Note

On a conjecture of Tuza about packing and covering of triangles

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Abstract

Zs. Tuza conjectured that if a simple graph $G$ does not contain more than $k$ pairwise edge disjoint triangles, then there exists a set of at most $2k$ edges which meets all triangles in $G$. We prove this conjecture for $K_{3,3}$-free graphs (graphs that do not contain a homeomorph of $K_{3,3}$). Two fractional versions of the conjecture are also proved.

1. Introduction

Let $G$ be a simple, undirected graph with vertex set $V(G) = V$ and edge set $E(G) = E$. Denote by $T = T(G) \subset E^3$ the collection of triangles of $G$, i.e. $(e_1, e_2, e_3) \in T$ if $e_1, e_2, e_3$ form a triangle in $G$. A triangle packing in $G$ is a set of pairwise edge disjoint triangles. A triangle edge cover in $G$ is a set of edges meeting all triangles. A fractional triangle packing is a function $f: T \rightarrow \mathbb{R}^+$ such that $\sum \{f(t): t \ni e\} \leq 1$ for every $e \in E$. A fractional triangle edge cover is a function $g: E \rightarrow \mathbb{R}^+$ such that $\sum \{g(e): e \in t\} \geq 1$ for every $t \in T$. We denote by $\tau_i(G)$ the maximum size of a triangle packing, by $\tau_i^*(G)$ the minimum size of a triangle edge cover, by $\tau^*_i(G)$ the maximum of $\sum \{f(t): t \ni e\}$ over all fractional triangle packings and by $\tau^*_i(G)$ the minimum of $\sum \{g(e): e \in E\}$ over all fractional triangle edge covers. Define also the hypergraph of triangles $H$ by $V(H) = E(G)$; $E(H) = T(G)$. Obviously,

$$\tau_i(G) = \tau(H), \quad \tau_i^*(G) = \tau^*(H),$$

$$\nu_i(G) = \nu(H), \quad \nu_i^*(G) = \nu^*(H).$$

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where $\tau(H), \gamma(H), \gamma_*(H), \tau_*(H)$ are the matching number, the covering number, the fractional matching number and the fractional covering number of $H$, respectively (for precise definitions see, e.g. [3]).

In [5] Tuza conjectured the following.

**Conjecture 1.** $\tau_t(G) \leq 2\nu_t(G)$ for every graph $G$.

In [6] Tuza proved it for some classes of graphs, in particular, for planar graphs. Here we make one step further, proving the conjecture of Tuza for $K_{3,3}$-free graphs (graphs that do not contain a homeomorph of $K_{3,3}$). In the second part of the article we prove the fractional versions of Tuza's conjecture, namely

$$\tau_t(G) \leq 2\tau_*(G) \quad \text{and} \quad \nu_t(G) \leq 2\nu_t(G).$$

### 2. Proof of the conjecture for $K_{3,3}$-free graphs

If a graph $G$ is not 2-connected, it can be split into two parts $G_1$ and $G_2$, which have no common triangles, and if the conjecture is valid for each part, then it is valid for $G$. Thus we may assume that $G$ is 2-connected.

The key to the proof is the following result of Hall [4].

**Theorem 2** (Hall [4], see also Asano [1]). Each 3-connected component of a $K_{3,3}$-free graph is either planar or exactly the graph $K_5$.

As a basis of our proof we shall use the result of Tuza and Proposition 4 below.

**Theorem 3** (Tuza [6]). $\tau_t(G) \leq 2\nu_t(G)$ for every planar graph $G$.

**Proposition 4.** $\tau_t(G) \leq 2\nu_t(G)$ for every subgraph $G$ of $K_5$.

This is easily verified.

Let us begin with a simple technical lemma.

**Lemma 5.** Let $G_1, G_2$ be two graphs such that

$V(G_1) \cap V(G_2) = \{u, v\}$

and assume that Conjecture 1 is true for $G_1$ and $G_2$, that is

$$\tau_t(G_1) \leq 2\nu_t(G_1),$$  \hspace{1cm} (1)

$$\tau_t(G_2) \leq 2\nu_t(G_2).$$  \hspace{1cm} (2)
Consider the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. Then

1. If $(u, v) \notin E(G)$ then $\tau_t(G) \leq 2v_t(G)$;
2. If $e_0 = (i, v) \in E(G_1) \cap E(G_2)$ and

   $\tau_t(G_1\setminus e_0) \leq 2v_t(G_1\setminus e_0)$, (3)

   $\tau_t(G_2\setminus e_0) \leq 2v_t(G_2\setminus e_0)$ (4)

(i.e. Conjecture 1 is true for graphs $G_1\setminus e_0, G_2\setminus e_0$), then $\tau_t(G) \leq 2v_t(G)$).

Proof. (1) The statement is obvious, since $G_1$ and $G_2$ have no common triangles.
(2) Obviously,

\[
\tau_t(G) \leq \tau_t(G_1) + \tau_t(G_2),
\]

\[
v_t(G_1) + v_t(G_2) - 1 \leq v_t(G) \leq v_t(G_1) + v_t(G_2).
\] (5)

If $v_t(G) = v_t(G_1) + v_t(G_2)$, then from (1), (2), and (5) it follows that $\tau_t(G) \leq 2v_t(G)$, so we may assume that

\[
v_t(G) = v_t(G_1) + v_t(G_2) - 1.
\] (6)

In fact, (6) states, that if $T_1$ is a maximal triangle packing in $G_1$ and $T_2$ is a maximal triangle packing in $G_2$, then $e_0 \in E(T_1 \cap E(T_2))$, where $E(T_i) = \{e \in E(G_i) \mid \exists t \in T_i, e \in t\}, i = 1, 2$. Hence we have

\[
v_t(G_1\setminus e_0) = v_t(G_1) - 1, \quad v_t(G_2\setminus e_0) = v_t(G_2) - 1.
\]

It follows from (3) and (4) that

\[
\tau_t(G_1\setminus e_0) \leq 2v_t(G_1\setminus e_0) = 2v_t(G_1) - 2,
\]

\[
\tau_t(G_2\setminus e_0) \leq 2v_t(G_2\setminus e_0) = 2v_t(G_2) - 2.
\]

But $\tau_t(G) \leq \tau_t(G_1\setminus e_0) + \tau_t(G_2\setminus e_0) + 1$. Hence

\[
\tau_t(G) \leq 2v_t(G_1) + 2v_t(G_2) - 3 < 2v_t(G).
\]

Now we are ready to prove the main result of this section.

**Theorem 6.** Conjecture 1 is true for $K_{3,3}$-free graphs.

Proof. By induction on the number of vertices in $G$. If $G$ is 3-connected, then the assertion follows from Theorems 2 and 3 and Proposition 4. Otherwise $G$ contains a separating pair $\{u, v\}$. Let $K$ be one of the connected components of $G\setminus \{u, v\}$. Denote

\[
G_1 = G[V(K) \cup \{u, v\}], \quad G_2 = G\setminus K.
\]
For $G_1$ and $G_2$ the conditions of Lemma 5 are satisfied by the induction hypothesis, so for $G = G_1 \cup G_2$ it follows from Lemma 5 that

$$r_1(G) \leq 2v_1(G).$$

\[\square\]

3. Proof of the fractional versions of Conjecture 1

Our aim is to prove two fractional relaxations of Conjecture 1:

$$r^+(G) \leq 2r^+(G) \quad \text{and} \quad v^+(G) \leq 2v^+(G),$$

where $r^+, v^+, r^*_1, v^*_1$ are defined as described in Section 1. The duality theorem of linear programming states that $v^*_1 = v^+_1$ and that if $f: T \to R^+$ and $g: E \to R^+$ are a maximum fractional triangle packing and a minimum fractional triangle edge cover respectively, then

$$f(t) > 0 \quad \text{implies} \quad \sum \{g(e) : e \in t\} = 1,$$

$$g(e) > 0 \quad \text{implies} \quad \sum \{f(t) : t \ni e\} = 1,$$

where $t \in T, e \in E$.

**Theorem 7.** $v^+(G) \leq 2v_1(G)$.

**Proof.** Consider the hypergraph $H$ of triangles. $H$ is 3-uniform, and we can use the following result of Füredi ([2]): if an $r$-uniform hypergraph $H$ does not contain a projective plane of order $r - 1$ as a partial hypergraph, then $v^+(H) \leq (r - 1)v(H)$. So we have only to check that no hypergraph of triangles contains the Fano plane (the projective plane of order 2) as a partial hypergraph. Denote the Fano plane by $H_0$ and its vertex set by $\{1, \ldots, 7\}$. Suppose to the contrary that $H_0 \subseteq H$. For $i = 1, \ldots, 7$ let $e_i \in E(G)$ be the graph edge corresponding to the vertex $i$ in $H_0$. Suppose also that $(1, 2, 3) \in E(H_0)$, so $(e_1, e_2, e_3)$ form a triangle in $G$. There are in $H_0$ edges, that contain the pairs $(4, 1), (4, 2), (4, 3)$. This means that the pairs of edges $(e_4, e_1), (e_4, e_2), (e_4, e_3)$ are contained in some triangles in $G$, so each of these pairs is intersecting, which is impossible. We have shown that $H_0 \not\subseteq H$. \[\square\]

The bound on the ratio between $v^*_1$ and $v_1$ is best possible, since for $G = K_4$ we have $v^*_1(G) = 2, v_1(G) = 1$.

**Theorem 8.** $\tau_1(G) \leq 2\tau^*_1(G)$.

**Proof.** Suppose to the contrary that there exist graphs which contradict the statement, and let $G$ be a minimal graph such that $\tau_1(G) > 2\tau^*_1(G)$. Then $\tau_1(G') \leq 2\tau^*_1(G')$ for every proper subgraph $G'$ of $G$. 

Let $f: T(G) \to \mathbb{R}^+$ be a maximum fractional triangle packing and $g: E(G) \to \mathbb{R}^+$ be a minimum fractional triangle edge cover of $G$. Consider two possible cases:

**Case 1:** $g(e) > 0$ for every $e \in E(G)$: Then it follows from the complementary slackness condition (7b) that

$$|E(G)| = \sum_{e \in E} 1 = \sum_{e \in E} \sum_{t \in T} f(t) = \sum_{t \in T} f(t)|E| = 3 \sum_{t \in T} f(t) = 3\tau^*_t(G),$$

so

$$\tau^*_t(G) = \frac{|E(G)|}{3}. \quad (8)$$

On the other hand, there is a bipartite graph $B$ in $G$ with at least $|E(G)|/2$ edges. Since $B$ contains no triangles, $E(G) \setminus E(B)$ meets all triangles in $G$, so for all $G$

$$\tau_t(G) \leq \frac{|E(G)|}{2}. \quad (9)$$

Comparing (8) and (9), we conclude that $\tau_t(G) \leq \frac{1}{2} \tau^*_t(G)$, contradicting the assumption on $G$.

**Case 2:** There exists $e_0 \in E(G)$ such that $g(e_0) = 0$: Since $G$ is a minimal graph which contradicts the statement, every edge in $G$ belongs to some triangle. Suppose that $(e_0, e_1, e_2) \in T(G)$. Since $g$ is the fractional triangle edge cover, $g(e_0) + g(e_1) + g(e_2) \geq 1$, but $g(e_0) = 0$, so $g(e_1) \geq 1/2$ or $g(e_2) \geq 1/2$, say, $g(e_1) \geq 1/2$. Consider the graph $G' = G \setminus e_1$, $V(G') = V(G)$, $E(G') = E(G) \setminus \{e_1\}$. Obviously,

$$\tau_t(G') \geq \tau_t(G) - 1 \quad (10)$$

(if $E_0 \subseteq E(G')$ is a triangle edge cover for $G'$, then $E_0 \cup \{e_0\}$ is a triangle edge cover for $G$). Due to the choice of $G$ for $G'$ we have $\tau_t(G') \leq 2\tau^*_t(G')$. But $g': E(G') \to \mathbb{R}^+$, $g'(e) = g(e)$ for all $e \in E(G')$, is a fractional triangle edge cover for $G'$, so

$$\tau^*_t(G') \leq \sum_{e \in E(G')} g'(e) = \tau^*_t(G) - g(e_1) \leq \tau^*_t(G) - 1/2. \quad (11)$$

It follows from (10) and (11) that

$$\tau_t(G) \leq \tau_t(G') + 1 \leq 2\tau^*_t(G') + 1 \leq 2(\tau^*_t(G) - 1/2) + 1 = 2\tau^*_t(G),$$

again a contradiction. \qed

We have no example which realizes the equality $\tau_t(G) = 2\tau^*_t(G)$, and perhaps this result is not best possible.

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References