We study the rate of approximation by Nörlund means for Walsh–Fourier series of a function in $L^p$ and, in particular, in $\text{Lip}(\alpha, p)$ over the unit interval $[0, 1)$, where $\alpha > 0$ and $1 \leq p < \infty$. In case $p = \infty$, by $L^p$ we mean $C_w$, the collection of the uniformly $W$-continuous functions over $[0, 1)$. As special cases, we obtain the earlier results by Yano, Jastrebova, and Skvorcov on the rate of approximation by Cesàro means. Our basic observation is that the Nörlund kernel is quasi-positive, under fairly general assumptions. This is a consequence of a Sidon type inequality. At the end, we raise two problems.

1. INTRODUCTION

We consider the Walsh orthonormal system $\{\psi_k(x): k \geq 0\}$ defined on the unit interval $I = [0, 1)$ in the Paley enumeration (see [4]). To be more specific, let

$$r_0(x) := \begin{cases} 1 & \text{if } x \in [0, 2^{-1}), \\ -1 & \text{if } x \in [2^{-1}, 1), \end{cases}$$

$$r_0(x + 1) := r(x),$$

$$r_j(x) := r_0(2^j x), \quad j \geq 1 \text{ and } x \in I,$$

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be the well-known Rademacher functions. For \( k = 0 \) set \( w_0(x) = 1 \), and if
\[
k := \sum_{j=0}^{\infty} k_j 2^j, \quad k_j = 0 \text{ or } 1,
\]
is the dyadic representation of an integer \( k \geq 1 \), then set
\[
w_k(x) := \prod_{j=0}^{\infty} [r_j(x)]^{k_j}.
\]
We denote by \( \mathcal{P} \), the collection of Walsh polynomials of order less than \( n \), that is, functions of the form
\[
P(x) := \sum_{k=0}^{n-1} a_k w_k(x),
\]
where \( n \geq 1 \) and \( \{a_k\} \) is any sequence of real (or complex) numbers.

Denote by \( \Sigma_m \) the finite \( \sigma \)-algebra generated by the collection of dyadic intervals of the form
\[
I_m(k) := [k 2^{-m}, (k + 1) 2^{-m}), \quad k = 0, 1, \ldots, 2^m - 1,
\]
where \( m \geq 0 \). It is not difficult to see that the collection of \( \Sigma_m \)-measurable functions on \( I \) coincides with \( \mathcal{P}_m, \ m \geq 0 \).

We will study approximation by means of Walsh polynomials in the norm of \( L^p = L^p(I) \), \( 1 \leq p < \infty \), and \( C_W = C_W(I) \). We remind the reader that \( C_W \) is the collection of functions \( f: I \to \mathbb{R} \) that are uniformly continuous from the dyadic topology of \( I \) to the usual topology of \( \mathbb{R} \), or in short, uniformly \( W \)-continuous. The dyadic topology is generated by the union of \( \Sigma_m \) for \( m = 0, 1, \ldots \).

As is known (see, e.g., [6, p. 9]), a function belongs to \( C_W \) if and only if it is continuous at every dyadic irrational of \( I \), is continuous from the right on \( I \), and has a finite limit from the left on \( (0, 1] \), all these in the usual topology. Hence it follows immediately that if the periodic extension of a function \( f \) from \( I \) to \( \mathbb{R} \) with period 1 is classically continuous, then \( f \) is also uniformly \( W \)-continuous on \( I \). The converse statement is not true. For example, the Walsh functions \( w_k \) belong to \( C_W \), but they are not classically continuous for \( k \geq 1 \).

For the sake of brevity in notation, we agree to write \( L^\infty \) instead of \( C_W \) and set
\[
\|f\|_\infty := \sup \{|f(x)|: x \in I\}.
\]
After these preliminaries, the best approximation of a function $f \in L^p$, $1 \leq p \leq \infty$, by polynomials in $\mathcal{P}_n$ is defined by

$$E_n(f, L^p) := \inf_{P \in \mathcal{P}_n} \|f - P\|_p.$$ 

Since $\mathcal{P}_n$ is a finite dimensional subspace of $L^p$ for any $1 \leq p \leq \infty$, this infimum is attained.

From the results of [6, pp. 142 and 156–158] it follows that $I_{\omega_1}$ is the closure of the Walsh polynomials when using the norm $\|\cdot\|_p$, $1 \leq p \leq \infty$. In particular, $C_W$ is the uniform closure of the Walsh polynomials.

Next, define the modulus of continuity in $L^p$, $1 \leq p \leq \infty$, of a function $f \in L^p$ by

$$\omega_p(f, \delta) := \sup_{|\tau| < \delta} \|\tau f - f\|_p, \quad \delta > 0,$$

where $\tau$, means dyadic translation by $t$:

$$\tau f(x) := f(x + t), \quad x, t \in I.$$ 

Finally, for each $\alpha > 0$, Lipschitz classes in $L^p$ are defined by

$$\text{Lip}(\alpha, p) := \{f \in L^p : \omega_p(f, \delta) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$ 

Unlike the classical case, $\text{Lip}(\alpha, p)$ is not trivial when $\alpha > 1$. For example, the function $f := w_0 + w_1$ belongs to $\text{Lip}(\alpha, p)$ for all $\alpha > 0$ since

$$\omega_p(f, \delta) = 0 \quad \text{when} \quad 0 < \delta < 2^{-1}.$$ 

2. **Main Results**

Given a function $f \in L^1$, its Walsh–Fourier series is defined by

$$\sum_{k=0}^{\infty} a_k w_k(x), \quad \text{where} \quad a_k := \int_0^1 f(t) w_k(t) \, dt. \quad (2.1)$$

The $n$th partial sums of series in (2.1) are

$$s_n(f, x) := \sum_{k=0}^{n-1} a_k w_k(x), \quad n \geq 1.$$ 

As is well known,

$$s_n(f, x) = \int_0^1 f(x + t) D_n(t) \, dt,$$
where

\[ D_n(t) := \sum_{k=0}^{n-1} w_k(t), \quad n \geq 1, \]

is the Walsh-Dirichlet kernel of order \( n \).

Let \( \{q_k : k \geq 0\} \) be a sequence of nonnegative numbers. The Nörlund means for series (2.1) are defined by

\[ t_n(f, x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} s_k(f, x), \]

where

\[ Q_n := \sum_{k=0}^{n-1} q_k, \quad n \geq 1. \]

We always assume that \( q_0 > 0 \) and

\[ \lim_{n \to \infty} Q_n = \infty. \quad (2.2) \]

In this case, the summability method generated by \( \{q_k\} \) is regular if and only if

\[ \lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0. \quad (2.3) \]

As to this notion and result, we refer the reader to [2, pp. 37–38].

We note that in the particular case when \( q_k = 1 \) for all \( k \), these \( t_n(f, x) \) are the first arithmetic or \( (C, 1) \)-means. More generally, when

\[ q_k = A^\beta_k := \binom{\beta+k}{k} \quad \text{for} \quad k \geq 1 \quad \text{and} \quad q_0 = A^\beta_0 := 1, \]

where \( \beta \neq -1, -2, \ldots \), the \( t_n(f, x) \) are the \( (C, \beta) \)-means for series (2.1).

The representation

\[ t_n(f, x) = \int_0^1 f(x + t) L_n(t) \, dt \quad (2.4) \]

plays a central role in the sequel, where

\[ L_n(t) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k(t), \quad n \geq 1, \quad (2.5) \]

is the so-called Nörlund kernel.
Our main results read as follows.

**Theorem 1.** Let $f \in L^p$, $1 \leq p \leq \infty$, let $n = 2^m + k$, $1 \leq k \leq 2^m$, $m \geq 1$, and let \{$q_k: k \geq 0$\} be a sequence of nonnegative numbers such that

\[ \frac{n^\gamma - 1}{Q_n} \sum_{k=0}^{n-1} q_k = O(1) \quad \text{for some} \quad 1 < \gamma \leq 2. \quad (2.6) \]

If \{$q_k$\} is nondecreasing, then

\[ \|t_n(f) - f\|_p \leq \frac{5}{2Q_n} \sum_{j=0}^{m-1} 2^j q_{n-2^j} \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}, \quad (2.7) \]

while if \{$q_k$\} is nonincreasing, then

\[ \|t_n(f) - f\|_p \leq \frac{5}{2Q_n} \sum_{j=0}^{m-1} \left( Q_{n-2^{j+1}} - Q_{n-2^{j+1}+1} \right) \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}. \quad (2.8) \]

Clearly, condition (2.6) implies (2.2) and (2.3).

We note that if \{$q_k$\} is nondecreasing, in sign $q_k \uparrow$, then

\[ \frac{nq_{n-1}}{Q_n} = O(1) \quad (2.9) \]

is a sufficient condition for (2.6). In particular, (2.9) is satisfied if

\[ q_k \asymp k^\beta \quad \text{or} \quad (\log k)^\beta \quad \text{for some} \quad \beta > 0. \]

Here and in the sequel, $q_k \asymp r_k$ means that the two sequences \{$q_k$\} and \{$r_k$\} have the same order of magnitude; that is, there exist two positive constants $C_1$ and $C_2$ such that

\[ C_1 r_k \leq q_k \leq C_2 r_k \quad \text{for all} \quad k \text{ large enough.} \]

If \{$q_k$\} is nonincreasing, in sign $q_k \downarrow$, then condition (2.6) is satisfied if, for example,

\[ \begin{align*}
(i) & \quad q_k \asymp k^{-\beta} \quad \text{for some} \quad 0 < \beta < 1, \text{ or} \\
(ii) & \quad q_k \asymp (\log k)^{-\beta} \quad \text{for some} \quad 0 < \beta.
\end{align*} \quad (2.10) \]

Namely, it is enough to choose $1 < \gamma < \min(2, \beta^{-1})$ in case (i), and $\gamma = 2$ in case (ii).
THEOREM 2. Let \( \{q_k : k \geq 0\} \) be a sequence of nonnegative numbers such that in case \( q_k \uparrow \) condition (2.9) is satisfied, while in case \( q_k \downarrow \) condition (2.10) is satisfied. If \( f \in \text{Lip}(\alpha, p) \) for some \( \alpha > 0 \) and \( 1 \leq p \leq \infty \), then

\[
|t_n(f) - f|_p = \begin{cases} 
\mathcal{O}(n^{-\frac{\alpha}{2}}) & \text{if } 0 < \alpha < 1, \\
\mathcal{O}(n^{-1} \log n) & \text{if } \alpha = 1, \\
\mathcal{O}(n^{-1}) & \text{if } \alpha > 1.
\end{cases}
\]  

(2.11)

Now we make a few historical comments. The rate of convergence of \((C, \beta)\)-means for functions in \( \text{Lip}(\alpha, p) \) was first studied by Yano [10] in the cases when \( 0 < \alpha < 1, \, \beta > \alpha \), and \( 1 \leq p \leq \infty \); then by Jastrebova [1] in the case when \( \alpha = \beta = 1 \) and \( p = \infty \). Later on, Skvorcov [7] showed that these estimates hold for \( 0 < \beta \leq \alpha \) as well, and also studied the cases when \( \alpha = 1, \, \beta > 0 \), and \( 1 \leq p \leq \infty \). In their proofs, the above authors rely heavily on the specific properties of the binomial coefficients \( A_n^k \).

Watari [8] proved that a function \( f \in L^p \) belongs to \( \text{Lip}(\alpha, p) \) for some \( \alpha > 0 \) and \( 1 \leq p \leq \infty \) if and only if

\[
E_n(f, L^p) = \mathcal{O}(n^{-\frac{\alpha}{2}}).
\]

Thus, for \( 0 < \alpha < 1 \) the rate of approximation to functions \( f \) in \( \text{Lip}(\alpha, p) \) by \( t_n(f) \) is as good as the best approximation.

3. AUXILIARY RESULTS

Yano [9] proved that the Walsh–Fejér kernel

\[
K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t) = \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n} \right) w_k(t), \quad n \geq 1,
\]

is quasi-positive, and \( K_{2^n}(t) \) is even positive. These facts are formulated in the following

**Lemma 1.** Let \( m \geq 0 \) and \( n \geq 1 \); then \( K_{2^n}(t) \geq 0 \) for all \( t \in I \),

\[
\int_0^1 |K_n(t)| \, dt \leq 2 \quad \text{and} \quad \int_0^1 K_{2^n}(t) \, dt = 1.
\]

A Sidon type inequality proved by Schipp and the author (see [3]) implies that the Nörlund kernel \( L_n(t) \) is also quasi-positive. More exactly,

\[
C = \left[ \mathcal{O}(1) \right]^{1/2\gamma/(\gamma - 1)}
\]

in the next lemma, where \( \mathcal{O}(1) \) is from (2.6).
Lemma 2. If condition (2.6) is satisfied, then there exists a constant $C$ such that

$$\int_0^1 |L_n(t)| \, dt \leq C, \quad n \geq 1.$$ 

Now, we give a specific representation of $L_n(t)$, interesting in itself.

**Lemma 3.** Let $n = 2^m + k$, $1 \leq k \leq 2^m$, and $m \geq 1$; then

$$Q_n L_n(t) = - \sum_{j=0}^{2^{m-1}-1} r_j(t) w_{2^j-1}(t) \sum_{i=1}^{2^j-1} \delta(q_n - 2^{j+1}i + q_n - 2^{j+1}i + 1) K_i(t)$$

$$- \sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) 2^j q_n - 2^i K_{2^j}(t)$$

$$+ \sum_{j=0}^{m-1} (Q_{n-2^{j+1}} - Q_{n-2^{j+1}+1}) D_{2^j+1}(t)$$

$$+ Q_{k+1} D_{2^m}(t) + Q_k r_m(t) L_k(t). \quad (3.1)$$

**Proof.** The technique applied in the proof is essentially due to Skvorcov [7]. By (2.5),

$$Q_n L_n(t) = \sum_{i=1}^{2^m} q_n - i D_i(t) + q_{n-2^m} D_{2^m}(t) + \sum_{i=2^m+1}^{2^m+k} q_{n-i} D_i(t)$$

$$= \sum_{j=0}^{2^{m-1}-1} \sum_{i=0}^{2^j-1} q_{n-2^j-i} (D_{2^j+1}(t) - D_{2^j}(t))$$

$$+ \sum_{j=0}^{m-1} \left( \sum_{i=0}^{2^j-1} q_{n-2^j-i} \right) D_{2^j+1}(t)$$

$$+ q_{n-2^m} D_{2^m}(t) + \sum_{i=1}^{k} q_{n-2^m-i} D_{2^m+i}(t). \quad (3.2)$$

As is well known (see, e.g., [6, p. 46]),

$$D_{2^m+i}(t) = D_{2^m}(t) + r_m(t) D_i(t), \quad 1 \leq i \leq 2^m. \quad (3.3)$$

Furthermore, by (1.1), it is not difficult to see that

$$w_{2^j-1-i}(t) = w_{2^j-1}(t) w_i(t), \quad 0 \leq i < 2^j.$$
Hence we deduce that

\[ D_{2^j+1}(t) - D_{2^j+1}(t) = r_j(t) \sum_{l=0}^{2^j-1} w_j(t) = r_j(t) \sum_{l=0}^{2^j-1} w_{2^j-1-l}(t) = r_j(t) w_{2^j-1}(t) D_{2^j-1}(t), \quad 0 \leq i < 2^j. \]  

(3.4)

Substituting (3.3) and (3.4) into (3.2) yields

\[
Q_n L_n(t) = - \sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) \sum_{l=0}^{2^j-1} q_{n-2^j+l} D_{2^j-1}(t) \\
+ \sum_{j=0}^{m-1} (Q_{n-2^j+l} - Q_{n-2^j+l+1}) D_{2^j+1}(t) \\
+ Q_{k+1} D_{2^m}(t) + Q_k r_m(t) L_k(t). \tag{3.5}
\]

Performing a summation by part gives

\[
\sum_{i=0}^{2^j-1} q_{n-2^j-i} D_{2^j-i}(t) = \sum_{i=1}^{2^j-1} iK(t)(q_{n-2^j+i} - q_{n-2^j+i+1}) + 2^j K_{2^j}(t) q_{n-2^j}.
\]

(3.6)

Substituting this into (3.5) results in (3.1).

**Lemma 4.** If \( g \in \mathcal{D}_2^m \), \( f \in L^p \), where \( m \geq 0 \) and \( 1 \leq p < \infty \), then for \( 1 \leq p < \infty \)

\[
\left\{ \int_0^1 \left| \int_0^1 r_m(t) g(t)[f(x + t) - f(x)] \, dt \right|^p \, dx \right\}^{1/p} \leq 2^{-1} \omega_p(f, 2^{-m}) \int_0^1 |g(t)| \, dt, \tag{3.6}
\]

while for \( p = \infty \)

\[
\sup \left\{ \left| \int_0^1 r_m(t) g(t)[f(x + t) - f(x)] \, dt : x \in I \right| \right\} \leq 2^{-1} \omega_\infty(f, 2^{-m}) \int_0^1 |g(t)| \, dt. \tag{3.7}
\]

**Proof.** Since \( g \in \mathcal{D}_2^m \), it takes a constant value, say \( g_m(k) \) on each dyadic interval \( I_m(k) \), where \( 0 \leq k < 2^m \). We observe that if \( t \in I_m(k) \) then \( t + 2^{-m-1} \in I_m(k) \).

We will prove (3.6). By Minkowski's inequality in the usual and in the generalized form, we obtain that
\[
\left\{ \int_0^1 \left| \int_0^1 r_m(t) g(t) \left[ f(x + t) - f(x) \right] dt \right|^p \ dx \right\}^{1/p} = \sum_{k=0}^{2^{m-1}} \left\{ \int_0^1 \left| \int_{m+1(2k)}^1 [f(x + t) - f(x + t + 2^{-m-1})] dt \right|^p \ dx \right\}^{1/p} \\
\leq \sum_{k=0}^{2^{m-1}} \left\{ \int_0^1 \left| \int_{m+1(2k)}^1 |f(x + t) - f(x + t + 2^{-m-1})| dt \right|^p \ dx \right\}^{1/p} \\
= \sum_{k=0}^{2^{m-1}} \left\{ \int_0^1 \left| \int_{m+1(2k)}^1 |f(x + t) - f(x + t + 2^{-m-1})| dt \right|^p \ dx \right\}^{1/p} \\
\leq \sum_{k=0}^{2^{m-1}} \left\{ \int_0^1 \left| \int_{m+1(2k)}^1 \omega_p(f, 2^{-m}) \ dx \right|^p \ dx \right\}^{1/p}.
\]
This is equivalent to (3.6).
Inequality to (3.7) can be proved analogously.

4. PROOFS OF THEOREMS 1 AND 2

We carry out the proof of Theorem 1 for \(1 \leq p < \infty\). The proof for \(p = \infty\) is similar and even simpler.

By (2.4), (3.1), and the usual Minkowski inequality, we may write that

\[
Q_n \| t_n(f) - f \|_p := \left\{ \int_0^1 \left| \int_0^1 Q_n L_n(t) \left[ f(x + t) - f(x) \right] dt \right|^p \ dx \right\}^{1/p} \\
\leq \sum_{j=0}^{m-1} \left\{ \int_0^1 \left| \int_0^1 r_j(t) g_j(t) \left[ f(x + t) - f(x) \right] dt \right|^p \ dx \right\}^{1/p} \\
+ \sum_{j=0}^{m-1} \left\{ \int_0^1 \left| \int_0^1 r_j(t) h_j(t) \left[ f(x + t) - f(x) \right] dt \right|^p \ dx \right\}^{1/p} \\
+ \sum_{j=0}^{m-1} \left( Q_{n-2j+1} - Q_{n-2j+1+1} \right) \\
\times \left\{ \int_0^1 \left| \int_0^1 D_{2j+1}(t) \left[ f(x + t) - f(x) \right] dt \right|^p \ dx \right\}^{1/p} \\
+ Q_{k+1} \left\{ \int_0^1 \left| \int_0^1 D_{2k}(t) \left[ f(x + t) - f(x) \right] dt \right|^p \ dx \right\}^{1/p} \\
+ Q_{k} \left\{ \int_0^1 \left| \int_0^1 r_m(t) L_k(t) \left[ f(x + t) - f(x) \right] dt \right|^p \ dx \right\}^{1/p} \\
=: A_{1n} + A_{2n} + A_{3n} + A_{4n} + A_{5n},
\]
(4.1)
say, where
\[ g_j(t) := \sum_{i=1}^{2^j - 1} \left| q_n \cdot 2^j \cdot i - q_n - 2^j \cdot i + 1 \right| K_i(t), \]
\[ h_j(t) := w_{2^j} \cdot 1(t) \cdot 2^j q_n \cdot 2^j q_n \cdot 2^j K_2(t), \quad 0 \leq j < m. \]

Applying Lemma 1, in the case when \( q_k \uparrow \), we get that
\[
\int_0^1 |g_j(t)| \, dt \leq 2 \sum_{i=1}^{2^j \cdot 1} i \left| q_n \cdot 2^j \cdot i - q_n - 2^j \cdot i + 1 \right|
= 2 \left( 2^j q_n - 2^j - \sum_{i=1}^{2^j} q_n \cdot 2^j \cdot i \right) \leq 2^{j+1} q_n \cdot 2^j,
\]
while in the case when \( q_k \downarrow \)
\[
\int_0^1 |g_j(t)| \, dt \leq 2 \left( 2^j q_n - 2^j - \sum_{i=1}^{2^j} q_n \cdot 2^j \cdot i \right) \leq 2(2^j q_n - 2^j - \sum_{i=1}^{2^j} q_n \cdot 2^j \cdot i).
\]
Thus, by Lemma 4, in the case \( q_k \uparrow \)
\[
A_{1n} \leq \sum_{j=0}^{m-1} 2^j q_n \cdot 2^j \omega_p(f, 2^j), \quad (4.2)
\]
while in the case \( q_k \downarrow \)
\[
A_{1n} \leq \sum_{j=0}^{m-1} (2^j q_n - 2^j - \sum_{i=1}^{2^j} q_n \cdot 2^j \cdot i) \omega_p(f, 2^j). \quad (4.3)
\]
By virtue of Lemmas 1 and 4 again, we obtain that
\[
A_{2n} \leq 2 \sum_{j=0}^{m-1} 2^j q_n - 2^j \omega_p(f, 2^j). \quad (4.4)
\]
Obviously, in the case \( q_k \downarrow \)
\[
2^j q_n - 2^j \leq Q_n - 2^j + 1 - Q_n - 2^j + 1 \cdot \quad (4.5)
\]
Since
\[
D_{2n}(t) = \begin{cases} 2^m & \text{if } t \in [0, 2^m), \\ 0 & \text{if } t \in [2^m, 1) \end{cases}
\]
(see, e.g., [6, p. 7]), by the generalized Minkowski inequality, we find that

\[
A_{3n} \leq \sum_{j=0}^{m-1} (Q_{n-2j+1} - Q_{n-2j+1+1}) \\
\times \int_0^1 D_{2j+1}(t) \left\{ \int_0^1 |f(x+t) - f(x)|^p dx \right\}^{1/p} dt \\
\leq \sum_{j=0}^{m-1} (Q_{n-2j+1} - Q_{n-2j+1+1}) \omega_p(f, 2^{-j}), \tag{4.6}
\]

\[
A_{4n} \leq Q_{k+1} \omega(f, 2^{-m}). \tag{4.7}
\]

Clearly, in the case \(q_k \uparrow\)

\[
Q_{n-2j+1} - Q_{n-2j+1+1} \leq 2^j q_{n-2j}. \tag{4.8}
\]

Finally, by Lemmas 2 and 4, in a similar way to the above we deduce that

\[
A_{5n} \leq 2^{-1} Q_{k} \omega_p(f, 2^{-m}) \int_0^1 |L_k(t)| dt \leq C Q_{n} \omega_p(f, 2^{-m}). \tag{4.9}
\]

Combining (4.1)–(4.9) yields (2.7) in the case \(q_k \uparrow\) and (2.8) in the case \(q_k \downarrow\).

**Proof of Theorem 2.** Case (a). \(q_k \uparrow\). We have

\[
n - 2^j \geq 2^{m-1} \quad \text{for } 0 \leq j \leq m - 1.
\]

Consequently, for such \(j\)’s

\[
\frac{2^j q_{n-2j}}{Q_n} = \frac{(n-2^j+1) q_{n-2j} / Q_{n-2j+1}}{Q_n} \frac{2^j}{n-2^j+1} \leq C 2^{j-m+1},
\]

where \(C\) equals \(\mathcal{O}(1)\) from (2.9). Since \(f \in \text{Lip}(\alpha, p)\), from (2.7) it follows that

\[
\|t_n(f) - f\|_p = \mathcal{O}(1) \sum_{j=0}^{m-1} 2^j q_{n-2j} 2^{-j\alpha} + \mathcal{O}(2^{-mx})
\]

\[
= \mathcal{O}(1) 2^m \sum_{j=0}^{m} 2^j \alpha \left\{ \begin{array}{ll}
\mathcal{O}(2^{-mx}) & \text{if } 0 < \alpha < 1, \\
\mathcal{O}(m 2^{-m}) & \text{if } \alpha = 1, \\
\mathcal{O}(2^{-m}) & \text{if } \alpha > 1.
\end{array} \right.
\]

This is equivalent to (2.11).
Case (b). $q_k \downarrow$. For example, we consider case (i) in (2.10). Then $Q_n \propto n^{1-\beta}$. This time we have

$$n - 2^{j+1} \geq 2^{m-1} \quad \text{for} \quad 0 \leq j \leq m-2.$$  

Since $f \in \text{Lip}(\alpha, p)$, from (2.8) it follows that

$$\|t_n(f) - f\|_p \leq \frac{5}{2} \sum_{j=0}^{m-2} 2^j q_{n-2j+1} \omega_p(f, 2^{-j})$$

$$+ \frac{5}{2} \omega_p(f, 2^{-m}) + \mathcal{O}\{\omega_p(f, 2^{-m})\}$$

$$= \frac{O(1)}{Q_n} \sum_{j=0}^{m-2} 2^j q_{n-2j+1} 2^{-j} x + \mathcal{O}(2^{-mx})$$

$$= \frac{\mathcal{O}(1)}{n^{1-\beta}} \sum_{j=0}^{m-2} 2^j (1-x) + \mathcal{O}(2^{-mx})$$

$$= \begin{cases} \mathcal{O}(n^{-1}2^m(1-x)) & \text{if} \quad 0 < \alpha < 1, \\ \mathcal{O}(n^{-1}m) & \text{if} \quad \alpha = 1, \\ \mathcal{O}(n^{-1}) & \text{if} \quad \alpha > 1. \end{cases}$$

Clearly, this is equivalent to (2.11).

Case (ii) in (2.10) can be proved analogously.

5. CONCLUDING REMARKS AND PROBLEMS

(A) We have seen that condition (2.6) is satisfied when $q_k = (k+1)^\beta$ for some $\beta > -1$, and Theorems 1 and 2 apply. If $q_k$ increases faster than a positive power of $k$, then relation (2.6) is no longer true in general. But the case, for example, when $q_k$ grows exponentially is not interesting, since then condition (2.3) of regularity is not satisfied. On the other hand, the case when $\beta = -1$ is of special interest.

Problem 1. Find substitutes of (2.8) and (2.11) when $q_k = (k+1)^{-1}$. In this case, the $t_n(f)$ are called the logarithmic means for series (2.1).

(B) It is also of interest that Theorems 1 and 2 remain valid when

$$q_k \asymp k^\beta \varphi(k),$$

where $\beta > -1$ and $\varphi(k)$ is a positive and monotone (nondecreasing or nonincreasing) functions in $k$, slowly varying in the sense that

$$\lim_{k \to \infty} \frac{\varphi(2k)}{\varphi(k)} = 1.$$
It is not difficult to check that in this case
\[ Q_n \approx n^{1+\beta} \varphi(n). \]

(C) Now, we turn to the so-called saturation problem concerning the Nörlund means \( t_n(f) \). We begin with the observation that the rate of approximation by \( t_n(f) \) to functions in \( \text{Lip}(a, p) \) cannot be improved too much as \( a \) increases beyond 1. Indeed, the following is true.

**Theorem 3.** If \( \{q_k\} \) is a sequence of nonnegative numbers such that
\[ \liminf_{m \to \infty} q_{2m} > 0, \quad (5.2) \]
and if for some \( f \in L^p, 1 \leq p \leq \infty \),
\[ \|t_{2m}(f) - f\|_p = o(Q_{2m}^{-1}) \quad \text{as} \quad m \to \infty, \quad (5.3) \]
then \( f \) is constant.

We note that condition (5.2) is certainly satisfied if \( q_k \uparrow \) or \( q_k \downarrow \) and \( \lim q_k > 0 \).

**Proof.** Since by definition
\[ E_{2m}(f, L^p) \leq \|t_{2m}(f) - f\|_p, \]
and by a theorem of Watari [8]
\[ \|s_{2m}(f) - f\|_p \leq 2E_{2m}(f, L^p), \]
it follows from (5.3) that
\[ \|s_{2m}(f) - f\|_p = o(Q_{2m}^{-1}) \quad \text{as} \quad m \to \infty. \quad (5.4) \]

A simple computation gives that
\[ Q_{2m}\{s_{2m}(f, x) - t_{2m}(f, x)\} = \sum_{k=1}^{2^m-1} (Q_{2^m} - Q_{2^{m-k}}) a_k w_k(x). \]

Now, (5.3) and (5.4) imply that
\[ \lim_{m \to \infty} \sum_{k=1}^{2^m-1} (Q_{2^m} - Q_{2^{m-k}}) a_k w_k(x) = 0. \]
Since \( \| \cdot \|_1 \leq \| \cdot \|_p \), for any \( p \geq 1 \) it follows that

\[
\lim_{m \to \infty} \left| (Q_{2^m} - Q_{2^{m-j}}) a_j \right| = \lim_{m \to \infty} \left| \int_0^1 w_j(x) \left\{ \sum_{k=1}^{2^m-1} (Q_{2^m} - Q_{2^{m-k}}) a_k w_k(x) \right\} dx \right| 
\leq \lim_{m \to \infty} \left\| \sum_{k=1}^{2^m-1} (Q_{2^m} - Q_{2^{m-k}}) a_k w_k(w) \right\|_1 = 0
\]

Hence, by (5.2), we conclude that \( a_j = 0 \) for all \( j \geq 1 \). Therefore, \( f = a_0 \) is constant.

In the particular case when \( q_k = 1 \) for all \( k \), the \( t_n(f) \) are the \((C, 1)\)-means for series (2.1) defined by

\[
\sigma_n(f, x) := \frac{1}{n} \sum_{k=1}^{n} s_k(f, x), \quad n \geq 1,
\]

and Theorem 3 is known (see, e.g., [6, p. 191]). It says that if for some \( f \in L^p \), \( 1 \leq p \leq \infty \),

\[
\|\sigma_{2^m}(f) - f\|_p = o(2^{-m}) \quad \text{as} \quad m \to \infty,
\]

then \( f \) is necessarily constant.

**Problem 2.** How can one characterize those functions \( f \in L^p \) such that

\[
\|\sigma_n(f) - f\|_p = o(n^{-1}) \quad \text{for some} \quad 1 \leq p \leq \infty?
\]  \hspace{2cm} (5.5)

We conjecture that (5.5) holds if and only if

\[
\sum_{m=0}^{\infty} 2^m \omega_p(f, 2^{-m}) < \infty, \quad \text{or equivalently} \quad \sum_{k=1}^{\infty} \omega_p(k^{-1}) < \infty.
\]

The "if" part can be proved in the same manner as in the case when \( \omega_p(f, \delta) = O(\delta^x) \) for some \( x > 1 \) (cf. [6, p. 190]). The proof (or disproof) of the "only if" part is a problem.

(D) Finally, we note that the results of this paper can be carried over to the systems that are obtained from the Walsh Paley system \( \{w_k(x)\} \) by means of the so-called piecewise linear rearrangements introduced by Schipp [5]. (See also [7].) In particular, the Walsh–Kaczmarz system is among them.
REFERENCES

5. F. SCHIPP, On certain rearrangements of series with respect to the Walsh system, *Mat. Zametki* 18 (1975), 193–201. [Russian]