1}{2} \sum_{j=1}^{2} \overline{N}_{(2}\left(r, \frac{1}{f_j - 1}\right) \\ &\leqslant N_{1)}(r) + \frac{1}{2} \sum_{j=1}^{2} N\left(r, \frac{1}{f_j - 1}\right) + \frac{1}{2} \sum_{j=1}^{2} \overline{N}_{(2}\left(r, \frac{1}{f_j - 1}\right) \\ &\leqslant N_{1)}(r) + \frac{1}{2} \sum_{j=1}^{2} T(r, f_j) + \frac{1}{2} \sum_{j=1}^{2} \overline{N}_{(2}\left(r, \frac{1}{f_j - 1}\right) + S(r, f_j). \end{aligned} \tag{4.7}$$

However

$$\overline{N}_{(2}\left(r, \frac{1}{f_j - 1}\right) \leqslant N\left(r, \frac{f_j}{f_j'}\right) \leqslant T\left(r, \frac{f_j'}{f_j}\right) \leqslant \overline{N}(r, f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) + S(r, f_j). \tag{4.8}$$

The second main theorem together with (4.5) implies that

$$N_{1)}(r) + \sum_{j=1}^{2} T(r, f_j) \leqslant \sum_{j=1}^{2}\left(2\overline{N}(r, f_j) + 2\overline{N}\left(r, \frac{1}{f_j}\right) + \overline{N}\left(r, \frac{1}{f_j - 1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f_j - 1}\right) + S(r, f_j)\right).$$

Substituting here (4.7) and (4.8), we obtain

$$\frac{1}{2}\sum_{j=1}^{2} T(r, f_j) \leqslant \frac{7}{2}\sum_{j=1}^{2}\left\{\overline{N}(r, f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) + S(r, f_j)\right\},$$

outside a set E with finite linear measure, which is a contradiction to the condition (4.4). □

Proof of Theorem 4.1. Suppose that $a_1, a_2, a_3 \in S(f)$. Defining

$$g(z) := \frac{f(z) - a_1}{f(z) - a_2} \cdot \frac{a_3 - a_2}{a_3 - a_1},$$

it is immediate to see that $T(r, f) = T(r, g) + S(r, g)$. Therefore, (4.3) may be expressed as

$$N\left(r, \frac{1}{g}\right) + N(r, g) \leqslant \big(\lambda + o(1)\big)T(r, g), \quad \lambda \in \left[0, \frac{1}{7}\right). \tag{4.9}$$

Assume $g(z_0) = 1$. Then either $f(z_0) = a_3$ or $f(z_0) = \infty$. In the former case, we easily obtain $g(z_0 + c) = 1$, since $f(z)$ and $f(z + c)$ share a_3 IM. In the latter case, we conclude that $a_1(z_0) = a_2(z_0)$, and hence $g(z_0 + c) = 1$. Conversely, if $g(z_0 + c) = 1$, then $g(z_0) = 1$. So we conclude that $g(z)$ and $g(z + c)$ share 1 IM. The following, we will prove $T(r, g) \leqslant (1 + o(1))T(r, g(z + c))$. From Lemma 3.2

$$\begin{aligned} T(r, g) &= m(r, g) + N(r, g) \\ &\leqslant m\left(r, g(z + c)\frac{g(z)}{g(z + c)}\right) + N\big(r + |c|, g(z + c)\big) \\ &\leqslant m\big(r, g(z + c)\big) + N\big(r + |c|, g(z + c)\big) + o\Big(T\big(r, g(z + c)\big)\Big), \end{aligned}$$

outside of an exceptional set of finite logarithmic measure, and combining [8, Lemma 2.1], we get $N(r + |c|, g(z + c)) = N(r, g(z + c)) + o(N(r, g(z + c)))$, again outside of an exceptional set of finite logarithmic measure. Thus

$$T(r, g) \leqslant \big(1 + o(1)\big)T\big(r, g(z + c)\big). \tag{4.10}$$

Using the idea due to [11, Theorem 8], by a simple geometric observation and [8, Lemma 2.1], thus (4.9) and (4.10) imply that

$$\begin{aligned} \overline{N}\left(r, \frac{1}{g(z + c)}\right) + \overline{N}\big(r, g(z + c)\big) &\leqslant \overline{N}\left(r + |c|, \frac{1}{g}\right) + \overline{N}\big(r + |c|, g\big) \\ &\leqslant \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g) + o\big(T(r, g)\big) \\ &\leqslant \big(\lambda + o(1)\big)T(r, g) \\ &\leqslant \big(\lambda + o(1)\big)T\big(r, g(z + c)\big). \end{aligned} \tag{4.11}$$

Combining (4.9), (4.11) with Theorem G, $g(z) = g(z + c)$ or $g(z) \cdot g(z + c) = 1$ follows.

If $g(z) \cdot g(z+c) = 1$, then $g^2(z) = \frac{g(z)}{g(z+c)}$. From Lemma 3.2, we get $m(r, g) = S(r, g)$. Therefore $T(r, g) < \frac{1}{7}T(r, g) + S(r, g)$, a contradiction. Thus, we must have $g(z + c) = g(z)$, meaning that $f(z + c) = f(z)$ for all $z \in \mathbb{C}$.

It remains to consider the case, say, when $a_1 = \infty$, while $a_2(z), a_3(z) \in S(f)$. Take $d \in \mathbb{C} \setminus \{a_2(z), a_3(z)\}$ and denote $h(z) := \frac{1}{f(z) - d}$, $b_2 := \frac{1}{a_2(z) - d}$ and $b_3 := \frac{1}{a_3(z) - d}$. Then $b_2(z), b_3(z) \in S(f)$ are two distinct periodic functions with period c. Hence $h(z)$ and $h(z + c)$ share b_3 IM and satisfy the following

$$N\left(r, \frac{1}{h - b_2}\right) + N\left(r, \frac{1}{h}\right) \leqslant \big(\lambda + o(1)\big)T(r, h), \quad \lambda \in \left[0, \frac{1}{7}\right).$$

Using the similar proof as above, thus we have completed the proof. □

From Theorem 4.1, we easily obtain the following result.

Corollary 4.2. *Let f be a transcendental entire function of finite order, $c \in \mathbb{C}$, and let $a(z), b(z) \in S(f)$ be two distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share $a(z)$ IM, and if*

$$\limsup_{r\to\infty} \frac{N(r, \frac{1}{f-b(z)})}{T(r,f)} < \frac{1}{7},$$

then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

5. Some applications to non-linear difference equations

We first give a simple application of Theorem 2.1. From Eq. (5.1) below, we observe that $f(z)$ and $f(z+c)$ share the set $\{\frac{a(z)}{\sqrt{2}}, -\frac{a(z)}{\sqrt{2}}\}$ CM. From Theorem 2.1, $f(z)$ must satisfy the case (iii), for otherwise $T(r,f) = S(r,f)$, which is a contradiction. From Eq. (5.1), we have $e^{\gamma} = i$ or $e^{\gamma} = -i$, and hence we get the following result.

Proposition 5.1. *Let f be a non-constant finite order entire solution of the non-linear difference equation*

$$f(z)^2 + f(z+c)^2 = a(z)^2, \tag{5.1}$$

then $f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, where $\frac{h_1(z+c)}{h_1(z)} = i$ and $\frac{h_2(z+c)}{h_2(z)} = -i$, $h_1(z)h_2(z) = a(z)^2$, where $a(z)$ is a non-vanishing small function to $f(z)$ with period c.

Remark 5.2. It is easy to verify that $f(z) = a(z)\sin z$ is a solution of Eq. (5.1), provided $c = \frac{\pi}{2}$. At the same time, we see that the case (iii) in Theorem 2.1 may appear. Indeed, taking $a(z) \equiv 1$, we may write f in the form $f(z) = \frac{1}{2}(-ie^{iz} + ie^{-iz})$.

Proposition 5.3. *There is no non-constant finite order entire solution of the non-linear difference equation*

$$f(z)^2 + (\Delta_c f)^2 = a^2, \tag{5.2}$$

where a is a non-zero constant.

Proof. Assume that $f(z)$ is a non-constant finite order entire solution of (5.2). From (5.2), we observe that $f(z)$ and $\Delta_c f$ share the set $\{\frac{a(z)}{\sqrt{2}}, -\frac{a(z)}{\sqrt{2}}\}$ CM. Thus $f(z+c) \equiv 2f(z)$, which implies $T(r,f) = S(r,f)$, a contradiction. This completes the proof. □

The following theorem is related to a conjecture proposed by Yang [19]. Namely, he conjectured that there does not exist an entire function f of infinite order that satisfies the difference equation

$$f(z)^n + bf(z+c) = h(z), \tag{5.3}$$

where $n \geqslant 2$, $b \in \mathbb{C} \setminus \{0\}$ and $h(z)$ is an entire function of finite order.

Theorem 5.4. *Eq. (5.3) has no entire solutions of infinite order, when $\overline{N}(r, \frac{1}{f(z+c)}) \leqslant T(r,f)$, $n \geqslant 3$ and $h(z)$ is a polynomial.*

Proof. Assume that $f(z)$ is an infinite order entire solution of Eq. (5.3). Define $f_1 := f(z)^n$ and $f_2 := bf(z+c)$. Then $f_1 + f_2 = h(z)$. Since $h(z)$ is a polynomial, it is a small function to $f(z)$. Applying the second main theorem for three small target functions [9, Theorem 2.5], we get

$$\begin{aligned} nT(r,f) = T(r,f_1) &\leqslant \overline{N}(r,f_1) + \overline{N}\left(r,\frac{1}{f_1}\right) + \overline{N}\left(r,\frac{1}{f_1 - h(z)}\right) + S(r,f) \\ &\leqslant \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f(z+c)}\right) + S(r,f) \\ &\leqslant 2T(r,f) + S(r,f). \end{aligned}$$

Since $n \geqslant 3$, we get $T(r,f) = S(r,f)$, which is a contradiction. □

Remark 5.5. From Theorem 2.1 in [1], we know that $\overline{N}(r, \frac{1}{f(z+c)}) \leqslant T(r, f) + O(r^{\rho-1+\varepsilon}) + S(r, f)$, provided that $f(z)$ is of finite order ρ, while this may false if $f(z)$ is of infinite order. The function $f(z) = e^{e^z} - 1$ is an example of the infinite order case: If $e^c = 4$, then $f(z+c) = e^{4e^z} - 1$. By the Valiron–Mohon'ko theorem [15], we have

$$T\big(r, f(z+c)\big) = 4T(r, f) + S(r, f).$$

From the second main theorem, we obtain

$$\begin{aligned} T\big(r, f(z+c)\big) &\leqslant \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}\left(r, \frac{1}{f(z+c)+1}\right) + \overline{N}\big(r, f(z+c)\big) + S\big(r, f(z+c)\big) \\ &\leqslant \overline{N}\left(r, \frac{1}{f(z+c)}\right) + S(r, f). \end{aligned}$$

Hence, $\overline{N}(r, \frac{1}{f(z+c)}) \geqslant 4T(r, f) + S(r, f)$. In fact, it is not difficult to construct an example of a function f that satisfies $\overline{N}(r, \frac{1}{f(z+c)}) \geqslant nT(r, f) + S(r, f)$, provided that $f(z)$ is of infinite order.

Remark 5.6. Suppose $h = 0$. Then Eq. (5.3) has no entire solutions of finite order, since a contradiction $m(r, f) = S(r, f)$ is immediate. Equation $f(z)^n - f(z+1) = 0$ of type (5.3) admits an entire solution of infinite order $f(z) = e^{e^{z\log n}}$, see [18, p. 124].

Remark 5.7. If $h(z)$ is non-zero constant and $n = 2$, Eq. (5.3) may have an infinite order solution. Indeed, $f(z) = \frac{1}{e^{e^z}} + e^{e^z}$ is an entire function of infinite order and solves equation $f(z)^2 - f(z+c) = 2$, where $e^c = -2$. Unfortunately, we have not been able to give an example of infinite order solutions of Eq. (5.3), if h is a non-constant entire function.

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References

[1] Y.M. Chiang, S.J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, Ramanujan J. 16 (2008) 105–129.
[2] M.L. Fang, L. Zalcman, Normal families and uniqueness theorems for entire functions, J. Math. Anal. Appl. 280 (2003) 273–283.
[3] F. Gross, C.F. Osgood, Entire functions with common preimages, in: Factorization Theory of Meromorphic Functions and Related Topics, in: Lect. Notes Pure Appl. Math., vol. 78, Dekker, New York, 1982, pp. 19–24.
[4] G.G. Gundersen, Meromorphic functions that share three or four values, J. London Math. Soc. 20 (1979) 457–466.
[5] G.G. Gundersen, Meromorphic functions that share four values, Trans. Amer. Math. Soc. 277 (1983) 545–567.
[6] R.G. Halburd, R.J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, J. Math. Anal. Appl. 314 (2006) 477–487.
[7] R.G. Halburd, R.J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math. 31 (2006) 463–478.
[8] R.G. Halburd, R.J. Korhonen, Finite-order meromorphic solutions and the discrete Painlevé equations, Proc. Lond. Math. Soc. 94 (2007) 443–474.
[9] W.K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[10] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, Uniqueness of meromorphic functions sharing values with their shifts, Complex Var. Elliptic Equ., in press.
[11] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, J.L. Zhang, Value sharing results for shifts of meromorphic function, and sufficient conditions for periodicity, J. Math. Anal. Appl. 355 (2009) 352–363.
[12] P. Li, Value sharing and differential equations, J. Math. Anal. Appl. 310 (2005) 412–423.
[13] P. Li, C.C. Yang, Value sharing of an entire function and its derivatives, J. Math. Soc. Japan 51 (1999) 781–799.
[14] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.
[15] A.Z. Mohon'ko, The Nevanlinna characteristics of certain meromorphic functions, Teor. Funktsii Funktsional. Anal. i Prilozhen. 14 (1971) 83–87 (in Russian).
[16] R. Nevanlinna, Le Théorème de Picard–Borel et la Théorie des Fonctions Méromorphes, Gauthiers–Villars, Paris, 1929.
[17] F.S. Sun, Y. Xu, Shared values for several meromorphic functions, J. Nanjing Norm. Univ. 23 (2000) 18–21 (in Chinese).
[18] N. Yanagihara, Meromorphic solutions of some difference equations II, Funkcial. Ekvac. 24 (1981) 113–124.
[19] C.C. Yang, On the analogues of non-linear differential and difference equations, unpublished communication.
[20] C.C. Yang, H.X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, Science Press, 2003.