Corrigendum

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“On 2-arc-transitivity of Cayley graphs”


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A characterization of 2-arc-transitive dihedrants, that is, Cayley graphs of dihedral groups, is given in [3, Theorem 8.4], describing such graphs either explicitly or as cyclic covers of certain basic dihedrants. The proof is carried out via a reduction scheme based on the classical results due to Schur and Wielandt, saying that cyclic groups of composite order and dihedral groups are all B-groups [4, Theorems 25.3 and 25.6]. Consequently, the automorphism group \( A \) of a connected 2-arc-transitive Cayley graph \( X \neq K_{2n} \) of a dihedral group \( D_{2n} = \langle \rho, \tau \mid \rho^n = \tau^2 = (\rho \tau)^2 = 1 \rangle \) is necessarily imprimitive, allowing the above mentioned reduction with respect to blocks of minimal length. The various possibilities that may occur for such blocks are covered in [3, Lemmas 7.1–7.5]. Unfortunately, in Lemma 7.2 (and consequently in Lemma 7.3) one of the cases needed to be considered was missed out. This case essentially leads to the situation, where the group \( A \) has an imprimitivity block system with blocks of size 2 having nonempty intersection with both orbits of \( \rho \), and a single edge between any two adjacent blocks.

In this note we fill out the gap by proving a new lemma (Lemma 3 below) taking care of this case too, and thus replacing [3, Lemmas 7.2 and 7.3]. In doing so we obtain the following strengthening (Theorem 1 below) of [3, Theorem 8.4] which reduces the class of basic dihedrants that may give rise (via cyclic covers) to 2-arc-transitive dihedrants.

Let \( n \geq 3 \). We let \( \mathcal{G} \) denote the class of graphs containing complete bipartite graphs \( K_{n,n} \), complete bipartite graphs minus a matching \( K_{n,n} - nK_2 \), incidence and nonincidence graphs \( B(H_{11}) \) and \( B'(H_{11}) \) of the Hadamard design on 11 points, and incidence and nonincidence graphs \( B(PG(d, q)) \) and \( B'(PG(d, q)) \), with \( d \geq 2 \) and \( q \) a prime power, of projective spaces; and we let \( \mathcal{H} \) denote the class of graphs containing cycles \( C_{2n} \), complete graphs \( K_{2n} \), and graphs...
of index 2 obtained as regular $\mathbb{Z}_2$-covers of $K_{q+1,q+1} - (q + 1)K_2$, $q$ an odd prime power, by identifying the vertex set of the base graph with two copies of the projective line $PG(1, q)$ where the missing matching consists of all pairs $[x, x']$, $x \in PG(1, q)$, and the edge $[x, y']$ carries voltage 1 if $x - y$ is a nonsquare in $GF(q)$, and voltage 0 in all other cases.

**Theorem 1.** Let $n \geq 3$, and let $X$ be a connected, 2-arc-transitive Cayley graph of a dihedral group $D = D_{2n} = \langle \rho, \tau \mid \rho^n = \tau^2 = (\rho \tau)^2 = 1 \rangle$ of order $2n$. Then one of the following occurs:

(i) either $X \in \mathcal{G} \cup \mathcal{H}$; or

(ii) $X$ is a regular cyclic cover of a graph in $\mathcal{G}$; more precisely: there exists a proper divisor $m$ of $n$ such that the set $\mathcal{B}$ of orbits of $\langle \rho^m \rangle$ is an imprimitivity block system of $Aut X$ relative to which $X$ is a regular $\mathbb{Z}_{n/m}$-cover of $X_\mathcal{B}$, the latter being a graph in $\mathcal{G}$ admitting a regular dihedral group $D/\langle \rho^m \rangle$.

Let $X \neq K_{2n}$ be a connected dihedrant of order $2n$, $n \geq 3$, relative to the group $D_{2n} = \langle \rho, \tau \mid \rho^n = \tau^2 = (\rho \tau)^2 = 1 \rangle$. A block $\mathcal{B}$ of $Aut X$ is said to be cyclic if there are $w \in V(X)$ and $m \in \mathbb{Z}_n^\#$ such that $\mathcal{B}$ coincides with the orbit $\langle \rho^m \rangle w$, and is said to be dihedral if there are vertices $u$ and $v$ belonging to distinct orbits of $\rho$ and $m \in \mathbb{Z}_n \setminus \mathbb{Z}_n^*$ such that $\mathcal{B} = \langle \rho^m \rangle u \cup \langle \rho^m \rangle v$.

The first lemma below corrects an error—having no consequence to [3, Theorem 8.4]—in the statement of [3, Lemma 7.1].

**Lemma 2.** (Correction to [3, Lemma 7.1].) Let $n \geq 3$, $X \neq K_{2n}$ be a connected dihedrant of order $2n$, let $\mathcal{B}$ be an imprimitivity block system of $A = Aut X$, let $\rho$ generate the cyclic subgroup of index 2 in a regular dihedral subgroup $D_{2n} = \langle \rho, \tau \mid \rho^n = \tau^2 = (\rho \tau)^2 = 1 \rangle$ of $A$, and let $K$ be the kernel of the action of $A$ on $\mathcal{B}$. Then one of the following occurs:

(i) the blocks in $\mathcal{B}$ are all cyclic and $K \cap \langle \rho \rangle = \langle \rho^m \rangle$ for some $m \in \mathbb{Z}_n^\#$; or

(ii) the blocks in $\mathcal{B}$ are all dihedral and $K \cap \langle \rho \rangle = \langle \rho^m \rangle$ for some $m \in \mathbb{Z}_n \setminus \mathbb{Z}_n^*$.

**Lemma 3.** (Replacement of [3, Lemmas 7.2 and 7.3].) Let $n \geq 3$, $X \neq K_{2n}$ be a connected 2-arc-transitive dihedrant of order $2n$ and valency at least 3, let $D = D_{2n} = \langle \rho, \tau \mid \rho^n = \tau^2 = (\rho \tau)^2 = 1 \rangle$ be a regular dihedral subgroup of $A = Aut X$, let $\mathcal{B}$ be an imprimitivity block system of $A$ with blocks of length $k \geq 2$, and let $K$ denote the kernel of the action of $A$ on $\mathcal{B}$. If $|\mathcal{B}| \geq 3$, then the following hold:

(i) if the blocks in $\mathcal{B}$ are cyclic, then $X$ is a regular $\mathbb{Z}_k$-cover of a 2-arc-transitive dihedrant $X_\mathcal{B}$, the latter admitting a 2-arc-transitive action of the group $A/K$;

(ii) if the blocks in $\mathcal{B}$ are dihedral, then there exists an associated imprimitivity block system $\mathcal{B}'$ for the group $A$ with kernel $K'$ such that either:

(a) the blocks in $\mathcal{B}'$ are cyclic, in which case (i) holds for $\mathcal{B}'$; or

(b) there are vertices $u$ and $v$ from different orbits of $\rho$ such that either $\{u, v\}$ or $\{u, v, \rho^{n/2}u, \rho^{n/2}v\}$ is a block inducing $\mathcal{B}'$, and $X$ is, respectively, a regular $\mathbb{Z}_2$-cover or a regular $\mathbb{Z}_2^2$-cover of a 2-arc-transitive circulant $X_{\mathcal{B}'}$, the latter admitting a 2-arc-transitive action of the group $A/K'$. 
Proof. First note that, because of 2-arc-transitivity of $X$, given any two blocks $B, B′ ∈ B$, a vertex in $B$ has at most one neighbour in $B′$. Also, part (i) follows directly by Lemma 2 and the argument used in the proof of [3, Lemma 7.2(i)].

We may therefore assume that the blocks in $B$ are dihedral. By Lemma 2 there exists $m ∈ \mathbb{Z}_n \setminus \mathbb{Z}_n^*$ such that $K \cap \langle ρ \rangle = \langle ρ^m \rangle$. Let $u$ and $v$ be two vertices from different orbits of $ρ$ belonging to the same block $B ∈ B$, and let $[S,T]$, where $S = -S ∈ \mathbb{Z}_n^0$, and $T ⊆ \mathbb{Z}_n$, be the symbol of $X$ relative to the triple $(u,v,ρ)$, that is, for each $i ∈ \mathbb{Z}_n$ we have that $ui = ρ^i u$ and $vi = ρ^i v$ are adjacent, respectively, with $ui+s, s ∈ S$, and $vi+s, s ∈ S$, and, moreover, $ui$ is adjacent with $vi+t, t ∈ T$. Then $B$ coincides with the set $\langle ρ^m \rangle u \cup \langle ρ^m \rangle v = \{u_0, u_m, \ldots, u_{(k/2−1)m}, v_0, v_m, \ldots, v_{(k/2−1)m}\}$. Let $C$ be an imprimitivity block system contained in $B$ consisting of minimal blocks, and let $H$ be the kernel of the action of $A$ on $C$. If the blocks in $C$ are cyclic, then $B′ = C$ satisfies part (a) of (ii). We may thus assume that the blocks in $C$ are dihedral, too.

Let $C ∈ C$ be the block containing $u_0$. Since the restriction $A_C^C$ of the setwise stabilizer $A_C$ to $C$ is, by minimality of $C$, primitive and contains the group $D_C^C$, it transpires that there are only two possibilities for $C$: Either $C = \{u_0, v_0\}$ or $C = \{u_0, v_0, u_{n/2}, v_{n/2}\}$. Moreover, if the latter occurs then the kernel of the action of $A/C$ on $C$ is isomorphic to $\mathbb{Z}_2^2$, and so $X$ is a regular $\mathbb{Z}_2^2$-cover of $X_C$ and thus every 2-arc in $X_C$ is a projection of a 2-arc in $X$. Hence $B′ = C$ satisfies part (b) of (ii). This leaves us with the case $C = \{u_0, v_0\}$. For each $i ∈ \mathbb{Z}_n$ let $C_i = ρ^i C$. If for any two adjacent blocks $C_i$ and $C_j$, the bigraph $X[C_i, C_j]$ is isomorphic to $2K_2$, then $B′ = C$ satisfies part (b) of (ii). We may therefore assume that, for any two adjacent blocks $C_i$ and $C_j$, the bigraph $X[C_i, C_j]$ is isomorphic to $K_2 + 2K_1$. By deriving a contradiction we shall see that this case cannot occur. First, it may be easily seen that the symbol of $X$ is $[0,T]$ with $T \cap (-T) = \emptyset$ and $|T| ≥ 3$. Clearly, $X_C ≃ \text{Circ}(n,T∪−T)$, is an $n$-circuit with symbol $T∪−T$, admitting an arc-transitive action of $A/H$. In addition, since $A$ acts transitively on 2-arcs of $X$, it follows that for any pair of neighbours $v_0$ and $v′$, $t, t′ ∈ T$, of $u_0$, there is an automorphism in $A_{u_0}$ switching $v_0$ and $v′$. Thus $A/H$ (or rather a subgroup of index 2 which fixes $\{C_t | t ∈ T\}$) acts doubly transitively on $\{C_t | t ∈ T\}$ (as well as on $\{C_{−t} | t ∈ T\}$). In summary,

$$A/H \text{ is doubly transitive on } \{C_t | t ∈ T\} \text{ and on } \{C_{−t} | t ∈ T\}. \quad (1)$$

Moreover, an automorphism of $X$ switches $u_0$ with $v_0$, and so its image in $A/H$ interchanges the sets $\{C_t | t ∈ T\}$ and $\{C_{−t} | t ∈ T\}$. We now use the classification of arc-transitive circulants (see [1, Theorem 1] or [2, Theorem 1.3]) to analyze the possible structure of $X_C$. By this classification, $X_C$ can be the complete graph $K_n$, a normal circulant, a lexicographic product $Γ \times \bar{K}_r$ where $Γ$ is an arc-transitive circulant of order $d$ and $dr = n$, or a deleted lexicographic product $Γ \times \bar{K}_r−rΓ$, with $Γ$, $d$ and $r$ as above.

Case 1. $X_C ≃ K_n$.

This case is covered in the last paragraph of the proof of [3, Lemma 7.3] and cannot occur.

Case 2. $X_C$ is a normal circulant.

In this case $A/H$ has a normal cyclic group $\langle ρ \rangle/H \cap \langle ρ \rangle ≃ \mathbb{Z}_n$ and so $A/H ≃ \mathbb{Z}_n : AC_0$, where $AC_0$ is the vertex stabilizer in the quotient graph $X_C$. Now $AC_0$ can be identified with a subgroup in Aut $\mathbb{Z}_n$. But the latter is abelian, and so $AC_0$ is abelian, too. Its action on $\{C_t | t ∈ T\} \cup \{C_{−t} | t ∈ T\}$ is therefore regular and so it cannot act doubly transitively on $\{C_t | t ∈ T\}$ (and $\{C_{−t} | t ∈ T\}$), proving that this case cannot occur.

Case 3. $X_C$ is a lexicographic product.
There exist \( d, r \geq 2 \) such that \( dr = n \) and \( X_C \cong \Gamma \wr \bar{K}_r \), where \( \Gamma \) is an arc-transitive \( d \)-circulant. So we may assume that there is an imprimitivity block system of \( X_C \) which superimposed on \( C \) gives rise to a new imprimitivity block system of \( X \), call it \( D \), generated by the block \( \{u_0, u_d, \ldots, u_{(r-1)d}, v_0, v_d, \ldots, v_{(r-1)d}\} \). Now there is a natural orientation of the circulant \( X_C \) induced by \( X \), with an arrow pointing from \( C_i \) to \( C_{i+t} \) for all \( t \in T \). Using (1) and the fact that \( X \) is 2-arc-transitive, we can reduce our analysis to the following three possibilities: Either \( d = 3 \) (and \( \Gamma \cong K_3 \)) and all edges between two blocks in \( X_C \) have the same orientation; or \( d = 2 \), in which case \( X_C \cong K_{n/2, n/2} \); or \( r = 2 \) and the bigraph between two adjacent blocks in \( X_C \) is a directed 4-cycle. It is immediate that the first possibility gives rise to a disconnected graph \( X \), and applying the argument, used in the penultimate paragraph of the proof of [3, Lemma 7.3], so does the second possibility. Finally, if \( r = 2 \) then \( D \) is generated by \( \{u_0, u_{n/2}, v_0, v_{n/2}\} \). If the kernel of the action of \( A \) on \( D \) is \( \langle \rho_{n/2} \rangle \), then \( \{u_0, u_{n/2}\} \) gives rise to a new imprimitivity block system \( B' \) satisfying part (a) of (ii). If however the kernel of the action of \( A \) on \( D \) is transitive on the blocks, then \( B' = D \) satisfies part (b) of (ii).

**Case 4.** \( X_B \) is a deleted lexicographic product.

Using a similar argument as in Case 3, we end up with two possibilities. Either \( d = 2 \), and so \( X_C \cong K_{n/2, n/2} - n/2K_2 \), or \( r = 3 \) and the bigraph between two adjacent blocks in \( X_C \) is a directed 6-cycle. Again, the first possibility is covered in the penultimate paragraph of the proof of [3, Lemma 7.3]. As for the second possibility, we see that \( \langle \rho_{n/3} \rangle \) is normal in \( A \) and the corresponding orbits give rise to an imprimitivity block system \( B' \) satisfying part (a) of (ii), thus completing the proof of Lemma 3.

**Proof of Theorem 1.** With Lemma 3 in hand, the proof literally follows that of [3, Theorem 8.4]. Part (b) of Lemma 3(ii) is taken care by [3, Lemmas 7.4 and 7.5], whereas the argument in the case of cyclic blocks, given at the end of the proof of [3, Theorem 8.4], may be extended to show that a 2-arc-transitive dihedrant which is a cyclic cover of \( K_{q+1}^4 \) (arising from cyclic blocks) is also a cyclic cover of \( K_{q+1, q+1} - (q + 1)K_2 \) (arising from cyclic blocks).

**References**