ACADEMIC PRESS
Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT®
Journal of Algebra 265 (2003) 221–228
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Characterization of completions of excellent domains of characteristic zero

S. Loepp

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, USA

Received 15 July 2002
Communicated by Craig Huneke

Abstract

We show that a complete local ring T containing the integers is the completion of a local excellent integral domain if and only if it is reduced, equidimensional, and no integer of T is a zero divisor.

1. Introduction

In [3], Lech gives necessary and sufficient conditions for a complete local ring to be the completion of a local integral domain. Specifically, he shows that a complete local ring T with maximal ideal M is the completion of a local integral domain if and only if

(1) unless $M = (0)$, M is not an associated prime ideal of T, and
(2) no integer of T is a zero divisor.

In this paper, we explore the corresponding question for excellent local integral domains. In other words, our goal is to find necessary and sufficient conditions for a complete local ring T to be the completion of an excellent local integral domain. We have achieved our goal in the characteristic zero case, but unfortunately, our proof falls a bit short for characteristic p. Immediately, one can see that if a complete local ring is the completion of an excellent integral domain, then it must be reduced, equidimensional, and satisfy the condition that no integer is a zero divisor. Surprisingly, these three conditions are also sufficient in the characteristic zero case. In other words, we show that a complete local

E-mail address: sloepp@williams.edu.

0021-8693/03/$ – see front matter
doi:10.1016/S0021-8693(03)00239-4

ring T containing the integers is the completion of an excellent local integral domain if and only if it is reduced, equidimensional, and no integer of T is a zero divisor. To show that these conditions are sufficient, we consider the case where $\dim T = 0$ and the case where $\dim T > 0$ separately. If T is a complete local reduced ring of dimension zero, then it is a field, so is the completion of an excellent local integral domain (namely, itself). For the other case, suppose T is a complete local reduced equidimensional ring containing the integers where no integer is a zero divisor and T is of dimension greater than zero. The bulk of this paper is dedicated to this case. In other words, given such a ring T we work to construct a local excellent integral domain whose completion is T. The idea of the construction is based on the methods used in [1,5]. We start with the prime subring of T (which will be $\mathbb{Z}$ in our case) localized at the appropriate prime ideal. Then, we use a process of successively adjoining elements of T to this ring. Of course, we want to adjoin these elements carefully. More specifically, we will always avoid zero divisors of T. This will force the ring we construct to be an integral domain. Also, we will adjoin elements of T until we have that for every ideal J of T such that J is not contained in an associated prime ideal of T, our ring contains an element of every coset in the ring T/JT. In other words, the ring that we construct, call it A, will satisfy the property that if J is an ideal of T with $J \nsubseteq Q$ for every $Q \in \operatorname{Ass} T$, then the map $A \to T/JT$ is onto. In addition, we will construct A so that $IT \cap A = I$ for every finitely generated ideal I of A. It turns out that these two properties are enough to force the completion of A to be T and A to be excellent.

We now make a few comments on notation. All rings in this paper are assumed to be commutative with unity. When we use the term local ring, we mean a ring that is Noetherian with exactly one maximal ideal. We will call a ring that is not necessarily Noetherian but that has exactly one maximal ideal quasi-local. We use (T, M) to denote a (quasi-) local ring T with maximal ideal M. Finally, we use c to denote the cardinality of the real numbers.

2. The construction

Recall that the bulk of our proof will be dedicated to showing that if T is a complete local ring that contains the integers, is reduced, equidimensional, is of dimension at least one, and satisfies the property that no integer is a zero divisor, then it is the completion of an excellent integral domain, call it A. We now begin the construction of A.

The following proposition is Proposition 1 from [2]. It will be used to show that the ring A we construct has the desired completion.

Proposition 1. *If $(R, M \cap R)$ is a quasi-local subring of a complete local ring (T, M), the map $R \to T/M^2$ is onto and $IT \cap R = I$ for every finitely generated ideal I of R, then R is Noetherian and the natural homomorphism $\widehat{R} \to T$ is an isomorphism.*

The following two elementary lemmas are needed for the proof of Lemma 4—a key lemma in our proof. Although they are no doubt well-known, we state them here and include a proof of Lemma 3.

Lemma 2. *Let T be an integral domain and I a nonzero ideal of T. Then, $|I| = |T|$.*

Lemma 3. *Let (T, M) be a complete local reduced ring of dimension at least one. Let $Q \in \operatorname{Ass} T$. Then, $|T/Q| = |T| \geqslant c$.*

Proof. Since T is reduced, complete and $\dim T \geqslant 1$, we have $|T| \geqslant c$. Now, T/Q is a complete local domain with $\dim(T/Q) \geqslant 1$. It follows that $|T/Q| \geqslant c$. Define a map $f : T \to \prod_{i=1}^{\infty} T/M^i$ by $f(t) = (t + M, t + M^2, t + M^3, \ldots)$. It is easy to see that f is injective and hence $|T| = \sup\{c, |T/M|\}$. Now, $|T/Q| \leqslant |T|$ and $|T/Q| \geqslant \sup\{c, |T/M|\} = |T|$, so $|T/Q| = |T|$. □

Lemma 4 is really the breakthrough lemma of this paper. We use it to adjoin elements to a specific subring of T so that the resulting ring contains no zero divisors of our complete local ring T.

Lemma 4. *Let T be a complete local reduced ring of dimension at least one and let I be an ideal of T such that $I \nsubseteq Q$ for every $Q \in \operatorname{Ass} T$. Let D be a subset of T such that $|D| < |T/Q|$ where $Q \in \operatorname{Ass} T$. Then $I \nsubseteq \bigcup\{r + Q \mid Q \in \operatorname{Ass} T,\ r \in D\}$.*

Proof. Let $\operatorname{Ass} T = \{Q_1, Q_2, \ldots, Q_n\}$. By the Prime Avoidance Theorem, $I \nsubseteq \bigcup_{i=1}^{n} Q_i$. Let $x \in I$, $x \notin \bigcup_{i=1}^{n} Q_i$. Define a family of maps $f_i : \{Q_i\} \times D \to T$ as follows. Let $(Q_i, r) \in \{Q_i\} \times D$. If $r + Q_i \notin (x + Q_i)(T/Q_i)$, define $f_i(Q_i, r) = 0$. On the other hand, if $r + Q_i \in (x + Q_i)(T/Q_i)$, then choose an $s_i \in T$ such that $r + Q_i = (x + Q_i)(s_i + Q_i)$. Define $f_i(Q_i, r) = s_i$. We note here that the map f_i is not unique. The important point is that one element of the coset $s_i + Q$ is chosen and for the proof it does not matter which one. Let $S_i = \operatorname{Image} f_i$. Now, $|S_i| \leqslant |D| < |T/Q_i| = |(Q_i + \bigcap_{j=1,\ j\neq i}^{n} Q_j)/Q_i|$. Note that the last equality holds by Lemma 2 and the fact that $Q_i + \bigcap_{j=1,\ j\neq i}^{n} Q_j$ is not the zero ideal of T/Q_i. So, there exists $t_i \in \bigcap_{j=1,\ j\neq i}^{n} Q_j$ such that $t_i + Q_i \neq s_i + Q_i$ for every $s_i \in S_i$. We claim that $x \sum_{j=1}^{n} t_j \notin \bigcup\{r + Q \mid Q \in \operatorname{Ass} T,\ r \in D\}$. To see this, suppose $x \sum_{j=1}^{n} t_j \in r + Q_i$ for some $i \in \{1, 2, \ldots, n\}$ and some $r \in D$. Then, $x \sum_{j=1}^{n} t_j + Q_i = r + Q_i$ and so $xt_i + Q_i = r + Q_i$. It follows that $(x + Q_i)(t_i + Q_i) = r + Q_i$, so $r + Q_i \in (x + Q_i)T/Q_i$. Hence, $(x + Q_i)(t_i + Q_i) = r + Q_i = (x + Q_i)(s_i + Q_i)$ for some $s_i \in S_i$. So, we have that $t_i + Q_i = s_i + Q_i$ for some $s_i \in S_i$, a contradiction. □

Definition. Let (T, M) be a complete local ring. Suppose that $(R, R \cap M)$ is a quasi-local subring of T such that $|R| < |T|$ and $R \cap Q = (0)$ for every $Q \in \operatorname{Ass} T$. Then we call R a small Q-avoiding subring of T and will denote it by SQA-subring.

SQA-subrings will be essential in our proof. If R is an SQA-subring of T then note that the condition $R \cap Q = (0)$ for every $Q \in \operatorname{Ass} T$ gives us that R contains no zero divisors of T—certainly a condition that the excellent domain we wish to construct must enjoy. The cardinality condition will allow us to adjoin an element to R so that the resulting ring will also not contain zero divisors of T. Note that if T is a complete local reduced ring of dimension at least one and if R is an SQA-subring of T, then by Lemma 3, we have that $|R| < |T/Q|$ for every $Q \in \operatorname{Ass} T$.

Recall that we want the ring A under construction to satisfy the property that if J is an ideal of T with $J \nsubseteq Q$ for every $Q \in \operatorname{Ass} T$, then the map $A \to T/JT$ is onto. Lemma 5 allows us to adjoin an element of an arbitrary coset of T/JT, which eventually will help us make the map $A \to T/JT$ onto as desired. We note here that the proof of Lemma 5 follows closely the proof of Lemma 3 in [5].

Lemma 5. *Let (T, M) be a complete local reduced ring of dimension at least one. Let J be an ideal of T such that $J \nsubseteq Q$ for every $Q \in \operatorname{Ass} T$. Let R be an SQA-subring of T and $u + J \in T/J$. Then there exists an infinite SQA-subring S of T such that $R \subseteq S \subseteq T$ and $u + J$ is in the image of the map $S \to T/J$. Moreover, if $u \in J$, then $S \cap J \neq (0)$.*

Proof. Let $Q \in \operatorname{Ass} T$. Let $D_{(Q)}$ be a full set of coset representatives of the cosets $t + Q$ that make $(u + t) + Q$ algebraic over R. Note that as $|R| < |T| = |T/Q| \geqslant c$, we have $|D_{(Q)}| < |T/Q|$. Let $D = \bigcup_{Q \in \operatorname{Ass} T} D_{(Q)}$, and note that $|D| < |T/Q|$. Now, use Lemma 4 with $I = J$ to find an $x \in J$ such that $x \notin \bigcup\{Q + r \mid Q \in \operatorname{Ass} T,\ r \in D\}$. We claim that $S = R[u + x]_{(R[u+x] \cap M)}$ is the desired SQA-subring. Clearly, $|S| < |T|$ for $Q \in \operatorname{Ass} T$. Now, suppose $f \in R[u + x] \cap Q$. Then $f = r_n(u + x)^n + \cdots + r_1(u + x) + r_0 \in Q$ where $r_i \in R$. But, by the way x was chosen, we have that $(u + x) + Q$ is transcendental over R. Hence, $r_i \in R \cap Q = (0)$ for every $i = 1, 2, \ldots, n$, and it follows that $f = 0$. So, $S \cap Q = (0)$ and S is an SQA-subring. Note that if $u \in J$, then $u + x \in J$. Since $(u + x) + Q$ is transcendental over R, we have that $u + x \neq 0$. It follows that $S \cap J \neq (0)$. □

Lemma 6 will help us to ensure that $IT \cap A = I$ for every finitely generated ideal I of A. Recall that this property is needed to apply Proposition 1. We note here that the proof of Lemma 6 follows the proof of Lemma 6 in [4].

Lemma 6. *Let (T, M) be a complete local reduced ring of dimension at least one. Let R be an SQA-subring of T. Suppose I is a finitely generated ideal of R and $c \in IT \cap R$. Then there exists an SQA-subring of T such that $R \subseteq S \subseteq T$ and $c \in IS$.*

Proof. We will induct on the number of generators of I. Suppose $I = aR$. Now, if $a = 0$, then $c = 0$ and so $S = R$ is the desired SQA-subring of T. So, suppose $a \neq 0$. Then, $c = au$ for some $u \in T$. We claim $S = R[u]_{(R[u] \cap M)}$ is the desired SQA-subring. Clearly, $|S| < |T|$. Now, suppose $f \in R[u] \cap Q$ where $Q \in \operatorname{Ass} T$. Then, $f = r_n u^n + \cdots + r_1 u + r_0 \in Q$. So, $a^n f = r_n(au)^n + \cdots + r_1 a^{n-1}(au) + r_0 a^n$ and it follows that $a^n f = r_n c^n + \cdots + r_1 a^{n-1} c + r_0 a^n \in Q \cap R = (0)$. But, a is not a zero divisor as $a \in R$ and $R \cap Q = (0)$ for every $Q \in \operatorname{Ass} T$. It follows that $f = 0$ and hence S is an SQA-subring of T. So, the lemma holds if I is principal.

Now, let I be an ideal of R that is generated by $m > 1$ elements and suppose that the lemma holds for ideals of R that are generated by $m - 1$ elements. Let $I = (y_1, \ldots, y_m)R$. Then we have $c = y_1 t_1 + \cdots + y_m t_m$ for some $t_i \in T$. Note that for any $t \in T$, we have $c = y_1 t_1 + y_1 y_2 t - y_1 y_2 t + y_2 t_2 + \cdots + y_m t_m$. So, $c = y_1(t_1 + y_2 t) + y_2(t_2 - y_1 t) + y_3 t_3 + \cdots + y_m t_m$. We let $x_1 = t_1 + y_2 t$ and $x_2 = t_2 - y_1 t$ where the element t will be chosen later. Now, let $Q \in \operatorname{Ass} T$. If $(t_1 + y_2 t) + Q = (t_1 + y_2 t') + Q$, then $y_2(t - t') \in Q$. But, $y_2 \in R$ and $y_2 \neq 0$, so $t - t' \in Q$. Hence $t + Q = t' + Q$. It follows that if $t + Q \neq t' + Q$, then

$(t_1 + y_2t) + Q \neq (t_1 + y_2t') + Q$. Let $D_{(Q)}$ be a full set of coset representatives of the cosets $t + Q$ that make $x_1 + Q$ algebraic over R. Let $D = \bigcup_{Q \in \operatorname{Ass} T} D_{(Q)}$. Note that $|D| < |T/Q|$ for $Q \in \operatorname{Ass} T$. Now, use Lemma 4 with $I = T$ to find an element $t \in T$ such that $x_1 + Q$ is transcendental over R for every $Q \in \operatorname{Ass} T$. It is easy to see that $R' = R[x_1]_{(R[x_1] \cap M)}$ is an SQA-subring of T. Let $J = (y_2, \dots, y_m)R'$ and $c^* = c - y_1x_1$. Now, $c^* \in JT \cap R'$, so we may use our induction assumption to conclude that there is an SQA-subring S of T such that $R' \subseteq S \subseteq T$ and $c^* \in JS$. So, $c^* = y_2s_2 + \cdots + y_ms_m$ for some $s_i \in S$. It follows that $c = y_1x_1 + y_2x_2 + \cdots + y_ms_m \in IS$ and so S is the desired SQA-subring. □

Definition. Let Ω be a well-ordered set and $\alpha \in \Omega$. We define $\gamma(\alpha) = \sup\{\beta \in \Omega \mid \beta < \alpha\}$.

Lemma 7 allows us to put many of our desired conditions together. We note here that the proof of Lemma 7 is based on the proof of Lemma 12 in [4].

Lemma 7. *Let (T, M) be a complete local reduced ring of dimension at least one. Let J be an ideal of T with $J \nsubseteq Q$ for every $Q \in \operatorname{Ass} T$ and let $u + J \in T/J$. Suppose R is an SQA-subring. Then, there exists an SQA-subring S of T such that*

(1) $R \subseteq S \subseteq T$;
(2) *if $u \in J$ then $S \cap J \neq (0)$;*
(3) *$u + J$ is in the image of the map $S \to T/J$; and*
(4) *for every finitely generated ideal I of S, we have $IT \cap S = I$.*

Proof. First apply Lemma 5 to find an infinite SQA-subring R' of T such that $R \subseteq R' \subseteq T$, $u + J$ is in the image of the map $R' \to T/J$, and if $u \in J$ then $R' \cap J \neq (0)$. We will construct the desired S so that $R' \subseteq S \subset T$, which will ensure that conditions (1)–(3) will hold. Now, let

$$\Omega = \bigl\{(I, c) \mid I \text{ is a finitely generated ideal of } R' \text{ and } c \in IT \cap R'\bigr\}.$$

Letting $I = R'$, we see that $|\Omega| \geqslant |R'|$. Now, as R' is infinite, the number of finitely generated ideals of R' is $|R'|$. Hence, $|R'| \geqslant |\Omega|$ and we have $|R'| = |\Omega|$. As R' is an SQA-subring of T, we have $|\Omega| = |R'| < |T|$. Well-order Ω so that it does not have a maximal element and let 0 denote its first element. We now work to inductively define a family of SQA-subrings of T—one for each element of Ω. Let $R_0 = R'$. Let $a \in \Omega$. If $\gamma(\alpha) < \alpha$ and $\gamma(\alpha) = (I, c)$ then define R_α to be the SQA-subring obtained from Lemma 6 so that $R_{\gamma(\alpha)} \subseteq R_\alpha \subseteq T$ and $c \in IR_\alpha$. If $\gamma(\alpha) = \alpha$, define $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$. Note that in either case we have R_α is an SQA-subring of T. Let $R_1 = \bigcup_{\alpha \in \Omega} R_\alpha$. Now, $|\Omega| < |T|$ and $|R_\alpha| < |T|$ for every $\alpha \in \Omega$, so we have that $|R_1| < |T|$. Also, since $R_\alpha \cap Q = (0)$ for every $Q \in \operatorname{Ass} T$ and for every $\alpha \in \Omega$, we have that $R_1 \cap Q = (0)$ for every $Q \in \operatorname{Ass} T$. It follows that R_1 is an SQA-subring. Note that if I is a finitely generated ideal of R_0 and $c \in IT \cap R_0$, then $(I, c) = \gamma(\alpha)$ for some $\alpha \in \Omega$ with $\gamma(\alpha) < \alpha$. It follows by the construction that $c \in IR_\alpha \subseteq IR_1$. Hence $IT \cap R_0 \subseteq IR_1$ for every I which is a finitely generated ideal of R_0.

In the same manner, construct an SQA-subring R_2 of T so that $R_1 \subseteq R_2 \subseteq T$ and $IT \cap R_1 \subseteq IR_2$ for every finitely generated ideal I of R_1. Continue to form a chain $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ of SQA-subrings of T so that $IT \cap R_n \subseteq IR_{n+1}$ for every finitely generated ideal I of R_n.

We claim that $S = \bigcup_{i=1}^{\infty} R_i$ is the desired SQA-subring. Clearly, S is an SQA-subring and $R \subseteq S \subseteq T$. Now let $I = (y_1, \ldots, y_k)S$ and let $c \in IT \cap S$. Then there exists an N such that $c, y_1, \ldots, y_k \in R_N$. Hence $c \in (y_1, \ldots, y_k)T \cap R_N \subseteq (y_1, \ldots, y_k)R_{N+1} \subset IS$. It follows that $IT \cap S = I$, so condition (4) of the lemma holds. □

Now we are in a position to construct a domain A that not only has the desired completion but also satisfies other interesting properties that will give us our result.

Lemma 8. *Let (T, M) be a complete local reduced ring of dimension at least one. Suppose no integer of T is a zero divisor. Then there exists a local domain A such that*

(1) $\widehat{A} = T$;
(2) *if P is a nonzero prime ideal of A, then $T \otimes_A k(P) \cong k(P)$ where $k(P) = A_P/PA_P$;*
(3) *the generic formal fiber ring of A is semilocal with maximal ideals the associated prime ideals of T;*
(4) *if I is a nonzero ideal of A, then A/I is complete.*

Proof. Define

$$\Omega = \{u + J \in T/J \mid J \text{ is an ideal of } T \text{ with } J \not\subseteq Q \text{ for every } Q \in \operatorname{Ass} T\}.$$

We claim $|\Omega| \leqslant |T|$. Since T is infinite and Noetherian, $|\{J \mid J$ is an ideal of T with $J \not\subseteq Q$ for every $Q \in \operatorname{Ass} T\}| \leqslant |T|$. Now, if J is an ideal of T, then $|T/J| \leqslant |T|$. It follows that $|\Omega| \leqslant |T|$.

The rest of the proof only involves minor adjustments to the proof of Lemma 6 in [5]. But, the proof is short and so we include it here.

Well order Ω so that each element has fewer than $|\Omega|$ predecessors. Let 0 denote the first element of Ω. We define R_0' to be the prime subring of T and R_0 to be R_0' localized at $R_0' \cap M$. Note that R_0 is an SQA-subring.

We recursively define a family of SQA-subrings as follows. R_0 is already defined. Let $\lambda \in \Omega$ and assume R_β has been defined for every $\beta < \lambda$. Then $\gamma(\lambda) = u + J$ for some ideal J of T with $J \not\subseteq Q$ for all $Q \in \operatorname{Ass} T$. If $\gamma(\lambda) < \lambda$, use Lemma 7 to obtain an SQA-subring R_λ so that $R_{\gamma(\lambda)} \subseteq R_\lambda \subseteq T$, $u + J \in \operatorname{Image}(R_\lambda \to T/J)$ and for every finitely generated ideal I of R_λ we have $IT \cap R_\lambda = I$. Moreover, if $u \in J$, we have $R_\lambda \cap J \neq (0)$. If $\gamma(\lambda) = \lambda$, define $R_\lambda = \bigcup_{\beta<\lambda} R_\beta$. Then, R_λ is an SQA-subring for every $\lambda \in \Omega$. We claim that $A = \bigcup_{\lambda \in \Omega} R_\lambda$ is the desired domain.

Now, as each R_λ is an SQA-subring, we have $R_\lambda \cap Q = (0)$ for all $Q \in \operatorname{Ass} T$. Hence, $A \cap Q = (0)$ for all $Q \in \operatorname{Ass} T$. Also, if J is an ideal of T with $J \not\subseteq Q$ for all $Q \in \operatorname{Ass} T$ then $0 + J \in \Omega$. So, $\gamma(\lambda) = 0 + J$ for some $\lambda \in \Omega$ with $\gamma(\lambda) < \lambda$. By construction, $R_\lambda \cap J \neq (0)$. It follows that $J \cap A \neq (0)$. Hence, the generic formal fiber ring of A is semilocal with maximal ideals the associated prime ideals of T.

We will now show that the completion of A is T. To do this, we make use of Proposition 1. Note that as T is reduced and of dimension at least one, we have that M^2 is not contained in an associated prime ideal of T. Hence, by the construction, the map $A \to T/M^2$ is surjective. Let I be a finitely generated ideal of A with $I = (y_1, \dots, y_k)$. Let $c \in IT \cap A$. Then $\{c, y_1, \dots, y_k\} \subseteq R_\lambda$ for some $\lambda \in \Omega$ with $\gamma(\lambda) < \lambda$. By construction, $(y_1, \dots, y_k)T \cap R_\lambda = (y_1, \dots, y_k)R_\lambda$. As $c \in (y_1, \dots, y_k)T \cap R_\lambda$, we have $c \in (y_1, \dots, y_k)R_\lambda \subseteq I$. Hence, $IT \cap A = I$. It follows by Proposition 1 that A is Noetherian and the completion of A is T.

Now suppose I is a nonzero ideal of A. Let $J = IT$. If $J \subseteq Q$ for some $Q \in \operatorname{Ass} T$, then $I \subseteq J \cap A \subseteq Q \cap A = (0)$, a contradiction. Hence, $J \nsubseteq Q$ for all $Q \in \operatorname{Ass} T$. It follows by the construction that the map $A \to T/J$ is surjective. Hence, the map $A/I \to T/J$ is an isomorphism and so A/I is complete.

We now claim that if $J \subseteq Q$ for all $Q \in \operatorname{Ass} T$, then $J = (A \cap J)T$. To see this, let $J \nsubseteq Q$ for all $Q \in \operatorname{Ass} T$ and note that since M is not contained in an associated prime ideal of T, we have $JM \nsubseteq Q$ for all $Q \in \operatorname{Ass} T$. So, $A \to T/JM$ is onto by our construction. Consider the T-modules J and $(A \cap J)T$. We will show that $J = MJ + (A \cap J)T$. Note that $MJ + (A \cap J)T \subseteq J$ is clear. Now, let $x \in J$. Then since $A \to T/JM$ is onto, there is an $a \in A$ such that $a + JM = x + JM$. So, $x = a + \sum_{i=1}^{n} j_i m_i$ where $j_i \in J$ and $m_i \in M$. Also, $a = x - \sum_{i=1}^{n} j_i m_i \in J$, so $a \in (A \cap J)T$. Hence, $x \in MJ + (A \cap J)T$ and it follows that $MJ + (A \cap J)T = J$. By Nakayama's Lemma, we have $J = (A \cap J)T$. So, our claim that $J = (A \cap J)T$, if $J \nsubseteq Q$ for all $Q \in \operatorname{Ass} T$, holds.

Now, suppose that P is a nonzero prime ideal of A and q is a prime ideal of T such that $q \cap A = P$. Then $q \subseteq Q$, where Q is an associated prime ideal of T, implies that $q \cap A = Q \cap A = (0)$, a contradiction. So, we must have $q \nsubseteq Q$ for all $Q \in \operatorname{Ass} T$. Hence, $q = (q \cap A)T = PT$ and it follows that the only prime ideal of T that lies over P is PT. Now, by the construction, $A \to T/PT$ is onto and since $A \cap PT = P$, we have that $A/P \cong T/PT$. So, $T \otimes_A k(P) \cong (T/PT)_{\overline{A-P}} \cong (A/P)_{\overline{A-P}} \cong A_P/PA_P = k(P)$.

It is interesting to note that we have also shown that there exists a one-to-one correspondence between nonzero prime ideals of A and prime ideals of T that are not in the generic formal fiber of A. □

We note here that we did not use the full power of T being reduced. We only needed that T have no embedded prime ideals. In fact, the previous results hold with the condition that T be reduced replaced with the weaker condition that T have no embedded prime ideals. For our final result, however, it will be necessary that T be reduced. So to make the statements of the previous results less cumbersome, we used the stronger condition that T be reduced.

Finally, we show that the domain constructed in the previous lemma gives us our result.

Theorem 9. *Let (T, M) be a complete local ring containing the integers. Then T is the completion of a local excellent domain if and only if it is reduced, equidimensional and no integer of T is a zero divisor.*

Proof. Assume T is the completion of an excellent domain, A. Then T is clearly reduced. Now, since A is universally catenary, it is formally catenary (see [6, Theorem 31.7]).

It follows that $A/(0) \cong A$ is formally equidimensional. Hence, its completion, T, is equidimensional. Now, A must contain the integers. As A is a domain, it follows that no integer can be a zero divisor in A and hence no integer of T can be a zero divisor in T.

Conversely, suppose T is reduced, equidimensional, and no integer of T is a zero divisor. If $\dim T = 0$, then T is a field and so is the completion of itself. If $\dim T \geqslant 1$, use Lemma 8 to construct the domain A. We claim that A is excellent. To see this, suppose that P is a nonzero prime ideal of A. Then, by Lemma 8, we have that $T \otimes_A k(P) \cong k(P)$. Let L be a finite field extension of $k(P)$. Then $T \otimes_A L \cong T \otimes_A k(P) \otimes_{k(P)} L \cong (k(P) \otimes_{k(p)} L) \cong L$. So, the fiber over P is geometrically regular. Now the maximal ideals of $T \otimes_A k(0)$ are the associated prime ideals of T. Let $Q \in \operatorname{Ass} T$. Then $T \otimes_A k(0)$ localized at Q is isomorphic to T_Q. Since T is reduced, it satisfies Serre's (R_0) condition. Hence, T_Q is regular. It follows that $T \otimes_A k(0)$ is regular. Now, since T contains the integers, so does A. It follows that $k((0))$ is a field of characteristic zero. Hence $T \otimes_A L$ is regular for every finite field extension L of $k((0))$. So, all the formal fibers of A are geometrically regular. Since A is formally equidimensional, it is universally catenary. It follows that A is excellent. □

Note that if T has characteristic $p > 0$, then the A we construct has the property that if P is a nonzero prime ideal of A then the fiber over P is geometrically regular. So, the only obstacle to A being excellent is that the generic formal fiber may not be geometrically regular. This author sees no reason to believe that the generic formal fiber of the domain A we have constructed is geometrically regular. However, it seems believable that one could modify this construction to ensure that this be true. Since at this point, this author does not know how to do this, we leave it as an open question.

Acknowledgment

I thank Ray Heitmann for his numerous helpful suggestions regarding the work in this paper.

References

[1] R. Heitmann, Characterization of completions of unique factorization domains, Trans. Amer. Math. Soc. 337 (1993) 379–387.
[2] R. Heitmann, Completions of local rings with an isolated singularity, J. Algebra 163 (1994) 538–567.
[3] C. Lech, A method for constructing bad Noetherian local rings, in: Lecture Notes in Math., Vol. 1183, Springer-Verlag, Berlin, 1986, pp. 241–247.
[4] S. Loepp, Excellent rings with local generic formal fibers, J. Algebra 201 (1998) 573–583.
[5] S. Loepp, C. Rotthaus, On the completeness of factor rings, Proc. Amer. Math. Soc. 130 (2002) 2189–2195.
[6] H. Matsumura, Commutative Ring Theory, Cambridge University Press, 1986.