# On Equality of Some Elements in Matrices

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ABSTRACT

If the non-zero entries of a matrix S satisfy certain sum restrictions, then it is shown that all these entries must be equal. This paper is prompted by a conjecture of Wang [2]. Not only is the conjecture proved, but the basic underlying idea is abstracted and the result extended considerably. The techniques given here have been successfully employed by the author in obtaining significant results on (0,1) matrices and Latin squares (to appear elsewhere).

## 1. PRELIMINARY IDEAS

Suppose $A$ is an $m\times n$ matrix. Let $S$ denote the set of all positions in $A$, i.e., $S=\{(i,j)|1\leqslant i\leqslant m,\ i\leqslant j\leqslant n\}$. For $T\subset S$ define the incidence matrix $E_T=(e_{ij})$ of order $m\times n$ by setting $e_{ij}=1$ if $(i,j)\in T$ and $e_{ij}=0$ otherwise. Then $\sum_{(i,j)\in T}a_{ij}=\operatorname{tr}(E_T^tA)$, where tr denotes trace and the superscript $t$ stands for transpose.

## 2. RESULTS

LEMMA 1. *Let $\alpha_i\geqslant 0$, $i=1,\dots,n$, and $\sum_{i=1}^n\alpha_i>0$. Then for any set of real numbers $x_1,\dots,x_n$,*

$$\sum_{i=1}^n\alpha_ix_i^2=\frac{\left(\sum_{i=1}^n\alpha_ix_i\right)^2}{\sum_{i=1}^n\alpha_i}$$




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*implies*

$$\alpha_i x_i \equiv \frac{\alpha_i \sum_{j=1}^{n} \alpha_j x_j}{\sum_{j=1}^{n} \alpha_j}.$$

*Proof.* According to Cauchy's inequality, $(\Sigma\alpha_i x_i)^2 = (\Sigma\sqrt{\alpha_i}\cdot x_i\sqrt{\alpha_i}\,)^2 \leqslant (\Sigma\alpha_i)(\Sigma\alpha_i x_i^2)$, equality holding if and only if for some $K$, $\sqrt{\alpha_i}\; x_i = K\sqrt{\alpha_i}$ for $i = 1,2,\ldots,n$ (i.e., $\alpha_i x_i = K\alpha_i$ or $\Sigma\alpha_i x_i = K\Sigma\alpha_i$). Hence $\alpha_i x_i = \alpha_i \Sigma\alpha_j x_j / \Sigma\alpha_j$ for $i = 1,2,\ldots,n$, when $\Sigma\alpha_i x_i^2 = (\Sigma\alpha_i x_i^2)/\Sigma\alpha_i$. ■

THEOREM 1 (Wang's Conjecture [2]). *Let $A$ be an $n\times n$ doubly stochastic matrix with $m$ disjoint zero diagonals, $1 \leqslant m \leqslant n-1$. If each diagonal that is disjoint from these zero diagonals has a sum $n/(n-m)$, then all the entries of $A$ off the zero diagonals are equal to $1/(n-m)$. [By a diagonal we mean the set of entries $\{a_{1\sigma(1)},\ldots,a_{n\sigma(n)}\}$ where $\sigma$ is a permutation of $(1,\ldots,n)$.]*

*Proof.* Write $A = \Sigma_{r=1}^{s}\beta_r P_r$, where $\beta_r > 0$, $\Sigma_1^s \beta_r = 1$, and the $P_r$'s are permutation matrices. That this is possible is well known [1]. Obviously the 1's in a $P_r$ are situated on a diagonal of $A$. It is also quite obvious that these $s$ diagonals are disjoint from the zero diagonals of $A$. Let us assume *only* that the diagonal sums of $A$ corresponding to these $s$ diagonals are each equal to $n/(n-m)$, $0 \leqslant m \leqslant n-1$. Thus

$$\operatorname{tr}(P_r^t A) = \frac{n}{(n-m)} \qquad \text{for} \quad r = 1,\ldots,s. \tag{3.1}$$

Let $T$ be the set of all positions in $A$ off the $m$ zero diagonals, and let $E_T$ be as in Sec. 1. Obviously, $\Sigma_i\Sigma_j e_{ij} = n(n-m)$. Also,

$$n = \sum_i \sum_j a_{ij} = \sum_i \sum_j e_{ij} a_{ij}. \tag{3.2}$$

Now,

$$\operatorname{tr}(A^t A) = \operatorname{tr} \sum_{r=1}^{s} \beta_r P_r^t A = \sum_{r=1}^{s} \beta_r \operatorname{tr}(P_r^t A)$$

$$= \sum_{r=1}^{s} \beta_r \frac{n}{n-m} = \frac{n}{n-m},$$

by (3.1). But

$$\operatorname{tr}(A^tA)=\sum_i\sum_j a_{ij}^2=\sum_i\sum_j e_{ij}a_{ij}^2,$$

whence

$$\sum_i\sum_j e_{ij}a_{ij}^2=\frac{n}{n-m}=\frac{n^2}{n(n-m)}=\frac{\left(\sum_i\sum_j e_{ij}a_{ij}\right)^2}{\sum_i\sum_j e_{ij}}.$$

So, by Lemma 1, we get

$$e_{ij}a_{ij}=e_{ij}\frac{\sum_i\sum_j e_{ij}a_{ij}}{\sum_i\sum_j e_{ij}}=\frac{e_{ij}}{n-m},$$

and the proof is complete. ∎

THEOREM 2 (An extension of Theorem 1). *Suppose $A$ is a $k\times n$ row stochastic matrix, i.e., $A$ is a non-negative matrix of order $k\times n$ with each row sum unity. Let $A=\sum_1^s\beta_rB_r$, where $\beta_r>0$, $\sum\beta_r=1$, and the $B_r$'s are $k\times n$ row stochastic $(0,1)$ matrices (they exist). Suppose in each row of $A$ there are $m$ positions, $0\leqslant m\leqslant n-1$, where $A$ has zero entry. If the sum of all entries in $A$ corresponding to entries 1 in $B_r$ is $k/(n-m)$ for $r=1,\dots,s$, then every entry of $A$ off the $mk$ zero positions is equal to $1/(n-m)$.*

*Proof.* Let $T$ be the set of all the $k(n-m)$ positions off the $km$ zero positions, and let $E_T$ be as in Sec. 1. We have

$$\sum_i\sum_j e_{ij}=k(n-m), \tag{4.1}$$

$$\operatorname{tr}(B_r^tA)=\frac{k}{n-m}\qquad\text{for}\quad r=1,\dots,s, \tag{4.2}$$

$$k=\sum_i\sum_j a_{ij}=\sum_i\sum_j e_{ij}a_{ij}. \tag{4.3}$$

Now,

$$\sum_i\sum_j e_{ij}a_{ij}^2=\sum_i\sum_j a_{ij}^2=\operatorname{tr}(A^tA)=\operatorname{tr}\sum_{r=1}^s\beta_rB_r^tA=\sum_{r=1}^s\beta_r\operatorname{tr}(B_r^tA)$$
$$=\sum_{r=1}^s\beta_r\frac{k}{n-m}=\frac{k}{n-m},$$

using (4.2). Thus

$$\sum_i \sum_j e_{ij} a_{ij}^2 = \frac{k}{n-m} = \frac{k^2}{k(n-m)} = \frac{\left(\sum_i \sum_j e_{ij} a_{ij}\right)^2}{\sum_i \sum_j e_{ij}},$$

using (4.1) and (4.3). Now an application of Lemma 1 yields

$$e_{ij} a_{ij} = e_{ij} \frac{\sum_i \sum_j a_{ij} e_{ij}}{\sum_i \sum_j e_{ij}} = \frac{e_{ij}}{n-m},$$

and the proof is complete. ■

## 3. CONCLUDING REMARKS

From the proof of Theorem 2, it is clear that the results here can be extended much further. Obviously the condition of row stochasticity can be relaxed. Also , the $B_r$'s can be assumed to be any (0, 1) matrices so long as $A$ can be expressed as $\sum_1^s \beta_r B_r$. The $\beta_r$'s need not even be positive. But the extensions so obtained, though good, appear to be algebraic monstrosities. What we wish to stress is the fact that our proof shows that equality of row sums, column sums or diagonal sums is not germane to the main idea of the result proved. Numbers arranged in a matrix form effectively conceal the basic result which is Lemma 1. Hence it is easy to extend the results here to a collection of numbers not necessarily arranged in a matrix form. The reader is invited to do this, though the result so obtained is not very profound.

## NOTE ADDED IN PROOF

Since the above was written, a proof of Wang's conjecture had appeared in R. Sinkhorn, Doubly stochastic matrices which have certain diagonals with constant sums, *Linear Algebra and Appl.* **16** (1977), 79–82. Some related results may be found in Eva Achilles, Doubly stochastic matrices with some equal diagonal sums, *Linear Algebra and Appl.* **22** (1978), 293–296, this issue.

## REFERENCES

1 G. Birkhoff, Tres observaciones sobre el algebra lineal, *Univ. Nac. de Tucuman, Rev. Ser. A* 5 (1946), 147–151.
2 Edward Tzu-Hsia Wang, Maximum and minimum diagonal sums of doubly stochastic matrices, *Linear Algebra and Appl.* **8**, No. 6 (1974), 483–507.

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