On $W_{1+\infty}$ 3-algebra and integrable system

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Abstract

We construct the $W_{1+\infty}$ 3-algebra and investigate its connection with the integrable systems. Since the $W_{1+\infty}$ 3-algebra with a fixed generator $W_0^0$ in the operator Nambu 3-bracket recovers the $W_{1+\infty}$ algebra, it is intrinsically related to the KP hierarchy. For the general case of the $W_{1+\infty}$ 3-algebra, we directly derive the KP and KdV equations from the Nambu–Poisson evolution equation with the different Hamiltonian pairs of the KP hierarchy. Due to the Nambu–Poisson evolution equation involves two Hamiltonians, the deep relationship between the Hamiltonian pairs of KP hierarchy is revealed. Furthermore we give a realization of the $W_{1+\infty}$ 3-algebra in terms of a complex bosonic field. Based on the Nambu 3-brackets of the complex bosonic field, we derive the (generalized) nonlinear Schrödinger equation and give an application in optical soliton.

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1. Introduction

The infinite-dimensional algebras play a very important role in the study of integrable systems. This is due to the fact that they have a deep intrinsic connection to the integrable systems. The Virasoro algebra is an important infinite-dimensional algebra which has been turned out to be a universal symmetry algebra in two-dimensional conformal field theory. Gervais and Neveu [1,2] found that this infinite-dimensional algebra is intrinsically related to the Korteweg–de Vries (KdV) equation where the Poisson bracket of the KdV equation is indeed the classical Virasoro algebra. The lattice Virasoro algebra appears in the study of the Toda field theory and Toda integrable systems [3,4]. Furthermore the intrinsic connection between the super Virasoro algebra and super KdV equation has been studied in [5–10]. As a higher-spin extension of the Virasoro algebra, the $W_N$ algebra has been found to be associated with the generalized KdV hierarchies [11,12].

The KP hierarchy is an integrable system consisting of an infinite number of non-linear differential equations. This hierarchy has proven to be bi-Hamiltonian and to possess an infinite number of conserved quantities. All the (generalized) KdV hierarchies can be incorporated into the KP hierarchy. The $W_{1+\infty}$ algebra [13] can be regarded as a linear deformation of the $W_N$ algebras in the large $N$ limit [14]. It has very important applications in physics, such as quantum Hall effect [15], topological string [16], Alday–Gaiotto–Tachikawa (AGT) relation [17], 2-dimensional quantum gravity [18] and crystal melting [19]. In the context of integrable systems, it was shown that the $W_{1+\infty}$ algebra is intrinsically related to the first Hamiltonian structure of the KP hierarchy [20,21]. The second Hamiltonian structure of the KP hierarchy was then shown to be isomorphic to $\tilde{W}_{\infty}$ [22], a centerless deformation of $W_{\infty}$.

Nambu mechanics [23] is a generalization of classical Hamiltonian mechanics in which the usual binary Poisson bracket is replaced by the Nambu bracket. Based on the Nambu bracket, a notion of a Nambu 3-algebra was introduced in [24] as a natural generalization of a Lie algebra for higher-order algebraic operations. Recently 3-algebras have been paid great attention due to a world-volume description of multiple M2-branes proposed by Bagger and Lambert [25], and Gustavsson [26] (BLG). There has been an increasing interest in the applications of 3-algebras to the string/M-theory [27–34]. The properties of various 3-algebras especially the infinite-dimensional cases have been widely investigated, such as Virasoro–Witt 3-algebra [35–37], Kac–Moody 3-algebra [38] and $w_{\infty}$ 3-algebra [39,40]. Furthermore the q-deformation of Virasoro–Witt 3-algebra has also been investigated. The nontrivial q-deformed Virasoro–Witt 3-algebra constructed in [41] is the so-called sh-3-Lie algebra.

More recently, the relation between the infinite-dimensional 3-algebras and the integrable systems has been investigated. Chen et al. [42] investigated the classical Heisenberg and $w_{\infty}$ 3-algebras and established the relation between the dispersionless KdV hierarchy and these two infinite-dimensional 3-algebras. They found that the dispersionless KdV system is not only a bi-Hamiltonian system, but also a bi-Nambu–Hamiltonian system. It should be pointed out that the dispersionless KdV hierarchy is a very simple integrable system. An interesting open question is whether we can derive other integrable nonlinear evolution equations from the infinite-dimensional 3-algebras. We have mentioned that the first Hamiltonian structure of the KP hierarchy is indeed corresponding to the $W_{1+\infty}$ algebra. In this paper, we shall construct the $W_{1+\infty}$ 3-algebra and discuss the relation between this infinite-dimensional 3-algebra and the integrable systems.

This paper is organized as follows. In Section 2, we construct the $W_{1+\infty}$ 3-algebra. In Section 3, we establish the relation between the $W_{1+\infty}$ 3-algebra and the KP hierarchy in the
framework of Nambu mechanics. In Section 4, by means of the $W_{1+\infty}$ 3-algebra, we derive the KP and KdV equations from the Nambu–Poisson (N-P) evolution equation with the different Hamiltonian pairs. In Section 5, we investigate the relation between the $W_{1+\infty}$ 3-algebra and the dispersionless KdV hierarchy. A conjecture is presented. In Section 6, we give a realization of the $W_{1+\infty}$ 3-algebra and investigate the (generalized) nonlinear Schrödinger equation. Moreover we give an application of the generalized nonlinear Schrödinger equation derived from the N-P evolution equation. We end this paper with the concluding remarks in Section 7.

2. $W_{1+\infty}$ 3-algebra

Let us take \( W^r_m = (-1)^r e^{i\pi x \frac{dr}{dx}} [m, r \in \mathbb{Z}, r \geq 0, i = \sqrt{-1}] \). We then obtain the algebra

\[
[W^r_m, W^s_n] = \left( \sum_{p=0}^{r} C^p_r (-im)^p - \sum_{p=0}^{s} C^p_s (-im)^p \right) W^{r+s-p}_{m+n},
\]

where \( C^p_r = \frac{r(r-1) \cdots (r-p+1)}{p!} \). The first three commutators of (1) are

\[
[W^0_m, W^0_n] = 0, \quad [W^0_m, W^1_n] = imW^0_{m+n}, \quad [W^1_m, W^1_n] = i(m-n)W^1_{m+n}.
\]

Eq. (2) can be recognized as the semidirect product of the Witt algebra with an abelian current algebra.

Taking the Fourier transformation field (FTF)

\[
v_r(x) = \frac{-i}{2\pi} \sum_{m=-\infty}^{\infty} W^r_m e^{-imx},
\]

and defining \( \{ , \} = -i[ , ] \), we may rewrite (1) as the following Poisson bracket algebra derived in [20]:

\[
\{ v_r(x), v_s(y) \} = \sum_{p=0}^{s} C^p_r (-1)^p v_{r+s-p}(y) \partial_x^p \delta(x-y) - \sum_{p=0}^{r} C^p_s (-1)^p v_{r+s-p}(x) \partial_x^p \delta(x-y)
\]

\[
= \sum_{p=0}^{r} \sum_{l=0}^{p} C^p_s C^l_s \partial_x^{p-l} v_{r+s-p}(y) \partial_x^l \delta(x-y) - \sum_{p=0}^{r} C^p_s (-1)^p v_{r+s-p}(x) \partial_x^p \delta(x-y).
\]

The $W_{1+\infty}$ algebra is an infinite-dimensional higher-spin algebra with generators having all integer conformal spins $s \geq 1$ [13]. It is worth noting that the Poisson bracket algebra (4) has been proved to be isomorphic to the $W_{1+\infty}$ algebra [20]. Thus we call (1) the $W_{1+\infty}$ algebra.

The operator Nambu 3-bracket was defined to be a sum of commutators of two operators multiplying the remaining one [23], i.e.,

\[
\]

where \([A, B] = AB - BA\).

Based on the operator Nambu 3-bracket (5) and the commutation relation (1), after a straightforward calculation we obtain the $W_{1+\infty}$ 3-algebra.
\[
\begin{align*}
[W_m, W_n, W_k] & = \left( \sum_{p=0}^{r} (-i)^p C_r^p n^p - \sum_{p=0}^{s} (-i)^p C_s^p m^p \right) \sum_{q=0}^{r+s-p} (-i)^q C_{r+s-p}^q q^q W^{r+s+h-p-q}_{m+n+k} \\
& + \left( \sum_{p=0}^{s} (-i)^p C_s^p k^p - \sum_{p=0}^{h} (-i)^p C_h^p n^p \right) \sum_{q=0}^{s+h-p} (-i)^q C_{s+h-p}^q m^q W^{s+h-p-q}_{m+n+k} \\
& + \left( \sum_{p=0}^{h} (-i)^p C_h^p m^p - \sum_{p=0}^{r} (-i)^p C_r^p k^p \right) \sum_{q=0}^{h+r-p} (-i)^q C_{h+r-p}^q n^q W^{h+r-p-q}_{m+n+k}. \quad (6)
\end{align*}
\]

Taking \( k = h = 0 \) in (6), we note that the Nambu 3-bracket \([W_m, W_n, W_0]\) can be regarded as the parametrized bracket relation \([W_m, W_n, W_0]\) which satisfies the Jacobi identity. This parametrized bracket relation gives rise to the \( W_{1+\infty} \) algebra (1)

\[
[W_m, W_n] W_0 \equiv [W_m, W_n, W_0] = [W_m, W_n].
\]

(7)

It is known that all three types of 3-brackets do satisfy a 3-on-3-on-3 multiple bracket relation, i.e., so-called BI condition [43,36]

\[
\epsilon_{i_1 i_2 \ldots i_6} \left[ [A, [B_{i_1}, B_{i_2}, B_{i_3}], B_{i_4}], B_{i_5}, B_{i_6} \right] = \epsilon_{i_1 i_2 \ldots i_6} \left[ [A, B_{i_1}, B_{i_2}], [B_{i_3}, B_{i_4}], B_{i_5}, B_{i_6} \right], \quad (8)
\]

where \( i_1, \ldots, i_6 \) are implicitly summed from 1 to 6. Thus the BI may be confirmed to hold for the ternary algebra (6).

It is worthwhile to mention another condition, i.e., Filippov condition or fundamental identity (FI) condition [44]

\[
[A, B, [C, D, E]] = [ [A, B, C], D, E ] + [ C, [A, B, D], E ] + [ C, D, [A, B, E] ] = 0. \quad (9)
\]

The FI condition (9) is not an operator identity. It holds only in special circumstances. For the \( W_{1+\infty} \) 3-algebra (6), we find that it does not satisfy the FI condition (9).

In order to achieve a better understanding of the \( W_{1+\infty} \) 3-algebra (6), we list the first few brackets as follows:

\[
\begin{align*}
[W^0_0, W^0_n, W^0_k] & = [W^1_0, W^1_n, W^1_k] = 0, \quad [W^1_m, W^0_n, W^0_k] = i(k - n)W^0_{m+n+k}, \\
[W^1_m, W^1_n, W^0_k] & = i(m - n)W^1_{m+n+k} + (km - kn)W^0_{m+n+k}, \\
[W^2_m, W^0_n, W^0_k] & = i(2k - 2n)W^1_{m+n+k} + (k^2 - n^2)W^0_{m+n+k}, \\
[W^3_m, W^0_n, W^0_k] & = i(3k - 3n)W^2_{m+n+k} + (3k^2 - 3n^2)W^1_{m+n+k} + i(n^3 - k^3)W^0_{m+n+k}, \\
[W^1_m, W^2_n, W^0_k] & = i(2m - n - k)W^2_{m+n+k} + (m^2 - k^2 + 2km - 2kn)W^1_{m+n+k} \\
& + i(nk^2 - 2km^2 - km^2)W^0_{m+n+k}, \\
[W^2_m, W^2_n, W^0_k] & = i(2m - 2n)W^3_{m+n+k} + (m^2 - n^2 + 2km - 4kn)W^2_{m+n+k} \\
& + i(2kn^2 - 2km^2 + 6nk^2 - 6mk^2)W^1_{m+n+k} \\
& + (k^2n^2 - k^2m^2 + 2nk^2 - 2km)W^0_{m+n+k},
\end{align*}
\]
\[
\begin{align*}
\left[W_m^1, W_n^1, W_k^2\right] &= i(n - m)W_{m+n+k}^3 + (n^2 - m^2 - kn - km)W_{m+n+k}^1 + i(km^2 - kn^2 - m^2n + mn^2)W_{m+n+k}^1, \\
\left[W_m^1, W_n^2, W_k^2\right] &= i(n - k)W_{m+n+k}^4 + (n^2 - k^2 + 2mn - 2mk)W_{m+n+k}^3 + i(5km^2 - 5nm^2 + k^2 - kn)W_{m+n+k}^2 + (k^2m^2 - m^2n^2 + 2km^2 - 2m^3n)W_{m+n+k}^1, \\
\left[W_m^2, W_n^2, W_k^2\right] &= 2(k^3n - k^3m + m^3k - m^3n + n^3m - n^3k)W_{m+n+k}^2 + 4i(k^2n - k^2m + m^2k - m^2n + n^2m - n^2k)W_{m+n+k}^1.
\end{align*}
\]

We observe that these brackets do not close except for the generators \(W_m^i, i = 0, 1\). The Nambu 3-bracket (5) with respect to the generators \(W_m^0\) and \(W_n^1\) indeed give the so-called the Virasoro–Witt 3-algebra \([35,37]\). However this 3-algebra does not satisfy the FI condition (9). The general Virasoro–Witt 3-algebra is given by Curtright et al. \([35]\)

\[
\begin{align*}
[R_p, R_q, R_k] &= 0, \\
[Q_p, R_q, R_k] &= (k - q)R_{k+p+q}, \\
[Q_p, Q_q, R_k] &= (p - q)(Q_{k+p+q} + zkR_{k+p+q}), \\
[Q_k, Q_m, Q_n] &= (k - m)(m - n)(k - n)R_{k+m+n},
\end{align*}
\]

where \(z\) is a parameter. They showed that the Virasoro–Witt 3-algebra (11) does not satisfy the FI condition (9) except when \(z = \pm 2i\).

3. \(W_{1+\infty}\) 3-algebra and KP hierarchy

The KP hierarchy is a paradigm of the hierarchies of integrable systems. It is defined as an infinite system of equations given in Lax form by

\[
\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, \cdots,
\]

(12)

under the compatible condition, (12) is equivalent to the Zakharov–Shabat (ZS) equation

\[
\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0,
\]

(13)

where the pseudo-differential operator \(L\) is given by

\[
L = \partial + \sum_{i=0}^{+\infty} v_i \partial^{-i-1},
\]

(14)

and \(B_n = (L)^n_+\) is the differential part of \(L^n\).

The Hamiltonians of the KP hierarchy are given by the general expression

\[
H_n = \text{Tr} \frac{L^{n+1}}{n+1} = \frac{1}{n+1} \int \text{Res} L^{n+1} dx.
\]

(15)

The first few members are
\[ H_0 = \int v_0 \, dx, \]
\[ H_1 = \int v_1 \, dx, \]
\[ H_2 = \int (v_2 + v_0^2) \, dx, \]
\[ H_3 = \int (v_3 + 3v_0v_1) \, dx, \]
\[ H_4 = \int (v_4 + 4v_0v_2 + 2v_1^2 + 2v_0^3 + 2v_0v_{1,x} + v_0v_{0,xx}) \, dx, \]
\[ H_5 = \int (v_5 + 5v_0v_3 + 5v_1v_2 + 10v_1v_0^2 - 5v_2v_{0,x} + 4v_1v_{0,xx} - v_0v_{0,xxx} + v_0v_{1,xx}) \, dx, \]
\[ \vdots, \quad (16) \]

where the subscript \(x\) denotes the derivative with respect to the variable \(x\).

It is known that the Poisson bracket (4) leads to the Hamiltonian description of the KP flow equations [20,21]. Substituting the Hamiltonians (15) into the Poisson evolution equation
\[
\frac{\partial v_i(t,x)}{\partial t_n} = \{v_i, H_n\}, \quad (17)
\]
and using the Poisson bracket (4), we may derive the KP hierarchy.

In order to establish the relation between the \(W_{1+\infty}\) 3-algebra and integrable systems, let us introduce the FTF as follows:
\[
v_r(x) = \frac{1 + i}{2\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} W_re^{-imx}, \quad (18)
\]
and define the classical Nambu bracket \(\{, , \} = -i[ , , ]\). Then we can rewrite the \(W_{1+\infty}\) 3-algebra (6) as the following Nambu 3-bracket relation:
\[
\{v_r(x), v_s(y), v_h(z)\} = \sum_{p=0}^{r} \sum_{q=0}^{r+s-p} (-1)^{p+q} C_r^p C_{r+s-p}^q v_{r+s+h-p-q}(x) \partial_x^p \delta(x-y) \partial_y^q \delta(y-z)
\]
\[ - \sum_{p=0}^{s} \sum_{q=0}^{r+s-p} (-1)^{p+q} C_s^p C_{r+s-p}^q v_{r+s+h-p-q}(y) \partial_y^p \delta(y-x) \partial_x^q \delta(x-z) \]
\[ + \sum_{p=0}^{s} \sum_{q=0}^{s+h-p} (-1)^{p+q} C_s^p C_{s+h-p}^q v_{r+s+h-p-q}(y) \partial_y^p \delta(y-z) \partial_x^q \delta(y-x) \]
\[ - \sum_{p=0}^{h} \sum_{q=0}^{s+h-p} (-1)^{p+q} C_h^p C_{s+h-p}^q v_{r+s+h-p-q}(z) \partial_z^p \delta(z-y) \partial_x^q \delta(z-x) \]
\[ + \sum_{p=0}^{h} \sum_{q=0}^{h+r-p} (-1)^{p+q} C_h^p C_{h+r-p}^q v_{r+s+h-p-q}(z) \partial_z^p \delta(z-x) \partial_y^q \delta(z-y) \]
\[- \sum_{p=0}^{r} \sum_{q=0}^{b+r-p} (-1)^{p+q} C_r^p C_q^q v_{r+s+h-p-q}(x) \partial_x^p \delta(x-z) \partial_x^q \delta(x-y). \] (19)

Not as the Poisson evolution equation, the N-P evolution equation involves two Hamiltonians. It is given by

\[ \frac{\partial v_i}{\partial t_{mn}} = \{v_i, H_m, H_n\}. \] (20)

In terms of the Nambu 3-bracket relation (19), we have

\[ \{v_r, H_0, v_3\} = \sum_{p=0}^{s} C_s^p (-1)^p v_{r+s-p}(y) \partial_y^p \delta(x-y) \]

\[ - \sum_{p=0}^{r} C_r^p (-1)^p v_{r+s-p}(x) \partial_x^p \delta(x-y). \] (21)

Comparing (21) with (4), it gives the equivalent relation

\[ \{v_r, H_0, v_3\} \cong \{v_r, v_3\}, \] (22)

where the FTFs in Nambu 3-bracket and Poisson bracket are given by (18) and (3), respectively. This equivalent relation is indeed corresponding to the relation (7) with \{ , , \} = -i[ , , ] and \{ , , \} = -i[ , , ]. Due to (7), it can be verified that the triple Hamiltonians \( (H_0, H_n, H_m) \) satisfy \( \{H_0, H_n, H_m\} = 0. \) But for the triple Hamiltonians \( (H_l, H_n, H_m), l, n, m \in Z_+ \), by means of (19), we note that they are not in involution with the Nambu 3-bracket structure, i.e., \( \{H_l, H_n, H_m\} \neq 0. \)

For the Hamiltonian pairs \( (H_0, H_n) \), by means of (22), it is not difficult to show that

\[ \{v_i, H_0, H_n\} \cong \{v_i, H_n\}. \] (23)

Thus based on (19) and (23), it is natural to derive the KP hierarchy from the N-P evolution equation (20) with \( H_m = H_0. \) For later convenience, we list the first few members of the hierarchy here.

- To give the KP equation, let us consider the following equations:

\[ \frac{\partial v_0}{\partial t_{03}} = \{v_0, H_0, H_3\} = 3v_{2,x} + 3v_{1,xx} + v_{0,xxx} + 6v_0 v_{0,x}, \] (24a)

\[ \frac{\partial v_0}{\partial t_{02}} = \{v_0, H_0, H_2\} = 2v_{1,x} + v_{0,xx}, \] (24b)

\[ \frac{\partial v_1}{\partial t_{02}} = \{v_1, H_0, H_2\} = 2v_{2,x} + v_{1,xxx} + 2v_0 v_{0,x}. \] (24c)

From (24b) and (24c), we obtain

\[ v_1 = \frac{1}{2} (\partial_x^{-1} v_{0,t_02} - v_{0,x}), \]

\[ v_2 = \frac{1}{4} (\partial_x^{-2} v_{0,t_02t_02} - 2v_{0,t_02} + v_{0,xx} - 2v_0^2). \] (25)

Substituting (25) into (24a), we obtain the usual KP equation

\[ 3v_{0,yy} = (4v_{0,t} - v_{0,xxx} - 12v_0 v_{0,x})x, \] (26)
where \( y = t_{02} \) and \( t = t_{03} \). The KP hierarchy can also be obtained from (13). It is easy to derive the KP equation (26) from (13) by taking \( m = 2 \) and \( n = 3 \).

- To give the integrable equation corresponding to the case of \( m = 2 \) and \( n = 4 \) in (13), we consider the following equations:

\[
\begin{align*}
\frac{\partial v_0}{\partial t_{04}} &= \{ v_0, H_0, H_4 \} = 4v_{3,x} + 6v_{2,xx} + 4v_{1,xxx} + v_{0,xxxx} \\
&+ 6(v_{0,x})^2 + 6v_0v_{0,xx} + 12v_0v_{1,x} + 12v_1v_{0,x}, \\
\frac{\partial v_0}{\partial t_{02}} &= \{ v_0, H_0, H_2 \} = 2v_{1,x} + v_{0,xx}, \\
\frac{\partial v_1}{\partial t_{02}} &= \{ v_1, H_0, H_2 \} = 2v_{2,x} + v_{1,xx} + 2v_0v_{0,x}, \\
\frac{\partial v_2}{\partial t_{02}} &= \{ v_2, H_0, H_2 \} = 2v_{3,x} + v_{2,xx} - 2v_0v_{0,xx} + 4v_1v_{0,x}.
\end{align*}
\]  

From (27b), (27c) and (27d) we may give the expressions of \( v_1, v_2 \) and \( v_3 \) with respect to \( v_0 \), respectively. Then substituting them into (27a), we obtain

\[
\frac{\partial^3 v_0}{\partial y \partial x^2} - 2 \frac{\partial v_0}{\partial t} + \frac{\partial^3 v_0}{\partial y^3} + 4v_0 \frac{\partial^2 v_0}{\partial x \partial y} + 8v_0 \frac{\partial v_0}{\partial y} = 0,
\]

where \( y = t_{02} \) and \( t = t_{04} \).

- The integrable equation corresponding to the case of \( m = 2 \) and \( n = 5 \) in (13) can be derived from the following equations:

\[
\begin{align*}
\frac{\partial v_0}{\partial t_{05}} &= \{ v_0, H_0, H_5 \} = 5v_{4,x} + 10v_{3,xx} + 10v_{2,xxx} + 5v_{1,xxxx} + v_{0,xxxxx} + 20v_2v_{0,x} \\
&+ 20v_0v_{2,xx} + 20v_1v_{1,xx} + 30v_0v_{1,xx} + 20v_0v_{1,xx} + 10v_1v_{0,xx} \\
&+ 20v_0v_{0,xx} + 10v_0v_{0,xxx} + 30v_0^2v_{0,x}, \\
\frac{\partial v_0}{\partial t_{02}} &= \{ v_0, H_0, H_2 \} = 2v_{1,x} + v_{0,xx}, \\
\frac{\partial v_1}{\partial t_{02}} &= \{ v_1, H_0, H_2 \} = 2v_{2,x} + v_{1,xx} + 2v_0v_{0,x}, \\
\frac{\partial v_2}{\partial t_{02}} &= \{ v_2, H_0, H_2 \} = 2v_{3,x} + v_{2,xx} - 2v_0v_{0,xx} + 4v_1v_{0,x}, \\
\frac{\partial v_3}{\partial t_{02}} &= \{ v_3, H_0, H_2 \} = 2v_{4,x} + v_{3,xx} + 6v_2v_{0,x} - 6v_1v_{0,xx} + 2v_0v_{0,xxx}.
\end{align*}
\]  

From the last four equations of (29), we may give the expressions of \( v_i, i = 1, 2, 3, 4 \) with respect to \( v_0 \). Then substituting them into the first equation of (29), we obtain

\[
\begin{align*}
\frac{\partial v_0}{\partial t} &= \frac{5}{16} v_{0,yyyy} + \frac{5}{8} v_{0,yyy} + \frac{5}{2} v_{0,y} \partial_x^{-1} v_{0,y} + \frac{5}{2} v_{0,x} v_{0,xx} \\
&+ \frac{5}{4} v_0 v_{0,xxx} + \frac{1}{16} v_{0,xxxx} + \frac{5}{2} v_0 \partial_x^{-1} v_{0,yy} + \frac{5}{4} \partial_x^{-1} (v_{0,y})^2 \\
&+ \frac{5}{4} \partial_x^{-1} (v_0 v_{0,yy}) + \frac{5}{4} v_{0,x} \partial_x^{-2} v_{0,yy} + \frac{15}{2} v_0^2 v_{0,x},
\end{align*}
\]  

where \( y = t_{02} \) and \( t = t_{05} \).
4. KP equation, KdV equation and $W_{1+\infty}$ 3-algebra

One has already known that the $W_{1+\infty}$ algebra is related to the KP hierarchy. Since the $W_{1+\infty}$ 3-algebra with a fixed generator $W_0^0$ in the operator Nambu 3-bracket (6) recovers the $W_{1+\infty}$ algebra, it is natural to derive the KP hierarchy from the $W_{1+\infty}$ 3-algebra. To explore the deep connection between the $W_{1+\infty}$ 3-algebra and the integrable equations, we need to deal with new case studies. Hereafter we focus on the general case of the $W_{1+\infty}$ 3-algebra. Due to the N-P evolution equation involving two Hamiltonians, this may hide some still unknown relations between these Hamiltonians. Thus it would be quite interesting to understand these Hamiltonians from the N-P evolution equation.

To start with let us assign the weights as follows:

$$[x] = -1, \quad [\partial_x] = 1, \quad [v_i] = i + 2, \quad [\partial t_{mn}] = m + n.$$  \hspace{1cm} (31)

**Theorem.** For any Hamiltonian pair $(H_m, H_n)$ of the KP hierarchy, $0 \leq m < n$, from the N-P evolution equations

$$\frac{\partial v_0}{\partial t_{mn}} = \{v_0, H_m, H_n\}, \hspace{1cm} (32a)$$

$$\frac{\partial v_i}{\partial t_{02}} = \{v_i, H_0, H_2\}, \hspace{1cm} (32b)$$

where $i = 0, 1, \cdots, s - 2$ and $s = m + n \geq 1$, it gives the evolution equation with respect to $v_0$. The weight of equation is $s + 2$.

**Proof.** By weight counting, it is easy to see that the pseudo-differential operator $L$ (14) is homogeneous of weight 1, from which we conclude that the weight of $L^{n+1}$ is $[L^{n+1}] = n + 1$. For any Hamiltonian $H_n$ (15), due to the weight $[\text{Res} \, L^{n+1}] = n + 2$, it implies that there are only finite $v_i$, $i = 0, 1, 2, \cdots, n$ being permitted to emerge in $H_n$. Note that for these finite $v_i$, $v_n$ has the highest weight $n + 2$. Thus to determine how many $v_i$ appearing in (32a), let us consider $\iint (v_0(x), v_m(y), v_n(z)) dy dz$. By a direct calculation, we have

$$\iint \{v_0(x), v_m(y), v_n(z)\} dy dz = \sum_{p=1}^{n} C_n^p \partial_x^p v_{s-p} - \sum_{p=1}^{m} C_m^p \partial_x^p v_{s-p}.$$  \hspace{1cm} (33)

In terms of (33), we find that there are only finite $v_i$, $i = 0, 1, 2, \cdots, s - 1$ being permitted to emerge in the evolution equation of $v_0$ (32a).

After a straightforward calculation, (32b) gives its final form

$$\frac{\partial v_i}{\partial t_{02}} = \{v_i, H_0, H_2\} = 2v_{i+1,x} + v_{i,xx} + 2v_i v_0 - 2 \sum_{p=0}^{i} (-1)^p C_i^p v_{i-p} \partial_x^p v_0, \hspace{0.5cm} i = 0, 1, \cdots, s - 2.$$  \hspace{1cm} (34)

Thus we have a recursion relation from (34)

$$v_{i+1}(x) = \frac{1}{2} \partial_x^{-1} \left( v_{i,t_02} - v_{i,xx} - 2v_i v_0 + 2 \sum_{p=0}^{i} (-1)^p C_i^p v_{i-p} \partial_x^p v_0 \right). \hspace{1cm} (35)$$
By means of the recursion relation (35), we may express \( v_i, i = 1, 2, \ldots, s - 1 \) as the function of \( v_0 \). Then substituting these \( v_i \) into (32a), we obtain the evolution equation with respect to \( v_0 \). Since the Nambu 3-bracket relation (19) and the Hamiltonian (15) are homogeneous of weight, in terms of (33), it immediately gives the weight \( s + 2 \) of the evolution equation with respect to \( v_0 \). \( \square \)

According to above theorem, we know that there are \( \frac{m+n+1}{2} \) different Hamiltonian pairs \( (H_{m+n+1-i}, H_{m+n+1+i}) \), \( i = 0, 1, \ldots, \frac{m+n-1}{2} \) such that (32) gives the evolution equations of \( v_0 \) with the same weight \( m + n + 2 \).

Now let us turn our attention to the KP equation. Taking the Hamiltonian pairs \( (H_m, H_n) \) to be \( (H_0, H_3) \) and \( (H_1, H_2) \), respectively, we may derive the evolution equations of \( v_0 \) with the same weight from (32). For the case of \( (H_0, H_3) \), we have derived the KP equation from (24).

Let us replace the Hamiltonian pair \( (H_0, H_3) \) in (24a) by \( (H_1, H_2) \), thus we have

\[
\frac{\partial v_0}{\partial t_{12}} = \{v_0, H_1, H_2\} = v_{2,x} + v_{1,xx} - 6v_0v_{0,x}.
\] (36)

Substituting (25) into (36), we obtain

\[
v_{0,yy} = (-4v_{0,t} + v_{0,xxx} + 28v_0v_{0,x})_x,
\] (37)

where \( y = t_{02} \) and \( t = t_{12} \). Under the scaling transformations \( y \rightarrow \frac{1}{\sqrt[3]{3}}y \) and \( v_0 \rightarrow \frac{3}{\sqrt[3]{3}}v_0 \), (37) becomes the usual KP equation (26). What is surprising here is that we derive the KP equation from the N-P evolution equation with the different Hamiltonian pairs. This is an intriguing nontrivial result, since there is not such kind of remarkable property in the framework of Poisson evolution equation. It is worth to emphasize that the \( W_{1+\infty} \) 3-algebra plays a pivotal role in deriving the KP equation here. Due to the N-P evolution equation involving two Hamiltonians, our analysis indicates that there is the deep relationship between the Hamiltonians of KP hierarchy.

Comparing (36) with (24a), we note that the coefficients of \( v_{1,xx} \) and \( v_{2,x} \) in (24a) are three times than those of (36). To explore more relationships between the Hamiltonian pairs \( (H_0, H_3) \) and \( (H_1, H_2) \), we consider the following evolution equation:

\[
\frac{\partial v_0}{\partial t_{03}} - 3\frac{\partial v_0}{\partial t_{12}} = \{v_0, H_0, H_3\} - 3\{v_0, H_1, H_2\}.
\] (38)

Substituting the expressions of (24a) and (36) into (38), we obtain

\[
-2\frac{\partial v_0}{\partial t} = v_{0,xxx} + 24v_0v_{0,x},
\] (39)

where \( t = t_{03} = t_{12} \). Under the scaling transformations \( t \rightarrow -2t \) and \( v_0 \rightarrow \frac{1}{4}v_0 \), (39) becomes the well-known KdV equation

\[
\frac{\partial v_0}{\partial t} = v_{0,xxx} + 6v_0v_{0,x}.
\] (40)

The connection between the KdV equation and the Virasoro algebra was first pointed out by Gervais and Neveu [1,2]. The KdV equation is a bi-hamiltonian system. It makes contact with the Virasoro algebra via the second hamiltonian structure. We have derived the KP equation from the N-P evolution equation with the different Hamiltonian pairs. It is well-known that the KP hierarchy is introduced as a generalization of KdV hierarchies. Under the low dimensional reduction, the KP equation reduces to the KdV equation. As one of intriguing results in this paper,
we find that the KdV equation may be obtained from the N-P evolution equation (38) without taking the low dimensional reduction. More precisely, we may directly derive the KdV equation from (38) by means of the \( W_{1+\infty} \) 3-algebra, where \( H_i, i = 0, 1, 2, 3 \) are the Hamiltonians of KP hierarchy. It should also be stressed that without the condition of low dimensional reduction, this result cannot be achieved from the Poisson evolution equation with the Hamiltonians of KP hierarchy. Therefore our investigation also provides new insight into the KdV equation.

Let us turn to consider the case of the Hamiltonian pairs \( (H_m, H_n) \) with \( m + n = 4 \). For the case of \( (H_0, H_4) \), we have derived the integrable equation (28) from (27). If the first equation in (27) is replaced by

\[
\frac{\partial v_0}{\partial t_{13}} = \{v_0, H_1, H_3\} = 2v_{3,x} + 3v_{2,xx} + v_{1,xxx} - 6v_{1,0,x} - 6v_0v_{1,x} - 3v_0v_{0,xx} - 3(v_{0,x})^2,
\]

then from (41) and the last three equations in (27), we obtain

\[
\frac{\partial^3 v_0}{\partial y \partial x^2} + 4\frac{\partial v_0}{\partial t} - \frac{\partial^2}{\partial x^2} \frac{\partial^3 v_0}{\partial y^3} + 20v_{0,xx} \frac{\partial^2 v_0}{\partial y} + 16v_0 \frac{\partial v_0}{\partial y} = 0,
\]

where \( y = t_{02} \) and \( t = t_{13} \). Its single soliton solution is

\[
v_0 = \frac{k^2}{12} \text{sech}^2 \frac{\xi}{2},
\]

where

\[
\xi = k(\omega t + x + y) + c,
\]

in which \( \omega = \frac{1}{4}(-k^2 + 1) \), \( k \) and \( c \) are the constants. Although (42) has the single soliton solution, by performing a Painlevé analysis of this equation, we note that it does not pass the test. Hence (42) does not have the Painlevé integrable property. In spite of this negative result it is instructive to pursue the analysis of (42). Taking the reduction \( x = y \), it is interesting to note that (42) may reduce to the KdV equation (40) with the appropriate scaling transformations.

Under the low dimensional reduction, we may also derive the KdV equation from the following N-P evolution equations with respect to \( v_0 \):

\[
\frac{\partial v_0}{\partial t_{04}} - 2\frac{\partial v_0}{\partial t_{13}} = \{v_0, H_0, H_4\} - 2\{v_0, H_1, H_3\} = 2v_{1,xxx} + v_{0,xxxx} + 12(v_{0,x})^2 + 12v_0v_{0,xx} + 24v_0v_{1,x} + 24v_1v_{0,x},
\]

(45a)

\[
\frac{\partial v_0}{\partial t_{02}} = \{v_0, H_0, H_2\} = 2v_{1,x} + v_{0,xx}.
\]

(45b)

Comparing (45a) with (27a), we observe that there are no such fields \( v_2 \) and \( v_3 \) in (45a). Thus not as the case of (27), we do not need the evolution equations of \( v_1 \) and \( v_2 \) here. Eliminating \( v_1 \) via the evolution equations of \( v_0 \) (45b), we can rewrite (45a) as

\[
-\frac{\partial v_0}{\partial t} = \frac{\partial^3 v_0}{\partial y \partial x^2} + 12v_0 \frac{\partial v_0}{\partial y} + 12v_{0,x} \left( \partial^{-1} \frac{\partial v_0}{\partial y} \right),
\]

(46)
where \( t = t_{04} = t_{13} \) and \( y = t_{02} \). Its single soliton solution is

\[
v_0 = \frac{k^2}{8} \text{sech}^2 \frac{\xi}{2},
\]

(47)

where \( \xi = k(-k^2t + x + y) + c \), \( k \) and \( c \) are the constants. By performing a Painlevé analysis of this equation, we note that it does not pass the test. Taking the reduction \( x = y \) and the scaling transformations \( t \to -t \) and \( v_0 \to \frac{1}{4}v_0 \), (46) reduces to the KdV equation (40).

5. \( W_{1+\infty} \) 3-algebra and dispersionless KdV hierarchy

The dispersionless KdV hierarchy is a well studied integrable system \([45,46]\), which has many applications in physics. It is given by

\[
\frac{\partial u(t, x)}{\partial t} = A_{n-1}u^{n-1}u_x = \begin{cases} 
  u_x, & n = 1, \\
  \frac{3}{2}u_x, & n = 2, \\
  \frac{15}{8}u_x, & n = 3, \\
  \frac{35}{16}u_x, & n = 4, \\
  \vdots 
\end{cases}
\]

(48)

where \( A_n \) is given by

\[
A_n = \frac{(2n + 1)!!}{(2n)!!}.
\]

(49)

Let us start from (32) to investigate the relation between the \( W_{1+\infty} \) 3-algebra and the dispersionless KdV hierarchy.

When \( s = 1 \), from (32a), it is easy to obtain

\[
\frac{\partial v_0}{\partial t_{01}} = \{v_0, H_0, H_1\} = v_{0,x}.
\]

(50)

This is the first equation in (48).

When \( s = m + n \) is odd and \( s \geq 3 \), let us choose the following \( \frac{s+1}{2} \) different Hamiltonian pairs \((H_{\frac{s-1}{2} - i}, H_{\frac{s+1}{2} + i})\), \( i = 0, 1, \cdots, \frac{s-1}{2} \). Based on the theorem in the previous section, we can derive \( \frac{s+1}{2} \) different evolution equations with respect to \( v_0 \) with odd weight \( s + 2 \). Under the low dimensional reduction \( t_{02} = 0 \) and the dispersionless limitation, i.e., \( \partial_{t_{0n}} \to \epsilon \partial_{t_{0n}} \) and \( \partial_x \to \epsilon \partial_x \) with \( \epsilon \to 0 \), we note that all higher derivation terms of \( v_0 \) in the evolution equations are vanishing. Thus we have the following equation with the weight \( s + 2 \):

\[
\frac{\partial v_0}{\partial t_{mn}} = \alpha_{mn}v_0^{\frac{s-1}{2}}v_{0,x},
\]

(51)

where \( \alpha_{mn} \) is a constant.

When \( \alpha_{mn} \neq 0 \), taking the scaling transformation \( t_{mn} \to \alpha_{mn}/A_{\frac{s-1}{2} - i} \), we see that (51) leads to the dispersionless KdV hierarchy (48). However how to prove \( \alpha_{mn} \neq 0 \) is the most challenging problem. A more detailed description of the difficulty will be provided later. Based on above analysis, we present a conjecture as follows:

**Conjecture.** For any Hamiltonian pair \((H_m, H_n)\) of the KP hierarchy, \( 0 \leq m < n \), when \( s = m + n \) is odd, we may choose \( \frac{s+1}{2} \) different Hamiltonian pairs \((H_m, H_n)\) for (32a) such that under
the low dimensional reduction with the dispersionless limit, (32) leads to the dispersionless KdV hierarchy.

In order to achieve a better understanding of above conjecture, as an example we consider the case of the Hamiltonian pairs \((H_m, H_n)\) with \(m + n = 5\). Taking the Hamiltonian pairs \((H_0, H_5)\), \((H_1, H_4)\) and \((H_2, H_3)\), respectively, according to the theorem in the previous section, we may derive the evolution equations of \(v_0\) with the same weight from (32).

For the case of the Hamiltonian pair \((H_0, H_5)\), we have given the integrable equation (30) from (29). Let us now replace the first equation in (29) by

\[
\frac{\partial v_0}{\partial t_{14}} = \{v_0, H_1, H_4\} \\
= 3v_{4,x} + 6v_{3,xx} + 4v_{2,xxx} + v_{1,xxxx} - 4v_2v_{0,x} - 4v_0v_{2,x} - 4v_1v_{1,x} \\
- 6v_{0,x}v_{1,x} - 4v_0v_{1,xx} - 2v_{1,x}v_{0,xx} - 12v_{0,x}v_{0,xx} - 6v_0v_{0,xxx} - 30v_0^2v_{0,x}. 
\]

Then from (52) and the last four equations in (29), we obtain

\[
\frac{\partial v_0}{\partial t} = \frac{3}{16}\partial_x^{-3}v_{0,yyyy} - \frac{1}{8}v_{0,xyy} - \frac{5}{2}v_{0,y}\partial_x^{-1}v_{0,y} - \frac{17}{2}v_{0,x}v_{0,xx} \\
- \frac{21}{4}v_{0,v_{0,xxx}} - \frac{1}{16}v_{0,v_{0,xxxxxx}} - \frac{5}{2}v_{0}\partial_x^{-1}v_{0,yy} + \frac{3}{4}\partial_x^{-1}(v_{0,y})^2 \\
+ \frac{3}{4}\partial_x^{-1}(v_{0,v_{0,yy}}) - \frac{13}{4}v_{0,x}\partial_x^{-2}v_{0,yy} - \frac{39}{2}v_0^2v_{0,x},
\]

where \(y = t_{02}\) and \(t = t_{14}\). Its single soliton solution is

\[
v_0 = \frac{1}{78}(3790 - 110\sqrt{1249})k^2 + 531 - 15\sqrt{1249} - (11370 - 330\sqrt{1249})k^2 \sech^2 \xi, \]

where \(\xi\) is given by (44) with \(\omega = -\frac{1}{624}(14144104\sqrt{1249} - 4998879128)k^4 + 134406117 - 3805749\sqrt{1249}

To test the integrability of this equation, we perform the Painlevé analysis of (53). Unfortunately it does not pass the test.

If the first equation in (29) is replaced by

\[
\frac{\partial v_0}{\partial t_{23}} = \{v_0, H_2, H_3\} \\
= v_{4,x} + 2v_{3,xx} + v_{2,xxx} + 3v_2v_{0,x} + 3v_0v_{2,x} - 18v_{1,xx} \\
- 6v_{0,x}v_{1,x} + 3v_0v_{1,xx} - 9v_1v_{0,xx} - 3v_{0,x}v_{0,xx} + 3v_0v_{0,xxx} + 24v_0^2v_{0,x},
\]

we can derive the following equation:

\[
\frac{\partial v_0}{\partial t} = \frac{1}{16}\partial_x^{-3}v_{0,yyyy} - \frac{1}{8}v_{0,xyy} - 5v_{0,y}\partial_x^{-1}v_{0,y} + \frac{9}{4}v_{0,x}v_{0,xx} + \frac{5}{2}v_{0,v_{0,xxx}} \\
+ \frac{1}{16}v_{0,v_{0,xxxx}} - \frac{1}{4}v_0\partial_x^{-1}v_{0,yy} + \frac{1}{4}\partial_x^{-1}(v_{0,y})^2 + \frac{1}{4}\partial_x^{-1}(v_{0,v_{0,yy}}) + 21v_0^2v_{0,x},
\]

where \(y = t_{02}\) and \(t = t_{23}\). By applying the Painlevé analysis to (56), we find that it does not pass the test. As the case of (53), (56) has the single soliton solution as follows:
\[
v_0 = \frac{31}{140} - \frac{5}{14} k^2 + \frac{15}{14} k^2 \text{sech}^2 \xi,
\]

(57)

where \( \xi \) is given by (44) with \( \omega = \frac{3}{28} k^4 + \frac{67}{700} \).

Under the reduction \( y = 0 \), (30), (53) and (56) reduce to the following equations:

\[
\begin{align*}
\frac{\partial v_0}{\partial t} &= \frac{1}{16} v_{0,xxxxx} + \frac{5}{4} v_0 v_{0,xxx} + \frac{5}{2} v_{0,x} v_{0,xx} + \frac{15}{2} v_0^2 v_{0,x}, \\
\frac{\partial v_0}{\partial t} &= -\frac{1}{16} v_{0,xxxxx} - \frac{21}{4} v_0 v_{0,xxx} - \frac{17}{2} v_{0,x} v_{0,xx} - \frac{39}{2} v_0^2 v_{0,x}
\end{align*}
\]

(58)

and

\[
\frac{\partial v_0}{\partial t} = \frac{1}{16} v_{0,xxxxx} + \frac{5}{2} v_0 v_{0,xxx} + \frac{9}{4} v_{0,x} v_{0,xx} + 21 v_0^2 v_{0,x},
\]

(59)

respectively.

Before discussing the dispersionless limit of (58), (59) and (60), let us pause here to look at the property of these equations. It should be noted that (58) is the completely integrable Lax’s fifth order KdV equation [47], but (59) and (60) are not integrable. The fifth order KdV equations describe motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice and has wide applications in quantum mechanics and nonlinear optics. Thus a great deal of research work has been invested for the study of the fifth order KdV equations. Recently the following general fifth order KdV equation has been investigated [48]:

\[
\frac{\partial v}{\partial t} = \omega v_{xxxxx} + \alpha v_{xxx} + \beta v_x v_{xx} + \gamma v^2 v_x,
\]

(61)

where \( \alpha, \beta, \gamma \) and \( \omega \) are the arbitrary real parameters. By using the exp function method, the solutions of (61) have also been presented. It can be very easily seen that (59) and (60) are the special cases of (61).

Let us now turn our attention to the case of the dispersionless limit. Comparing the coefficients of \( v_0^2 v_{0,x} \) in (58), (59) and (60), we observe that they are completely different. This is due to the following fact. Under the low dimensional reduction, there are many different terms in the first equation of (29), (52) and (55), such as \( v_{4,x}, v_2 v_{0,x} \) and \( v_0 v_{2,x} \), which may produce the term \( v_0^2 v_{0,x} \) besides itself. This is the reason that makes it difficult to prove \( a_{mn} \neq 0 \) in (51) due to the contribution of too many terms.

Taking the dispersionless limit \( \partial_t \to \epsilon \partial_t \) and \( \partial_x \to \epsilon \partial_x \) with \( \epsilon \to 0 \), it is easy to see that (58), (59) and (60) reduce to the following dispersionless KdV equation with the appropriate scaling transformations:

\[
v_{0,t} = \frac{15}{8} v_0^2 v_{0,x}.
\]

(62)

We also check the cases of the Hamiltonian pairs \((H_m, H_n)\) with \( m + n = 7, 9 \) and confirm that the conjecture holds for these cases. From above examples, it is of interest to note that we derive the dispersionless KdV equations from the N-P evolution equation with the different Hamiltonian pairs. We call this intrinsic relation between the Hamiltonian pairs “degeneration”. Such a degeneration does not exist in the framework of Poisson evolution equation.
6. $W_{1+\infty}$ 3-algebra and non-linear Schrödinger equations

6.1. Non-linear Schrödinger equation

The non-linear Schrödinger equation is given by

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2 \psi = 0,$$

(63)

where $\psi$ is a complex bosonic field.

The non-linear Schrödinger equation possesses an infinite set of conserved quantities $\hat{H}_n$, given by the formula

$$\hat{H}_n = -(-i)^n \int \psi \Gamma_n (\psi, \psi^*) dx,$$

(64)

where

$$\Gamma_0 = \psi^*, \quad \Gamma_1 = \Gamma_{0,x}, \quad \Gamma_{n+1} = \Gamma_{n,x} + \psi \sum_{p=0}^{n-1} \Gamma_p \Gamma_{n-1-p}, \quad n \geq 1.$$

(65)

The first few such Hamiltonians are

$$\hat{H}_0 = -\int \psi^* \psi dx,$$

$$\hat{H}_1 = -i \int \psi^* \psi_x dx,$$

$$\hat{H}_2 = \int (\psi^* \psi_{xx} + \psi^* \psi^2_{x^2}) dx,$$

$$\hat{H}_3 = i \int (\psi^* \psi_{xxx} + 3 \psi^* \psi^2 \psi_x) dx,$$

$$\hat{H}_4 = -\int (\psi^* \psi_{xxxx} + 4 \psi^* \psi_{x} \psi_{xx} + 2 \psi^* \psi^2 \psi^2_{x} - 2 \psi^* \psi (\psi^* \psi_x)_x$$

$$+ \psi^* \psi (\psi^* \psi)_{xx} + 2 (\psi^* \psi)^3) dx,$$

\ldots$$

(66)

The non-linear Schrödinger equation (63) can be obtained from

$$\frac{\partial \psi}{\partial t} = \{ \psi(x), \hat{H}_2 \},$$

(67)

where the Poisson brackets of the complex bosonic field are given by [49]

$$\{ \psi^*(x), \psi(y) \} = -i \delta(x - y), \quad \{ \psi(x), \psi(y) \} = \{ \psi^*(x), \psi^*(y) \} = 0.$$

(68)

It is known that there is a realization of $W_{1+\infty}$ algebra in terms of a complex bosonic field with Poisson brackets (68). Taking

$$v_r(x) = i(-1)^r \psi^*(x) \psi^{(r)}(x),$$

(69)
where $\psi^{(r)}(x) = \frac{d^r \psi(x)}{dx^r}$, by means of (68), it is easy to verify that the generators (69) satisfy (4). Due to this kind of realization of $W_{1+\infty}$ algebra, the connection between the non-linear Schrödinger equation and the KP hierarchy has been established by Freeman and West [49].

For the case of $W_{1+\infty}$ 3-algebra (19), we find that its realization can be achieved in terms of a complex bosonic field with Nambu 3-brackets given by

$$\{\psi^{(r)}(x), \psi^*(y)\psi^{(s)}(y), \psi^*(z)\} = -i\partial_x^r \delta(x - y)\partial_y^s \delta(y - z),$$

$$\{\psi^*(x), \psi^*(y)\psi^{(s)}(y), \psi^*(z)\} = \{\psi^{(r)}(x), \psi^*(y)\psi^{(s)}(y), \psi^{(h)}(z)\} = 0.$$  (70)

The proof is straightforward. One can confirm the $W_{1+\infty}$ 3-algebra (19) by substituting $v_r(x) = \frac{\sqrt{2}}{2}(1 + i)(-1)^r \psi^*(x)\psi^{(r)}(x)$ and using the Nambu 3-brackets (70).

As the cases of (22) and (23), in terms of the Nambu 3-bracket relations (70), we have also the similar equivalence relations

$$\{\psi, \hat{H}_0, \psi^*\} \cong \{\psi, \psi^*\}, \quad \{\psi, \hat{H}_0, \hat{H}_n\} \cong \{\psi, \hat{H}_n\}. \quad (71)$$

Thus we may also derive the non-linear Schrödinger equation from the N-P evolution equation

$$\frac{\partial \psi}{\partial t} = \{\psi, \hat{H}_0, \hat{H}_2\} = i\left(\frac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2 \psi\right). \quad (72)$$

Based on the $W_{1+\infty}$ 3-algebra (19), we have derived the KP and KdV equations from the N-P evolution equation with the first few Hamiltonian pairs of the KP hierarchy. Note that the $W_{1+\infty}$ 3-algebra (19) can be realized in terms of the complex bosonic field with the Nambu 3-bracket relations (70). Based on (70), here we should like to draw attention to investigate the N-P evolution equations with the first few Hamiltonian pairs of (66). A list of the corresponding nonlinear evolution equations is given as follows:

- **Hamiltonian pair $(\hat{H}_0, \hat{H}_3)$**

  $$\frac{\partial \psi}{\partial t} = \{\psi, \hat{H}_0, \hat{H}_3\} = -\left(\frac{\partial^3 \psi}{\partial x^3} + 6|\psi|^2 \frac{\partial \psi}{\partial x}\right). \quad (73)$$

  This equation is the so-called integrable complex mKdV equation.

- **Hamiltonian pair $(\hat{H}_1, \hat{H}_2)$**

  $$\frac{\partial \psi}{\partial t} = \{\psi, \hat{H}_1, \hat{H}_2\} = -\left(\frac{\partial^3 \psi}{\partial x^3} + 2\frac{\partial |\psi|^2 \psi}{\partial x}\right). \quad (74)$$

  Its single soliton solution is $\psi = k_1 \text{sech}(k_1 x - k_1^3 t)e^{ik_2}$, where $k_1$ and $k_2$ are the real constants.

- **Hamiltonian pair $(\hat{H}_1, \hat{H}_3)$**

  $$\frac{\partial \psi}{\partial t} = \{\psi, \hat{H}_1, \hat{H}_3\} = -i\left(\frac{\partial^4 \psi}{\partial x^4} + 6 \frac{\partial}{\partial x}\left(|\psi|^2 \frac{\partial \psi}{\partial x}\right)\right). \quad (75)$$

  Its single soliton solution is $\psi = k \text{sech}(k x - \frac{8\sqrt{7}}{9}k^4 t)e^{i(-\frac{2\sqrt{7}}{3}x + \frac{8}{3}k^4 t)}$, where $k$ is a real constant.

To test the integrability of (74) and (75), we perform the Painlevé analysis and find that they do not have the Painlevé integrable property.

For the Hamiltonian pairs $(\hat{H}_0, \hat{H}_3)$ and $(\hat{H}_1, \hat{H}_2)$ of the KP hierarchy, we have found that there is the deep relationship between them. As done in that of (38), let us consider the following evolution equation with two Hamiltonian pairs of the non-linear Schrödinger equation:
\[ \frac{\partial \psi}{\partial t} = \{\psi, \hat{H}_0, \hat{H}_3\} + 3\{\psi, \hat{H}_1, \hat{H}_2\}. \]  

(76)

Substituting the expressions of (73) and (74) into (76), we obtain

\[ \frac{\partial \psi}{\partial t} = -4 \frac{\partial^3 \psi}{\partial x^3} - 12|\psi|^2 \frac{\partial \psi}{\partial x} - 6\psi \frac{\partial |\psi|^2}{\partial x}. \]  

(77)

Under the scaling transformations \( \psi \rightarrow \sqrt{2}\psi \) and \( t \rightarrow \frac{1}{4}t \), we may rewrite (77) as

\[ \frac{\partial \psi}{\partial t} + \frac{\partial^3 \psi}{\partial x^3} + 6|\psi|^2 \frac{\partial \psi}{\partial x} + 3\psi \frac{\partial |\psi|^2}{\partial x} = 0. \]  

(78)

We immediately recognize that (78) is nothing but the so-called integrable Sasa–Satsuma equation [50].

6.2. An application in optical soliton

It is well-known that the non-linear Schrödinger equation plays a critical role in the study of optical solitons for optical fiber communications. To describe a large number of nonlinear effects in optical fibers, the effects of the higher-order terms on the nonlinear Schrödinger equation have been paid more attention. The propagation of optical field in a monomode fiber may be described by the following generalized nonlinear Schrödinger equation [51]:

\[ \frac{\partial A}{\partial Z} + \frac{\alpha}{2} A + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial T^2} - \frac{\beta_3}{6} \frac{\partial^3 A}{\partial T^3} = i\gamma \left( |A|^2 A + \frac{i}{\omega_0} \frac{\partial |A|^2 A}{\partial T} - T_R A \frac{\partial |A|^2}{\partial T} \right), \]  

(79)

where \( Z \) is the normalized propagation distance, \( T \) is the normalized retarded time, i.e., time in the frame of reference moving with the wave packet, \( A(Z, T) \) denotes the normalized slowly varying complex wave packet envelope in the electric field, and \( |A|^2 \) represents the optical power. Eq. (79) is an important nonlinear equation for studying higher-order nonlinear effects in optical fibers. The second and third terms on the left side of (79) describe the effects of fiber loss and chromatic dispersion, respectively. The term proportional to \( \beta_3 \) governs the effects of third-order dispersion and becomes important for ultrashort pulses because of their wide bandwidth. The first term on the right side of (79) describes the nonlinear effects of self-phase modulation. The terms proportional to \( \omega_0^{-1} \) and \( T_R \) are responsible for the phenomenon of self-steepening and the Raman-induced frequency shift, respectively.

Based on the results in the previous subsection, let us turn to derive generalized nonlinear Schrödinger equation in the framework of Nambu mechanics. For the Hamiltonian pairs \( (\hat{H}_0, \hat{H}_2) \) and \( (\hat{H}_1, \hat{H}_2) \), we consider the following N-P evolution equation:

\[ \frac{\partial \psi}{\partial t} = -\frac{\beta_2}{2} \{\psi, \hat{H}_0, \hat{H}_2\} - \frac{\beta_3}{6} \{\psi, \hat{H}_1, \hat{H}_2\}, \]  

(80)

where the parameters \( \beta_2 \) and \( \beta_3 \) in (79) are introduced into (80).

By means of (72) and (74), (80) leads to

\[ \frac{\partial \psi}{\partial t} = -i\beta_2 \frac{\partial^2 \psi}{\partial x^2} + \beta_3 \frac{\partial^3 \psi}{\partial x^3} - i\beta_2 |\psi|^2 \psi + \frac{\beta_3}{3} \frac{\partial |\psi|^2 \psi}{\partial x}. \]  

(81)

Its single soliton solution is

\[ \psi = k\text{sech} \left( kx + \frac{\beta_3}{6} k^3 t \right) e^{-4i\beta_3 k^2 t}, \]  

(82)

where \( k \) is a real constant.
Comparing (79) with (81), we observe that in absence of the fiber loss and Raman-induced frequency shift terms, i.e., \(\alpha = T_R = 0\), (79) reduces to (81) with \(\gamma = -\beta_2\), and \(\omega_0 = \frac{3\beta_3}{2\beta_2}\). The corresponding optical soliton solution is given by (82).

We have already seen that the generalized nonlinear Schrödinger equation can be derived in terms of the Hamiltonian pairs \((\hat{H}_0, \hat{H}_2)\) and \((\hat{H}_1, \hat{H}_2)\). It should be pointed out that we may also derive the similar result just from the N-P evolution equation with the Hamiltonian pair \((\hat{H}_1, \hat{H}_2)\) under the appropriate variable transformations. For the Hamiltonian pair \((\hat{H}_1, \hat{H}_2)\), the corresponding N-P evolution equation is given by (74). Let us take the variable transformations as follows:

\[
\begin{align*}
t &= \frac{\gamma \beta_3}{2\beta_2} \sqrt{-\frac{3\gamma}{\beta_2}} Z, \\
x &= \frac{1}{2\beta_3} \sqrt{-\frac{3\gamma}{\beta_2}} (2\beta_2 T + \beta_2^2 Z), \\
A &= \psi(t, x) \exp i B, \\
B &= \frac{\beta_2}{\beta_3} \left( T + \frac{\beta_2^2}{3\beta_3} Z \right) + \frac{\alpha}{2} i Z.
\end{align*}
\]

It is not difficult to prove that under the variable transformations (83), (74) becomes (79) with \(T_R = 0\) and \(\omega_0 = \frac{\beta_2}{\beta_3}\).

7. Concluding remarks

The KP hierarchy is an important integrable system which can be regarded as the generalization of KdV hierarchies. It is well-known that there is a remarkable connection between the integrable systems and the infinite-dimensional conformal algebra and its extensions. For the KP hierarchy, one has known that its first Hamiltonian structure is related to the \(W_{1+\infty}\) algebra. To discuss the relation between the infinite-dimensional 3-algebra and the KP hierarchy, we constructed the \(W_{1+\infty}\) 3-algebra which does not satisfy the FI condition, but the BI condition holds for this ternary algebra. By introducing the FTF, we rewrote the \(W_{1+\infty}\) 3-algebra as the Nambu 3-bracket structure of the FTF. We noted that when there is a fixed generator \(W^{(0)}_0\) in the operator Nambu 3-bracket, the \(W_{1+\infty}\) 3-algebra may reduce to the \(W_{1+\infty}\) algebra. Thus based on this special \(W_{1+\infty}\) 3-algebra, we derived the KP hierarchy from the N-P evolution equation with the Hamiltonian pairs \((H_0, H_1)\).

To explore the relation between the general case of the \(W_{1+\infty}\) 3-algebra and integrable equations, we replaced the Hamiltonian pair \((H_0, H_2)\) by \((H_1, H_2)\) in the N-P evolution equation (24). An intriguing feature is that it still leads to the KP equation. Moreover without taking the low dimensional reduction, we directly derived the KdV equation by means of the N-P evolution equation with the Hamiltonian pairs \((H_0, H_3)\) and \((H_1, H_2)\). It did provide new insight into the KP and KdV equations. Furthermore we investigated the N-P evolution equation (32) with the different Hamiltonian pairs and presented a conjecture with respect to the relation between the \(W_{1+\infty}\) 3-algebra and the dispersionless KdV hierarchy. For the Hamiltonians of KP hierarchy, one did only know that these Hamiltonians are in involution. Our investigation turned out that the intrinsic relationship between these Hamiltonians is actually more complicated than has been previously recognized.
For the $W_{1+\infty}$ 3-algebra, we also presented its realization in terms of a complex bosonic field with Nambu 3-brackets (70). Based on (70), we derived the (generalized) nonlinear Schrödinger equation, complex mKdV equation and Sasa–Satsuma equation from the N-P evolution equation. Moreover an application in optical soliton has been given for the generalized nonlinear Schrödinger equation derived from the N-P evolution equation. It is worthwhile to mention the $(2+1)$-dimensional non-integrable evolution equations presented in this paper, which have the single soliton solutions. Since the (higher-order) KP and KdV equations have been widely used in various fields such as fluid physics, plasma physics and quantum field theory, the applications of the $(2+1)$-dimensional evolution equations mentioned above might be of interest in these fields.

Our analysis suggested that there exist the much deeper connections between the $W_{1+\infty}$ 3-algebra and the integrable equations. More properties with respect to their relations still deserve further study. We believe that the infinite-dimensional 3-algebras may shed new light on the integrable systems.

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