A New Model for Linguistic Modifiers

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ABSTRACT

This paper presents a new model for the representation of linguistic modifiers. Each modifier is characterized by three parameters and is identified by a single positive real number n. The number n fully characterizes the generic term of a boolean linguistic variable. The representation of the terms maintains the order relation with which they are used in natural language. Lastly, it is shown how this new model allows the fuzzy set of each boolean expression to be evaluated through suitably simple fuzzy functions.

KEYWORDS: linguistic modifiers, fuzzy logic, boolean fuzzy term evaluation.

1. INTRODUCTION

In 1623 Galileo in his dissertation Il Saggiatore [1] pointed out the difference between the primary and secondary qualities of objects. The primary qualities characterize the objective qualitative models, which are independent of the subjective entities. On the other hand, the sensitive qualities, or secondary qualities, are features of the cultural and cognitive state of the subject, as well as of the interactions of the reasoning subject with the related geotemporal context in which the judgement is expressed. Historically, this dilemma has been studied by a multitude of scientists, including George Berkeley (1685–1753), who stressed the relativity of assertions, and in particular those related to sensations.

Zadeh [2] explained that a proposition such as “The sea is very rough” can be interpreted as “It is very true that the sea is rough.” Consequently, the sentences “The sea is very rough,” “It is very true that the sea is rough,” and “It is very true that the sea is rough” are all valid expressions of the same proposition. This illustrates the power of the fuzzy set theory to handle imprecise and uncertain information.
rough,” “(The sea is rough) is very true” can be considered as being equivalent. In fact, truth function modification allows an algorithmic approach to the calculus of deduction in approximate reasoning [3], by deepening the liaison with the classical logic. Since in traditional propositional logic the validity of a reasoning depends on the simple truth proof of logic propositions [4], in a fuzzy logic we have the truth values that determine the fuzzy set associated with the conclusion of a deduction [5]. Hence, the transformation of a proposition “X is mA” into “(X is A) is mTrue” stresses the dependence of the conclusion on the initial conditions, as happens in traditional binary logic. For this reason, in a deduction process the analytic representation of expressions such as “very true,” “more or less true,” “absolutely true” plays an important role.

The adverbial locutions “very,” “more or less,” “absolutely” modify the truth value of the words “true” and “false.” The first are named linguistic modifiers, the second linguistic truth values. These two categories generate different problems, such as how to build the related characteristic function, for a given combination of modifiers with logic connectives, or how to name the resulting fuzzy set.

In this paper, we start out from the assumption that a linguistic label can be defined by three parameters and shaped via a function underlying an area which is directly proportioned to the truth meaning of the label, and we propose a new model suitable for functions mA, where m is a modifier and A is a generic linguistic term. In particular we will apply this new model to the linguistic variable “truth”. We will show that our approach “area vs. truth values” guarantees the same results obtained by other authors [6].

The paper is organized as follows. In Section 2, we provide the basic details of linguistic variables, boolean linguistic variables, and linguistic modifiers [2, 7–9]. In Section 3 we show how each term mA can be specified by three parameters without using Zadeh’s CON and DIL operators. The mathematical framework for the values of the truth linguistic variable is discussed in Section 4. The model allows a continuous and ordered set of labels to be obtained; to each of these labels, it is possible to associate a number n which is related to equivalent models already known in literature. Furthermore, we explore some simple relationships between area variables and boolean linguistic variables. In Section 5 we show how it is possible to build a fuzzy propositional logic on our labels, as initially discussed in [7] and [3]. The correspondence between the subsets of R+ and the term set of the truth variable is introduced and explained in Section 6, showing how it is possible to evaluate a generic fuzzy boolean expression through a calculus in R+. The extension of our model, previously defined in the interval [0, 1], towards a generic linguistic variable and a generic interval in R is presented in Section 7. Lastly, we conclude the
2. LINGUISTIC VARIABLES

Basic notions of linguistic variables were formalized in different works by Zadeh in the mid 1970s [8-10]. These papers represent the mathematical attempt to define a theory for linguistic variables, for which a linguistic variable is a quintuple \((x, T(x), U, G, M)\) where:

- \(x\) is the name of the variable;
- \(T(x)\) is the term set of the variable \(x\), i.e. the set of names of linguistic values of \(x\);
- \(U\) is the universe of discourse;
- \(G\) is a syntactic rule for generating the name \(X\) of values of \(x\);
- \(M\) is the semantic rule for associating the meaning \(\bar{M}(X)\) to each \(X\).

A particular \(X\), which is the name generated by \(G\), is called a term.

We report a well-known example taken from [7], for which \(X = \text{"Age,}" \(U = [0, 100]\), and the terms of the linguistic variable are \("old," "young," "very old," etc. The rule which assigns a meaning to each fuzzy set of \(T\) is

\[
\bar{M}(\text{old}) = \{(u, \mu_{\text{old}}(u)) | u \in [0, 100]\},
\]

where

\[
\mu_{\text{old}}(u) = \begin{cases} 
0 & \text{if } u \in [0, 50], \\
1 & \text{otherwise.}
\end{cases}
\]

The term set of \(T\) can be \(T(\text{Age}) = \{\text{old, very old, not so old, more or less young, quite young, ...}\}\), and each of the elements of \(T\) is generated by a syntactic rule \(G\).

The terms \("old" and "young" are primary terms, whereas \("very," "more or less," "not so," and "quite" are examples of linguistic hedges which act as meaning modifiers.

A linguistic modifier is an operation which modifies the meaning of a term or of a fuzzy set. For instance, if \(\tilde{A}\) is a fuzzy set, then the modifier \(m\) generates the new term \(\tilde{B} = m(\tilde{A})\). Some classical modifiers, with their
corresponding mathematical representations, are:

concentration: \[ \text{CON}(\tilde{A}) = \mu_{\text{con}}(\tilde{A})(x) = [\mu_\lambda(x)]^2, \]
dilatation: \[ \text{DIL}(\tilde{A}) = \mu_{\text{dil}}(\tilde{A})(x) = [\mu_\lambda(x)]^{0.5}. \]

Hence

\[ \text{very } \tilde{A} = \text{CON}(\tilde{A}) \]

more or less \[ \tilde{A} = \text{DIL}(\tilde{A}) \]

plus \[ \tilde{A} = \tilde{A}^{1.25} \]

slightly \[ \tilde{A} = \text{INT}[\text{plus } \tilde{A} \text{ and not(very } \tilde{A})], \]

where

\[ \text{INT}(\tilde{A}) = \mu_{\text{int}}(\tilde{A})(x) = \begin{cases} 
2[\mu_\lambda(x)]^2 & \text{if } \mu_\lambda(x) \in [0, 0.5], \\
1 - 2\{1 - [\mu_\lambda(x)]^2\} & \text{otherwise}
\end{cases} \]

represents the contrast intensification, whereas "and" and "not" are interpreted according to [11]. A more precise definition was introduced in [12, 13], where

\[ \mu_{m \tilde{A}} = \mu_m \circ \mu_\lambda \circ q_m, \]

with \( \mu_m : [0, 1] \to [0, 1] \) and \( q_m \) a translation [14].

For example, by adopting the above notation, \( \tilde{A} = \text{possible} \), and \( \mu_{\text{possible}}(x) \) defined in [15], we have (see Figure 1)

\[ \mu_{\text{very possible}}(x) = (\mu_{\text{very}} \circ \mu_{\text{possible}} \circ q_{\text{very}})(x) \]

\[ = (\mu_{\text{very}} \circ \mu_{\text{possible}})(x - \Delta) = [\mu_{\text{possible}}(x - \Delta)]^2. \]
where $\Delta \in R$;

$$\mu_{\text{more or less possible}}(x) = \left( \mu_{\text{more or less}} \circ \mu_{\text{possible}} \circ q_{\text{more or less}} \right)(x)$$

$$= \left( \mu_{\text{more or less}} \circ \mu_{\text{possible}} \right)(x + \Delta)$$

$$= \left[ \mu_{\text{possible}}(x + \Delta) \right]^{0.5},$$

$$\mu_{\text{ant(possible)}}(x) = \mu_{\text{possible}}(1 - x),$$

where $\text{ant(possible)} = \text{antonym of possible} = \text{impossible}.$

Lastly, a boolean linguistic variable $A$ is a linguistic variable whose terms are boolean expressions which have been defined as follows:

- $Xp$ is a term;
- $mXp$ is a term;
- if $X$ and $Y$ are terms, then not $X,$ $X$ and $Y,$ $X$ or $Y,$ and $mX$ are also terms,

where $m$ is a linguistic modifier, $Xp$ is a primary term and $mX$ is the name of a fuzzy set which is obtained by applying $m$ to $X$ [8].

For example, Truth is the name of a boolean linguistic variable whose primary terms are “true” and “false.” The linguistic values of this variable constitute the set of its truth values. The set of Truth terms is $T(\text{Truth}) = \{\text{true, not true, not very true and fairly false,} \ldots\}.$ This linguistic variable plays a crucial role when defining a fuzzy logic theory. As far as the definitions of the membership functions are concerned we are reminded of Zadeh’s s-shaped functions for the terms “true” and “false” and Baldwin’s and Pilsworth’s approaches, which use

$$\mu_{\text{true}}(x) = x \quad \text{and} \quad \mu_{\text{false}}(x) = 1 - x, \quad x \in [0, 1].$$

Cat Ho and Wechler in [16, 17] pointed out the discrepancy between the intuitive use made in the natural language of linguistic truth values and the numerical values obtained using \text{CON} and \text{DIL} operators. For example, from an intuitive point of view, it is always assumed that “true” is truer than “(very) $n$ approximately true” for any natural number $n.$ By such an interpretation, the function with the label “(very) $n$ approximately true” is greater than function with the label “true.” But when we interpret “very” as the \text{CON} operator, the order relation becomes inverted from a certain $n$ value upwards. The relation induced by \text{CON} thus contradicts the intuitive meaning of the truth values “true” and “(very) $n$ approximately true.” The same thing happens with the \text{DIL} operator.

In this paper we put together a mathematical model for the truth values of the Truth linguistic variable and, on a more general basis, for the values
of a boolean linguistic generic variable, which maintains the natural order relation existing between them. Particularly, in our model the following conditions hold [16]:

1. each modifier strengthens or weakens the "positive" or "negative" meaning of the primary terms of a linguistic variable;
2. each modifier strengthens or weakens the meaning of every other hedge.

For example, the primary terms of the linguistic variables "age" and "Truth" are, respectively, "old" and "young," "true" and "false". "Young" and "true" have a positive meaning, whereas "old" and "false" have a negative meaning. Furthermore, the modifier "very" strengthens the meaning of "more," whereas it weakens the meaning of "approximately."

Also, in the case of the Truth variable, a model which maintains the order established by intuition allows the residuated lattice properties of the linguistic truth value set to be used [18], and these prove essential when constructing a fuzzy logic [19].

3. THE THREE VALUE MODEL

The representation of vague concepts (e.g. "small," "less," "large") and linguistic modifiers (e.g. "very," "more or less"), which give rise to the terms "very small," "more or less," "true," etc., leaves a wide margin for vagueness in the corresponding fuzzy set which mathematizes them. This reflects both the objective vagueness of the concept represented and the subjectivity of a judgement and the meaning of the terms included [20].

However, the functions which are generally used to represent a vague concept (S-shaped, \( \pi \)-shaped, triangular, and trapezoidal curves, etc.) present common characteristics which can be outlined as follows:

- they are continuous on the universe of discourse \( U \);
- they are different from zero on an interval \([a, b]\) included in \( U \);
- the position of this interval is linked to the concept represented;

From the semantic point of view these characteristics correspond to the following aspects which are implicitly associated to each meaning:

- a position of the universe,
- an assertion of truth,
- a precision.

We give a quantitative dimension to these characteristics, and we introduce the following corresponding parameters: \( CT \), the central tendency; \( VT \), the value of truth; and \( VP \), the degree of precision, defined thus:

\[
CT = \left| \frac{b + a}{2} - c \right|, \text{ where } c < a \in U;
\]
VT = the average value of the area subtended by the curve representing the vague concept;

VP = the size of the interval \([a, b]\) on which the membership function is different from zero.

Since the meaning of a term always implicitly contains the reference to these parameters, we take it to be a representation of the triple \((CT, VT, VP)\) regardless of the function \(\mu(x)\). Therefore, if \(m\) is a linguistic modifier and \(X\) is a value of a boolean linguistic variable \(A\), the application of \(m\) to \(X\) modifies the triple \((CT, VT, VP)\).

There may be partial modifications in the parameters. We will see in Section 4 that only the parameter \(VT\) becomes modified with the Truth variable.

In general, once a universe of discourse \(U\) and a boolean linguistic variable \(A\) have been assigned, we indicate its primary terms with \(X_1, X_2, \ldots, X_k\). We associate each \(X_i\) with an interval \([a_i, b_i] \in U\) and a point \(c_i < a_i\). Each interval is chosen in such a way that assertions such as “\(A\) is \(mX_i\)” lead to a distribution of possibilities which is different from zero on the interval.

For example, let us consider \(A = \text{temperature}\), \(U = [0, 100]\), and the following propositions which express value concepts: “The water is cold”, “The water is hot,” “The water is boiling,” all of which can constitute a classification on \(U\).

In the present model a variation of the parameter \(CT\) leads to a change in class and a variation of the parameter \(VT\) increases or decreases the degree of truth associated to the meaning of a term of a given class, i.e. the area under the curve \(x \rightarrow \mu(x)\). Lastly, \(VP\) produces a reduction in the increase of the relative interval, which now takes on the meaning of “precision interval.”

The modifiers linked to the parameter \(CT\) are called modifiers of translation [21]. For the parameter \(VP\), the relation with the modifiers of precision such as “exactly,” “precisely” [12] is very clear. A modifier of precision always modifies \(VT\) but never \(CT\). For instance, if a triangular fuzzy number is used to represent “...is 1.70 meters tall...” in the universe \([0, 2]\), with \(VP = 2\) meters, then \(VP\) changes for sentences such as “...is exactly 1.70 meters tall...” If a modifier of precision \(m\) expresses a different semantic, then the value \(VT\) is different from the corresponding term \(m(\tilde{A})\). Figure 2 explains this concept, showing the effect deriving from two different \(VP\) values. The problem of how to define a fuzzy semantic for each significant value of \(VP\) is an important issue which we do not face in this work.

Let us consider two fuzzy sets with equal \(CT\) and \(VP\) values (Figure 3). The difference in height between \(A\) and \(B\) is interpreted in the
following way: $B$ is more true than $A$, or, similarly, $A$ is less true than $B$.

Now, let us slightly modify the previous figure to analyze the truth degree of $A$ and $B$, as shown in Figure 4. Also in this case we can state that "$B$ is more true than $A$.” This choice is confirmed by the fact that the mean of the membership degrees of $B$ is greater than that of $A$. Let us define

$$VT = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

Through this definition, we see that, for equal values of $\text{CT}$ and $\text{VP}$, the truer a term is the higher its $VT$ value becomes.

Now let us tackle the problem of how to represent the modifiers for the $VT$ values. In this work we use triangular and trapezoidal fuzzy numbers in order to obtain a simple and efficient treatment.

For instance, Figure 5 explains the effect of terms such as "very true” or “more or less true” when for the first term we use a $VT$ value greater than the second one. In this way, it is possible to associate a $VT$ value to each linguistic modifiers. In Table 1 we report the comparisons between the standard model and our model.
4. THE MATHEMATICAL MODEL

For the sake of simplicity, in this section we discuss our model assuming the interval $[0, 1]$ as universe of discourse, even though our proposal has been developed for the more general universe $[a, b]$ (see Section 7), and we only consider the Truth boolean linguistic variable.

The characteristic function of a generic term of the Truth linguistic variable for a given $n \in R^+$ is given by

$$
\mu_n(x) = \begin{cases} 
\min(1, nx) & \text{for } x \in [0, \frac{1}{2}], \\
\min(1, -n(x - 1)) & \text{for } x \in [\frac{1}{2}, 1].
\end{cases}
$$

The two extreme labels of our term set are as follows:

- for $n \to \infty$, for each $x \in [0, 1]$, we have the label Absolutely True,
- for $n = 0$, for each $x \in [0, 1]$, we have the label Absolutely False.

Figure 6 shows the graph of $\mu(x)$ for a given $n$ value.

All the other linguistic labels fall between these two extremes. More specifically, we will prove that the set $[2, +\infty[$ characterizes the terms with a linguistic truth value greater than or equal to True, whereas $[0, 1]$ characterizes the terms with a linguistic truth value less than or equal to False, and that $]1, 2[$ allows all the possible variants between True and False to be expressed.
Table 1

<table>
<thead>
<tr>
<th>Model</th>
<th>VP</th>
<th>CT</th>
<th>VT</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our model</td>
<td>Not modified</td>
<td>Not modified</td>
<td>Modified</td>
<td>Modified</td>
</tr>
<tr>
<td>A standard translation-based model</td>
<td>Not modified</td>
<td>Modified</td>
<td>Not modified</td>
<td>Modified</td>
</tr>
</tbody>
</table>

Associated to the two above-mentioned labels, the areas under the functions $\mu(x)$ referred to above are:

\[ S_{m_{\text{True}}} = \frac{(n - 1)}{n} \quad \text{for} \quad n \in [2, +\infty[, \quad \text{where} \quad m_{\text{True}} \geq \text{True}; \]

\[ S_{m_{\text{False}}} = \frac{n}{4} \quad \text{for} \quad n \in [0, 1], \quad \text{where} \quad m_{\text{False}} \leq \text{False}; \]

\[ S_t = \frac{n}{4} \quad \text{for} \quad n \in [1, 2], \quad \text{where} \quad \text{False} < t < \text{True}. \]

These areas supply the numerical values for the parameter VT.

One of the initial problems to be solved is how to define the labels. For this purpose, let us introduce the parameter $d = 1/n$ which measures the distance between points $A$ and $B$. Now let us focus our attention on Figure 6.

For $d = 0$, we obtain the label Absolutely True; thus we have no transformation.

Where $d$ is greater than 1, the distance between $A$ and $B$ is greater than the width of precision, and the point $B$ lies outside the universe; thus we can assume that $d = 1$ and $n = 1$ to identify the label False, with area $S = \frac{1}{4}$. Similarly, where $d = \frac{1}{2}$ the distance between $A$ and $B$ is half that for VP. More specifically, this distance is equal to the measure of CT, which, as is known, is strictly bound to the translation, and hence we are able to obtain several labels on the basis of the original one. By reasoning in a symmetric way, $d = \frac{1}{2}$ and $n = 2$ identify the label True (Figure 7).
It is very interesting to note that $S_{True} = 2S_{False}$. Nafarieh and Keller (NK) [6] use a similar area calculus (even though in a different perspective), but they obtain $n = 1$ for True and $n = 2$ for Very True. This difference could be eliminated by translating the values of $n$, but that makes it impossible to generate all the remaining labels from Absolutely False up to Absolutely True. In NK, since $n \geq 0$, it is not discussed how is possible to generate the labels $mFalse$.

We wanted our model to be compatible with the NK model, with the exception of one translation, having the same areas for the same labels. The areas of the labels which are generally less or more specific than True can be computed via the parameter $d$. In fact, we have $S_{mTrue} = 1 - d$, under the condition that $d$ belongs to $[0, \frac{1}{2}]$. Using the NK approach, we would obtain $1 - d = 1/(n + 1)$, with the constraint $d \in [0, \frac{1}{2}]$. By replacing $n$ with $1/n$, we obtain $1 - d = n/(n + 1)$, and in order to generate the labels $mFalse$, the above-mentioned substitution provides the previous formula $S_{mTrue} = (n - 1)/n$. In our model it is very easy to establish a one-to-one correspondence between each label and its antonym. In fact, it will suffice to reflect the set $[2, \infty]$ in the set $[0, 1]$, by applying the substitution $n \leftarrow 2/n$. Thus, if $n = 1$, we obtain False; if $n = 2$, True. Between the antonym labels and their corresponding values we have the following correspondence:

$$
\begin{align*}
0 & \rightarrow 1 \\
\uparrow & \\
2 & \rightarrow +\infty
\end{align*}
$$

Now let us compare the areas of two antonymous labels—$E$, whose
Table 2

<table>
<thead>
<tr>
<th>n</th>
<th>(d = 1/n) translation</th>
<th>Labels</th>
<th>Area (A_1)</th>
<th>(d = n/2) translation</th>
<th>Antonym</th>
<th>Area (A_2)</th>
<th>(A_1/A_2)</th>
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<tr>
<td>2</td>
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<td>True</td>
<td>1/2</td>
<td>1</td>
<td>False</td>
<td>1/4</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1/3</td>
<td>Very</td>
<td>2/3</td>
<td>2/3</td>
<td>Very</td>
<td>1/6</td>
<td>4</td>
</tr>
<tr>
<td>(k+1)</td>
<td>(1/(k+1))</td>
<td>Very (k)</td>
<td>(k/(k+1))</td>
<td>(2/(k+1))</td>
<td>((k+1)/2)</td>
<td>Very (k)</td>
<td>(1/(2k+2))</td>
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</tbody>
</table>

The parameter is assumed to be \(n \in [2, +\infty[, \text{ and ant}(E):

\[
S_E = \frac{n-1}{n}, \quad S_{\text{ant}(E)} = \frac{1}{2n}, \quad S_F = 2(n-1)S_{\text{ant}(E)}.
\]

For the sake of compatibility with the NK method, we chose

\[
n_{\text{Very True}} = n_{\text{Very True}, NK} + 1 = 2 + 1 = 3.
\]

The results are shown in Table 2, where \(k = 2^i\) is the NK coefficient with \(i = 1, \ldots, m\).

Now let us tackle the problem of characterizing and interpreting the labels with \(n \in ]1, 2[\). If we pursue our approach in constructing a symmetric generator of labels, we have the following correspondence:

\[
1 \ldots 1.5
\]

\[
\uparrow
\]

\[
2 \ldots 1.5
\]

In that case \(n_{m\text{False}} = -n_{m\text{True}} + 3\), where \(n \in [1,1.5]\) and \(n' \in [1.5,2]\). By applying the usual shift, \(n = 1.5 = 1 + 0.5\), we obtain the labels Fairly True and Fairly False. These terms are the only ones which have the same parameter \(n\). In order to distinguish the correct linguistic approximation, it will suffice to consider the related context. For such terms, the following relation is always satisfied:

\[
S_{\text{Fairly True}} = \frac{2^k + 1}{2^{k+1} - 1} S_{\text{Fairly False}}.
\]

5. AN APPROPRIATE FUZZY LOGIC

Since each term is identified by a single positive real number, let us indicate with \(n_A\) the number \(n\) which characterizes the term \(A\), and with \(E[n]\) (the value of a linguistic variable) the term associated with the
number \( n \). In the rest of the paper we use the symbol \( \neg \) to denote the unary connective “not.”

Let us examine the problem of a fuzzy evaluation for each connective [22–24]. In the following relations, \( n_A (m_B) \) is the number of the term \( A (B) \):

\[
\begin{align*}
\vee (A \text{ and } B) &= E[\min(n_A, m_B)], \\
\vee (A \text{ or } B) &= E[\max(n_A, m_B)], \\
\vee (\neg A) &= \begin{cases} E[3 - n_A] & \text{if } n_A \in [1, 2], \\ E[2/n_A] & \text{otherwise,} \end{cases} \\
\vee (A \rightarrow B) &= E[\max(n_{\neg A}, m_B)].
\end{align*}
\]

For instance, if \( A = \text{ Very True} \), \( n_A = 3 \), and \( B = \text{ Fairly False} \), \( n_B = 1.5 \), then we obtain

\[
\begin{align*}
\vee (A \text{ and } B) &= E[\min(3, 1.5)] = E[1.5] = \text{ Fairly False}, \\
\vee (A \text{ or } B) &= E[\max(3, 1.5)] = E[3] = \text{ Very True}, \\
\vee (A \rightarrow B) &= E[\max(\frac{3}{3}, 1.5)] = E[1.5] = \text{ Fairly False}, \\
\vee (\neg A) &= E[\frac{2}{3}] = \text{ Very False}, \\
\vee (\neg A \rightarrow B) &= E[\max(2/n_{\neg A}, 1.5)] = E[\max(2/3, 1.5)] \\
&= E[\max(3, 1.5)] = E[3] = \text{ Very True}.
\end{align*}
\]

From these results we can establish Table 3.

<table>
<thead>
<tr>
<th>( \vee (A) )</th>
<th>( \vee (B) )</th>
<th>( \vee (A \text{ and } B) )</th>
<th>( \vee (A \text{ or } B) )</th>
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<td>False</td>
<td>False</td>
<td>Abs. True</td>
</tr>
<tr>
<td>Abs. True</td>
<td>Abs. True</td>
<td>Abs. True</td>
<td>Abs. True</td>
</tr>
<tr>
<td>Abs. True</td>
<td>Abs. False</td>
<td>Abs. False</td>
<td>Abs. True</td>
</tr>
</tbody>
</table>
Now, considering Table 4, we can show that our approach is compatible with the one proposed in [3]. The correspondence between the first and the second column of Table 4 is natural in our model; it will suffice to consider that for each \( n \in [2, +\infty[ \) or \( n \in ]0, 1] \) (\( n \in [1, 2] \)) there exists an index \( 2/n \) (respectively, \( 3 - n \)) which identifies the same label present in the corresponding column. For instance, let \( A \) be "the sea is rough," with \( \lor(A) = \text{Very True} \), and \( n = 3 \). In order to identify \( \lor(\neg A) \), we compute the label with index \( 2/n = 2/3 \), which corresponds to \( \text{Very False} \), and hence \( \lor(\neg A) \) is equal to \( \text{Very False} \).

Now, we know how to find the antonymous label for a given label, and we also know the correspondence of the set of \( n \) in \( R^+ \). Reasoning by exclusion, we have that, given \( \lor(A) \) with \( n \in [2, +\infty[ \) (or, respectively, the set \([0, 1]\)), the label \( \neg \lor(A) \) is described by \( n \in [1, 1.5] \) (or, respectively, to \([1.5, 2]\)), and vice versa:

\[
\begin{array}{ccc}
2 & \ldots & +\infty \\
0 & \ldots & 1 \\
1 & \ldots & 1.5 \\
1.5 & \ldots & 2
\end{array}
\]

In this way we guarantee the compatibility of our space with Table 4.

Now let us give the formulas which allow the basic labels to be computed, given the parameter \( n \):

(a) \( n(\text{Very}^k \text{ True}) = 2^k + 1 \); and for \( k = 0, 1, \ldots, +\infty \),

\[
\begin{array}{cccc}
2 \ldots \infty \\
0 \ldots 1 \\
1 \ldots 1.5 \\
1.5 \ldots 2 \\
\end{array}
\]

Table 4

<table>
<thead>
<tr>
<th>( \lor(A) )</th>
<th>( \lor(\neg A) )</th>
<th>( \neg \lor(A) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>Fairly True</td>
<td>Fairly False</td>
<td>Very False</td>
</tr>
<tr>
<td>Very True</td>
<td>Very False</td>
<td>Fairly False</td>
</tr>
<tr>
<td>Abs. True</td>
<td>Abs. False</td>
<td>True</td>
</tr>
<tr>
<td>False</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>Fairly False</td>
<td>Fairly True</td>
<td>Very True</td>
</tr>
<tr>
<td>Very False</td>
<td>Very True</td>
<td>Fairly True</td>
</tr>
<tr>
<td>Abs. False</td>
<td>Abs. True</td>
<td></td>
</tr>
</tbody>
</table>
A New Model for Linguistic Modifiers

(b) \( n(\text{Very}^k \text{False}) = \frac{2}{(2^k + 1)} \); and for \( k = 0, 1, \ldots, +\infty \),

<table>
<thead>
<tr>
<th>( k = 1 )</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{Very False} \frac{2}{3}</td>
<td>\frac{2}{5}</td>
<td>\frac{2}{9}</td>
<td>\frac{2}{17}</td>
</tr>
</tbody>
</table>

(c) \( n(\text{Fairly}^k \text{True}) = 3 - [1 + (\frac{1}{2})^k] \); and for \( k = 1, \ldots, +\infty \),

<table>
<thead>
<tr>
<th>( k = 1 )</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{Fairly True} 3 - [1 + (\frac{1}{2})^k] = 1.5</td>
<td>1.75</td>
<td>1.875</td>
<td>1.9375</td>
</tr>
</tbody>
</table>

(d) \( n(\text{Fairly}^k \text{False}) = [1 + (\frac{1}{2})^k] \); and for \( k = 1, \ldots, +\infty \),

<table>
<thead>
<tr>
<th>( k = 1 )</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{Fairly False} 1 + (\frac{1}{2})^k = 1.5</td>
<td>1.25</td>
<td>1.125</td>
<td>1.0625</td>
</tr>
</tbody>
</table>

6. EVALUATION OF A BOOLEAN EXPRESSION THROUGH CRISP FUNCTIONS

The questions

Which fuzzy set describes a term as \( \neg(\text{Very True}) \) and \( \neg\text{False} \)?

Which fuzzy set represents \( \text{Plus Very True or Fairly False} \)?

are examples of boolean expressions characterized by fuzzy information. This section aims to prove how our model is suitable for computing the calculus of the fuzzy connectives. In particular, we show how such a calculus is based on an evaluation of a crisp function.

The essential problem is how to evaluate \( \neg \vee (A) \). The above relations (a), (b), (c), and (d) play a fundamental role in considering this question.

On the basis of Table 4, we know that our calculus framework is based on the logic rule \( \neg(\text{Very}^k \text{True}) = \text{Fairly}^k \text{False} \), where \( k \) is a positive integer. With \( k \geq 1 \), \( n(\text{Fairly}^k \text{False}) \in [1, 1.5] \), whereas \( n(\text{Very}^k \text{True}) \in [3, +\infty[ \). Thus, the problem is to define a function \( n : [3, +\infty[ \to ]1, 1.5] \) so that the above-mentioned logic rule is satisfied. Using relations (a) and (d),
the function is

\[ n : \left[3, +\infty \right] \rightarrow [1, 1.5], \quad n \rightarrow 1 + \left(\frac{1}{2}\right)^{(n-1)/2}. \quad (1) \]

The restriction of the \( f \) membership set to \( [3, +\infty] \) instead of \( [2, +\infty] \) is due to the constraint \( \neg(\text{Very}^k \text{ True}) = \text{Fairly}^k \text{ False} \). In fact, for \( n = 2 \), we obtain the term \( E[1.707] \), which does not belong to the codomain of the terms such as \( \text{Fairly}^k \text{ False} \). On the other hand, for values greater than 1.5, we obtain \text{False}.

We note that the codomain of the function matches with the required one. Let us verify this with a simple test:

\[ \neg(\text{Very True}) = E\left[1 + \left(\frac{1}{2}\right)^{(3-1)/2}\right] = E[1, 5] = \text{Fairly} \text{ False}, \]

\[ \neg(\text{Very Very True}) = E\left[1 + \left(\frac{1}{2}\right)^{(5-1)/2}\right] = E[1, 25] = \text{Fairly} \text{ Fairly} \text{ False}. \]

For \( n = 2^k + 1 \), we have \( E[n] = E[2^k + 1] = E[1 + \left(\frac{1}{2}\right)^{2^k - 1}] \). When \( k = 2, \ n = 5 \), we have \( \neg(\text{Very Very True}) = E[5] = E[1 + \left(\frac{1}{2}\right)^{2^2 - 1}] = E[1.25] = \text{Fairly} \text{ Fairly} \text{ False} \).

The inverse function

\[ n : [1, 1.5] \rightarrow [3, +\infty], \quad n \rightarrow 1 - 2 \log_2(n - 1), \]

must obey the opposite constraint, \( \neg(\text{Fairly}^k \text{ False}) = \text{Very}^k \text{ True} \). For instance, for \( n = 1.25 \), \( E[1.25] = \text{Fairly} \text{ Fairly} \text{ False} \), \( \neg(\text{Fairly} \text{ Fairly} \text{ False}) = E[1 - 2 \log_2(1.25 - 1)] = E[5] = \text{Very} \text{ Very} \text{ True} \).

The next step consists in defining the function which allows \( \neg(\text{Fairly}^k \text{ True}) = \text{Very}^k \text{ False} \) to be obtained. Our initial and natural attempt would define a function \( f \) defined in \([1.5, 2]\) with values in \([0, 1]\), but this leads to the same drawbacks previously discussed. The correct function is

\[ n : [1.5, 2] \rightarrow [0, \frac{3}{3}], \quad n \rightarrow \frac{2 - n}{3 - n}. \]

For instance, with \( E[1.5] = \text{Fairly True} \), we have \( E[2(2 - 1.5)/(3 - 1.5)] = E[\frac{2}{3}] = \text{Very False} \). Furthermore, with \( E[1.875] = \text{Fairly Fairly True} \), we obtain \( E[2(2 - 1.875)/(3 - 1.875)] = E[0.222] = E[\frac{2}{3}] = \text{Antonym} E[9] = \text{Antonym(Very Very Very True)} = \text{Very Very Very False} \) [by considering relation (d)].
The inverse correspondence is obtained through to the following function:

\[ n : \left[0, \frac{2}{3}\right] \to [1.5, 2], \quad n \to 2 - \left( \frac{1}{2} \right)^{\ln((2-n)/n)} \ln 2 \]

\[ = \frac{4 - 3n}{2 - n}. \]

For instance, with \( E[\frac{2}{3}] = \text{Very False} \), we have

\[ E\left[2 - \left( \frac{1}{2} \right)^{\ln((2-\frac{2}{3})/\frac{2}{3})/\ln 2}\right] = E[1.5] = \text{Fairly True}; \]

with \( E[\frac{2}{3}] = \text{Very Very False} \), we have

\[ E\left[2 - \left( \frac{1}{2} \right)^{\ln((2-\frac{2}{3})/\frac{2}{3})/\ln 2}\right] = E[1.75] = \text{Fairly Fairly False}. \]

To complete our discussion, we analyze the interval \([\frac{2}{3}, 1]\) and \([2, 3]\). The following function allows the set \([2, 3]\) to be associated with the interval \([\frac{2}{3}, 1]\):

\[ n : [2, 3] \to \left[\frac{2}{3}, 1\right], \quad n \to \frac{1}{2 - (\frac{1}{2})^{\log_2(n-1)}} = \frac{n - 1}{2n - 3}. \]

This function derives from the composition of the two functions

\[ f : [2, 3] \to [0, 1], \quad f(n) = \frac{\log_2(n-1)}{\log_2 2} = \log_2(n - 1), \]

\[ g : [0, 1] \to \left[\frac{2}{3}, 1\right], \quad g(n) = \frac{1}{2 - (\frac{1}{2})^n}. \]

For example, if \( n = 2 \) and \( E[2] = \text{True} \), then \( E[(2 - 1)/(4 - 3)] = E[1] = \text{False} \).

To understand these results better we must consider that the terms \( \text{Very}^k \text{ True} \) belong to the set \([2, 3]\), whereas the terms \( \text{Very}^k \text{ False} \) belong to \([\frac{2}{3}, 1]\), with \( k \in [0, 1] \).

The inverse correspondence is described by the inverse function

\[ n : \left[\frac{2}{3}, 1\right] \to [2, 3], \quad n \to \frac{3n - 1}{2n - 1}. \]

To illustrate the label-interval association better, we have drawn a graph which summarizes this situation (Figure 8).
The following correspondences exist among the preceding intervals:

\[ 0 \ldots \frac{2}{3} \quad \frac{2}{3} \ldots 1 \quad 1 \ldots 1.5 \quad 1.5 \ldots 2 \quad 2 \ldots 3 \quad 3 \ldots +\infty \]

The semantics at the basis of these correspondences is the same as reported in the literature [3].

Finally, with \( n = n_A \), we have for \( \lnot \lor (A) \)

\[
E\left[\frac{4 - 3n}{2 - 2}\right], \quad n \in [0, \frac{2}{3}],
\]

(1)

\[
E\left[\frac{3n - 1}{2n - 1}\right], \quad n \in [\frac{2}{3}, 1],
\]

(2)

\[
E[1 - 2 \log_2(n - 1)], \quad n \in [1, 1.5],
\]

(3)

\[
E\left[2 \cdot \frac{2 - n}{3 - n}\right], \quad n \in [1.5, 2],
\]

(4)

\[
E\left[\frac{n - 1}{2n - 3}\right], \quad n \in [2, 3],
\]

(5)

\[
E\left[1 + \left(\frac{1}{2}\right)^\frac{n-1}{2}\right], \quad n \in [3, +\infty[.
\]

(6)
Here are some useful examples:

\(-\neg(\text{False}) = E\left[\frac{3 \times 1 - 1}{2 \times 1 - 1}\right] = E[2] = \text{True},\)

since \(n(\text{False}) = 1,\) and by applying (2);

\(-\neg(\text{True}) = E\left[\frac{2 - 1}{2 \times 2 - 3}\right] = E[1] = \text{False},\)

since \(n(\text{True}) = 2,\) and by applying (5);

\(-\neg(\text{Very True}) = E[1 + 0.5^{2/2}] = E[1.5] = \text{Fairly False},\)

since \(n(\text{Very True}) = 3,\) and by applying (6);

\(-\neg(\text{Abs True}) = E[1 + 0.5^\infty] = E[1] = \text{False},\)

since \(n(\text{Abs True}) = +\infty,\) and by applying (6);

\(-\neg(\text{Abs False}) = E[2] = \text{True},\)

since \(n(\text{Abs False}) = 0,\) and by applying (1);

\(-\neg(\text{Plus True}) = E\left[\frac{2.25 - 1}{2 \times 2.25 - 3}\right] = E[0.833] \equiv \text{Almost False},\)

since \(n(\text{Plus True}) = 2.25,\) and by applying (5);

\(-\neg(\text{Plus False}) = E\left[\frac{3 \times \frac{2}{2.5} - 1}{2 \times \frac{2}{2.5} - 1}\right] = E[2.143] \equiv \text{Almost True},\)

since \(n(\text{Plus False}) = 2/2.5,\) and by applying (2);

\(-\neg(\text{Very True and (Very Very False or } \neg(\text{Plus True}))\))

\[= -\neg(\min(3, \max(\frac{2}{5}, 0.833)))\]

\[= -\neg(\min(3, 0.833)) = E\left[\frac{3 \times 0.833 - 1}{2 \times 0.833 - 1}\right] = E[2.250\ldots] \equiv \text{Plus True},\]

by applying (2).
Our model can be easily and immediately extended to an arbitrary universe. In this section, this is proved by considering a generic linguistic variable $X$, with primary terms $X_p$ and $X_q$, where $X_p$ is the antonym of $X_q$, with $X_p > X_q$.

If $U + [a, b]$, the representation of $mX$ is exactly the label on $[a, b]$, with $n$ which corresponds to the modifier $m$. The new functions are obtained by a composition of the previous ones and the linear functions of correspondence between $[a, b]$ and $[0, 1]$:

$$
\mu(x) = \begin{cases} 
0 & \text{for } x \in R^+ - [a, b], \\
\min \left( 1; \frac{x - a}{b - a} \right) & \text{for } x \in \left[ a, \frac{a + b}{2} \right], \\
\min \left( 1; \frac{x - b}{b - a} \right) & \text{for } x \in \left[ \frac{a + b}{2}, b \right]. 
\end{cases}
$$

In such a way, all our previous results still hold true. In particular, there is no change in the values of $n$ for the above-mentioned labels. In fact, we have that

$$
S_{mX_p} = \frac{(b - a)(n - 1)}{n} \quad \text{for } n \in [2, +\infty[ \text{ and } mX_p \geq X_p,
$$

$$
S_{mX_q} = \frac{(b - a)n}{4} \quad \text{for } n \in [0, 1] \text{ and } mX_p \leq X_p,
$$

$$
S_t = \frac{(b - a)n}{4} \quad \text{for } n \in [1, 2] \text{ and } X_q < t < X_p,
$$

where $t \in T(X)$.

With similar reasoning, $d_{xp} = (b - a)/2$ and $d_{xq} = b - a$, since the value of $d$ is the first which leads to point $A$, outside the universe of discourse. Hence $VT_{[a, b]} = VT_{[0, 1]}$.

As an example, let us consider the set $[10, 30]$, which contains all the daily summer temperatures measured in a temperate country, latitude 40°N. The term set $T$ (temperature) is described by the function $\mu(x)$ discussed before. Figure 9 displays the graphs associated with “normal temperature” ($n = 2$), “temperature much above normal” ($n = 3$), and “low temperature” ($n = 1$).
8. CONCLUSION

In our model a generic fuzzy term is characterized by being simple, efficient, and semantically in accord with human intuition. The model expresses an interesting relation between truth degree and geometrical form for a given term. Via the boolean linguistic variables we are able to build the prepositional fuzzy logic, by reducing the determination of a fuzzy set to the calculus of a crisp function which represents the evaluation of a generic boolean fuzzy expression. Our model is able to generate a set of labels which is ordered and continuous like the positive real numbers. The model herein discussed satisfies the hypotheses for which Cat Ho and Wechler worked out an abstract algebra for modifiers [16]. The main, "classical" problem remains the linguistic interpretation of $E[n]$ for each $n$. In fact, predicates such as "very, very, very, very true" are rare, and probably appear only in a comic strip.

We think that interesting results may be obtained if we were to develop the idea, which has been successfully tested when algebraic fuzzy structures were designed to handle modifiers [25], in order to obtain an appropriate "discretization" on the continuous set and thus to reduce infinitely long strips.

Lastly, we wish to explore a further research trend, which adopts basic concepts of approximate reasoning [18].

One important detail to develop in future work is to identify a mathematical theory for the modifiers of precision and to complete a theory of terms.
References


