A FACTORIZATION THEOREM FOR A CERTAIN CLASS OF GRAPHS

Sukhamay KUNDU

IBM Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, N.Y. 10598, USA

Received 27 March 1973

Abstract. In this note we give a necessary and sufficient condition for factorization of graphs satisfying the "odd cycle property". We show that a graph $G$ with the odd cycle property contains a $[k_i]$ factor if and only if the sequence $[H] + [k_i]$ is graphical for all subgraphs $H$ of the complement of $G$.

A similar theorem is shown to be true for all digraphs.

1. Introduction

The factorization problem for graphs is stated as follows. Suppose $G$ is a finite graph and $[k_i]: 1 \leq i \leq n$ are $n$ natural numbers, one for each vertex of $G$. Then under what conditions does there exist a subgraph of $G$ having $k_i$ lines incident with the $i$th vertex for all $i$? Tutte [14] has given a necessary and sufficient condition (n.s.c.) that a given graph shall have a $[k_i]$ factor. The special case of the factorization problem, where $k_i$ are equal, were studied by various authors [1, 4, 5, 6, 9]. Tutte [12, 13] and Gallai [4] have also studied the problem for the locally finite infinite graphs. It is the goal of this paper to obtain a set of necessary and sufficient conditions for the existence of $[k_i]$ factor based on the realizability of certain degree sequences. Our theorem is not completely general, and applies only to a certain class of graphs, namely, those having the "odd cycle property". The odd cycle property was introduced by Fulkerson et al. in [3]. As far as factorization of a digraph, con-
cerned, a corresponding theorem is shown to be true for all digraphs. Other n.s.c. for the existence of a factor of a digraph with prescribed outdegrees and indegrees were given by Ore [7, 8].

2. The odd cycle property

We consider finite graphs \( G = (V(G), E(G)) \) without multiple lines and loops. The vertices \( V = V(G) = \{v_1, v_2, ..., v_n\} \) are fixed throughout the discussion. If the lines of a graph \( H \) are contained in the set of lines of \( G \), then \( H \) is said to be a subgraph of \( G \) and denoted by \( H \subseteq G \). The complement of \( G \), \( G^c \), is defined by \( E(G^c) = E(K_n) - E(G) \), where \( K_n \) is the complete graph on \( V(G) \). The degree of a vertex is the number of lines incident with that vertex. We write \([G]\) for the degree sequence of graph \( G \) and say that the sequence \([G]\) is graphical. A subgraph \( H \) of \( G \) is called a \([k_i]\) factor if \([H] = [k_i] \). We shall denote by \([d_i] + [k_i] \) the sequence \([d_i] + [k_i] \).

A graph \( G \) is said to have the odd cycle property if any two of its odd (simple) cycles either have a vertex in common, or there exists a pair of vertices, one from each cycle, which are joined by a line. In other words, the distance between two odd cycles of \( G \) is at most one. Some examples of graphs with the odd cycle property are the following. Complete graphs and bipartite graphs trivially have the property. It is not difficult to show that the complement of a line graph has the odd cycle property, as does the complement of a graph which is planar or has girth (the size of the smallest cycle) greater than four. Later (Proposition 3.4) we shall prove other properties of the class of graphs satisfying the odd cycle property. The following factorization theorem was established by Fulkerson, Hoffman and McAndrew [3]. Let \( c_{ij} = 1 \) if \( (v_i, v_j) \) is a line of \( G \), and 0 otherwise. In particular, \( c_{ii} = 0 \) for all \( i \).

**Theorem 2.1 (Fulkerson, Hoffman, McAndrew).** Let \( G \) be a graph satisfying the odd cycle property. Then there exists a subgraph of \( G \) with degree sequence \([k_i]\) if and only if

(i) the sum of the \( k_i \)'s is even, and

(ii) for any three subsets \( S, T, U \) (empty sets are not excluded) which partition \( V \), we have

\[
\sum_{v_i \in S} k_i \leq \sum_{v_i \in T} k_i + \sum_{v_i \in S \cup U \setminus T} c_{ii}.
\]

We shall use Theorem 2.1 in the derivation of our theorem.
3. The factorization theorem:

We prove the following n.s.c. for the existence of factors.

**Theorem 3.1.** Let $G$ be a graph satisfying the odd cycle property. Then $G$ contains a $[k_i]$ factor if and only if the sequence $[k_i] + |H|$ is graphical for all subgraphs $H \subseteq G^c$.

**Proof.** The necessity part is trivial. To prove the sufficiency, we first note that $[k_i]$ is graphical (putting $E(H) = \emptyset$) and thus the sum of the $k_i$'s is even. To show that inequality (1) holds, we choose the subgraph $H$ as follows:

$$E(H) = \{ (v_i, v_j) : v_i \in S, v_j \in S \cup U \text{ and } v_i \neq v_j \} \cap E(G^c).$$

If we let $|d_i| = [k_i] + |H|$, then, since $|d_i|$ is graphical, we get

$$\sum_{v_i \in S} d_i \leq |S|(|S \cup U| - 1) + \sum_{v_i \in \bar{S}} d_i = |S|(|S \cup U| - 1) + \sum_{v_i = F} k_i.$$

We also have

$$\sum_{v_i \in S} d_i - \sum_{v_i \in S} k_i = |S|(|S \cup U| - 1) - \sum_{v_j \in S \cup U} c_{ij}$$

$$= 2 \text{ (the number of lines of } H \text{ joining vertices in } S) + \text{ (the number of lines of } H \text{ joining a vertex in } S \text{ to a vertex in } U).$$

Combining (2) and (3), we get immediately inequality (1). The theorem is proved.

The following corollary is essentially a restatement of Theorem 3.1.

**Corollary 3.2.** Let $F$ be a graph whose complement has the odd cycle property. Then there exists a realization of $[k_i]$ containing $F$ if and only if the sequence $[k_i] - [F]$ is graphical, and for every line $(v_i, v_j)$ of $F$, there exists a realizing graph of $[k_i]$ containing $F - (v_i, v_j)$.

Corollary 3.2 (equivalently, Theorem 3.1) is tight in the sense that the conclusion may not hold true if $F^c$ does not satisfy the odd cycle property. This is illustrated in the following example.
Example 3.3. Let \( F = K_{3,3} \) and \( \{ k_1 \} = \{ 4, 4, 4, 4, 4 \} \). Fig. 1 shows a realization of \( \{ k_1 \} \) which contains all but one line of \( F \). Since \( k_1 \) are equal and the graphs \( F - (v_i, u_j) \) are isomorphic to each other, the hypothesis of Corollary 3.2 is satisfied. Clearly, there is no regular graph of degree 4 containing the graph \( K_{3,3} \).

On the other hand, we need, at least in general, the full strength of the hypothesis in the corollary. This is illustrated in the following example. Let \( F \) consist of three lines \( \{(v_1, v_2), (v_3, v_4), (v_5, v_6)\} \) and \( \{ k_1 \} = \{ 2, 2, 1, 1, 3, 3 \} \). Indeed, there is no realization of the sequence \( \{ k_1 \} \) which contains \( F \). It may be explained by saying that \( \{ k_1 \} \) has no realization containing graph \( F - (v_5, u_6) \) although the sequence \( \{ k_1 \} \) is graphical.

We noted earlier that the complement of a line graph has the odd cycle property. It is also clear from the definition that if \( G \) has the odd cycle property, it implies that every point induced subgraph of \( G \) has the property. This suggests the following definition. Let \( O(G) = (T(G^c))^c \), where \( T(G) \) denotes the total graph of \( G \). The vertices of \( O(G) \) fall into two disjoint classes. There is a vertex \( v_i \) for each \( 1 \leq i \leq n \) and a vertex \( e \) corresponding to each line in \( G^c \). We denote the two classes of vertices by \( V \) and \( E \), respectively. Two vertices \( e, e' \) are adjacent in \( O(G) \) if the corresponding lines of \( G^c \) have no common vertex; a vertex \( v_i \) is adjacent to \( e \) if line \( e \) (in \( G^c \)) is not incident with \( v_i \) in graph \( G^c \), and finally, \( v_i, v_j \) are adjacent in \( O(G) \) if \( (v_i, u_j) \) belongs to \( E(G) \). We prove:

Proposition 3.4. A graph \( G \) has the odd cycle property if and only if the graph \( O(G) \) has the odd cycle property.

Proof. We prove the "only if" part. Suppose \( G \) satisfies the odd cycle property, and suppose \( C, D \) are two odd cycles in \( O(G) \) which are at a distance of 2 or more from each other. We can assume that \( C, D \) have no chords, i.e., a line joining two vertices in \( C \) (or \( D \)) belongs to the cycle itself. First we show that either all vertices of \( C \) are \( v_i \) type or they are
e type. Assume the contrary. Let $C = \{x_1, x_2, \ldots, x_r, x_1\}$ written as a closed sequence of vertices where consecutive vertices are adjacent. Let $y_1$ denote an arbitrary vertex in cycle $D$. If $x_1, x_2, x_3 \in V$ and $x_4 \in E$, then it follows that $x_4$ is adjacent to $x_1$, a contradiction. Therefore $x_4 \in V$, and it follows successively that each $x_i$ is in $V$. Now suppose that $x_1, x_2 \in V$ and $x_r, x_3 \in E$. In this case, $x_3 = (x_1, v_j)$ (as a line of $G^c$), where $v_j \neq x_2$ and $x_j = (x_2, v_j)$. The vertex $y_1$, not being adjacent to $x_1$, $x_2$, cannot be an $e$ type vertex and $y_1 = v_j$. But this is impossible since $y_1$ is an arbitrary vertex in $D$. Finally, let $x_1 \in V$ and $x_1, x_2 \in E$. If $x_1$ corresponds to a line in $G^c$, then we can write $x_2 = (v_i, v_j)$ and $x_j = (x_1, v_k)$ while $x_i = (v_j, v_k)$. Consequently, $y_1$ is adjacent to at least one of $x_i, x_1, x_2, x_3$. On the other hand, if $x_3$ is in $V$, then $y_1$ is the common vertex of the lines corresponding to $x_2$ and $x_i$. Once again this is impossible. Hence all vertices of each of $C, D$ consist entirely of vertices from $V$ or from $E$. It is easy to show now that the distance between $C$ and $D$ is at most one. The proof is complete.

4. The digraphs

In this section we provide a necessary and sufficient condition for factorization of digraphs in the form of Theorem 3.1. This time the proof is based on the concept of alternating chains, and will hold for all digraphs. Other n.s.c. are given in [3, 7, 8], which have the form of a system of inequalities. In view of Theorem 4.1, those conditions ought to be equivalent to the realizability conditions given below.

In the following, $(u_i, v_j)$ denotes the arc from $u_i$ to $v_j$. A digraph may have both the arcs $(u_i, v_j)$ and $(v_j, u_i)$. Multiple arcs and loops are excluded from the present discussion. The set of arcs of a digraph $G$ will be denoted by $E(G)$. The outdegree (indegree) of a vertex $u_i$ is defined as the number of arcs in $G$ incident from (into) $u_i$. If we let $d_i^+$ and $d_i^-$ denote, respectively, the outdegree and indegree of $u_i$, then the sequence $[d_i^+, d_i^-]$ is called the degree sequence of $G$ and written in short as $(d_i)$. The digraph $G$ is said to be a realization of the degree sequence. A digraph $F$ with the same vertices as $G$ and $E(F) \subseteq E(G)$ is called a subdigraph of $G$ and denoted by $F \subseteq G$. Symmetric difference of two sets $A, B$ is denoted by $A \Delta B$. The following factorization theorem (in the complement form) is proved.
Theorem 4.1. There exists a realization of the degree sequence \([d_1^*, d_1]\) containing the digraph \(F\) if and only if the sequence \([d_1^*, d_1]\) \([G]\) is realizable by a digraph for all subdigraphs \(G \subseteq F\).

Proof (by induction). If \(E(F) = \emptyset\), there is nothing to prove. Let the theorem be true for all digraphs with \(|E(F)| = 1\) arcs. Let \((u_i, v_j)\) be an arc of \(F\) and let \(F_1\) be the subdigraph obtained by removing \((u_i, v_j)\) from \(F\). Then there are digraphs \(H_1\) and \(H_2\), each containing \(F_1\), where \([H_1]\) = \([d_1^*, d_1]\) and \(H_2\) has the same degree sequence as that of \(H_1\) except that the outdegree (indegree) of \(u_i(v_j)\) in \(H_2\) is one less than the corresponding degree in \(H_1\). Assume that the quantity \(m = |E(H_1) \cap E(H_2)| - E(F_1)|\) is maximum among all choices of \(H_1, H_2\). We show that \((u_i, v_j)\) belongs to \(H_1\). First observe that there exists a (chain)sequence of vertices and arcs such as

\[P = [x_0, x_1, x_2, x_3, \ldots, x_{2q}, x_{2q+1}, x_{2q+2}]\]

where \(v_i = x_i\) and \(x_{2q+1} = v_j\), and the arcs \((x_{i}, x_{i+1})\) and \((x_{i}, x_{i-1})\) belong, respectively, to \(E(H_1) \cap E(H_2)\) and \(E(H_1) \cap E(\overline{F_1})\) for \(i = 0, 2, \ldots, 2q\). If possible, suppose \((u_i, v_j) \notin E(H_1)\). There are two cases. If \((u_i, v_j)\) is not an arc in \(P\), then one could define the digraph \(H_1'\) by \(E(H_1') = E(H_1) \cup E(P)\), which would contain \(F\) as a subdigraph. On the other hand, if \((u_i, v_j)\) belongs to \(P\), let it be the 2th arc or the \((2t+1)th\) arc. In either case, the first \(2t\) arcs in \(P\) define a closed "alternating" chain \(Q\). Define a digraph \(H_1'\) by \(E(H_1') = E(H_1) \cup E(Q)\), where \(H_1'\) has the same degree sequence as that of \(H_1\) but gives a larger value of \(m\), a contradiction. The theorem is proved.

Acknowledgment

I am indebted to the anonymous referee for suggesting the possibility of proving Theorem 3.1 in this generality. Originally, it was proved only for a very small subclass of graphs satisfying the odd cycle property using the theory of alternating paths.
References