Journal of Combinatorial Theory, Series A 119 (2012) 460–475

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

A combinatorial result with applications to self-interacting random walks

Mark Holmes [a], Thomas S. Salisbury [b]

[a] Department of Statistics, University of Auckland, New Zealand
[b] Department of Mathematics and Statistics, York University, Canada

ARTICLE INFO

Article history:
Received 10 February 2011
Available online 26 October 2011

Keywords:
Random walk
Coupling
Monotonicity

ABSTRACT

We give a series of combinatorial results that can be obtained from any two collections (both indexed by $\mathbb{Z} \times \mathbb{N}$) of left and right pointing arrows that satisfy some natural relationship. When applied to certain self-interacting random walk couplings, these allow us to reprove some known transience and recurrence results for some simple models. We also obtain new results for one-dimensional multi-excited random walks and for random walks in random environments in all dimensions.

1. Introduction

Coupling is a powerful tool for proving certain kinds of properties of random variables or processes. A coupling of two random processes X and Y typically refers to defining random processes X' and Y' *on a common probability space* such that $X' \sim X$ (i.e. X and X' are identically distributed) and $Y' \sim Y$. There can be many ways of doing this, but generally one wants to define the probability space such that the *joint distribution* of (X', Y') has some property. For example, suppose that $X = \{X_n\}_{n \geqslant 0}$ and $Y = \{Y_n\}_{n \geqslant 0}$ are two nearest-neighbour simple random walks in 1 dimension with drifts $\mu_X \leqslant \mu_Y$ respectively. One can define $X' \sim X$ and $Y' \sim Y$ on a common probability space so that X' and Y' are independent, but one can also define $X'' \sim X$ and $Y'' \sim Y$ on a common probability space so that $X''_n \leqslant Y''_n$ for all n with probability 1.

Consider now a nearest-neighbour random walk $\{X_n\}_{n \geqslant 0}$ on $\mathbb{Z}^d$ that has transition probabilities $(2d)^{-1}$ of stepping in each of the $2d$ possible directions, except on the *first departure from each site*. On the first departure, these are also the transition probabilities for stepping to the left and right in any coordinate direction other than the first. But in the first coordinate, the transition probabilities are instead $(2d)^{-1}(1 + \beta)$ (right) and $(2d)^{-1}(1 - \beta)$ (left), for some fixed parameter $\beta \in [0, 1]$. This is

E-mail addresses: holmes@stat.auckland.ac.nz (M. Holmes), salt@yorku.ca (T.S. Salisbury).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2011.10.004

known as an excited random walk [1] and the behaviour of these and more general walks of this kind has been studied in some detail since 2003. For this particular model, it is known [2] that for $d \geqslant 2$ and $\beta > 0$, there exists $v_\beta = (v_\beta^{[1]}, 0, \dots, 0) \in \mathbb{Z}^d$ with $v_\beta^{[1]} > 0$ such that $\lim_{n\to\infty} n^{-1} X_n = v_\beta$ with probability 1. When $d = 1$ the model is recurrent (0 is visited infinitely often) except in the trivial case $\beta = 1$. It is plausible that $v_\beta^{[1]}$ should be a non-decreasing function of β (i.e. increasing the local drift should increase the global drift) but this is not known in general.

A natural first attempt at trying to prove such a monotonicity result would be as follows: given $0 < \beta_1 < \beta_2 \leqslant 1$, construct a coupling of excited random walks X and Y with parameters β_1 and $\beta_2 > \beta_1$ respectively such that with probability 1, $X_n^{[1]} \leqslant Y_n^{[1]}$ for all n. Thus far no one has been able to construct such a coupling, and the monotonicity of $v_\beta^{[1]}$ as a function of β remains an open problem in dimensions $2 \leqslant d \leqslant 8$. In dimensions $d \geqslant 9$ this result has been proved [3] using a somewhat technical expansion method, as well as rigorous numerical bounds on simple random walk quantities. More general models in 1 dimension have been studied, and some monotonicity results [6] have been obtained via probabilistic arguments but without coupling. This raises the question of whether or not one can obtain proofs of these kinds of results using a coupling argument that has weaker aims e.g. such that $\max_{m\leqslant n} X_m^{[1]} \leqslant \max_{m\leqslant n} Y_m^{[1]}$ for all n, rather than $X_n^{[1]} \leqslant Y_n^{[1]}$ for all n.

This paper addresses this issue in 1 dimension. We study relationships between completely deterministic (non-random) 1-dimensional systems of arrows that may prove to be of independent interest in combinatorics. Each system $\mathcal{L}$ of arrows defines a sequence L of integers. We show that under certain natural local conditions on arrow systems $\mathcal{L}$ and $\mathcal{R}$, one obtains relations between the corresponding sequences such as $\max_{m\leqslant n} L_m^{[1]} \leqslant \max_{m\leqslant n} R_m^{[1]}$ for all n (while it's still possible that $L_n^{[1]} > R_n^{[1]}$ for some n).

These may be applied to certain random systems of arrows, to give self-interacting random walk couplings. Doing so, one can obtain results about the (now random) sequence R_n *if* L_n (also random) is well understood, and vice versa. This yields alternative proofs of some existing results, as well as new non-trivial results about the so-called multi-excited random walks in 1 dimension and some models of random walks in random environments in all dimensions – see e.g. [4]. To be a bit more precise, in [4] a projection argument applied to some models of random walks in random environments (in all dimensions) gives rise to a one-dimensional random walk Y, which can be coupled with a one-dimensional multi-excited random walk Z (both walks depending on a parameter p) so that for every $j \in \mathbb{Z}$ and every $r \geqslant 1$:

(i) if Y goes left on its rth visit to j then so does Z (if such a visit occurs), and therefore
(ii) if Z goes right on its rth visit to j then so does Y (if such a visit occurs).

Explicit conditions ($p > \frac{3}{4}$ in this case) governing when $Z_n \to \infty$ as $n \to \infty$ are given in [6]. One would like to conclude that also $Y_n \to \infty$ (whence the original random walk in d dimensions returns to its starting point only finitely many times) when $p > \frac{3}{4}$. This can be achieved by applying the result of this paper to the coupling mentioned above.

The main contributions of this paper are: combinatorial results concerning sequences defined by arrow systems satisfying certain natural local relationships (see Theorem 1.3); some non-trivial counterintuitive examples; and application of these combinatorial results with non-monotone couplings to obtain new results in the theory of random walks.

1.1. Arrow systems

A collection $\mathcal{E} = (\mathcal{E}(x,r))_{x\in\mathbb{Z}, r\in\mathbb{N}}$, where $\mathcal{E}(x,r) \in \{\leftarrow, \rightarrow\}$ is the arrow above the vertex $x \in \mathbb{Z}$ at level $r \in \mathbb{N}$, is called an *arrow system*. This should be thought of as an infinite (ordered) stack of arrows rising above each vertex in $\mathbb{Z}$.

In a given arrow system $\mathcal{E}$, let $\mathcal{E}_\leftarrow(j,r)$ denote the number of $\leftarrow$ arrows, out of the first r arrows above j. As r increases, this quantity counts the number of $\leftarrow$'s appearing in the arrow columns above j. Similarly define $\mathcal{E}_\rightarrow(j,r) = r - \mathcal{E}_\leftarrow(j,r)$. We can define a sequence $E = \{E_n\}_{n\geqslant 0}$ by setting $E_0 = 0$ and letting E evolve by taking one step to the left or right (at unit times), according to the lowest arrow of the $\mathcal{E}$-stack at its current location, and then deleting that arrow. In other words, if $\#\{0 \leqslant m \leqslant n\colon\ E_m = E_n\} = k$ then $E_{n+1} = E_n + 1$ if $\mathcal{E}(E_n, k) = \rightarrow$ (resp. $E_{n+1} = E_n - 1$ if $\mathcal{E}(E_n, k) = \leftarrow$).

Definition 1.1 *($\mathcal{L} \preccurlyeq \mathcal{R}$).* Given two arrow systems $\mathcal{L}$ and $\mathcal{R}$, we write $\mathcal{L} \preccurlyeq \mathcal{R}$ if for each $j \in \mathbb{Z}$ and each $r \in \mathbb{N}$,

$$\mathcal{L}_{\leftarrow}(j,r) \geqslant \mathcal{R}_{\leftarrow}(j,r) \quad \big(\text{and hence also } \mathcal{L}_{\rightarrow}(j,r) \leqslant \mathcal{R}_{\rightarrow}(j,r)\big).$$

Definition 1.2 *($\mathcal{L} \trianglelefteqslant \mathcal{R}$).* We write $\mathcal{L} \trianglelefteqslant \mathcal{R}$ if for each $j \in \mathbb{Z}$ and each $r \in \mathbb{N}$,

$$\mathcal{L}(j,r) = \rightarrow \quad \Rightarrow \quad \mathcal{R}(j,r) = \rightarrow .$$

It is easy to see that $\mathcal{L} \trianglelefteqslant \mathcal{R}$ implies $\mathcal{L} \preccurlyeq \mathcal{R}$.

Now define two paths/sequences $\{L_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$ in $\mathbb{Z}$ according to the arrows in $\mathcal{L}$ and $\mathcal{R}$ respectively as above (in particular $L_0 = R_0 = 0$). Since each arrow system determines a unique sequence, but a given sequence may be obtained from multiple different arrow systems, we write $L \preccurlyeq R$ (resp. $L \trianglelefteqslant R$) if there exist $\mathcal{L} \preccurlyeq \mathcal{R}$ (resp. $\mathcal{L} \trianglelefteqslant \mathcal{R}$) whose corresponding sequences are L and R respectively. Note that when $\mathcal{L} \trianglelefteqslant \mathcal{R}$, the paths $Z = L$ and $Y = R$ constructed from $\mathcal{L}$ and $\mathcal{R}$ as above automatically satisfy the conditions (i) and (ii) appearing at the beginning of Section 1.

An arrow system $\mathcal{E}$ is said to be 0-*right recurrent* if in the new system $\mathcal{E}_+$ defined by $\mathcal{E}_+(0,i) = \rightarrow$ for all $i \geqslant 1$, and $\mathcal{E}_+(x,i) = \mathcal{E}(x,i)$ for all $i \geqslant 1$ and $x > 0$, $E_{+,n} = 0$ infinitely often.

The main result of this paper is the following theorem, in which $n_{E,t}(x) = \#\{k \leqslant t : E_k = x\}$ (see also Corollary 3.10 in the case that L is transient to the right).

Theorem 1.3. *Suppose that $\mathcal{L} \preccurlyeq \mathcal{R}$. Then the following hold.*

(i) $\liminf_{n\to\infty} L_n \leqslant \liminf_{n\to\infty} R_n$.
(ii) $\limsup_{n\to\infty} L_n \leqslant \limsup_{n\to\infty} R_n$.
(iii) *Let $a_n \leqslant n$ be any increasing sequence, with $a_n \to \infty$. If there exists $x \in \mathbb{Z}$ such that $R \geqslant x$ infinitely often then* $\limsup_{n\to\infty} \frac{L_n}{a_n} \leqslant \limsup_{n\to\infty} \frac{R_n}{a_n}$.
(iv) *If $n_{R,t}(x) > n_{L,t}(x)$ then $n_{R,t}(y) \geqslant n_{L,t}(y)$ for every $y > x$.*
(v) *If $\mathcal{R}$ is 0-right recurrent then so is $\mathcal{L}$.*

As $\frac{L_n}{n}$ represents the average speed of the sequence L, up to time n, in many applications the sequence of interest in Theorem 1.3(iii) will be $a_n = n$. Part (ii) of Theorem 1.3 actually follows from part (i) by a simple mirror symmetry argument. There is a symmetric version of (iii), but one must be careful. Part (iii) obviously implies that if $u = \lim n^{-1} R_n$ and $l = \lim n^{-1} L_n$ both exist then $l \leqslant u$, however we show in Section 4.1 that $L \trianglelefteqslant R$ does not imply that $\liminf \frac{L_n}{n} \leqslant \liminf \frac{R_n}{n}$. The mirror image (about 0) of the counterexample in Section 4.1 also shows that (iii) is not true in general if we drop the condition that $L \geqslant x$ infinitely often, for some x. One might also conjecture that if $L \trianglelefteqslant R$ then the amount of time that $R > L$ is at least as large as the amount of time that $R < L$. This is also false as per a counterexample in Section 4.2.

The remainder of the paper is organised as follows. Section 2 contains the basic combinatorial relations which are satisfied by the arrow systems and their corresponding sequences. These will be needed in order to prove our first results. Section 3 gives various consequences of the relationship $\mathcal{L} \preccurlyeq \mathcal{R}$ between two arrow systems, and includes the proofs of the main results of the paper. Section 4 contains the counterexamples described above. Finally Section 5 contains applications of our results in the study of self-interacting random walks.

2. Basic relations

Given an arrow system $\mathcal{E}$ and $t \geqslant 0$, let $n_{E,t}(x) = \#\{k \leqslant t\colon\ E_k = x\}$ and $n_{E,t}(x,y) = \#\{k \leqslant t\colon\ E_{k-1} = x, E_k = y\}$. Then the following relationships hold:

$$n_{E,t}(x) = \delta_{x,0} + n_{E,t}(x-1,x) + n_{E,t}(x+1,x), \tag{2.1}$$

$$n_{E,t}(x) = \delta_{E_t,x} + n_{E,t}(x,x+1) + n_{E,t}(x,x-1), \tag{2.2}$$

$$t+1 = \sum_{i=-\infty}^{\infty} n_{E,t}(i). \tag{2.3}$$

Relation (2.1) says that every visit to x is either from the left or right, except for the first visit if $x=0$. Relation (2.2) is similar, but in terms of departures from x. The sum in (2.3) is in fact a finite sum since $n_{E,t}(i)=0$ for $|i|>t$.

Next

$$n_{E,t}(x,x+1) = \mathcal{E}_{\to}\bigl(x, n_{E,t}(x) - I_{E_t=x}\bigr), \tag{2.4}$$

$$n_{E,t}(x,x-1) = \mathcal{E}_{\leftarrow}\bigl(x, n_{E,t}(x) - I_{E_t=x}\bigr), \tag{2.5}$$

where e.g. relation (2.4) says that the number of departures from x to the right is the number of "used" right arrows at x.

Finally,

$$n_{E,t}(x,x+1) + I_{x+1\leqslant 0} I_{E_t \leqslant x} = n_{E,t}(x+1,x) + I_{x\geqslant 0} I_{E_t\geqslant x+1}, \tag{2.6}$$

which says that the number of moves from x to $x+1$ is closely related to the number of moves from $x+1$ to x. They may differ by 1 depending on the position of x relative to 0 and the current value of the sequence. For example, if $0 \leqslant x < E_t$ then the number of moves from x to $x+1$ up to time t is one more than the number of moves from $x+1$ to x up to time t.

3. Implications of $\mathcal{L} \preccurlyeq \mathcal{R}$

In this section we always assume that $\mathcal{L} \preccurlyeq \mathcal{R}$. The results typically have symmetric versions using the fact that $\mathcal{L} \preccurlyeq \mathcal{R} \iff -\mathcal{R} \preccurlyeq -\mathcal{L}$, which is equivalent to considering arrow systems reflected about 0. We divide the section into two subsections based roughly on the nature of the results and their proofs.

For $x \in \mathbb{Z}$ and $k \geqslant 0$, let $T_L(x,k) = \inf\{t \geqslant 0\colon\ n_{L,t}(x) = k\}$, and $T_R(x,k) = \inf\{t \geqslant 0\colon\ n_{R,t}(x) = k\}$.

3.1. Results obtained from the basic relations

The proofs in this section are based on applications of the basic relations of Section 2. The first few results are somewhat technical, but will be used in turn to prove some of the more appealing results. Roughly speaking they describe how the relative numbers of visits of L and R to neighbouring sites $x-1$ and x relate to each other.

Lemma 3.1. *If L hits x at least $k \geqslant 1$ times and R is eventually to the left of x after fewer than k visits to x, then there exists a site $y < x$ that R hits at least $n_{L,T_L(x,k)}(y)$ times.*

Proof. Fix x,k and let $T = T_L(x,k)$ and $y_0 := \inf\{z \leqslant x\colon\ n_{L,T}(z) > 0\} \leqslant 0$. If $y_0 = x$ then the first $k-1$ arrows at x are all right arrows, i.e. $\mathcal{L}_{\to}(y_0, k-1) = k-1$. Then also $\mathcal{R}_{\to}(y_0,k-1) = k-1$ so R cannot be to the left of x after fewer than k visits. Similarly if $y_0 < x$ then the first $n_{L,T}(y_0)$ arrows at y_0 are all right arrows, i.e. $\mathcal{L}_{\to}(y_0, n_{L,T}(y_0)) = n_{L,T}(y_0)$, and so also $\mathcal{R}_{\to}(y_0, n_{L,T}(y_0)) = n_{L,T}(y_0)$. Therefore either R visits y_0 at least $n_{L,T}(y_0)$ times or it stays in (y_0,x) infinitely often, whence it must visit some site $y \in (y_0,x)$ at least $n_{L,T}(y)$ times as required. $\square$

Let $n_L(x) = n_{L,\infty}(x)$ and $n_R(x) = n_{R,\infty}(x)$.

Lemma 3.2. *If R hits $x-1$ at least $n_L(x-1)$ times then either*

(a) $n_R(x) \geqslant n_L(x)$, *or*
(b) *R is always to the right of x after fewer than $n_L(x)$ visits* ($\Rightarrow \liminf_{n\to\infty} R_n > x$).

Proof. Assume that the first claim fails, so in particular $n_R(x) < \infty$. Let $T = \inf\{t\colon\ n_{L,t}(x) = n_R(x)+1\}$. Then $T < \infty$ so $L_T = x$. Choose r sufficiently large so that $R_t \neq x$ for any $t \geqslant r$, $R_r \neq x-1$, and $n_{R,r}(x-1) \geqslant n_{L,T}(x-1)$. Then by (2.1) applied to L at time T, and also to R at time r,

$$n_{R,r}(x) + 1 = n_R(x) + 1 = n_{L,T}(x) = n_{L,T}(x-1,x) + n_{L,T}(x+1,x) + \delta_{0,x},$$

$$n_{R,r}(x) = \delta_{x,0} + n_{R,r}(x-1,x) + n_{R,r}(x+1,x).$$

Subtracting one from the other and rearranging we obtain

$$n_{R,r}(x-1,x) - n_{L,T}(x-1,x) + n_{R,r}(x+1,x) + 1 = n_{L,T}(x+1,x).$$

Now $n_{L,T}(x+1,x) = n_{L,T}(x,x+1) + I_{x+1\leqslant 0}$ from (2.6), so

$$n_{R,r}(x+1,x) + 1 + \big[n_{R,r}(x-1,x) - n_{L,T}(x-1,x)\big] = n_{L,T}(x,x+1) + I_{x+1\leqslant 0}. \tag{3.1}$$

Using (2.4) and the fact that $R_r \neq x$, then $\mathcal{L} \preccurlyeq \mathcal{R}$, then the fact that $n_{L,T}(x) = 1 + n_{R,r}(x)$, and finally again using (2.4) and the fact that $L_T = x$ we obtain

$$n_{R,r}(x,x+1) = \mathcal{R}_{\rightarrow}\big(x, n_{R,r}(x)\big) \geqslant \mathcal{L}_{\rightarrow}\big(x, n_{R,r}(x)\big) = \mathcal{L}_{\rightarrow}\big(x, n_{L,T}(x) - 1\big) = n_{L,T}(x,x+1).$$

Using this bound in (3.1) yields

$$n_{R,r}(x+1,x) + 1 + \big[n_{R,r}(x-1,x) - n_{L,T}(x-1,x)\big] \leqslant n_{R,r}(x,x+1) + I_{x+1\leqslant 0}. \tag{3.2}$$

Using the fact that $R_r \neq x-1$ and applying (2.4) to R_r at $x-1$, then using $n_{R,r}(x-1) \geqslant n_{L,T}(x-1)$, then $\mathcal{L} \preccurlyeq \mathcal{R}$, and finally using the fact that $L_T \neq x-1$ and applying (2.4) to L_T at $x-1$, we have that

$$\begin{aligned} n_{R,r}(x-1,x) &= \mathcal{R}_{\rightarrow}\big(x-1, n_{R,r}(x-1)\big) \geqslant \mathcal{R}_{\rightarrow}\big(x-1, n_{L,T}(x-1)\big) \\ &\geqslant \mathcal{L}_{\rightarrow}\big(x-1, n_{L,T}(x-1)\big) = n_{L,T}(x-1,x). \end{aligned}$$

Therefore by (3.2), and then (2.6)

$$n_{R,r}(x+1,x) + 1 \leqslant n_{R,r}(x,x+1) + I_{x+1\leqslant 0} \leqslant n_{R,r}(x+1,x) + I_{R_r \geqslant x+1}.$$

Therefore $R_r \geqslant x+1$, so in fact $R_t > x$ for every $t \geqslant r$. Moreover $n_{R,r}(x) = n_R(x) < n_L(x)$, which shows (b). □

Lemma 3.3. *Let $x \in \mathbb{Z}$, and suppose that for some $k > 0$, $n_L(x) \geqslant k$ and $n_R(x) \geqslant k$. Then $n_{R,T_R(x,k)}(x-1) \leqslant n_{L,T_L(x,k)}(x-1)$.*

Proof. Let $T = T_L(x,k) < \infty$ and $S = T_R(x,k) < \infty$. Then $R_S = x > x-1$, so from (2.6) and (2.5)

$$n_{R,S}(x-1,x) = n_{R,S}(x,x-1) + I_{x\geqslant 1} = \mathcal{R}_{\leftarrow}(x,k-1) + I_{x\geqslant 1}.$$

Similarly

$$n_{L,T}(x-1,x) = n_{L,T}(x,x-1) + I_{x\geqslant 1} = \mathcal{L}_{\leftarrow}(x,k-1) + I_{x\geqslant 1}.$$

Since $\mathcal{R}_{\leftarrow}(x,k-1) \leqslant \mathcal{L}_{\leftarrow}(x,k-1)$ it follows that $n_{R,S}(x-1,x) \leqslant n_{L,T}(x-1,x)$. Finally,

$$\mathcal{R}_{\rightarrow}\big(x-1, n_{R,S}(x-1)\big) = n_{R,S}(x-1,x) \quad \text{and} \quad n_{L,T}(x-1,x) = \mathcal{L}_{\rightarrow}\big(x-1, n_{L,T}(x-1)\big)$$

whence $\mathcal{R}_{\rightarrow}(x-1, n_{R,S}(x-1)) \leqslant \mathcal{L}_{\rightarrow}(x-1, n_{L,T}(x-1))$. Since the $n_{R,S}(x-1)$th arrow at $x-1$ is $\rightarrow$ by definition of S (and similarly for $n_{L,T}(x-1)$ and T) this implies that $n_{R,S}(x-1) \leqslant n_{L,T}(x-1)$ as required. □

Lemma 3.4. *If $T = T_L(x,k) < \infty$ and R stays to the right of x after fewer than k visits to x then $n_R(x-1) \leqslant n_{L,T}(x-1)$.*

Proof. Assume that $n_R(x-1)>0$, otherwise there is nothing to prove. Let $S'=\sup\{t\colon\ R_t=x\}$. Then $R_{S'}=x$, $\mathcal{R}(x-1,n_{R,S'}(x-1))=\rightarrow$ and $\mathcal{R}(x,n_{R,S'}(x))=\rightarrow$. By (2.6) applied at $x-1$, and then using (2.5), and finally the fact that $\mathcal{R}(x,n_{R,S'}(x))=\rightarrow$,

$$\begin{aligned} n_{R,S'}(x-1,x) &= n_{R,S'}(x,x-1)+I_{x\geqslant 1}=\mathcal{R}_{\leftarrow}\big(x,n_{R,S'}(x)-1\big)+I_{x\geqslant 1}\\ &= \mathcal{R}_{\leftarrow}\big(x,n_{R,S'}(x)\big)+I_{x\geqslant 1}. \end{aligned}$$

Therefore by (2.4),

$$\mathcal{R}_{\rightarrow}\big(x-1,n_{R,S'}(x-1)\big)=n_{R,S'}(x-1,x)=\mathcal{R}_{\leftarrow}\big(x,n_{R,S'}(x)\big)+I_{x\geqslant 1}. \tag{3.3}$$

Since $n_{R,S'}(x)<k=n_{L,T}(x)$ we have $\mathcal{R}_{\leftarrow}(x,n_{R,S'}(x))\leqslant\mathcal{L}_{\leftarrow}(x,n_{L,T}(x)-1)$, therefore the right hand side of (3.3) is bounded above by

$$\begin{aligned} \mathcal{L}_{\leftarrow}\big(x,n_{L,T}(x)-1\big)+I_{x\geqslant 1} &= n_{L,T}(x,x-1)+I_{x\geqslant 1}\\ &= n_{L,T}(x-1,x)=\mathcal{L}_{\rightarrow}\big(x-1,n_{L,T}(x-1)\big), \end{aligned}$$

where we have used (2.5), followed by (2.6), and then (2.4). We have shown that

$$\mathcal{R}_{\rightarrow}\big(x-1,n_{R,S'}(x-1)\big)\leqslant\mathcal{L}_{\rightarrow}\big(x-1,n_{L,T}(x-1)\big).$$

Since $\mathcal{R}(x-1,n_{R,S'}(x-1))=\rightarrow$, this implies that $n_{R,S'}(x-1)\leqslant n_{L,T}(x-1)$ as required. □

3.2. Results obtained by contradiction

The results in this section include less technical results than those of the previous section. Roughly speaking their proofs will be based on contradiction arguments that proceed as follows. Suppose that we have already proved a statement A whenever $\mathcal{L}\preccurlyeq\mathcal{R}$. We now want to prove a statement B whenever $\mathcal{L}\preccurlyeq\mathcal{R}$. Assume that for some $\mathcal{L}$, $\mathcal{R}$ with $\mathcal{L}\preccurlyeq\mathcal{R}$, B is false. Construct two new systems $\mathcal{L}'\preccurlyeq\mathcal{R}'$ from $\mathcal{L}$ and $\mathcal{R}$ such that statement A is violated for $\mathcal{L}'$ and $\mathcal{R}'$. This gives a contradiction, hence there was no such example where $\mathcal{L}\preccurlyeq\mathcal{R}$ but B is false.

Lemma 3.5. *Let $x\in\mathbb{Z}$, and suppose that $n_R(x)<k\leqslant n_L(x)$. Then $n_R(x-1)\leqslant n_{L,T_L(x,k)}(x-1)$ and $\liminf R_n>x$ (i.e. R is forever to the right of x after fewer than k visits to x and at most $n_{L,T_L(x,k)}(x-1)$ visits to $x-1$).*

Proof. By Lemma 3.4, it is sufficient to prove that under the hypotheses of the lemma, R is to the right of x infinitely often. Suppose instead that R is forever to the left of x (after fewer than k visits to x). Then we may define two new systems $\mathcal{R}'$ and $\mathcal{L}'$ by forcing every arrow at x at level k and above to be $\rightarrow$. To be precise, given an arrow system $\mathcal{E}$ we'll define $\mathcal{E}'$ by $\mathcal{E}'(y,\cdot)=\mathcal{E}(y,\cdot)$ for all $y\neq x$, $\mathcal{E}'(x,j)=\mathcal{E}(y,j)$ for all $j<k$, and $\mathcal{E}'(x,j)=\rightarrow$ for every $j\geqslant k$. Clearly $\mathcal{L}'\preccurlyeq\mathcal{R}'$ and $T'=T_{L'}(x,k)=T$. The sequences R and R' are identical since we have not changed any arrow used by R anyway. The sequences L and L' agree up to time T, while $L'_n\geqslant x$ for all $n\geqslant T$, since L' can never go left from x after time T. It follows that $n_{L'}(z)=n_{L,T}(z)<\infty$ for every $z<x$.

Let $y_1:=\max\{z<x\colon\ n_{R'}(z)\geqslant n_{L',T}(z)\}$. By Lemma 3.1, $-\infty<y_1<x$. By Lemma 3.2 (applied to L', R') either R' hits y_1+1 at least $n_{L'}(y_1+1)\geqslant n_{L,T}(y_1+1)$ times, or R' is forever to the right of y_1+1 after fewer than $n_{L'}(y_1+1)$ visits. In either case, $y_1+1<x$ (as $n_{R'}(x)<k$ and R' lies eventually to the left of x). So there exists some $y_2\in(y_1,x)$ such that $n_{R'}(y_2)\geqslant n_{L'}(y_2)=n_{L',T}(y_2)$. This contradicts the definition of y_1. □

Corollary 3.6. *If $n_{R,t}(x-1)>n_{L,t}(x-1)$ then $n_{R,t}(x)\geqslant n_{L,t}(x)$.*

Proof. Suppose instead that $n_{R,t}(x)<n_{L,t}(x)$. Let $k=n_{R,t}(x)+1$, so that $T=T_L(x,k)\leqslant t$ and $S=T_R(x,k)>t$. Then

$$n_{R,S}(x-1)\geqslant n_{R,t}(x-1)>n_{L,t}(x-1)\geqslant n_{L,T}(x-1).$$

This violates Lemma 3.3 (if $n_R(x)\geqslant k$) or Lemma 3.5 (if $n_R(x)<k$). □

Corollary 3.7. *Fix $x > 0$, and let $T = T_L(x, 1) = \inf\{t\colon L_t = x\}$ and $S = T_R(x, 1)$. Then $S \leqslant T$.*

Proof. If $T = \infty$ then the result is trivial. So assume $T < \infty$. Lemma 3.5 with $k = 1$ implies that $S < \infty$ as well (R cannot be to the right of $x > 0$ without ever passing through x). For each $i < x$, the number of times that L hits i before T is $n_{L,T}(i)$, so $T = \sum_{i=-\infty}^{x-1} n_{L,T}(i)$. Moreover, $n_{L,T}(i)$ is the number of times that L hits i before hitting $i + 1$ for the $n_{L,T}(i + 1)$th time (by definition of T, the last visit to $i < x$ up to time T occurs before the last visit to $i + 1$ up to time T). By Lemma 3.3 with $k = 1$ we get that $n_{R,S}(x - 1) \leqslant n_{L,T}(x - 1)$. Set $k_0 = 1$.

Now apply Lemma 3.3 with $x - 1$ instead of x and with $k_1 = n_{R,S}(x - 1)$ to get

$$n_{R,T_R(x-1,k_1)}(x-2) \leqslant n_{L,T_L(x-1,k_1)}(x-2).$$

But $n_{R,T_R(x-1,k_1)}(x - 2) = n_{R,S}(x - 2)$ since R cannot visit $x - 2$ at times in $(T_r(x - 1, k_1), S]$ (in other words, the last visit to $x - 2$ occurs before the last visit to $x - 1$). Furthermore, $n_{L,T_L(x-1,k_1)}(x - 2) \leqslant n_{L,T}(x - 2)$ since $n_{L,T}(x - 1) \geqslant k_1 \Rightarrow T_L(x - 1, k_1) \leqslant T$. We have just shown that

$$n_{R,S}(x-2) = n_{R,T_R(x-1,k_1)}(x-2) \leqslant n_{L,T_L(x-1,k_1)}(x-2) \leqslant n_{L,T}(x-2).$$

Iterating this argument while $k_j = n_{R,S}(x - j) > 0$ by applying Lemma 3.3 at $x - j$ with $k = k_j$ (there is nothing to do once $n_{R,S}(x - j) = 0$ for some j), we obtain by induction that $n_{R,S}(i) \leqslant n_{L,T}(i)$ for every $i < x$. Thus $S = \sum_{i=-\infty}^{x-1} n_{R,S}(i) \leqslant \sum_{i=-\infty}^{x-1} n_{L,T}(i) = T$ as required. □

It follows immediately from Corollary 3.7 that

$$\bar{R}_n := \max_{k \leqslant n} R_k \geqslant \max_{k \leqslant n} L_k =: \bar{L}_n. \tag{3.4}$$

Of course by mirror symmetry we also have $\underline{R}_n := \min_{k \leqslant n} R_k \geqslant \min_{k \leqslant n} L_k = \underline{L}_n$. The following result extends this idea to the number of visits of the two paths to $\bar{R}_n$ by time n.

Lemma 3.8. *For each $t \geqslant 0$, $n_{R,t}(\bar{R}_t) \geqslant n_{L,t}(\bar{R}_t)$ and $n_{L,t}(\underline{L}_t) \geqslant n_{R,t}(\underline{L}_t)$.*

Proof. Let $\mathcal{L} \preccurlyeq \mathcal{R}$ and suppose the first claim fails. Let $T = \inf\{t \geqslant 0\colon\ n_{R,t}(\bar{R}_t) < n_{L,t}(\bar{R}_t)\} < \infty$. Let $\mathcal{N}_t = n_{L,t}(\bar{R}_t) - n_{R,t}(\bar{R}_t)$. Then $\mathcal{N}_{t+1} - \mathcal{N}_t \leqslant 1$ if $\bar{R}_{t+1} = \bar{R}_t$, and by (3.4), $\mathcal{N}_{t+1} = 0$ or -1 if $\bar{R}_{t+1} > \bar{R}_t$. Therefore by definition of T we must have $R_T < \bar{R}_T$, $L_T = \bar{R}_T$, and $n_{L,T}(\bar{R}_T) = 1 + n_{R,T}(\bar{R}_T)$. Moreover this happens regardless of the arrows of $\mathcal{L}$ or $\mathcal{R}$ at $\bar{R}_T$ above level $n_{R,T}(\bar{R}_T)$. Define new arrow systems $\mathcal{L}', \mathcal{R}'$ by setting all arrows at $\bar{R}_T$ at level $1 + n_{R,T}(\bar{R}_T)$ and above to be $\rightarrow$. By construction $\mathcal{L}' \preccurlyeq \mathcal{R}'$, and $(L_n, R_n) = (L'_n, R'_n)$ for $n \leqslant T$. However $\bar{L}'_{T+1} = \bar{R}_T + 1 > \bar{R}_T = \bar{R}'_{T+1}$ which violates the fact that $\bar{R}'_n \geqslant \bar{L}'_n$ for all $n \geqslant 0$.

The second result follows by mirror symmetry. □

For each $z \in \mathbb{Z}$, $t \in \mathbb{Z}_+$, let $\bar{z}_t = \max(n_{L,t}(z), n_{R,t}(z))$.

Lemma 3.9. *If there exist t, y such that $R_t \leqslant y < L_t$ and $n_{R,t}(y) > n_{L,t}(y)$ then $n_{R,t}(x) \geqslant n_{L,t}(x)$ for every $x \in [y, L_t]$.*

Proof. Suppose that t and y satisfy the above hypotheses, but the conclusion fails for some $x \in [y, L_t]$. In other words, $y < x \leqslant L_t$ and $n_{R,t}(x) < n_{L,t}(x)$. Define new arrow systems $\mathcal{L}'$ and $\mathcal{R}'$ by setting:

- all arrows at y at level $n_{R,t}(y) + I_{\{R_t \neq y\}}$ and above to be $\leftarrow$;
- all arrows at x at level $n_{L,t}(x) + I_{\{L_t \neq x\}}$ and above to be $\rightarrow$; and
- for each $z > x$ set all arrows above level $\bar{z}_t$ to be $\rightarrow$.

The resulting arrow systems satisfy $\mathcal{L}' \preccurlyeq \mathcal{R}'$ with $(L_n, R_n) = (L'_n, R'_n)$ for $n \leqslant t$. By construction $L'_n \to \infty$ as $n \to \infty$, since L'_n never again goes below x, and can make at most finitely many more $\leftarrow$ moves. But also $R'_n \leqslant y$ for all $n \geqslant t$, which contradicts the fact that $\bar{R}'_n \geqslant \bar{L}'_n$ for all $n \geqslant 0$. □

We say that a sequence $\{L_n\}_{n\geqslant 0}$ on $\mathbb{Z}$ is *transient to the right* if for every $x\in\mathbb{Z}$ there exists $n_x\geqslant 0$ such that $L_n>x$ for all $n\geqslant n_x$ (i.e. if $\liminf_{n\to\infty} L_n=+\infty$).

Corollary 3.10. *If* $\liminf_{n\to\infty} L_n=+\infty$ *then* $n_R(x)\leqslant n_L(x)$ *for every* x *and* $\liminf_{n\to\infty} R_n=+\infty$.

Proof. Suppose that L is transient to the right. Then $n_L(y)<\infty$ for each y. Suppose that for some x, $n_R(x)>n_L(x)$. Let $T=T_R(x,n_L(x)+1)$. Define new systems $\mathcal{L}'\preccurlyeq\mathcal{R}'$ by setting every arrow at x above level $n_L(x)$ to be $\leftarrow$. Then $L'=L$, so $L'\to\infty$, but $R'_t\leqslant x$ for every $t\geqslant T$. This violates (3.4) for L', R'. Therefore $n_R(x)\leqslant n_L(x)$ for every x, which establishes the first claim.

For the second claim, suppose that R is not transient to the right. Then R is either transient to the left or it visits some site x infinitely often. In either case there is some site x such that $n_R(x)>n_L(x)$ which cannot happen by the first claim. $\square$

Corollary 3.11. $R\geqslant L$ *infinitely often.*

Proof. If R is not bounded above, this follows by considering the times at which R extends its maximum. It follows similarly if L is not bounded below, using times at which L extends its minimum. The only remaining possibility is that R is bounded above and L is bounded below, in which case by (3.4) both paths visit only finitely many vertices. In this case consider the sets of vertices that R and L visit infinitely often. Let $x_\infty=\sup\{z\in\mathbb{Z}\colon\ n_R(z)=\infty\}$ and $y_\infty=\sup\{z\in\mathbb{Z}\colon\ n_L(z)=\infty\}$. If $x_\infty<y_\infty$ then Lemma 3.5 is violated (apply it to $x=y_\infty$ for $k>n_R(y_\infty)$). Therefore $x_\infty\geqslant y_\infty$, so $R_t\geqslant L_t$ at all sufficiently large t for which $R_t=x_\infty$. $\square$

3.2.1. Proof of Theorem 1.3

To prove (i) we show that if $L_n\geqslant x$ for all n sufficiently large, then $R_n\geqslant x$ for all n sufficiently large. Suppose instead that $R_n<x$ infinitely often. Then choose N sufficiently large so that $L_n\geqslant x$ for all $n\geqslant N$, but $R_N<x$ and $n_{R,N}(R_N)>n_{L,N}(R_N)$. Define two new arrow systems $\mathcal{L}',\mathcal{R}'$ by switching all arrows at R_N from level $n_{R,N}(R_N)$ and above to be $\leftarrow$. Then $\mathcal{L}'\preccurlyeq\mathcal{R}'$ but Lemma 3.9 is violated, as is Corollary 3.11. This establishes (i). Applying (i) to $-\mathcal{R}\preccurlyeq-\mathcal{L}$ establishes (ii).

If $R_n\geqslant x$ infinitely often then $\limsup R_n/a_n\geqslant\limsup x/a_n=0$. Thus the result is trivial unless there exists $0<M<\infty$ such that $\limsup L_n/a_n>M$. Then L_n visits infinitely many sites >0. Let T_i be the times at which L extends its maximum, i.e. $T_0=0$ and for $i\geqslant 1$, $T_i=\inf\{n>T_{i-1}\colon\ L_n=1+\max_{k<n}L_k\}$. We first verify the (intuitively obvious) statement that $\frac{L_{T_i}}{a_{T_i}}>M$ infinitely often. If $\frac{L_{T_i}}{a_{T_i}}>M$ only finitely often then for all i sufficiently large, $\frac{L_{T_i}}{a_{T_i}}\leqslant M$. But for all $n\in[T_i,T_{i+1})$, $\frac{L_n}{a_n}\leqslant\frac{L_{T_i}}{a_n}\leqslant\frac{L_{T_i}}{a_{T_i}}$. So $\frac{L_n}{a_n}\leqslant M$ for all but finitely many n, contradicting the fact that $\limsup L_n/a_n>M$. Let S_i be the times at which R extends its max. By definition, $L_{T_i}=i=R_{S_i}$ and from Corollary 3.7, $i\leqslant S_i\leqslant T_i$. It follows immediately that for infinitely many i,

$$\frac{R_{S_i}}{a_{S_i}}\geqslant\frac{L_{T_i}}{a_{T_i}}>M,$$

whence $\limsup_{n\to\infty}\frac{R_n}{a_n}\geqslant M$. This establishes part (iii)

To prove (iv), suppose that (iv) does not hold, and let τ be the first time at which this fails. In other words

$$\tau=\inf\bigl\{t\geqslant 0\colon\ \text{there exist } y, x<y \text{ such that } n_{R,t}(x)>n_{L,t}(x) \text{ and } n_{R,t}(y)<n_{L,t}(y)\bigr\}.$$

Let x_0 be the largest such x, i.e. $x_0=\sup\{x\in\mathbb{Z}\colon\ n_{R,\tau}(x)>n_{L,\tau}(x), \exists y>x \text{ such that } n_{R,\tau}(y)<n_{L,\tau}(y)\}$ and $y_0=\inf\{y>x_0\colon\ n_{R,\tau}(y)<n_{L,\tau}(y)\}$. Then $x_0\leqslant y_0-2$ or else Corollary 3.6 is violated. By definition of x_0 and y_0 we have $n_{R,\tau}(y_0-1)\geqslant n_{L,\tau}(y_0-1)$. Let $k=n_{L,\tau}(y_0)$. Then $n_{L,\tau}(y_0-1)\geqslant n_{L,T_L(y_0,k)}(y_0-1)$ so $n_{R,\tau}(y_0-1)\geqslant n_{L,T_L(y_0,k)}(y_0-1)$. On the other hand $n_{R,\tau}(y_0)<k$, so $\tau<T_R(y_0,k)$. If $R_\tau<y_0-1$ then $n_{R,T_R(y_0,k)}(y_0-1)\geqslant n_{R,\tau}(y_0-1)+1>n_{L,T_L(y_0,k)}(y_0-1)$. This contradicts one of Lemmas 3.3 or 3.5 (depending on whether $n_R(y_0)\geqslant k$), so we must have instead that

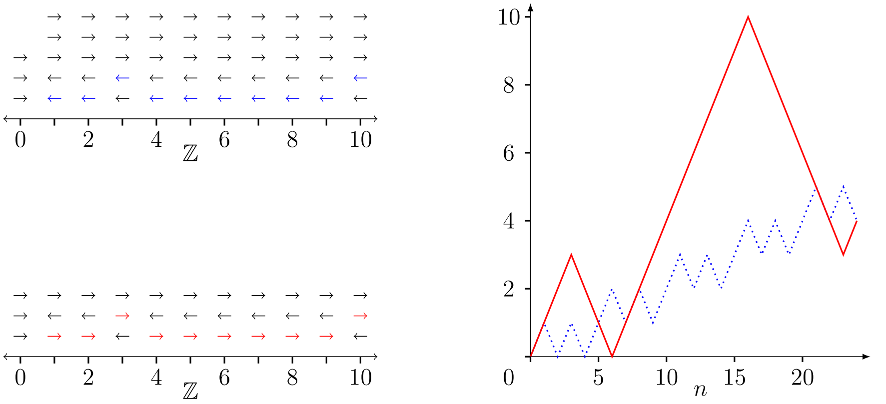

Fig. 1. On the left are parts of the systems $\mathcal{L}$ (top) and $\mathcal{R}(3)$ (bottom) and on the right are their corresponding sequences R_n (solid) and L_n (dotted), defined in Section 4 such that $\liminf n^{-1}L_n \geqslant \liminf n^{-1}R_n$. Each site in $\mathbb{N}$ appears five times in the sequence L and three times in the sequence R.

$R_\tau \geqslant y_0 - 1 > x_0$. Therefore $n_{R,\tau-1}(x_0) = n_{R,\tau}(x_0) > n_{L,\tau}(x_0) \geqslant n_{L,\tau-1}(x_0)$. Similarly if $L_\tau > x_0 + 1$ we get a contradiction to the symmetric versions of Lemmas 3.3 or 3.5, so we must have $L_\tau \leqslant x_0 + 1 < y_0$, and therefore $n_{L,\tau-1}(y_0) = n_{L,\tau}(y_0) > n_{R,\tau-1}(y_0)$. This contradicts the definition of τ.

Finally, to prove (v), note that if $\mathcal{L} \preccurlyeq \mathcal{R}$ then also $\mathcal{L}_+ \preccurlyeq \mathcal{R}_+$. If $\mathcal{R}$ is 0-right recurrent, then $R_{+,n} = 0$ infinitely often so $L_{+,n} = 0$ infinitely often by (i).

4. Counterexamples

4.1. $L \trianglelefteq R$ *does not imply that* $\liminf \frac{L_n}{n} \leqslant \liminf \frac{R_n}{n}$

In general, $L \trianglelefteq R$ does not imply that $\liminf \frac{L_n}{n} \leqslant \liminf \frac{R_n}{n}$, as we shall see in the following example.

Let us first define the two systems as follows, starting with $\mathcal{L}$. At 0 the first three arrows are $\rightarrow$. At every $x > 0$ the first two arrows are $\leftarrow$ and the next three arrows are $\rightarrow$. It is easy to check that such a system results in a sequence L that takes steps with the pattern $\rightarrow\leftarrow\rightarrow\leftarrow\rightarrow$ repeated indefinitely (without ever needing to look at arrows other than those specified above). Thus $\lim_{n\to\infty} \frac{L_n}{n} = \frac{3-2}{5} = \frac{1}{5}$.

Let us now define a system $\mathcal{R} = \mathcal{R}(N)$, according to a parameter N as follows. At 0 the first three arrows are $\rightarrow$. At each site $x_k = x_k(N)$ of the form

$$x_k = \sum_{m=1}^{k} N^m - \sum_{m=1}^{k-1}\sum_{r=0}^{m}(-1)^{m-r}N^r, \quad k \geqslant 1 \tag{4.1}$$

the first arrow is $\leftarrow$ and the next two arrows are $\rightarrow$. At all remaining sites $x > 0$, the first three arrows are $\rightarrow$, $\leftarrow$, $\rightarrow$. See Fig. 1 for parts of the systems $\mathcal{L}$ and $\mathcal{R}(3)$. By definition of these systems the arrows to the left of 0 and above those shown are irrelevant, so we can set them to be the same (for example, all $\rightarrow$).

By construction $L \trianglelefteq R$ for each $N \geqslant 1$, but we will show that $\liminf \frac{R_n}{n} \leqslant \frac{1}{2N+1} < \frac{1}{5}$ for $N \geqslant 3$ (also $\limsup \frac{R_n}{n} \geqslant \frac{N}{N+2}$).

The first site of the form (4.1) is $x_1 = N$. The walk R first encounters a $\leftarrow$ at its first visit to this site and then sees a $\rightarrow$ at site 0 (second visit to 0). The walk R then visits site x_1 for the second time, whence it sees a $\rightarrow$. It continues moving right, visiting every site between x_1 and x_2 exactly once before reaching x_2 at this point it sees a $\leftarrow$, moves to $x_2 - 1$ (for the second visit to that site) and continues seeing $\leftarrow$ at every site in (x_1, x_2) until reaching x_1 for the third time. It then sees $\rightarrow$ at every site in $[x_1, x_2)$ (third visit to each of those sites), but also at every site in $[x_2, x_3)$ (second visit

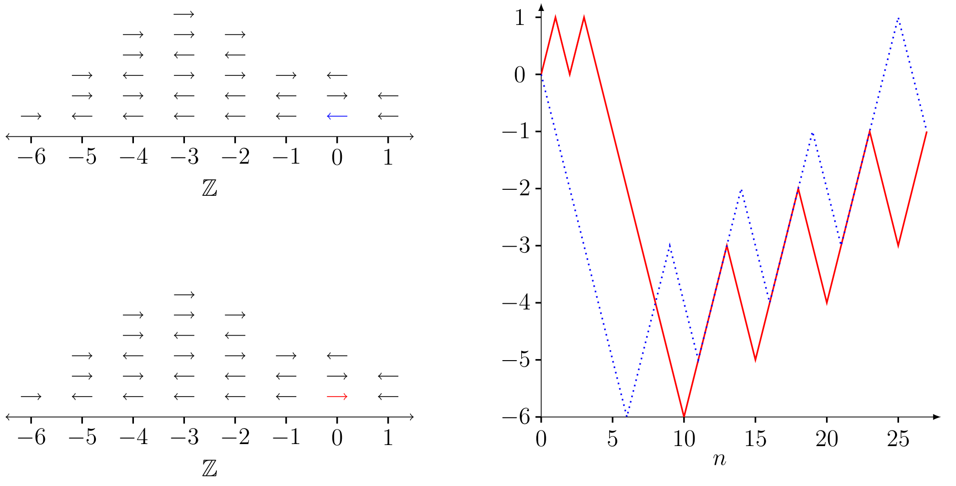

Fig. 2. On the left are parts of arrow systems $\mathcal{L}$ (top) and $\mathcal{R}$ (bottom) with $\mathcal{L} \trianglelefteq \mathcal{R}$, and on the right are the corresponding paths R_n (solid) and L_n (dotted). Here, $|A_{R,26}| = 7 < 9 = |A_{L,26}|$.

to x_3 and first visit to each site in (x_3, x_4)). Continuing in this way, the walk turns left at every x_i on the first visit, and continues left (second visit at interior sites) until reaching x_{i-1} for the third time, and then continues to go right until reaching x_{i+1} for the first time.

At time $t_k = \sum_{m=1}^{k} N^m + \sum_{m=1}^{k-1} \sum_{r=0}^{m} (-1)^{m-r} N^r$ the walk is at position $x_k = \sum_{m=1}^{k} N^m - \sum_{m=1}^{k-1} \sum_{r=0}^{m} (-1)^{m-r} N^r$ for the first time. Simple calculations then give

$$\lim_{k\to\infty} \frac{R_{t_k}}{t_k} = \frac{N}{N+2},$$

which gives rise to the limit supremum claimed.

Similarly at times $s_k = \sum_{m=1}^{k} N^m + \sum_{m=1}^{k} \sum_{r=0}^{m} (-1)^{m-r} N^r$ the walk is at position $x_{k-1} = \sum_{m=1}^{k} N^m - \sum_{m=1}^{k} \sum_{r=0}^{m} (-1)^{m-r} N^r$ for the last time. After some simple calculations we obtain

$$\lim_{n\to\infty} \frac{R_{s_k}}{s_k} = \frac{1}{2N+1},$$

which gives rise to the limit infimum claimed.

4.2. *L can be in the lead more than R*

Given two sequences L and R with $L \preccurlyeq R$, let $A_{R,t} = \{n \leqslant t\colon\ R_n > L_n\}$ and $A_{L,t} = \{n \leqslant t\colon\ R_n < L_n\}$. It is not unreasonable to expect that for every $t \in \mathbb{N}$, $|A_{R,t}| \geqslant |A_{L,t}|$ which essentially says that R is ahead of L more than L is ahead of R. It turns out that this does not hold even when $L \trianglelefteq R$.

To see this, consider the partial arrow systems $\mathcal{R}$ and $\mathcal{L}$ on the left hand side of Fig. 2. These two systems differ only at the first arrow at 0, whence $\mathcal{L} \trianglelefteq \mathcal{R}$ (if we set all other arrows to be equal, for example). The first 28 terms of the sequences L and R are plotted on the right of the figure. At any place where the solid line is above the dotted line, $R > L$. In particular $R_n > L_n$ only for $1 \leqslant n \leqslant 7$. Similarly $L > R$ when the dotted line lies above the solid line, which happens at times 9, 10, 14, 15, 19, 20, 24, 25, 26. Thus we have $|A_{R,25}| = 7 < 8 = |A_{L,25}|$ and similarly $|A_{R,26}| = 7 < 9 = |A_{L,26}|$.

We can modify these systems slightly to get another interesting example. Define $\mathcal{R}'$ from $\mathcal{R}$ by switching the second arrow at 0 to $\leftarrow$, the first arrow at 1 to be $\rightarrow$ and setting the first arrow at 2 to be $\leftarrow$. Define $\mathcal{L}'$ from $\mathcal{L}$ by switching the first arrow at 1 to be $\rightarrow$ and setting the first arrow at 2 to be $\leftarrow$. The resulting partial systems satisfy $\mathcal{L}' \preccurlyeq \mathcal{R}'$. At time $t = 28$, $|A_{R,28}| < |A_{L,28}|$, the number

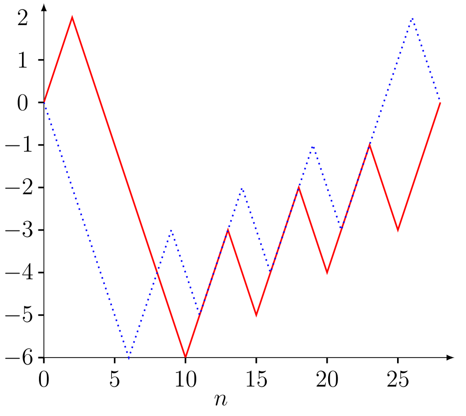

Fig. 3. Paths R'_n (solid) and L'_n (dotted) with $L'_n \preccurlyeq R'_n$ and $|A_{R,28}| = 7 < 10 = |A_{L,28}|$. The walks have visited each site the same number of times.

of visits to each site is identical, and $L_{28} = R_{28} = 0$ (see Fig. 3). This means we can define a system which repeats such a pattern indefinitely. We can add any common steps that we wish in between repetitions of this pattern and hence we can have recurrent, transient, or even ballistic sequences satisfying $L \preccurlyeq R$ but such that $t^{-1}(|A_{L,t}| - |A_{R,t}|) \to v > 0$ as $t \to \infty$.

5. Applications

In this section we describe some of the applications of our main results in the theory of nearest-neighbour self-interacting random walks, i.e. sequences $(X_n)_{n\geqslant 0}$ of $\mathbb{Z}$-valued random variables (which may include projections of higher dimensional walks), such that $X_{n+1} - X_n \in \{-1, 1\}$ a.s. for every n. For each application, what we actually do is show that there is a probability space on which the relevant random walks live and on which they are related via the property $\preccurlyeq$ or $\trianglelefteq$ almost surely. It is then clear that on that probability space the conclusions of Theorem 1.3 hold almost surely for the walks satisfying those relations.

Our original motivation for the present paper was in studying random walks in (non-elliptic) random environments in dimensions $d \geqslant 2$ (see e.g. [4]). In [4] the authors apply Theorem 1.3 to random walks in i.i.d. random environments such that for some diagonal direction u, with sufficiently large probability at each site there is a drift in direction u, and that almost surely there is no drift in direction $-u$. For such walks, the projection R in direction u can be coupled with the so-called 1-dimensional multi-excited random walk (see below) L so that $L \trianglelefteq R$, and transience and positive speed results can be obtained for this projection, when the strength of the drift is sufficiently large.

Our results can also be applied to recurrent models. For example, given $\beta > -1$, let X be a once-reinforced random walk (ORRW) on $\mathbb{Z}$ with reinforcement parameter β, i.e. $X_0 = 0$ and

$$\mathbb{P}(X_{n+1} - X_n = 1|\mathcal{F}_n) = \frac{1 + \beta I_{\{X_n+1\in\vec{X}_{n-1}\}}}{2 + \beta[I_{\{X_n+1\in\vec{X}_{n-1}\}} + I_{\{X_n-1\in\vec{X}_{n-1}\}}]}.$$

We can similarly define ORRW on $\mathbb{Z}^+$ by forcing the walk to step right when at 0. Then it is possible to define a probability space on which there is an ORRW $X^+(\beta)$ for each $\beta > -1$ and such that $X^+(\beta) \trianglelefteq X^+(\zeta)$ whenever $\beta \geqslant \zeta > -1$. On this probability space the corresponding local times processes then satisfy the monotonicity property Theorem 1.3(iv).

Most of our results, including that for random walks in random environments above, involve comparisons with the so-called *multi-excited random walks in i.i.d. cookie environments*. A cookie environment is an element $\boldsymbol{\omega} = (\omega(x, n))_{x\in\mathbb{Z},n\in\mathbb{N}}$ of $[0, 1]^{\mathbb{Z}\times\mathbb{N}}$. A (multi-)excited random walk in cookie environment $\boldsymbol{\omega}$, starting from the origin, is a sequence of random variables $X = \{X_n\}_{n\geqslant 0}$ defined on a probability space (and adapted to a filtration $\mathcal{F}_n$) such that $X_0 = 0$ a.s. and

$$P_{\boldsymbol{\omega}}(X_{n+1} = X_n + 1|\mathcal{F}_n) = \omega\big(x, \ell(n)\big) = 1 - P_{\boldsymbol{\omega}}(X_{n+1} = X_n - 1|\mathcal{F}_n),$$

where $\ell(n) = \ell_X(n) = \sum_{m=0}^{n} 1_{\{X_m=X_n\}}$. In other words, if you are currently at x and this is the kth time that you have been at x then your next step is to the right with probability $\omega(x,k)$, independent of all other information. A random cookie environment $\boldsymbol{\omega}$ is said to be i.i.d. if the random vectors $\omega(x,\cdot)$ are i.i.d. as x varies over $\mathbb{Z}$.

Let $\mathbf{U} = (U(x,n))_{x\in\mathbb{Z},n\in\mathbb{N}}$ be a collection of independent standard uniform random variables defined on some probability space. For each $x \in \mathbb{Z}$, $n \in \mathbb{N}$, and each cookie environment $\boldsymbol{\omega}$ let

$$\mathcal{E}_{\boldsymbol{\omega},\mathbf{U}}(x,n) = \begin{cases} \rightarrow, & \text{if } U(x,n) < \omega(x,n), \\ \leftarrow, & \text{otherwise.} \end{cases}$$

Then $\mathcal{E}_{\boldsymbol{\omega},\mathbf{U}}$ is an arrow system determined entirely by the pairs $(\omega(x,n), U(x,n))_{x\in\mathbb{Z},n\in\mathbb{N}}$, and the corresponding walk $E = E_{\boldsymbol{\omega},\mathbf{U}}$ is an excited random walk in cookie environment $\boldsymbol{\omega}$. Given two cookie environments $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ we write $\boldsymbol{\omega} \trianglelefteq \boldsymbol{\omega}'$ if $\omega(x,n) \leqslant \omega'(x,n)$ for every $x \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $\boldsymbol{\omega} \trianglelefteq \boldsymbol{\omega}'$, then on the above probability space $\mathcal{E}_{\boldsymbol{\omega},\mathbf{U}} \trianglelefteq \mathcal{E}_{\boldsymbol{\omega}',\mathbf{U}}$ so Theorem 1.3 applies to the corresponding excited random walks.

For excited random walks in i.i.d. cookie environments in 1 dimension, it is known up to a high level of generality that right transience and the existence of a positive speed $v > 0$ do not depend on the order of the cookies (see e.g. [5]). One might expect that the value of v should depend on this order. The main result of this section is Theorem 5.1 below, which essentially states that one cannot decrease the (lim sup)-speed of a cookie random walk by swapping stronger cookies in a pile with weaker cookies that appear earlier in the same pile (and doing this at each site). In order to state the result precisely we require some further notation.

For each $x \in \mathbb{Z}$, let $\mathcal{A}_x$ denote a partition of $\mathbb{N}$ into finite (non-empty) subsets. For any such partition we can order the elements of the partition as $\mathcal{A}_x = (A_x^1, A_x^2, \ldots)$ (e.g. according to the ordering of the smallest element in each A_x^i). Let $\mathcal{A} = (\mathcal{A}_x)_{x\in\mathbb{Z}}$ denote a particular collection of such partitions (indexed by $\mathbb{Z}$), and $\mathcal{P}$ denote the set of all such collections. Let $\mathcal{P}_n$ denote the set of such collections where every A_x^s is a set containing at most n elements.

Fix $\mathcal{A} \in \mathcal{P}$. Let $x \in \mathbb{Z}$, $s \in \mathbb{N}$, ω be a cookie environment, and $j,k \in A_x^s$ with $j \leqslant k$. We say that (j,k) is an *(x,s,ω)-favourable swap* if $\omega(x,j) \leqslant \omega(x,k)$. Let $\omega(x, A_x^s) = (\omega(x,r))_{r\in A_x^s}$, and let $b = (j,k)$ be an (x,s,ω)-favourable swap. Define $\omega_b(s, A_x^s)$ by,

$$\omega_b(s,r) = \begin{cases} \omega(s,k), & \text{if } r = j, \\ \omega(s,j), & \text{if } r = k, \\ \omega(s,r), & \text{if } r \in A_x^s \setminus \{j,k\}. \end{cases}$$

Then we say that $\omega_b(s, A_x^s)$ is the A_x^s-environment produced by the swap $b = (j,k)$, and write $\omega(x, A_x^s) \xrightarrow{b} \omega_b(x, A_x^s)$. Given two cookie environments $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$, we say that $\boldsymbol{\omega}'$ is *an $\mathcal{A}$-permutation of $\boldsymbol{\omega}$* if for each s and x, $\omega'(x, A_x^s)$ is a permutation of $\omega(x, A_x^s)$. If $\boldsymbol{\omega}'$ is an $\mathcal{A}$-permutation of $\boldsymbol{\omega}$ and if also on every A_x^s, $\boldsymbol{\omega}'$ can be generated from $\boldsymbol{\omega}$ from a finite sequence of favourable swaps then we write $\boldsymbol{\omega} \preccurlyeq^{\mathcal{A}} \boldsymbol{\omega}'$. More precisely $\boldsymbol{\omega} \preccurlyeq^{\mathcal{A}} \boldsymbol{\omega}'$ if for every $x \in \mathbb{Z}$, $s \in \mathbb{N}$, $j \leqslant k$, there exists a finite sequence of pairs of A_x^s indices $b_1, \ldots, b_K$ (for some $K \geqslant 0$), and A_x^s-environments $(\omega_i(x, A_x^s))_{i=0}^{K}$ with $\omega_0(x, A_x^s) = \omega(x, A_x^s)$ and $\omega_K(x, A_x^s) = \omega'(x, A_x^s)$ such that $\omega_i(x, A_x^s) \xrightarrow{b_{i+1}} \omega_{i+1}(x, A_x^s)$ are favourable swaps for each $i = 0, \ldots, K-1$.

Given $\mathcal{A} \in \mathcal{P}$ and an environment $\boldsymbol{\omega}$, let $\underline{\boldsymbol{\omega}}_{\mathcal{A}}$ denote the environment obtained by permuting ω on each A_x^s so that $\underline{\boldsymbol{\omega}}_{\mathcal{A}}(x,j) \leqslant \underline{\boldsymbol{\omega}}_{\mathcal{A}}(x,k)$ for all $j,k \in A_x^s$ such that $j < k$. Note that $\underline{\omega}_{\mathcal{A}}(x, A_x^s)$ can be obtained from $\omega(x, A_x^s)$ by a sequence consisting of at most $|A_x^s| - 1$ swaps that are not favourable: first perform the swap that moves the largest $\omega(x,k)$ for $k \in A_x^s$ to the highest location in A_x^s, then proceed iteratively, always moving the next largest value to the next highest location. Reversing this procedure generates $\omega(x, A_x^s)$ from $\underline{\omega}_{\mathcal{A}}(x, A_x^s)$ by a sequence of (at most $|A_x^s| - 1$) favourable swaps, so that $\underline{\omega}_{\mathcal{A}} \preccurlyeq^{\mathcal{A}} \omega$.

For fixed $\boldsymbol{\omega}$ and for any finite subset $A_x \subset \mathbb{N}$, let $\{V_i\}_{i\in A_x}$ be a collection of i.i.d. standard uniform random variables and define $N_{A_x} = \sum_{i\in A_x} I_{V_i \leqslant \omega(x,i)}$ (which can be thought of as the number of right arrows generated by $\omega(x, A_x)$). Note that the law of $N_{A_x}(\boldsymbol{\omega})$ is invariant under permutations of the indices in the set A_x, so that $q_{\boldsymbol{\omega},A_x}(y) = \mathbb{P}(N_{A_x}(\boldsymbol{\omega}) = y)$ is invariant under such permutations.

Theorem 5.1. *Let $\mathcal{A} \in \mathcal{P}_3$ and let $\boldsymbol{\omega}$ be a cookie environment. Then there exists a probability space on which: for each $\mathcal{A}$-permutation $\boldsymbol{\omega}'$ of $\underline{\boldsymbol{\omega}}_{\mathcal{A}}$ there is an excited random walk $E_{\boldsymbol{\omega}'}$ in environment $\boldsymbol{\omega}'$, defined such that $E_{\boldsymbol{\omega}'} \preccurlyeq E_{\boldsymbol{\omega}''}$ almost surely whenever $\boldsymbol{\omega}' \preccurlyeq^{\mathcal{A}} \boldsymbol{\omega}''$.*

Proof. Let $\mathbf{U} = \{U_{x,s}\}_{x\in\mathbb{Z},s\in\mathbb{N}}$ be i.i.d. standard uniform random variables, and $\mathbf{Y} = \{Y_{x,s}\}_{x\in\mathbb{Z},s\in\mathbb{N}}$ be independent random variables (independent of $\mathbf{U}$) where $Y_{x,s}$ has the law of $N_{A_x^s}(\boldsymbol{\omega})$ for each x, s.

Let $x \in \mathbb{Z}$ and $s \in \mathbb{N}$ and consider the set A_x^s, which contains $n = |A_x^s| \leqslant 3$ elements. Without loss of generality let us assume that $A_x^s = \{1, \dots, n\}$. Let $y = Y_{x,s}$ and note that (since $n \leqslant 3$) the set $S_{n,y}$ of n-stacks (an n-stack is any element of $\{\leftarrow, \rightarrow\}^n$) containing exactly y right arrows is a completely ordered set (under $\preccurlyeq$) of cardinality $n_y = \binom{n}{y}$. Let $(a_1^{(y)}, \dots, a_{n_y}^{(y)})$ be the reverse ordering of the set (so that $a_1^{(y)}$ is the element consisting of y right arrows underneath $n - y$ left arrows), and let $a_i^{(y)}(j)$ be the jth arrow of $a_i^{(y)}$.

Now for any $\mathcal{A}$-permutation $\boldsymbol{\omega}'$ of $\underline{\boldsymbol{\omega}}_{\mathcal{A}}$, define a probability measure $P_{\boldsymbol{\omega}'}$ on $S_{n,y}$ by setting

$$P_{\boldsymbol{\omega}'}\big(a_i^{(y)}\big) = \big(q_{\boldsymbol{\omega},A_x^s}(y)\big)^{-1} \prod_{j=1}^{n} \big[\omega'(x,j) I_{a_i^{(y)}(j)=\rightarrow} + \big(1-\omega'(x,j)\big) I_{a_i^{(y)}(j)=\leftarrow}\big], \quad i = 1, \dots, n_y.$$

This is the conditional probability of selecting (for the arrows corresponding to A_x^s) a particular configuration $a_i^{(y)}$ consisting of y right arrows and $n-y$ left arrows, given that the configuration contains exactly y right arrows and $n - y$ left arrows. Define $\mathcal{E}_{\boldsymbol{\omega}'}(x, A_x^s) = (\mathcal{E}_{\boldsymbol{\omega}'}(x, j))_{j\in A_x^s}$ by

$$\mathcal{E}_{\boldsymbol{\omega}'}\big(x, A_x^s\big) = a_m^{(y)}, \quad \text{if } \sum_{i=1}^{m-1} P_{\boldsymbol{\omega}'}\big(a_i^{(y)}\big) < U_{x,s} \leqslant \sum_{i=1}^{m} P_{\boldsymbol{\omega}'}\big(a_i^{(y)}\big).$$

Let $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}''$ be $\mathcal{A}$-permutations of $\underline{\boldsymbol{\omega}}_{\mathcal{A}}$ with $\boldsymbol{\omega}' \preccurlyeq^{\mathcal{A}} \boldsymbol{\omega}''$. Recall that $q_{\boldsymbol{\omega}'',A_x^s}(y) = q_{\boldsymbol{\omega}',A_x^s}(y)$ by invariance under permutations. Also note that for every $m \leqslant n_y$,

$$\sum_{i=1}^{m} P_{\boldsymbol{\omega}''}\big(a_i^{(y)}\big) \geqslant \sum_{i=1}^{m} P_{\boldsymbol{\omega}'}\big(a_i^{(y)}\big),$$

so that under this coupling, $\mathcal{E}_{\boldsymbol{\omega}'}(x, A_x^s) = a_m^{(y)} \Rightarrow \mathcal{E}_{\boldsymbol{\omega}''}(x, A_x^s) = a_k^{(y)}$ for some $k \leqslant m$. This means that $\mathcal{E}_{\boldsymbol{\omega}'}(x, A_x^s) \preccurlyeq \mathcal{E}_{\boldsymbol{\omega}''}(x, A_x^s)$ when we consider $\preccurlyeq$ on A_x^s only.

Let us now summarise what we have achieved. For fixed $\mathcal{A}$ and $\boldsymbol{\omega}$, we have coupled arrow systems (and hence the corresponding walks) defined from all $\mathcal{A}$-permutations of $\underline{\boldsymbol{\omega}}_{\mathcal{A}}$ (including $\boldsymbol{\omega}$ itself) so that $\mathcal{E}_{\boldsymbol{\omega}'}(x, A_x^s) \preccurlyeq \mathcal{E}_{\boldsymbol{\omega}''}(x, A_x^s)$ for each $x \in \mathbb{Z}$, $s \in \mathbb{N}$ when $\boldsymbol{\omega}' \preccurlyeq^{\mathcal{A}} \boldsymbol{\omega}''$, where the coupling took place independently (according to the variables $\mathbf{U}$ and $\mathbf{Y}$) for each x, s. It follows that for any such $\boldsymbol{\omega}'$, $\boldsymbol{\omega}''$, under this coupling, $\mathcal{E}_{\boldsymbol{\omega}'} \preccurlyeq \mathcal{E}_{\boldsymbol{\omega}''}$. The result follows since for each $\mathcal{A}$-permutation $\boldsymbol{\omega}'$, the corresponding walk $E_{\boldsymbol{\omega}'}$ has the law of an excited random walk in cookie environment $\boldsymbol{\omega}'$. $\square$

Note that in the statement (and proof) of Theorem 5.1 the probability space depends on $\mathcal{A}$ and $\boldsymbol{\omega}$ and is constructed in such a way that each A_x^s has the same number of right arrows under $\boldsymbol{\omega}$ as under $\boldsymbol{\omega}'$ (and likewise left arrows). If $\mathcal{A} \in \mathcal{P}_2$, which corresponds to considering only disjoint transpositions/swaps, then the above proof can be simplified slightly, and the probability space defined independently of $\boldsymbol{\omega}$. The coupling is then defined on $\mathcal{A}_x^s = (j, k)$ for each $\boldsymbol{\omega}$ by

$$\big(\mathcal{E}(x, j), \mathcal{E}(x, k)\big) = \begin{cases} (\rightarrow, \rightarrow), & \text{if } U_{x,k,j} < \omega(x, j)\omega(x, k), \\ (\rightarrow, \leftarrow), & \text{if } \omega(x, j)\omega(x, k) \leqslant U_{x,k,j} < \omega(x, j), \\ (\leftarrow, \rightarrow), & \text{if } \omega(x, j) \leqslant U_{x,k,j} < \omega(x, j) + \omega(x, k)(1 - \omega(x, j)), \\ (\leftarrow, \leftarrow), & \text{otherwise.} \end{cases} \tag{5.1}$$

This works because the set of 2-stacks is totally ordered according to $\preccurlyeq$ as

$$\substack{\rightarrow \\ \rightarrow} \quad \succcurlyeq \quad \substack{\leftarrow \\ \rightarrow} \quad \succcurlyeq \quad \substack{\rightarrow \\ \leftarrow} \quad \succcurlyeq \quad \substack{\leftarrow \\ \leftarrow}$$

so there is no need to define the random variables $Y_{x,s}$ whose laws depend on $\boldsymbol{\omega}$. If on the other hand we relax the condition that $\mathcal{A} \in \mathcal{P}_3$ to $\mathcal{A} \in \mathcal{P}_4$ the proof breaks down because e.g. the 4-stacks $\substack{\rightarrow \\ \leftarrow \\ \leftarrow \\ \rightarrow}$ and $\substack{\leftarrow \\ \rightarrow \\ \rightarrow \\ \leftarrow}$ are not ordered by $\preccurlyeq$. However, by considering finite sequences of favourable swaps, we can obtain the following theorem.

Theorem 5.2. *Let $\mathcal{A} \in \mathcal{P}$ and let $\boldsymbol{\omega} \preccurlyeq^{\mathcal{A}} \boldsymbol{\omega}'$ be two cookie environments. Then there exists a probability space on which there are excited random walks $E_{\boldsymbol{\omega}}$ and $E_{\boldsymbol{\omega}'}$ in environments $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ respectively, defined such that $E_{\boldsymbol{\omega}} \preccurlyeq E_{\boldsymbol{\omega}'}$ almost surely.*

Proof. Fix $x \in \mathbb{Z}$, $s \in \mathbb{N}$. Then $\boldsymbol{\omega}'(x, A_x^s)$ can be obtained from $\boldsymbol{\omega}(x, A_x^s)$ by a finite sequence of favourable swaps $\omega_i(x, A_x^s) \xrightarrow{b_{i+1}} \omega_{i+1}(x, A_x^s)$, $i = 0, \dots, K_x^s - 1$, with $\omega_0(x, A_x^s) = \omega(x, A_x^s)$ and $\omega_{K_x^s}(x, A_x^s) = \omega'(x, A_x^s)$. Using the coupling in Theorem 5.1 for a single favourable swap on A_x^s, for each i we can define a probability space with finite chunks of random arrow systems $(\mathcal{E}_i(x, A_x^s), \mathcal{E}_i'(x, A_x^s))$ with marginal laws defined by $\omega_i(x, A_x^s)$ and $\omega_{i+1}(x, A_x^s)$ respectively, and such that $\mathcal{E}_i(x, A_x^s) \preccurlyeq \mathcal{E}_i'(x, A_x^s)$.

Let (X_1, Y_1) and (Y_2, Z_2) be random quantities (not necessarily defined on the same probability space) such that Y_1 and Y_2 have the same distribution. Then we can construct X_3, Y_3, and Z_3 on a common probability space by letting $Y_3 \sim Y_1 \sim Y_2$, and letting X_3 and Z_3 be conditionally independent given Y_3, with conditional laws the same as X_1 given Y_1 and Z_2 given Y_2 respectively. Iterating this construction, and applying the resulting coupling to the random objects $\mathcal{E}_i(x, A_x^s)$, we can construct a probability space on which there are finite chunks of random arrow systems $\mathcal{E}_i(x, A_x^s)$ with marginal laws defined by $\omega_i(x, A_x^s)$, $i = 0, \dots, K_x^s$, such that $\mathcal{E}_i(x, A_x^s) \preccurlyeq \mathcal{E}_{i+1}(x, A_x^s)$ for each i. Taking the product probability space over $x \in \mathbb{Z}$ and $s \in \mathbb{N}$, and letting $\mathcal{E} = (\mathcal{E}_0(x, A_x^s))_{x\in\mathbb{Z}, s\in\mathbb{N}}$ and $\mathcal{E}' = (\mathcal{E}_{K_x^s}(x, A_x^s))_{x\in\mathbb{Z}, s\in\mathbb{N}}$, we have that $\mathcal{E} \preccurlyeq \mathcal{E}'$. Defining E_ω and $E_{\omega'}$ to be the corresponding walks gives the result. □

Since Theorems 5.1 and 5.2 are defined rather abstractly, we now give an explicit example. Suppose that $\boldsymbol{\omega}$ is an environment defined by $\omega(x, 2k - 1) = p_1$ and $\omega(x, 2k) = p_2$ for every $x \in \mathbb{Z}$, $k \in \mathbb{N}$, with $p_2 > p_1$. Suppose also that we wish to understand the effect (on the asymptotic properties of the corresponding excited random walk) of switching the order of the first two cookies at every even site, or instead, of switching the values of p_1 and p_2 at even sites. In the first case the environment of interest is $\boldsymbol{\omega}'$ where $\omega'(x, 1) = \omega(x, 2)$ and $\omega'(x, 2) = \omega(x, 1)$ for each $x \in 2\mathbb{Z}$ and otherwise $\omega'(x, k) = \omega(x, k)$, while in the second case we have $\boldsymbol{\omega}''$ defined by $\omega''(x, 2k - 1) = \omega(x, 2k)$ and $\omega''(x, 2k) = \omega(x, 2k - 1)$ for all $x \in 2\mathbb{Z}$, $k \in \mathbb{N}$ and otherwise $\omega'(x, k) = \omega(x, k)$. In this example the permutations of interest are composed of *disjoint* swaps/transpositions, and hence we can choose partitions consisting of sets containing at most 2 elements. For example, letting $A_x^s = \{2s - 1, 2s\}$ for each $x \in \mathbb{Z}$, $s \in \mathbb{N}$ defines one particular choice (among many) of $\mathcal{A}$ for which $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}''$ are $\mathcal{A}$-permutations of $\boldsymbol{\omega}$, and such that $\boldsymbol{\omega} \preccurlyeq^{\mathcal{A}} \boldsymbol{\omega}' \preccurlyeq^{\mathcal{A}} \boldsymbol{\omega}''$. Theorems 5.1 and 1.3 then imply that e.g. if $p_1 \geqslant \frac{1}{2}$ (so that the walks are not transient to the left) then the limsup speeds of the corresponding random walks satisfy $\bar{v}_\omega \leqslant \bar{v}_{\omega'} \leqslant \bar{v}_{\omega''}$.

The ORRW is an example of a walk whose drift can depend on more than just the number of visits to the current site. For example, on $\mathbb{Z}_+$ the drift encountered by the ORRW at a site x at time n

(so $X_n = x$) depends on whether the local time of the walk at $x+1$ is positive. Some of the known results for excited random walks in i.i.d. or ergodic environments can be extended to more general self-interacting random walks (where the drifts may depend on the history in an unusual way) with a bounded number of positive drifts per site.

Theorem 5.3. *Let X_n be a nearest-neighbour self-interacting random walk and $\mathcal{F}_n = \sigma(X_k, k \leqslant n)$. Suppose that there exist $M \in \mathbb{N}$ and $(\eta_k)_{k \leqslant M} \in [0,1)^M$ such that*

- $\mathbb{P}(X_{n+1} = X_n + 1 | \mathcal{F}_n) I_{\ell(n)=k} \leqslant \eta_k$ *for all $k \leqslant M$ and all $n \in \mathbb{Z}_+$ almost surely, and*
- $\mathbb{P}(X_{n+1} = X_n + 1 | \mathcal{F}_n) I_{\ell(n)=k} \leqslant \frac{1}{2}$ *for all $k > M$ and all $n \in \mathbb{Z}_+$, almost surely.*

If $\alpha = \sum_{k=1}^{M}(2\eta_k - 1) \leqslant 1$ then X is not transient to the right, almost surely. If $\alpha \leqslant 2$ then $\limsup n^{-1} X_n \leqslant 0$, almost surely. If $\alpha < -1$ then X is transient to the left, almost surely. If $\alpha < -2$ then $\liminf n^{-1} X_n < 0$ almost surely.

Proof. Define $\eta_k = \frac{1}{2}$ for $k > M$. For each $x \in \mathbb{Z}$, let $\omega(x,k) = \eta_k$ for $k \in \mathbb{N}$. Let $\mathbf{U} = (U(x,m))_{x \in \mathbb{Z}, m \in \mathbb{N}}$ be i.i.d. standard uniform random variables and define $\mathcal{R}$ by

$$\mathcal{R}(x,k) = \begin{cases} \rightarrow & \text{if } U(x,k) \leqslant \eta_k, \\ \leftarrow & \text{otherwise.} \end{cases}$$

The corresponding walk R_n has the law of an excited random walk in the (non-random) environment ω. By [5], the conclusions of the theorem hold for the walk R, e.g. if $\alpha = \sum_{k=1}^{M}(2\eta_k - 1) \leqslant 1$ then R is not transient to the right, almost surely.

For a nearest-neighbour sequence $x_0, \ldots, x_n$ define

$$P_{n,k}(x_0, \ldots, x_n) = \mathbb{P}(X_{n+1} = X_n + 1 \mid X_0 = x_0, \ldots, X_n = x_n) I_{\ell_x(n)=k}.$$

Define a nearest-neighbour self-interacting random walk L by setting $L_0 = 0$ and given that $\ell_L(n) = k$,

$$L_{n+1} = \begin{cases} L_n + 1, & \text{if } U(L_n, k) \leqslant P_{n,k}(L_0, \ldots, L_n), \\ L_n - 1, & \text{otherwise.} \end{cases}$$

Then L has the law of X. Since $P_{n,k} \leqslant \eta_k$ almost surely, we have that $L \trianglelefteq R$ almost surely. The result now follows by Corollary 3.10. The astute reader may have noticed that we have not defined the arrow system $\mathcal{L}$. We can do so, according to the walk L as follows. Given that $\ell_L(n) = k$, define

$$\mathcal{L}(L_n, k) = \begin{cases} \rightarrow, & \text{if } U(L_n, k) \leqslant P_{n,k}(L_0, \ldots, L_n), \\ \leftarrow, & \text{otherwise.} \end{cases}$$

In other words, this inductively defines $\mathcal{L}$ as the arrow system determined by the steps of the walk L. Since L does not define an entire arrow system at any site x visited only finitely often by L we can define $\mathcal{L}(x,k) = \leftarrow$ for each $k > n_L(x)$.

To be more precise, for each n we can define $\mathcal{L}^{(n)}$ according to the arrow system determined by $L_0, \ldots, L_n$ and adding $\leftarrow$ everywhere else. For each such n we have $\mathcal{L}^{(n)} \trianglelefteq \mathcal{R}$, so that Theorem 1.3(iv) holds for each n, and so does (3.4). The former result implies the claims about transience when $\alpha \leqslant 1$ and $\alpha < -1$, while (3.4) and its minimum equivalent imply the remaining results (see e.g. the proof of Theorem 1.3(iii)). $\square$

Acknowledgments

The authors would like to thank two anonymous referees and the editors for their helpful suggestions and the Fields Institute for hosting them while part of this work was carried out. This research was supported in part by the Marsden Fund (Holmes), and by NSERC (Salisbury).

References

[1] I. Benjamini, D.B. Wilson, Excited random walk, Electron. Comm. Probab. 8 (2003) 86–92.
[2] J. Bérard, A. Ramírez, Central limit theorem for excited random walk in dimension $d \geqslant 2$, Electron. Comm. Probab. 12 (2007) 300–314.
[3] R. van der Hofstad, M. Holmes, Monotonicity for excited random walk in high dimensions, Probab. Theory Related Fields 147 (2010) 333–348.
[4] M. Holmes, T. Salisbury, Random walks in degenerate random environments, preprint, 2010.
[5] E. Kosygina, M. Zerner, Positively and negatively excited random walks on integers, with branching processes, Electron. J. Probab. 13 (2008) 1952–1979.
[6] M. Zerner, Multi-excited random walks on integers, Probab. Theory Related Fields 133 (2005) 98–122.