Note

On a conjecture of Tuza about packing and covering of triangles

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Abstract

Zs. Tuza conjectured that if a simple graph \( G \) does not contain more than \( k \) pairwise edge disjoint triangles, then there exists a set of at most \( 2k \) edges which meets all triangles in \( G \). We prove this conjecture for \( K_{3,3} \)-free graphs (graphs that do not contain a homeomorph of \( K_{3,3} \)). Two fractional versions of the conjecture are also proved.

1. Introduction

Let \( G \) be a simple, undirected graph with vertex set \( V(G) = V \) and edge set \( E(G) = E \). Denote by \( T = T(G) \subseteq E^3 \) the collection of triangles of \( G \), i.e. \( (e_1, e_2, e_3) \in T \) if \( e_1, e_2, e_3 \) form a triangle in \( G \). A triangle packing in \( G \) is a set of pairwise edge disjoint triangles. A triangle edge cover in \( G \) is a set of edges meeting all triangles. A fractional triangle packing is a function \( f : T \rightarrow \mathbb{R}^* \) such that \( \sum \{ f(t) : t \in T \} \leq 1 \) for every \( e \in E \). A fractional triangle edge cover is a function \( g : E \rightarrow \mathbb{R}^* \) such that \( \sum \{ g(e) : e \in t \} \geq 1 \) for every \( t \in T \). We denote by \( v_t(G) \) the maximum size of a triangle packing, by \( \tau_t(G) \) the minimum size of a triangle edge cover, by \( v^*_t(G) \) the maximum of \( \sum \{ f(t) : t \in T \} \) over all fractional triangle packings and by \( \tau^*_t(G) \) the minimum of \( \sum \{ g(e) : e \in E \} \) over all fractional triangle edge covers. Define also the hypergraph of triangles \( H \) by \( V(H) := E(G) \); \( E(H) := T(G) \). Obviously,

\[
\tau_t(G) = \tau(H), \quad \tau^*_t(G) = \tau^*(H).
\]

\[
v_t(G) = v(H), \quad v^*_t(G) = v^*(H),
\]

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where \( v(H), \tau(H), v^*(H), \tau^*(H) \) are the matching number, the covering number, the fractional matching number and the fractional covering number of \( H \), respectively (for precise definitions see, e.g. [3]).

In [5] Tuza conjectured the following.

**Conjecture 1.** \( \tau_t(G) \leq 2v_t(G) \) for every graph \( G \).

In [6] Tuza proved it for some classes of graphs, in particular, for planar graphs. Here we make one step further, proving the conjecture of Tuza for \( K_{3,3} \)-free graphs (graphs that do not contain a homeomorph of \( K_{3,3} \)). In the second part of the article we prove the fractional versions of Tuza's conjecture, namely

\[
\tau_t(G) \leq 2\tau^*_t(G) \quad \text{and} \quad v^*_t(G) \leq 2v_t(G).
\]

### 2. Proof of the conjecture for \( K_{3,3} \)-free graphs

If a graph \( G \) is not 2-connected, it can be split into two parts \( G_1 \) and \( G_2 \), which have no common triangles, and if the conjecture is valid for each part, then it is valid for \( G \). Thus we may assume that \( G \) is 2-connected.

The key to the proof is the following result of Hall [4].

**Theorem 2** (Hall [4], see also Asano [1]). Each 3-connected component of a \( K_{3,3} \)-free graph is either planar or exactly the graph \( K_5 \).

As a basis of our proof we shall use the result of Tuza and Proposition 4 below.

**Theorem 3** (Tuza [6]). \( \tau_t(G) \leq 2v_t(G) \) for every planar graph \( G \),

**Proposition 4.** \( \tau_t(G) \leq 2v_t(G) \) for every subgraph \( G \) of \( K_5 \).

This is easily verified.

Let us begin with a simple technical lemma.

**Lemma 5.** Let \( G_1, G_2 \) be two graphs such that

\[
V(G_1) \cap V(G_2) = \{u, v\}
\]

and assume that Conjecture 1 is true for \( G_1 \) and \( G_2 \), that is

\[
\tau_t(G_1) \leq 2v_t(G_1), \quad \tau_t(G_2) \leq 2v_t(G_2).
\]
Consider the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. Then

1. If $(u, v) \notin E(G)$ then $\tau_i(G) \leq 2v_i(G)$;
2. If $e_0 = (i, v) \in E(G_1) \cap E(G_2)$ and
   \[ \tau_i(G_1 \setminus e_0) \leq 2v_i(G_1 \setminus e_0), \]
   \[ \tau_i(G_2 \setminus e_0) \leq 2v_i(G_2 \setminus e_0) \]  
   (i.e. Conjecture 1 is true for graphs $G_1 \setminus e_0, G_2 \setminus e_0$), then $\tau_i(G) \leq 2v_i(G)$).

**Proof.** (1) The statement is obvious, since $G_1$ and $G_2$ have no common triangles.

(2) Obviously,

\[ \tau_i(G) \leq \tau_i(G_1) + \tau_i(G_2), \]
\[ v_i(G_1) + v_i(G_2) - 1 \leq \tau_i(G) \leq v_i(G_1) + v_i(G_2). \]  

If $\tau_i(G) = v_i(G_1) + v_i(G_2)$, then from (1), (2), and (5) it follows that $\tau_i(G) \leq 2v_i(G)$, so we may assume that

\[ v_i(G) = v_i(G_1) + v_i(G_2) - 1. \]  

In fact, (6) states, that if $T_1$ is a maximal triangle packing in $G_1$ and $T_2$ is a maximal triangle packing in $G_2$ ($|T_1| = v_i(G_1), |T_2| = v_i(G_2)$), then $e_0 \in E(T_1) \cap E(T_2)$, where $E(T_i) = \{ e \in E(G_i) : \exists t \in T_i, e \in t \}$, $i = 1, 2$. Hence we have

\[ v_i(G_1 \setminus e_0) = v_i(G_1) - 1, \quad v_i(G_2 \setminus e_0) = v_i(G_2) - 1. \]

It follows from (3) and (4) that

\[ \tau_i(G_1 \setminus e_0) \leq 2v_i(G_1 \setminus e_0) = 2v_i(G_1) - 2, \]
\[ \tau_i(G_2 \setminus e_0) \leq 2v_i(G_2 \setminus e_0) = 2v_i(G_2) - 2. \]

But $\tau_i(G) \leq \tau_i(G_1 \setminus e_0) + \tau_i(G_2 \setminus e_0) + 1$. Hence

\[ \tau_i(G) \leq 2v_i(G_1) + 2v_i(G_2) - 3 < 2v_i(G). \]  

Now we are ready to prove the main result of this section.

**Theorem 6.** Conjecture 1 is true for $K_{3,3}$-free graphs.

**Proof.** By induction on the number of vertices in $G$. If $G$ is 3-connected, then the assertion follows from Theorems 2 and 3 and Proposition 4. Otherwise $G$ contains a separating pair $\{u, v\}$. Let $K$ be one of the connected components of $G \setminus \{u, v\}$. Denote

\[ G_1 = G[V(K) \cup \{u, v\}], \quad G_2 = G \setminus K. \]
For $G_1$ and $G_2$ the conditions of Lemma 5 are satisfied by the induction hypothesis, so for $G = G_1 \cup G_2$ it follows from Lemma 5 that

$$\tau_t(G) \leq 2v_t(G). \quad \square$$

3. Proof of the fractional versions of Conjecture 1

Our aim is to prove two fractional relaxations of Conjecture 1:

$$\tau_t(G) \leq 2\tau^*_t(G) \quad \text{and} \quad v_t(G) \leq 2v_t(G),$$

where $\tau_t, v_t, \tau^*_t, v^*_t$ are defined as described in Section 1. The duality theorem of linear programming states that $\tau^*_t = v_t^*$ and that if $f: T \rightarrow \mathbb{R}^+$ and $g: E \rightarrow \mathbb{R}^+$ are a maximum fractional triangle packing and a minimum fractional triangle edge cover respectively, then

$$f(t) > 0 \implies \sum \{g(e) : e \in t\} = 1,$$

$$g(e) > 0 \implies \sum \{f(t) : t \ni e\} = 1,$$

where $t \in T$, $e \in E$.

Theorem 7. $v^*_t(G) \leq 2v_t(G)$.

Proof. Consider the hypergraph $H$ of triangles. $H$ is 3-uniform, and we can use the following result of Füredi ([2]): if an $r$-uniform hypergraph $H$ does not contain a projective plane of order $r - 1$ as a partial hypergraph, then $v_t^*(H) \leq (r - 1)v(H)$. So we have only to check that no hypergraph of triangles contains the Fano plane (the projective plane of order 2) as a partial hypergraph. Denote the Fano plane by $H_0$ and its vertex set by $\{1, \ldots, 7\}$. Suppose to the contrary that $H_0 \subseteq H$. For $i = 1, \ldots, 7$ let $e_i \in E(G)$ be the graph edge corresponding to the vertex $i$ in $H_0$. Suppose also that $(1, 2, 3) \in E(H_0)$, so $(e_1, e_2, e_3)$ form a triangle in $G$. There are in $H_0$ edges, that contain the pairs $(4, 1), (4, 2), (4, 3)$. This means that the pairs of edges $(e_4, e_1), (e_4, e_2), (e_4, e_3)$ are contained in some triangles in $G$, so each of these pairs is intersecting, which is impossible. We have shown that $H_0 \not\subseteq H$. \square

The bound on the ratio between $v^*_t$ and $v_t$ is best possible, since for $G = K_4$ we have $v^*_t(G) = 2$, $v_t(G) = 1$.

Theorem 8. $\tau_t(G) \leq 2\tau^*_t(G)$.

Proof. Suppose to the contrary that there exist graphs which contradict the statement, and let $G$ be a minimal graph such that $\tau_t(G) > 2\tau^*_t(G)$. Then $\tau_t(G') \leq 2\tau^*_t(G')$ for every proper subgraph $G'$ of $G$. 
Let $f: T(G) \to \mathbb{R}^+$ be a maximum fractional triangle packing and $g: E(G) \to \mathbb{R}^+$ be a minimum fractional triangle edge cover of $G$. Consider two possible cases:

Case 1: $g(e) > 0$ for every $e \in E(G)$: Then it follows from the complementary slackness condition (7b) that

$$|E(G)| = \sum_{e \in E} 1 = \sum_{e \in E} \sum_{t \in T} f(t) = \sum_{t \in T} f(t)|E| = 3 \sum_{t \in T} f(t) = 3\tau^*_t(G),$$

so

$$\tau^*_t(G) = \frac{|E(G)|}{3}. \tag{8}$$

On the other hand, there is a bipartite graph $B$ in $G$ with at least $|E(G)|/2$ edges. Since $B$ contains no triangles, $E(G) \setminus E(B)$ meets all triangles in $G$, so for all $G$

$$\tau_i(G) \leq \frac{|E(G)|}{2}. \tag{9}$$

Comparing (8) and (9), we conclude that $\tau_i(G) \leq \frac{1}{3}\tau^*_t(G)$, contradicting the assumption on $G$.

Case 2: There exists $e_0 \in E(G)$ such that $g(e_0) = 0$: Since $G$ is a minimal graph which contradicts the statement, every edge in $G$ belongs to some triangle. Suppose that $(e_0, e_1, e_2) \in T(G)$. Since $g$ is the fractional triangle edge cover, $g(e_0) + g(e_1) + g(e_2) \geq 1$, but $g(e_0) = 0$, so $g(e_1) \geq 1/2$ or $g(e_2) \geq 1/2$, say, $g(e_1) \geq 1/2$. Consider the graph $G' = G \setminus e_1$, $V(G') = V(G)$, $E(G') = E(G) \setminus \{e_1\}$. Obviously,

$$\tau_i(G') \geq \tau_i(G) - 1 \tag{10}$$

(if $E_0 \subseteq E(G')$ is a triangle edge cover for $G'$, then $E_0 \cup \{e_0\}$ is a triangle edge cover for $G$). Due to the choice of $G$ for $G'$ we have $\tau_i(G') \leq 2\tau^*_t(G')$. But $g': E(G') \to \mathbb{R}^+$, $g'(e) := g(e)$ for all $e \in E(G')$, is a fractional triangle edge cover for $G'$, so

$$\tau^*_t(G') \leq \sum_{e \in E(G')} g'(e) = \tau^*_t(G) - g(e_1) \leq \tau^*_t(G) - 1/2. \tag{11}$$

It follows from (10) and (11) that

$$\tau_i(G) \leq \tau_i(G') + 1 \leq 2\tau^*_t(G') + 1 \leq 2(\tau^*_t(G) - 1/2) + 1 = 2\tau^*_t(G),$$

again a contradiction. $\square$

We have no example which realizes the equality $\tau_i(G) = 2\tau^*_t(G)$, and perhaps this result is not best possible.

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References