Coxeter and crystallographic arrangements are inductively free

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Abstract

Using the classification of finite Weyl groupoids we prove that crystallographic arrangements, a large subclass of the class of simplicial arrangements which was recently defined, are hereditarily inductively free. In particular, all crystallographic reflection arrangements are hereditarily inductively free, among them the arrangement of type E_8. With little extra work we prove that also all Coxeter arrangements are inductively free.

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1. Introduction

A hyperplane arrangement $\mathcal{A}$ is called free if the module of $\mathcal{A}$-derivations $D(\mathcal{A})$ is free. If no additional information about an arrangement $\mathcal{A}$ is known, then one can decide its freeness and in that case construct a free basis for $D(\mathcal{A})$ by intensive use of GRÖBNER basis techniques.

Fortunately, for certain arrangements there is a purely combinatorial method to prove their freeness[1] based on the Addition-Theorem [11, Theorem 4.50]: For a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ of arrangements where $\mathcal{A} = \mathcal{A}' \cup \{H\}$, $\mathcal{A}'' = \mathcal{A}^H$, the arrangement $\mathcal{A}$ is free if $\mathcal{A}'$, $\mathcal{A}''$ are free and

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[1] TERAO even conjectured that the freeness of an arrangement, over a fixed field, only depends on the intersection lattice, and is hence a purely combinatorial notion.

$\exp \mathcal{A}'' \subseteq \exp \mathcal{A}'$. This theorem naturally leads to the stronger notion of *inductive freeness*: The empty arrangement is inductively free, and $\mathcal{A}$ is inductively free if there is a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ as above where $\mathcal{A}', \mathcal{A}''$ are inductively free. So inductive freeness implies freeness, but the converse is false [11, 4.59].

A very important class of arrangements is the class of reflection arrangements. There is an elegant invariant theoretic proof (see e.g. [11, Theorem 6.60]) that reflection arrangements are free. In fact, it is conjectured [11, Conjecture 6.90, Conjecture 6.91] that reflection arrangements are inductively free or even *hereditarily* inductively free. ORLIK and TERAO [12] proved that COXETER arrangements are hereditarily free.

Here we prove the inductive freeness for the following large class of simplicial arrangements: Let $\mathcal{A}$ be a simplicial arrangement and let R be a set of nonzero covectors such that $\mathcal{A} = \{\alpha^{\perp} \mid \alpha \in R\}$. Assume that $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. The pair $(\mathcal{A}, R)$ is called *crystallographic*, see [4, Definition 2.3], if for any chamber K the elements of R are integer linear combinations of the covectors defining the walls of K. For example, all reflection arrangements from Weyl groups[2] are crystallographic arrangements. In rank greater than two the arrangements of type H_3 and H_4 are the only COXETER arrangements which are not crystallographic.

In this paper we prove that crystallographic arrangements are hereditarily inductively free. We treat the sporadic cases with the computer: The algorithm is mainly based upon the fact that the roots of a finite WEYL groupoid are real roots and that the finite WEYL groupoids are in one-to-one correspondence with the crystallographic arrangements [4]. For the non-crystallographic COXETER arrangements of type H_3 and H_4 we use a generic version of the algorithm.

The paper is organized as follows. In Section 2 we recall all necessary definitions following [11]. In Section 3 we briefly describe the classification of crystallographic arrangements [5]. In the next section we give a detailed description of the infinite series and prove that their arrangements are inductively free. Section 5 treats the sporadic cases using the computer. This is the most difficult part, in particular the arrangement of type E_8 requires significant extra work. All the algorithms we use to decide the inductive freeness by searching for an "inductive path" produce a *certificate* providing an a posteriori proof of correctness for the computed path. In the last section we describe algorithms to decide the freeness and compute a free basis for the module of derivations of a general central arrangement. Finally, in Appendices A and B we describe the computation of a free basis for the largest sporadic crystallographic arrangement of rank 7 and 8 and list the exponents of all sporadic crystallographic arrangements.

2. Preliminaries on arrangements

Let $r \in \mathbb{N}$, $V := \mathbb{R}^r$. We first recall the definition of a simplicial arrangement (compare [11, 1.2, 5.1]).

Definition 2.1. Let $\mathcal{A}$ be a simplicial arrangement in V, i.e. $\mathcal{A} = \{H_1, \ldots, H_n\}$ where $H_1, \ldots, H_n$ are distinct linear hyperplanes in V and every component of $V \setminus \bigcup_{H \in \mathcal{A}} H$ is an open simplicial cone. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components of $V \setminus \bigcup_{H \in \mathcal{A}} H$; they are called the *chambers* of $\mathcal{A}$.

We also need the concepts of a subarrangement and restriction:

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[2] Notice that for example the arrangements of type B_n and C_n are not isomorphic as crystallographic arrangements because of the additional datum R.

Definition 2.2. (See [11, 1.12–1.14].) Let $(\mathcal{A}, V)$ be an arrangement. We denote by $L(\mathcal{A})$ the set of all nonempty intersections of elements of $\mathcal{A}$.

If $\mathcal{B} \subseteq \mathcal{A}$ is a subset, then $(\mathcal{B}, V)$ is called a *subarrangement*. For $X \in L(\mathcal{A})$ define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

Define an arrangement $(\mathcal{A}^X, X)$ in X by

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A}\backslash\mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

We call $\mathcal{A}^X$ the *restriction* of $\mathcal{A}$ to X.

Let $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A}\backslash\{H_0\}$ and let $\mathcal{A}'' = \mathcal{A}^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a *triple* of arrangements and H_0 the *distinguished* hyperplane.

Recall the module of derivations of an arrangement:

Definition 2.3. (See [11, 4.1].) Let $(\mathcal{A}, V)$ be a real arrangement and $S = S(V^*)$ the symmetric algebra of the dual space V^* of V. We choose a basis $x_1, \ldots, x_r$ for V^* and identify S with $\mathbb{R}[x_1, \ldots, x_r]$ via the natural isomorphism $S \cong \mathbb{R}[x_1, \ldots, x_r]$. We write $\mathrm{Der}(S)$ for the set of derivations of S over $\mathbb{R}$. It is a free S-module with basis $D_1, \ldots, D_r$ where D_i is the usual derivation $\partial/\partial x_i$.

A nonzero element $\theta \in \mathrm{Der}(S)$ is *homogeneous of polynomial degree* p if $\theta = \sum_{k=1}^r f_k D_k$ and $f_k \in S_p$ for $a \leqslant k \leqslant r$. In this case we write $\mathrm{pdeg}\,\theta = p$.

Let $\mathcal{A}$ be an arrangement in V with defining polynomial

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

where $H = \ker \alpha_H$. Define the *module of $\mathcal{A}$-derivations* by

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

In this paper, we prove that certain arrangements are *free*:

Definition 2.4. An arrangement $\mathcal{A}$ is called a *free arrangement* if $D(\mathcal{A})$ is a free module over S.

If $\mathcal{A}$ is free and $\{\theta_1, \ldots, \theta_r\}$ is a homogeneous basis for $D(\mathcal{A})$, then $\mathrm{pdeg}\,\theta_1, \ldots, \mathrm{pdeg}\,\theta_r$ are called the *exponents* of $\mathcal{A}$ and we write

$$\exp \mathcal{A} = \{\mathrm{pdeg}\,\theta_1, \ldots, \mathrm{pdeg}\,\theta_r\}.$$

Remark that the pdegrees depend only on $\mathcal{A}$ (up to ordering).

We will use the following theorem:

Theorem 2.5 (*Addition–Deletion*). (*See [11, Theorem 4.51].*) *Suppose* $\mathcal{A} \neq \emptyset$. *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple. Any two of the following statements imply the third*:

$$\mathcal{A} \text{ is free with } \exp \mathcal{A} = \{b_1, \ldots, b_{r-1}, b_r\},$$

$$\mathcal{A}' \text{ is free with } \exp \mathcal{A}' = \{b_1, \ldots, b_{r-1}, b_r - 1\},$$

$$\mathcal{A}'' \text{ is free with } \exp \mathcal{A}'' = \{b_1, \ldots, b_{r-1}\}.$$

Inspired by this theorem, one defines:

Definition 2.6. (*See [11, Definition 4.53].*) The class $\mathcal{IF}$ of *inductively free* arrangements is the smallest class of arrangements which satisfies

(1) The empty arrangement Φ_ℓ of rank ℓ is in $\mathcal{IF}$ for $\ell \geqslant 0$,
(2) if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{IF}$, $\mathcal{A}' \in \mathcal{IF}$, and $\exp \mathcal{A}'' \subset \exp \mathcal{A}'$, then $\mathcal{A} \in \mathcal{IF}$.

We will say that $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is an *inductive chain* of arrangements if $\mathcal{A}_i \setminus \mathcal{A}_{i-1} = \{H_i\}$ for $i = 2, \ldots, n$ and suitable H_i, and if $(\mathcal{A}_i, \mathcal{A}_{i-1}, \mathcal{A}_i^{H_i})$ is a triple of inductively free arrangements for all $i = 2, \ldots, n$.

We further need:

Theorem 2.7. (*See [11, Theorem 6.60].*) *If G is a finite reflection group, then its reflection arrangement $\mathcal{A} = \mathcal{A}(G)$ is free and $\exp \mathcal{A}$ is the set of coexponents of G.*

Thus by [3, VI. Planche II, IV]:

Remark 2.8. Let $\mathcal{A}$ resp. $\mathcal{B}$ be the reflection arrangements of type B_r resp. D_r. Then

$$\exp \mathcal{A} = \{1, 3, \ldots, 2r - 3, 2r - 1\},$$

$$\exp \mathcal{B} = \begin{cases} \{1, 3, \ldots, r - 3, r - 1, r - 1, r + 1, \ldots, 2r - 5, 2r - 3\}, & r \text{ even}, \\ \{1, 3, \ldots, r - 4, r - 2, r - 1, r, r + 2, \ldots, 2r - 5, 2r - 3\}, & r \text{ odd}. \end{cases}$$

3. Crystallographic arrangements

Recall the definition of a crystallographic arrangement and the correspondence to CARTAN schemes:

Definition 3.1. (*See [4, Definition 2.3].*) Let $(\mathcal{A}, V)$ be a simplicial arrangement and $R \subseteq V$ a finite set such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. For a chamber K of $\mathcal{A}$ set

$$R_+^K = R \cap \sum_{\alpha \in B^K} \mathbb{R}_{\geqslant 0}\alpha,$$

where B^K is the set of normal vectors in R of the walls of K pointing to the inside. We call $(\mathcal{A}, R)$ a *crystallographic arrangement* if

(I) $R \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha$ for all chambers K.

Theorem 3.2. *(See [4, Theorem 1.1].) There is a one-to-one correspondence between crystallographic arrangements and connected simply connected* CARTAN *schemes for which the real roots are a finite root system* (*up to equivalence on both sides*).

We omit the definitions of CARTAN schemes and their root systems here because we will not need them. It suffices to know that there is a complete classification of those CARTAN schemes which correspond to crystallographic arrangements [5, Theorem 1.1]:

Theorem 3.3. *There are exactly three families of connected simply connected* CARTAN *schemes for which the real roots form a finite irreducible root system*:

(1) *The family of* CARTAN *schemes of rank two parametrized by triangulations of a convex n-gon by non-intersecting diagonals.*
(2) *For each rank $r > 2$, the standard* CARTAN *schemes of type A_r, B_r, C_r and D_r, and a series of $r - 1$ further* CARTAN *schemes described explicitly in Theorem 4.3.*
(3) *A family consisting of 74 further "sporadic"* CARTAN *schemes (including those of type F_4, E_6, E_7 and E_8).*

Definition 3.4. For a finite set $\Lambda \subset \mathbb{R}^r$ we will write

$$^\Lambda \mathcal{A} := \{\beta^\perp \mid \beta \in \Lambda\}.$$

Let $r, s \in \mathbb{N}$ with $r \leqslant s$. We will say that a finite set $\Lambda \subseteq \mathbb{Z}^s$ is a *root set of rank r* if there exists a CARTAN scheme $\mathcal{C}$ of rank r and an injective linear map $w : \mathbb{Z}^r \to \mathbb{Z}^s$ such that $w((R^{\mathrm{re}})^a) = \Lambda$ for some object a.

4. The infinite series

Let $r \in \mathbb{N}$. Denote by $\{\alpha_1, \ldots, \alpha_r\}$ the standard basis of $\mathbb{Z}^r$. We use the following notation: For $1 \leqslant i, j \leqslant r$, let

$$\eta_{i,j} := \begin{cases} \sum_{k=i}^{j} \alpha_k, & i \leqslant j, \\ 0, & i > j. \end{cases}$$

Definition 4.1. Let $Z \subseteq \{1, \ldots, r - 1\}$. Let $\Xi_{r,Z}$ denote the set of roots[3]

$$\eta_{i,j-1}, \quad 1 \leqslant i < j \leqslant r,$$

$$\eta_{i,r-2} + \alpha_r, \quad 1 \leqslant i < r,$$

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[3] This set is denoted by $\Phi_{r,Z}$ in [5]; we denote it by $\Xi_{r,Z}$ here to avoid confusing it with the empty arrangement.

$$\eta_{i,r} + \eta_{j,r-2}, \quad 1 \leqslant i < j < r,$$

$$\eta_{j,r} + \eta_{j,r-2}, \quad j \in Z.$$

Let $Y \subseteq \{1, \ldots, r-1\}$. Let $\Psi_{r,Y}$ denote the set of roots

$$\eta_{i,j}, \quad 1 \leqslant i \leqslant j \leqslant r,$$

$$\eta_{i,r} + \eta_{j,r-1}, \quad 1 \leqslant i < j < r,$$

$$\eta_{j,r} + \eta_{j,r-1}, \quad j \in Y.$$

Further, denote by $\Psi'_{r,Y}$ the set obtained from $\Psi_{r,Y}$ by exchanging α_{r-1} and α_r.

Remark 4.2. The sets $\Xi_{r,\emptyset}$ resp. $\Psi_{r,\{1,\ldots,r-1\}}$ are the sets of positive roots of the WEYL groups of type D_r resp. C_r, compare [3, VI. 4.6, 4.8].

The following holds [5, Theorem 3.21]:

Theorem 4.3. *Let $\mathcal{C}$ be a connected simply connected* CARTAN *scheme of rank $r > 8$ for which the real roots form a finite irreducible root system and let*

$$\mathcal{R}_+ := \{ (R^{\mathrm{re}})^a_+ \mid a \in \mathcal{C} \}.$$

Then there are two possibilities:

(1) *The* CARTAN *scheme $\mathcal{C}$ is standard ($|\mathcal{R}_+| = 1$) of type A, B, C, D.*
(2) *Up to equivalence the root sets of $\mathcal{C}$ are given by*

$$\mathcal{R}_+ = \{ \Xi_{r,Z}, \Psi_{r,Y}, \Psi'_{r,Y} \mid Z, Y \subseteq \{1, \ldots, r-1\}, \ |Z| = s, \ |Y| = s-1 \}$$

for some $s \in \{1, \ldots, r-1\}$.

In particular, if $\mathcal{C}$ is not standard then it has

$$\binom{r-1}{s-1} + \binom{r}{s}$$

different root sets and $2^{r-1}(m+r)(r-1)!$ objects.

Since the sets in $\mathcal{R}_+$ are equal up to a base change, we obtain:

Corollary 4.4. *Let $\mathcal{A}$ be an irreducible crystallographic arrangement of rank $r \geqslant 3$ which is not sporadic. Then up to a base change*

$$\mathcal{A} = \{ \alpha^\perp \mid \alpha \in R_+ \}$$

where R_+ is either a set of positive roots of type A_r, B_r, C_r or D_r, or $R_+ = \Xi_{r,Z}$ for some $Z \subset \{1, \ldots, r-1\}$.

We now treat the inductive freeness of the series:

Proposition 4.5. *Let $\mathcal{A}$ be an irreducible crystallographic arrangement of rank r which is not sporadic. Then $\mathcal{A}$ is an inductively free arrangement.*

Proof. If $\mathcal{A}$ is a reflection arrangement of type A, then it is inductively free by [11, Example 4.55]. So assume that $\mathcal{A}$ is not of type A and that $\mathcal{A}$ is not sporadic. By Corollary 4.4 we may assume $\mathcal{A} = \{\alpha^{\perp} \mid \alpha \in \varXi_{r,Z}\}$ for a certain subset $Z \subseteq \{1, \ldots, r-1\}$, $Z \neq \emptyset$, or that $\mathcal{A}$ is of type C. Using the base change

$$\alpha_i \mapsto \alpha_i - \alpha_{i+1}, \qquad \alpha_r \mapsto \alpha_{r-1} + \alpha_r$$

for $i = 1, \ldots, r-1$, one obtains that the arrangement $\varXi_{r,Z}$ is isomorphic to the arrangement denoted by D_r^k in [10] defined by

$$x_1 \cdots x_k \prod_{1 \leqslant i < j \leqslant r} (x_i \pm x_j)$$

for $k = |Z|$. Further, D_r^r is isomorphic to the arrangements of type B and C. JAMBU and TERAO showed [10, Example (2.6)] that D_r^k are inductively free for all r, k. $\quad\square$

5. Inductive freeness of the sporadic crystallographic arrangements

5.1. Ranks three to seven

The key to the algorithm is the fact that the root sets "are" sets of real roots for a CARTAN scheme: Let $\mathcal{C}$ be a CARTAN scheme and a an object. Then for each root $\alpha \in R_+^a$ there exists an object b and a morphism $w \in \mathrm{Hom}(a, b)$ such that $w(\alpha)$ is a simple root. In particular, we get:

Definition 5.1. Let $\mathcal{C}$ be a CARTAN scheme, a an object and R^a the set of real roots at a. Then there exist maps (not unique)

$$\mu_{R^a} : R^a \to \mathrm{Hom}(a, \mathcal{W}(\mathcal{C})), \qquad \iota_{R^a} : R^a \to \{1, \ldots, r\}$$

such that for all $\alpha \in R^a$ the set $\mu_{R^a}(\alpha)(R^a)$ is a root set and

$$\mu_{R^a}(\alpha)(\alpha) = \alpha_{\iota_{R^a}(\alpha)}$$

is simple. Denote $g_i(\lambda_1, \ldots, \lambda_r) := \gcd(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_r)$ where "$\widehat{}$" means that we omit this element and let $\alpha \in R^a$. We set

$$^{\alpha}R^a := \left\{ \frac{1}{g_{\iota_R(\alpha)}(\lambda_1, \ldots, \lambda_r)} \sum_{\iota_R(\alpha) \neq i = 1}^{r} \lambda_i \alpha_i \,\middle|\, \sum_{i=1}^{r} \lambda_i \alpha_i \in \mu(R^a), \quad \sum_{\iota_R(\alpha) \neq i = 1}^{r} \lambda_i \alpha_i \neq 0 \right\}$$

and view $^{\alpha}R^a \subseteq \langle \alpha_1, \ldots, \widehat{\alpha_{\iota(\alpha)}}, \ldots, \alpha_r \rangle$ as a subset of $\mathbb{R}^{r-1}$.

Proposition 5.2. *Let $\mathcal{A}$ be a crystallographic arrangement. Then $\mathcal{A} = {}^R\mathcal{A}$ for some root set R and*

$$
{}^{\alpha R}\mathcal{A} = \mathcal{A}^{\alpha^\perp}
$$

for all $\alpha \in R$.

Proof. We use the correspondence to WEYL groupoids: Up to a base change, $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R_+^a\}$ where R_+^a is the set of real roots at an object a of a CARTAN scheme. Now we use maps μ, ι as in Definition 5.1. Explicitly, $\mathcal{A}^{\alpha^\perp}$ is the arrangement given by R_+^a where we erase the $\iota(\alpha)$-th coordinate, collect the resulting vectors and possibly divide them by the greatest common divisor of their coordinates. □

We also need the following proposition which is a corollary to the classification:

Proposition 5.3. *Let $\mathcal{A}$ be a crystallographic arrangement and $H \in \mathcal{A}$. Then $\mathcal{A}^H$ is crystallographic.*

Proof. Let R be a root set with $\mathcal{A} = {}^R\mathcal{A}$ and $\alpha \in R$ such that $H = \alpha^\perp$. It suffices to check that ${}^\alpha R$ is a root set. We use the classification. For the 74 sporadic crystallographic arrangements we use the computer. The infinite series are easy to treat: Restricting from type A, B yields type A, B respectively. Restricting from type C gives type C except for one coordinate which yields type B.

For type D or the arrangements from $\varXi_{r,Z}$, $\varPsi_{r,Y}$, $\varPsi'_{r,Y}$ one checks all restrictions: In the standard ordering, restricting a root system of type D_r to $\alpha_1^\perp, \ldots, \alpha_{r-2}^\perp$ gives a root set of the form $\varXi_{r-1,Z}$ and restricting to $\alpha_{r-1}^\perp$ or $\alpha_r^\perp$ gives a root set of the form $\varPsi_{r-1,Y}$. Thus the only roots one has to consider when restricting from $\varXi_{r,Z}$, $\varPsi_{r,Y}$ or $\varPsi'_{r,Y}$ are those parametrized by Z resp. Y. But it is easy to check in each case that one again obtains such a set. □

Remark that Proposition 5.3 is only used in the algorithm and in Corollary 5.15, where we state that the crystallographic arrangements are hereditarily inductively free.

Let L be a global variable which is a sequence of inductively free arrangements $\mathcal{A}$ stored as pairs $(\varLambda, \exp \mathcal{A})$ where $\mathcal{A} = {}^\varLambda\mathcal{A}$. The following algorithm treats almost all sporadic crystallographic arrangements:

Algorithm 5.4. IsInductivelyFree$(r, R_0, R_1, e_0, \hat{R}, \mu)$
Test if R_1 is inductively free.
Input: $r \in \mathbb{N}$, $R_0 \subseteq R_1 \subseteq \mathbb{N}_0^r$, $e_0 = \exp {}^{R_0}\mathcal{A}$, a root set $\hat{R}$ with $R_1 \subseteq \hat{R}$, a map $\mu = \mu_{\hat{R}} : \hat{R} \to \mathrm{GL}(\mathbb{Z}^r)$.
Output: True or **False**, a sequence of exponents if **True**.

1. If $|R_0| = |R_1|$ then return (**True**, e_0).
2. Set $F \leftarrow R_1 \backslash R_0$. Sort F in a "good" way: Call **HeuristicGoodOrdering**(F) (see below).
3. For each $\alpha \in F$, perform steps 4–12.
4. Set $R_2 = R_0 \cup \{\pm\alpha\}$. Let $R := \mu(\alpha)(R_2)$ and compute $R'' := {}^{\mu(\alpha)(\alpha)}R$.
5. Initialize $e \leftarrow \emptyset$.

6. If the rank of R'' is 2, then $e := (1, |R''| - 1)$, go to step 12.

7. If $R'' \in L$, then R'' is known to be inductively free, set $e := \exp^{R''} \mathcal{A}$, go to step 12.

8. If $|R''| = r - 1$ and the rank of R'' is $r - 1$ then set $e := (1, \ldots, 1)$ ($|e| = r$), go to step 12.

9. Search in L for a largest set Λ with $\Lambda \subseteq R''$. If there is no such set, then choose a linearly independent $\Lambda \in R''$ with $R'' \subseteq \langle \Lambda \rangle$.

10. Let $\hat{R}'' := {}^\alpha \hat{R}$. By Proposition 5.3, $\hat{R}''$ is crystallographic and we obtain a new map $\mu_{\hat{R}''}$.

11. Call **IsInductivelyFree**($r - 1, \Lambda, R'', \exp^\Lambda \mathcal{A}, \hat{R}'', \mu_{\hat{R}''}$). If R'' is inductively free, then include it with its exponents into L and set $e := \exp^{R''} \mathcal{A}$.

12. If $|e| > 0$ and $e \subseteq e_0$, then (R_0, R_2, R'') is a triple of arrangements and thus by Theorem 2.5 we know that R_2 is inductively free and we know $\exp^{R_2} \mathcal{A}$. Call **IsInductivelyFree**($r, R_2, R_1,$ $\exp^{R_2} \mathcal{A}, \hat{R}, \mu$) and return the result (**True**, $\exp^{R_1} \mathcal{A}$) if it is **True**.

13. Return (**False**, $\emptyset$).

Remark 5.5. To use the above algorithm to show that all sporadic crystallographic arrangements are inductively free, we start with the arrangements of rank three and continue up to rank seven. After each call of the function, we store the result in L as well as all the root sets from other objects of the same WEYL groupoid.

The runtime of the algorithm strongly depends on good hash and search functions for L. Further, when looking for an arrangement in L, we also consider arrangements with permuted coordinates. So we need a good function that recognizes whether two matrices are equal up to permutations of columns and rows.

Remark 5.6. We also keep track of pairs (R_0, R_1) for which no inductively free chain from ${}^{R_0} \mathcal{A}$ to ${}^{R_1} \mathcal{A}$ was found during the algorithm to avoid testing them again in future. As for L, we need good hash functions and perform all tests up to permutations of rows and columns.

The following function has proven to give good orderings (although we have to admit that we do not know why).

Algorithm 5.7. HeuristicGoodOrdering(F)

Sort $F = \{\beta_1, \ldots, \beta_n\}$ in such a way that $({}^{\{\beta_1\}} \mathcal{A}, \ldots, {}^{\{\beta_1, \ldots, \beta_n\}} \mathcal{A})$ is hopefully almost (up to very few transpositions) an inductive chain.

Input: $F \subseteq \mathbb{N}_0^r$.

Output: An ordering $F = \{\beta_1, \ldots, \beta_n\}$.

1. $T \leftarrow F$.

2. Compute a graph Γ having $1, \ldots, r$ as vertices and for which (i, j) is an edge if and only if $\alpha_i + \alpha_j \in T$ (this is almost the Dynkin diagram of T when T is a root system).

3. If possible, choose a path $i_1, \ldots, i_r$ in Γ that passes each vertex exactly once.

4. Permute the coordinates of the elements of T to $i_1, \ldots, i_r$.

5. Sort T lexicographically and apply the same exchanges to F.

Algorithm 5.4 works very well for all crystallographic arrangements of rank up to 7 except for the largest arrangement $\mathcal{A}_{7,2}$ of rank 7 with 91 hyperplanes. After several experiments one

$F_+ := \{(0,0,0,0,0,0,1), (0,0,0,0,0,1,0), (0,0,0,0,1,0,0), (0,0,0,1,0,0,0), (0,0,1,0,0,0,0),$
$(0,1,0,0,0,0,0), (1,0,0,0,0,0,0), (0,0,0,0,0,1,1), (0,0,0,0,1,0,1), (0,0,0,0,1,1,1),$
$(0,0,0,0,1,1,0), (0,0,0,1,0,0,1), (0,0,0,1,0,1,1), (0,0,0,1,1,1,1), (0,0,0,1,1,0,1),$
$(0,0,0,1,1,1,2), (0,0,1,0,0,1,0), (0,0,1,0,0,1,1), (0,0,1,0,1,1,1), (0,0,1,0,1,1,0),$
$(0,0,1,0,1,2,1), (0,0,1,1,1,1,1), (0,0,1,1,1,2,2), (0,0,1,1,0,1,1), (0,0,1,1,1,1,2),$
$(0,0,1,1,1,2,1), (0,0,1,1,2,2,2), (0,1,0,1,0,0,0), (0,1,0,1,0,0,1), (0,1,0,1,0,1,1),$
$(0,1,0,1,1,1,1), (0,1,0,1,1,0,1), (0,1,0,1,1,1,2), (0,1,0,2,1,1,2), (0,1,1,1,1,1,1),$
$(0,1,1,1,1,2,2), (0,1,1,2,1,2,2), (0,1,1,1,0,1,1), (0,1,1,1,1,2,1), (0,1,1,2,2,2,3),$
$(0,1,1,1,1,1,2), (0,1,1,1,2,2,2), (0,1,1,2,1,1,2), (0,1,1,2,1,2,3), (0,1,1,2,2,2,2),$
$(0,1,1,2,2,3,3), (0,1,2,2,2,3,3), (1,1,0,0,0,0,0), (1,1,0,1,0,0,0), (1,1,0,1,0,0,1),$
$(1,1,0,1,0,1,1), (1,1,0,1,1,1,1), (1,1,0,1,1,0,1), (1,1,0,1,1,1,2), (1,1,0,2,1,1,2),$
$(1,1,1,1,1,1,1), (1,1,1,1,1,2,2), (1,1,1,2,1,2,2), (1,1,1,1,0,1,1), (1,1,1,2,2,2,3),$
$(1,1,1,1,1,1,2), (1,1,1,1,1,2,1), (1,1,1,1,2,2,2), (1,1,1,2,1,1,2), (1,1,1,2,1,2,3),$
$(1,1,1,2,2,2,2), (1,1,1,2,2,3,3), (1,1,2,2,2,3,3), (1,2,1,2,1,2,2), (1,2,1,2,2,2,3),$
$(1,2,1,3,2,2,3), (1,2,1,3,2,3,4), (1,2,0,2,1,1,2), (1,2,2,3,2,3,4), (1,2,1,2,1,1,2),$
$(1,2,1,2,1,2,3), (1,2,1,3,1,2,3), (1,2,1,3,2,2,4), (1,2,2,3,3,4,4), (1,2,1,2,2,2,2),$
$(1,2,1,2,2,3,3), (1,2,1,3,2,3,3), (1,2,1,3,3,3,4), (1,2,2,2,2,3,3), (1,2,2,3,2,3,3),$
$(1,2,2,3,2,4,4), (1,2,2,3,3,3,4), (1,2,2,3,3,4,5), (1,2,2,4,3,4,5), (1,3,2,4,3,4,5),$
$(2,3,2,4,3,4,5)\}$

Fig. 1. A set of normal vectors for $\mathcal{A}_{7,2}$.

also finds a good ordering for $\mathcal{A}_{7,2}$: Let F_+ be defined as in Fig. 1 and denote by $\gamma_1, \ldots, \gamma_{91}$ the elements of F_+ in the ordering of Fig. 1. Then

$$\{\gamma_i^\perp \mid i = 1, \ldots, k\}, \quad k = 1, \ldots, 91,$$

is an inductive chain of arrangements. The proof is just an application of Algorithm 5.4 (notice that all computations are now of rank six).

5.2. The arrangement of type E_8

In rank 8 there is only one sporadic crystallographic arrangement, the reflection arrangement of type E_8. We will denote this arrangement by $\mathcal{A}_{8,1}$. Unfortunately, Algorithm 5.4 is not good enough for this last case (we stopped it after a month of computation). So we have to look more closely at the structure of $\mathcal{A}_{8,1}$. Experiments with Algorithm 5.4 lead to the conjecture that "**HeuristicGoodOrdering**" yields indeed an inductive chain for $\mathcal{A}_{8,1}$. So the set R_+ we will consider is the one given in Fig. 2 (in this ordering), and $\mathcal{A}_{8,1} = {}^{R_+}\mathcal{A}$.

If we write $R_+ = \{\beta_1, \ldots, \beta_{120}\}$, then we obtain arrangements $\mathcal{A}_1, \ldots, \mathcal{A}_{120}$ where $\mathcal{A}_i := \{\beta_1^\perp, \ldots, \beta_i^\perp\}$ for $i = 1, \ldots, 120$. We claim that $(\mathcal{A}_1, \ldots, \mathcal{A}_{120})$ is an inductive chain of arrangements. To prove this, we need to check that $\mathcal{A}_2^{\beta_2^\perp}, \ldots, \mathcal{A}_{120}^{\beta_{120}^\perp}$ are inductively free and that the exponents of the triples satisfy the assumptions of Theorem 2.5.

First notice that each restriction $\mathcal{A}_{8,1}^H$, $H \in \mathcal{A}_{8,1}$, comes from a root set of the sporadic finite Weyl groupoid $\mathcal{W}$ of rank 7 with 91 positive roots. So whenever we consider $\mathcal{A}^H$ for some $\mathcal{A} \subseteq \mathcal{A}_{8,1}$ and $H \in \mathcal{A}_{8,1}$, we have an action of $\mathcal{W}$ on the corresponding "roots" and in particular the automorphisms of the chosen object act as well. More precisely:

$R_+ := \{(0,0,0,0,0,0,0,1), (0,0,0,0,0,0,1,0), (0,0,0,0,0,1,0,0), (0,0,0,0,1,0,0,0),$
$(0,0,0,1,0,0,0,0), (0,0,1,0,0,0,0,0), (0,1,0,0,0,0,0,0), (1,0,0,0,0,0,0,0),$
$(0,0,0,0,0,0,1,1), (0,0,0,0,0,1,1,0), (0,0,0,0,0,1,1,1), (0,0,0,0,1,1,0,0),$
$(0,0,0,0,1,1,1,0), (0,0,0,0,1,1,1,1), (0,0,0,1,1,0,0,0), (0,0,0,1,1,1,0,0),$
$(0,0,0,1,1,1,1,0), (0,0,0,1,1,1,1,1), (0,0,1,1,0,0,0,0), (0,0,1,1,1,0,0,0),$
$(0,0,1,1,1,1,0,0), (0,0,1,1,1,1,1,0), (0,0,1,1,1,1,1,1), (0,1,0,1,0,0,0,0),$
$(0,1,0,1,1,0,0,0), (0,1,0,1,1,1,0,0), (0,1,0,1,1,1,1,0), (0,1,0,1,1,1,1,1),$
$(0,1,1,1,0,0,0,0), (0,1,1,1,1,0,0,0), (0,1,1,1,1,1,0,0), (0,1,1,1,1,1,1,0),$
$(0,1,1,1,1,1,1,1), (0,1,1,2,1,0,0,0), (0,1,1,2,1,1,0,0), (0,1,1,2,1,1,1,0),$
$(0,1,1,2,1,1,1,1), (0,1,1,2,2,1,0,0), (0,1,1,2,2,1,1,0), (0,1,1,2,2,1,1,1),$
$(0,1,1,2,2,2,1,0), (0,1,1,2,2,2,1,1), (0,1,1,2,2,2,2,1), (1,0,1,0,0,0,0,0),$
$(1,0,1,1,0,0,0,0), (1,0,1,1,1,0,0,0), (1,0,1,1,1,1,0,0), (1,0,1,1,1,1,1,0),$
$(1,0,1,1,1,1,1,1), (1,1,1,1,0,0,0,0), (1,1,1,1,1,0,0,0), (1,1,1,1,1,1,0,0),$
$(1,1,1,1,1,1,1,0), (1,1,1,1,1,1,1,1), (1,1,1,2,1,0,0,0), (1,1,1,2,1,1,0,0),$
$(1,1,1,2,1,1,1,0), (1,1,1,2,1,1,1,1), (1,1,1,2,2,1,0,0), (1,1,1,2,2,1,1,0),$
$(1,1,1,2,2,1,1,1), (1,1,1,2,2,2,1,0), (1,1,1,2,2,2,1,1), (1,1,1,2,2,2,2,1),$
$(1,1,2,2,1,0,0,0), (1,1,2,2,1,1,0,0), (1,1,2,2,1,1,1,0), (1,1,2,2,1,1,1,1),$
$(1,1,2,2,2,1,0,0), (1,1,2,2,2,1,1,0), (1,1,2,2,2,1,1,1), (1,1,2,2,2,2,1,0),$
$(1,1,2,2,2,2,1,1), (1,1,2,2,2,2,2,1), (1,1,2,3,2,1,0,0), (1,1,2,3,2,1,1,0),$
$(1,1,2,3,2,1,1,1), (1,1,2,3,2,2,1,0), (1,1,2,3,2,2,1,1), (1,1,2,3,2,2,2,1),$
$(1,1,2,3,3,2,1,0), (1,1,2,3,3,2,1,1), (1,1,2,3,3,2,2,1), (1,1,2,3,3,3,2,1),$
$(1,2,2,3,2,1,0,0), (1,2,2,3,2,1,1,0), (1,2,2,3,2,1,1,1), (1,2,2,3,2,2,1,0),$
$(1,2,2,3,2,2,1,1), (1,2,2,3,2,2,2,1), (1,2,2,3,3,2,1,0), (1,2,2,3,3,2,1,1),$
$(1,2,2,3,3,2,2,1), (1,2,2,3,3,3,2,1), (1,2,2,4,3,2,1,0), (1,2,2,4,3,2,1,1),$
$(1,2,2,4,3,2,2,1), (1,2,2,4,3,3,2,1), (1,2,2,4,4,3,2,1), (1,2,3,4,3,2,1,0),$
$(1,2,3,4,3,2,1,1), (1,2,3,4,3,2,2,1), (1,2,3,4,3,3,2,1), (1,2,3,4,4,3,2,1),$
$(1,2,3,5,4,3,2,1), (1,3,3,5,4,3,2,1), (2,2,3,4,3,2,1,0), (2,2,3,4,3,2,1,1),$
$(2,2,3,4,3,2,2,1), (2,2,3,4,3,3,2,1), (2,2,3,4,4,3,2,1), (2,2,3,5,4,3,2,1),$
$(2,2,4,5,4,3,2,1), (2,3,3,5,4,3,2,1), (2,3,4,5,4,3,2,1), (2,3,4,6,4,3,2,1),$
$(2,3,4,6,5,3,2,1), (2,3,4,6,5,4,2,1), (2,3,4,6,5,4,3,1), (2,3,4,6,5,4,3,2)\}$

Fig. 2. A set of normal vectors for $\mathcal{A}_{8,1}$.

Let F denote the root set of $\mathcal{W}$ for which we know an inductive chain, i.e. an ordering of $F_+ = \{\gamma_1, \ldots, \gamma_{91}\}$. For each $i = 1, \ldots, 120$, $^{\mu(\beta_i)}R_+$ is some set of positive roots for $\mathcal{W}$, so there is an automorphism φ with

$$\varphi\big(^{\mu(\beta_i)}R_+ \cup -^{\mu(\beta_i)}R_+\big) = F.$$

Now we consider $\mathcal{A}_i^{\beta_i^\perp} \subseteq {}^{\mu(\beta_i)}R_+$. The automorphism group of the object F in $\mathcal{W}$ is a subgroup of the symmetric group S_{182} and it acts on $\varphi(\mathcal{A}_i^{\beta_i^\perp} \cup -\mathcal{A}_i^{\beta_i^\perp})$. A computation yields:

Lemma 5.8. *Let O_i be the orbit of $\varphi(\mathcal{A}_i^{\beta_i^\perp} \cup -\mathcal{A}_i^{\beta_i^\perp})$ under the action of* Aut(F) *for $i = 43, \ldots, 120$. Then $O_i = O_j$ whenever $|\mathcal{A}_i^{\beta_i^\perp}| = |\mathcal{A}_j^{\beta_j^\perp}|$.*

Remark 5.9. The lemma is certainly also true for $i < 43$ but for these cases Algorithm 5.4 is good enough.

Since these automorphisms are linear maps, it suffices to check inductive freeness for one representative of each orbit. These are arrangements with

$$42, 46, 49, 51, 52, 58, 60, 61, 65, 66, 68, 74, 75, 77, 80, 84, 90, 91$$

hyperplanes. Let O be such an orbit and assume that $\{\gamma_{i_1}, \ldots, \gamma_{i_m}\}$ are the positive elements corresponding to a representative. We choose this representative in such a way that $\max\{i_1, \ldots, i_m\}$ is minimal. This way we ensure that the resulting ordering is very close to the preferred ordering for F. Indeed, using all these techniques we can prove the inductive freeness of the reflection arrangement of type E_8 in less than 5 min on a usual PC with GAP.

5.3. Certificates for inductive freeness

The above algorithm is quite complicated when implemented and it is very hard to completely exclude coding errors. Therefore we use a second very short and simple program to check that the results are correct.

Definition 5.10. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an inductively free hyperplane arrangement of rank at least 2. A *certificate for inductive freeness* $C_{\mathcal{A}}$ for $\mathcal{A}$ is

$$C_{\mathcal{A}} = \begin{cases} 2, & \text{rank } \mathcal{A} = 2, \\ ((i_1, \ldots, i_n), (C_1, \ldots, C_n)), & \text{rank } \mathcal{A} > 2, \end{cases}$$

where $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$, $\mathcal{A}_j := \{H_{i_1}, \ldots, H_{i_j}\}$, $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is an inductive chain and C_j is a certificate for $\mathcal{A}_j^{H_{i_j}}$.

After several modifications to the above algorithms one obtains as output the exponents and a certificate as well. We can then check the certificate via the following:

Algorithm 5.11. CheckCertificate$(\mathcal{A}, C)$
Check whether C is a certificate for $\mathcal{A}$.
Input: A hyperplane arrangement $\mathcal{A}$, an object C.
Output: Exponents of $\mathcal{A}$ or **False**.

1. Let r be the rank of $\mathcal{A}$.
2. If $r = 2$ then return $(1, |\mathcal{A}| - 1)$.
3. Denote $\mathcal{A} = (H_1, \ldots, H_n)$ and $C = ((i_1, \ldots, i_n), (C_1, \ldots, C_n))$.
4. $e \leftarrow (1, \ldots, 1)$ (r-times).
5. For j from $r + 1$ to $|\mathcal{A}|$ do steps 6–8.
6. $\tilde{\mathcal{A}} \leftarrow \{H_{i_1}, \ldots, H_{i_{j-1}}\}^{H_{i_j}}$.
7. $\tilde{e} \leftarrow$ **CheckCertificate**$(\tilde{\mathcal{A}}, C_{j-r})$.
8. If $\tilde{e} \neq$ **False** and $\tilde{e} \subseteq e$ then $e \leftarrow \tilde{e} \cup (j - |\mathcal{A}|)$, else return **False**.
9. Return e.

Remark 5.12. The "$\cup$"-symbol in step 8 is a union of multisets, i.e. e, $\tilde{e}$ are in fact sets with multiplicities.

Remark 5.13. Step 6 is the time consuming part. Since we are only dealing with small integers this is a function which is very easy to implement in C. A certificate for the arrangement $\mathcal{A}_{8,1}$ takes depending on the format between 300 kB and 500 MB.

CheckCertificate takes about 15 min for $\mathcal{A}_{8,1}$ using GAP with a dynamic C-module for restrictions. It may seem surprising that it takes longer to check the certificate than to create it. There are two reasons for this: First, Algorithm 5.4 descends only once into each branch it has already computed to be inductively free (remember the global variable L). Secondly, in Algorithm 5.4 we use the information on WEYL groupoids and morphisms and can therefore always restrict to a simple root which amounts to erasing a coordinate and compute gcd's. In **Check-Certificate** we want to keep everything as short as possible and in particular we transfer no information on the morphisms.

Certificates for the sporadic crystallographic arrangements are available at [1].

5.4. The arrangements of type H_3 and H_4

To prove that all COXETER arrangements are inductively free, we still need to compute certificates for the non-crystallographic cases. The case of rank two being trivial, there are two arrangements left, the arrangements of type H_3 and H_4. Fortunately these cases are of rank three and four and are small enough to be treated by a generic version of Algorithm 5.4 which does not use the structure of the groupoids.

5.5. Summary

Theorem 5.14. *Crystallographic arrangements are inductively free.*

Proof. This is Proposition 4.5 and a computation with Algorithm 5.4. □

Corollary 5.15. *Crystallographic arrangements are hereditarily inductively free.*

Proof. Let $\mathcal{A}$ be a crystallographic arrangement and $X \in L(\mathcal{A})$. Then $X = H_1 \cap \cdots \cap H_k$ for certain hyperplanes $H_1, \ldots, H_k \in \mathcal{A}$. Applying Proposition 5.3 k-times, we obtain that $\mathcal{A}^X$ is crystallographic and thus inductively free by Theorem 5.14. □

Corollary 5.16. COXETER *arrangements are inductively free.*

Proof. This is Theorem 5.14 and Section 5.4. □

6. Freeness of the graded module $D(\mathcal{A})$

In this section we describe algorithms to compute a free basis of $D(\mathcal{A})$ for a free *central* arrangement $\mathcal{A}$. These algorithms can also be used to *decide* the freeness of finitely presented *graded* S-modules and, in particular, the freeness of central arrangements.

The first two subsections describe well-known algorithms. The algorithm in the third subsection enabled us to compute a free basis for the E_8-arrangement given as the product of 120 sparse matrices.

6.1. $D(\mathcal{A})$ as a kernel

Expressing $D(\mathcal{A})$ as a kernel of an S-module homomorphism allows the use of GRÖBNER basis techniques to compute a set of generators $D(\mathcal{A})$ as a subset of the free module $\mathrm{Der}(S)$ of rank r, the latter being identified with $S^{1 \times r}$ using the standard basis $(\theta_i := \frac{\partial}{\partial x_i} \mid i = 1 \ldots r)$.

By definition, $D(\mathcal{A})$ is the kernel of the map

$$\psi_{\mathcal{A}} : \begin{cases} \mathrm{Der}(S) \to S/\langle Q(\mathcal{A}) \rangle_S, \\ \theta \mapsto \theta(Q(\mathcal{A})) + \langle Q(\mathcal{A}) \rangle_S. \end{cases}$$

If $Q(\mathcal{A})$ is a complicated polynomial of large degree the GRÖBNER basis computations quickly become unfeasible. Fortunately, there exists an alternative description of $D(\mathcal{A})$ as a kernel of some other map $\phi_{\mathcal{A}}$ and it turns out that various GRÖBNER basis implementations scale much better when performing this kernel computation. First recall the identity [11, Proposition 4.8]

$$D(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} D(\alpha_H)$$

expressing the module of $\mathcal{A}$-derivations as the intersection of $|\mathcal{A}|$ free submodules of $\mathrm{Der}(S)$ with

$$D(\alpha_H) := \left\{ \theta \in \mathrm{Der}(S) \mid \theta(\alpha_H) \in \langle \alpha_H \rangle_S \right\} = \ker \phi_H,$$

where ϕ_H is the S-module map

$$\phi_H : \begin{cases} \mathrm{Der}(S) \to S/\langle \alpha_H \rangle_S, \\ \theta \mapsto \theta(\alpha_H) + \langle \alpha_H \rangle_S \end{cases}$$

between $\mathrm{Der}(S)$ and the cyclic torsion module $S/\langle \alpha_H \rangle_S$. The intersection $D(\mathcal{A})$ of these kernels can now be computed as the kernel of the product map $\phi_{\mathcal{A}} := \prod_{H \in \mathcal{A}} \phi_H$

$$\phi_{\mathcal{A}} : \mathrm{Der}(S) \to T_{\mathcal{A}}$$

with values in the torsion S-module

$$T_{\mathcal{A}} := \prod_{H \in \mathcal{A}} S/\langle \alpha_H \rangle_S = S^n / \prod_{H \in \mathcal{A}} \langle \alpha_H \rangle_S.$$

With respect to the standard generating system $(\bar{e}_j \mid j = 1 \ldots n)$ of $T_{\mathcal{A}} = S^n / \prod_{H \in \mathcal{A}} \langle \alpha_H \rangle_S$ we identify

$$T_{\mathcal{A}} \equiv \mathrm{coker}\big(S^{1 \times n} \xrightarrow{t_{\mathcal{A}}} S^{1 \times n}\big),$$

where $t_{\mathcal{A}}$ is the $n \times n$ diagonal matrix $(\delta_{ij} \alpha_{H_j})$ with diagonal entries. The map $\phi_{\mathcal{A}}$ can be represented by the constant coefficients $r \times n$ matrix $f_{\mathcal{A}} = (\theta_i(\alpha_{H_j}))$:

$$\phi_{\mathcal{A}} : S^{1\times r} \xrightarrow{f_{\mathcal{A}}} \operatorname{coker}\left(S^{1\times n} \xrightarrow{t_{\mathcal{A}}} S^{1\times n}\right),$$

with $\operatorname{Der}(S)$ identified with $S^{1\times r}$ as above.

Computing $D(\mathcal{A})$ as the kernel of ϕ_A amounts to determining a generating set of solutions of the homogeneous linear system of equations over S

$$\chi f_{\mathcal{A}} + \eta t_{\mathcal{A}} = 0,$$

or equivalently

$$(\,\chi \,|\, \eta\,)\left(\frac{f_{\mathcal{A}}}{t_{\mathcal{A}}}\right) = 0, \tag{6.1}$$

with $\chi \in S^{1\times r}$ and $\eta \in S^{1\times n}$. It follows that $\{(X_1 \mid \eta_1), \dots, (X_q \mid \eta_q)\}$ is a generating set of solutions of (6.1) iff $\{X_1, \dots, X_q\}$ is a generating set of $D(\mathcal{A})$ as a subspace of $\operatorname{Der}(S) \equiv S^{1\times r}$. A generating set of solutions is thus nothing but the rows of a matrix $(X \mid Y) \in S^{q\times(r+n)}$ of row syzygies of the matrix $\left(\frac{f_{\mathcal{A}}}{t_{\mathcal{A}}}\right)$ and, as such, can be computed using a modern computer algebra system supporting GRÖBNER basis. Most such systems even provide faster procedures to compute X without computing (a normal form of) Y explicitly. The desired matrix X is called the matrix of relative row syzygies of $f_{\mathcal{A}}$ modulo $t_{\mathcal{A}}$. Summing up: The rows $(X_1, \dots, X_q)$ of X generate $D(\mathcal{A}) \leqslant \operatorname{Der}(S) \equiv S^{1\times r}$.

If $\mathcal{A}$ is central then all modules in this section are graded, all maps are graded of degree 0, and the relative syzygies algorithm will produce a matrix X with homogeneous rows $(X_1, \dots, X_q)$.

6.2. Deciding the freeness of the graded submodule $D(\mathcal{A})$

Since $T_{\mathcal{A}}$ is torsion and hence of rank zero the short exact sequence

$$0 \to D(\mathcal{A}) \to \operatorname{Der}(S) \xrightarrow{\phi_{\mathcal{A}}} \operatorname{im} \phi_{\mathcal{A}} \to 0$$

implies that $\operatorname{rank}_S D(\mathcal{A}) = \operatorname{rank}_S \operatorname{Der}(S) = r$, by the additivity of the rank. This means that constructing a generating set of $D(\mathcal{A})$ with r elements implies the freeness of $D(\mathcal{A})$.

But since the number q of computed generators of $D(\mathcal{A})$ will generally exceed the rank r, the above argument cannot be directly applied and one needs another way to decide the freeness $D(\mathcal{A})$.

The QUILLEN–SUSLIN theorem states that the freeness of an S-module ($S = k[x_1, \dots, x_r]$) is equivalent to its projectiveness. But an algorithm to decide projectiveness has usually a major drawback. It does not produce a free basis.

In any case, all these algorithms take a presentation matrix of the module as their input (see [2, Section 3.4] for a short survey). In our situation, where the submodule $D(\mathcal{A})$ is only given by a set of generators $\{X_1, \dots, X_q\} \subset S^{1\times r}$, this means that we would still need to compute a generating set of S-relations among the q generators before entering any of these algorithms. These relations are again computable as the rows of a matrix $Z \in S^{p\times q}$ of row syzygies of the matrix X, and $D(\mathcal{A}) \cong \operatorname{coker}(S^{1\times p} \xrightarrow{Z} S^{1\times q})$.

In the large examples of interest to us the presentation matrix Z usually contains huge entries. An algorithms that performs nontrivial operations on Z would significantly be slowed down by the size of such entries. So it would be desirable to have an algorithm that only uses Z in the cheapest possible way.

There does exist an algorithm that uses Z in a very cheap way to detect obsolete rows in X, i.e. the redundant generators of $D(\mathcal{A})$ among the rows of X. And fortunately, in the graded case this leads to an algorithm deciding freeness.

For the rest of the subsection let $S = \bigoplus_{i=0}^{\infty} S_i$ be a positively graded commutative ring with one, finitely generated as an algebra over the field[4] $S_0 = k$ (i.e. $S = k[x_1, \ldots, x_m]/I$, where I is a homogeneous ideal). Denote by $\mathfrak{m} = \bigoplus_{i \geqslant 1} S_i \triangleleft S$ the unique maximal homogeneous ideal.

Proposition 6.1. *Let M be a graded submodule of the graded free module $S^{1 \times r}$, $\mathcal{X} = \{X_1, \ldots, X_q\}$ a finite set of homogeneous generators of M, and $X = (X_i)_{i=1,\ldots,q} \in S^{q \times r}$ the matrix with i-th row X_i. The following conditions are equivalent*:

(1) *$\mathcal{X}$ is minimal, i.e. M cannot be generated by a proper subset of $\mathcal{X}$.*
(2) *All entries of a matrix Z of row syzygies of X lie in $\mathfrak{m}$.*
(3) *A matrix Z of row syzygies of X with homogeneous entries has no unit entries, i.e. no entries in $S_0 \setminus \{0\} = k^*$.*

Moreover, any two minimal set of generators $\mathcal{X}$ and $\mathcal{X}'$ have the same cardinality q.

Proof. The equivalence is a special case of the content of [6, Section 20.1]. $\quad\square$

Corollary 6.2. *Let M be a graded submodule of $S^{1 \times r}$ of rank r. Then M is free if and only if the cardinality of any minimal set of generators is r.*

This corollary combined with the last condition of Proposition 6.1 suggest a simple algorithm to decide freeness of the graded module $M \leqslant S^{1 \times r}$.

Algorithm 6.3. GetColumnIndependentUnitPositions(Z)
Determine the position of the "column independent" units in the matrix Z.
Input: A matrix $Z \in S^{p \times q}$.
Output: A subset K of $\{1, \ldots, q\}$.

1. Set $J := \{1, \ldots, q\}$ and $K := \emptyset$.
2. For $i \in \{1, \ldots, p\}$ and $j \in J$ do:
    if $Z_{ij} \in k^*$, i.e. is a unit, then
    redefine $K := K \cup \{j\}$, redefine $J := \{l \in J \mid Z_{il} = 0\}$, and break the j loop.
3. Return K (after finishing the i-loop).

---

[4] A NOETHERian local ring S_0 would suffice, cf. [6, Exercise 20.1].

Algorithm 6.4. LessGenerators(X)

Compute a minimal set of homogeneous[5] generators.

Input: $X \in S^{q \times r}$ with homogeneous rows. The rows $X_1, \ldots, X_q$ of X form a set of homogeneous generators of a graded submodule $M \leqslant S^{1 \times r}$.

Output: A submatrix of X with some rows eventually deleted. The set of rows of this submatrix is a minimal generating set of the graded submodule $M \leqslant S^{1 \times r}$.

1. Compute a matrix Z of row syzygies of X with homogeneous entries.
2. Compute $K := \textbf{GetColumnIndependentUnitPositions}(Z)$.
3. If $K = \emptyset$ then return X.
4. Define $\widetilde{X}$ as the matrix with rows $(X_i)_{i \in \{1, \ldots, q\} \setminus K}$.
5. Return **LessGenerators**($\widetilde{X}$).

The graded submodule $M \leqslant S^{1 \times r}$ of rank r is free if and only if **LessGenerators**(X) has r rows, i.e. is a square matrix.

6.3. Descending chains of free submodules ending with $D(\mathcal{A})$

Set $S = k[x_1, \ldots, x_r]$. Let $\beta_1, \ldots, \beta_n \in S$ be of degree 1, $\mathcal{A} = \{\ker \beta_1, \ldots, \ker \beta_n\} \subset k^r$ be a central arrangement with a fixed order of hyperplanes, and $\Phi_\ell =: \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n := \mathcal{A}$ the ascending maximal chain of central subarrangements with $\mathcal{A}_j := \{\ker \beta_1, \ldots, \ker \beta_j\}$.

The following algorithm decides the freeness of $D(\mathcal{A}_j) \leqslant S^{1 \times r}$ for all $j \leqslant n$: It returns fail if $D(\mathcal{A}_j)$ is not free for some j. Otherwise it constructs a free basis $(X_1^{(j)}, \ldots, X_r^{(j)})$ of $D(\mathcal{A}_j)$ written in the free basis $(X_1^{(j-1)}, \ldots, X_r^{(j-1)})$ of $D(\mathcal{A}_{j-1})$ for all $1 \leqslant j \leqslant n$, starting with the standard basis $(X_1^{(0)}, \ldots, X_r^{(0)}) := (\alpha_1, \ldots, \alpha_r)$ as the free basis of $S^{1 \times r} = \text{Der}(S) = D(\Phi_\ell) = D(\mathcal{A}_0)$. In other words, the $r \times r$-matrix $X^{(j)}$, with $X_i^{(j)}$ being the i-th row, describes the embedding of $D(\mathcal{A}_j)$ in $D(\mathcal{A}_{j-1})$. Hence, in case all $D(\mathcal{A}_j)$ are free, the algorithm constructs the descending maximal chain of free modules

$$S^{1 \times r} = D(\mathcal{A}_0) > \cdots > D(\mathcal{A}_n) = D(\mathcal{A})$$

and returns the tuple of successive embeddings $(X^{(j)})_{j=1 \ldots n}$. It follows that the total embedding of $D(\mathcal{A})$ in $S^{1 \times r}$ is the product matrix $T := \prod_{j=1}^{n} X^{(j)}$, the rows of which form a free basis of $D(\mathcal{A})$ expressed in the standard basis $(\alpha_1, \ldots, \alpha_r)$ of $S^{1 \times r}$.

Algorithm 6.5. ConstructFreeChain($\mathcal{A}$)

Compute the list of successive embeddings $(X^{(j)})_{j=1 \ldots n}$.

Input: $\mathcal{A} = \{\ker \beta_1, \ldots, \ker \beta_n\}$ with fixed order of hyperplanes.

Output: A list of matrices $(X^{(j)})_{j=1 \ldots n}$ describing the successive embedding of $D(\mathcal{A}_j)$ in $D(\mathcal{A}_{j-1})$.

1. For all $j = 1 \ldots n$ compute the morphism $\phi_{H_j} : S^{1 \times r} \to S/\langle \beta_j \rangle$ represented by the $r \times 1$-matrix ϕ_j (cf. Section 6.1).

---

[5] The algorithm can be used to reduce the number of generators of a non-graded submodule given by a non-homogeneous matrix. But in that case the number of rows of the output matrix has no intrinsic meaning.

2. Compute the matrix X of relative row syzygies of $\mu := \phi_{H_1} : S^{1 \times r} \to S/\langle \beta_1 \rangle$ (cf. Section 6.1).
3. If $X^{(1)} := \mathbf{LessGenerators}(X)$ is not a square matrix then return fail.
4. Set $t = I_r$, the identity matrix of rank r.
5. For $j = 2, \ldots, n$ perform steps 6–9:
6. Set $t := X^{(j-1)} \cdot t$. It is the $r \times r$-matrix representing the embedding $D(\mathcal{A}_{j-1}) \equiv S^{1 \times r} \to \mathrm{Der}(S) \equiv S^{1 \times r}$.
7. Set $\mu := t \cdot \phi_j$. It is the $r \times 1$-matrix representing the morphism $D(\mathcal{A}_{j-1}) \equiv S^{1 \times r} \to S/\langle \beta_j \rangle$.
8. Compute the matrix X of relative row syzygies of μ (the rows of which form a generating set of $\ker \mu = D(\mathcal{A}_j) < D(\mathcal{A}_{j-1})$).
9. If $X^{(j)} := \mathbf{LessGenerators}(X)$ is not a square matrix then return fail.
10. Return $(X^{(j)})_{j=1\ldots n}$.

Remark 6.6. This algorithm has the advantage of computing relative syzygies of morphisms represented by *one-column* matrices μ (see step 8) as opposed to the algorithms in Section 6.1. The rows of the product matrix $T := \prod_{j=1}^{n} X^{(j)}$ form a basis of $D(\mathcal{A})$ and another advantage of this algorithm is that it returns the n much simpler factors $(X^{(j)})_{j=1\ldots n}$ instead of T itself (cf. [11, Theorem 4.46]).

We succeeded to compute such a descending maximal chain of free modules $S^{1 \times 8} = D(\mathcal{A}_0) > \cdots > D(\mathcal{A}_{120}) = D(\mathcal{A}_{8,1})$ for the arrangement $\mathcal{A}_{8,1}$ of type E_8 with the roots as in Fig. 2 but sorted degree reverse-lexicographically. The first 8 roots are then the standard basis vectors $(\alpha_1, \ldots, \alpha_8)$.

In particular, the deleted arrangement $\mathcal{A}_{119} = \mathcal{A}'_{8,1}$ is free. Since E_8 is free[6] by Theorem 2.7 we obtain another proof for the freeness of the restricted arrangement $\mathcal{A}_{7,2} \cong \mathcal{A}''_{8,1}$ by Theorem 2.5.

Appendix A. A free basis of $D(\mathcal{A}_{7,2})$ and of $D(\mathcal{A}_{8,1})$

Here we shortly describe the computation of a free basis of $D(\mathcal{A}_{7,2})$ and of $D(\mathcal{A}_{8,1}) = D(\mathcal{A}(E_8))$. The algorithms in Section 6 are implemented (see [1]) using some packages of the `homalg` project [9], written in GAP4 [7]. `homalg` used SINGULAR [8] as the GRÖBNER basis engine.

The matrix X of generators of $D(\mathcal{A}_{7,2})$, computed as a matrix of relative syzygies of $f_{\mathcal{A}_{7,2}}$ modulo $t_{\mathcal{A}_{7,2}}$, is a 1.9 GB 17×7-matrix. The matrix Z of row syzygies of X is a 178 MB 10×17 matrix. **GetColumnIndependentUnitPositions**(Z) returned a subset of $\{1, \ldots, 17\}$ of cardinality 10. **LessGenerators**(X) deleted these 10 rows from X and returned a 169 MB quadratic 7×7-matrix $\widetilde{X}$. It follows that the seven homogeneous rows of $\widetilde{X}$ form a free basis of $D(\mathcal{A}_{7,2})$. Their degrees are $(1, 7, 11, 13, 17, 19, 23)$. The computations took 742 h, i.e. around 30 days, and dropped to 6 GB of RAM at the end of the computation.

Applying **ConstructFreeChain** to the E_8-arrangement took only 8 h and 30 min but needed 130 GB of RAM. These basis computations were performed on an Opteron-8356 machine with 128 GB of RAM. The free basis is given as the rows of an 8×8-matrix computed as the product of 120 sparse matrices [1].

Please note that proving the inductive freeness and producing the certificates for *all* sporadic arrangements only took about 5 min on a usual PC as mentioned at the end of Section 5.2.

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[6] And even inductively free by Theorem 5.14.

Appendix B. Exponents of the sporadic crystallographic arrangements

In this appendix we list the exponents of all sporadic crystallographic arrangements $\mathcal{A}_{r,m}$ of rank r and number m as numbered in [5].

r	m	$\exp \mathcal{A}_{r,m}$	r	m	$\exp \mathcal{A}_{r,m}$	r	m	$\exp \mathcal{A}_{r,m}$
3	1	1, 4, 5	3	26	1, 9, 10	4	1	1, 4, 5, 5
3	2	1, 4, 5	3	27	1, 9, 11	4	2	1, 4, 5, 7
3	3	1, 5, 5	3	28	1, 9, 11	4	3	1, 5, 5, 7
3	4	1, 5, 6	3	29	1, 9, 11	4	4	1, 5, 7, 8
3	5	1, 5, 6	3	30	1, 10, 11	4	5	1, 5, 7, 9
3	6	1, 5, 7	3	31	1, 11, 13	4	6	1, 5, 7, 11
3	7	1, 5, 7	3	32	1, 11, 13	4	7	1, 7, 8, 9
3	8	1, 5, 7	3	33	1, 11, 13	4	8	1, 7, 9, 11
3	9	1, 5, 7	3	34	1, 11, 13	4	9	1, 7, 11, 11
3	10	1, 6, 7	3	35	1, 12, 13	4	10	1, 7, 11, 13
3	11	1, 7, 7	3	36	1, 12, 13	4	11	1, 7, 11, 13
3	12	1, 7, 8	3	37	1, 13, 13	5	1	1, 4, 5, 7, 8
3	13	1, 7, 8	3	38	1, 13, 13	5	2	1, 5, 7, 8, 9
3	14	1, 7, 9	3	39	1, 13, 13	5	3	1, 5, 7, 9, 11
3	15	1, 7, 9	3	40	1, 13, 14	5	4	1, 7, 9, 11, 13
3	16	1, 7, 9	3	41	1, 13, 14	5	5	1, 7, 11, 13, 14
3	17	1, 8, 9	3	42	1, 13, 14	5	6	1, 7, 11, 13, 17
3	18	1, 8, 9	3	43	1, 13, 15	6	1	1, 4, 5, 7, 8, 11
3	19	1, 9, 9	3	44	1, 13, 15	6	2	1, 5, 7, 9, 11, 13
3	20	1, 7, 11	3	45	1, 13, 15	6	3	1, 7, 11, 13, 14, 17
3	21	1, 9, 9	3	46	1, 13, 16	6	4	1, 7, 11, 13, 17, 19
3	22	1, 9, 9	3	47	1, 13, 17	7	1	1, 5, 7, 9, 11, 13, 17
3	23	1, 7, 11	3	48	1, 13, 17	7	2	1, 7, 11, 13, 17, 19, 23
3	24	1, 8, 11	3	49	1, 16, 17	8	1	1, 7, 11, 13, 17, 19, 23, 29
3	25	1, 9, 10	3	50	1, 17, 19			

References

[1] M. Barakat, M. Cuntz, Some certificates for inductive freeness, http://www.mathematik.uni-kl.de/~cuntz/indcert.html, 2010.

[2] Mohamed Barakat, Markus Lange-Hegermann, An axiomatic setup for algorithmic homological algebra and an alternative approach to localization, J. Algebra Appl. 10 (2) (2011) 269–293.

[3] N. Bourbaki, Éléments de mathématique. Groupes et algèbres de Lie, Hermann, Paris, 1968 (Chapitres 4, 5 et 6).

[4] M. Cuntz, Crystallographic arrangements: Weyl groupoids and simplicial arrangements, Bull. London Math. Soc. 43 (4) (2011) 734–744.

[5] M. Cuntz, I. Heckenberger, Finite Weyl groupoids, arXiv:1008.5291v1, 2010, 35 pp.

[6] David Eisenbud, Commutative Algebra, Grad. Texts in Math., vol. 150, Springer-Verlag, New York, 1995, with a view toward algebraic geometry. MR MR1322960 (97a:13001).

[7] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4, http://www.gap-system.org, 2006.

[8] G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 3-1-1, A computer algebra system for polynomial computations, Centre for Computer Algebra, University of Kaiserslautern, http://www.singular.uni-kl.de, 2009.

[9] The homalg project authors, The homalg project, http://homalg.math.rwth-aachen.de/, 2003–2011.

[10] Michel Jambu, Hiroaki Terao, Free arrangements of hyperplanes and supersolvable lattices, Adv. Math. 52 (3) (1984) 248–258.

[11] P. Orlik, H. Terao, Arrangements of Hyperplanes, Grundlehren Math. Wiss., vol. 300, Springer-Verlag, Berlin, 1992.

[12] P. Orlik, H. Terao, Coxeter arrangements are hereditarily free, Tohoku Math. J. (2) 45 (3) (1993) 369–383.