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Saddle-Point Optimality Criteria of Continuous Time Programming without Differentiability

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Saddle-point optimality criteria of Kuhn–Tucker and Fritz Johns are established in the case of continuous time programming problems. The functions involved are not assumed to be differentiable. In the process, an important theorem of the alternative is also proven.

1. INTRODUCTION

Tyndall [10] treated rigorously the following two continuous time programming problems, which originated from Bellman's bottleneck problems [1].

PRIMAL PROBLEM Maximize

$$\int_0^T a'(t)\, z(t)\, dt$$

subject to

$$B(t)\, z(t) \leqslant c(t) + \int_0^t K(t, s)\, z(s)\, ds \qquad 0 \leqslant t \leqslant T$$

$$z(t) \geqslant 0 \qquad 0 \leqslant t \leqslant T$$

where $z(t)$ is an $n \times 1$ vector-valued function, bounded and measurable on $[0, T]$.

DUAL PROBLEM. Minimize

$$\int_0^T c'(t)\, w(t)\, dt$$

subject to

$$B'(t)\, w(t) \geqslant a(t) + \int_t^T K'(s, t)\, w(s)\, ds \qquad 0 \leqslant t \leqslant T$$

$$w(t) \geqslant 0 \qquad 0 \leqslant t \leqslant T$$

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where $w(t)$ is an $m \times 1$ vector-valued function, bounded and measurable on $[0, T]$.

Assuming B and K are nonnegative constant matrices, $a(t)$, $c(t)$ are continuous vector-valued functions defined on $[0, T]$, and under the regularity condition

$$\{z(t)\colon Bz(t) \leqslant 0,\, z(t) \geqslant 0 \ \forall t \in [0, T]\} = \{0\},$$

Tyndall [10] proved a duality theorem. Levinson [8] generalized and considerably shortened Tyndall's results. He let $a(t)$, $c(t)$, $B(t)$ be piecewise continuous on $[0, T]$ and $K(t, s)$ be piecewise continuous on $[0, T] \times [0, T]$. Grinold [4, 5] relaxed Tyndall's regularity assumptions and required that $a(t)$, $c(t)$, $B(t)$, $K(t, s)$ be bounded and measurable. Hanson and Mond [6] generalized the duality theory by considering a concave objective function of the primal problem. They also established the complementary slackness principle and the Kuhn–Tucker conditions. Farr and Hanson [2, 3] further generalized the continuous time programming problem by introducing nonlinear, differentiable constrains and establishing the complementary slackness principle and Kuhn–Tucker theorem in their setup. In [3]. they introduced the time lag effect, thus extending the results of Larsen and Polak [7].

The purpose of this paper is to consider the continuous time programming problem (linear and nonlinear) in an entirely different direction. We establish the Kuhn–Tucker and Fritz Johns saddle-point optimality criteria without assuming differentiability of the functions involved. In the process we prove a theorem of the alternative (Theorem 7 below) which is of great value in itself. This puts the continuous time programming problem in the same perspective as the classical nonlinear programming problem, as presented in [9, Chap. 5].

2. Sufficient Optimality Criteria

We consider the following:

Maximization Problem (MP). Maximize

$$l(z) = \int_0^T \phi(z(t))\, dt$$

subject to

$$\begin{aligned} f(z(t)) &\leqslant c(t) + \int_0^t g(s, t, z(s))\, ds \qquad && 0 \leqslant t \leqslant T \\ z(t) &\geqslant 0 && 0 \leqslant t \leqslant T \end{aligned} \tag{1}$$

where $z(\cdot)$ is an $n \times 1$ vector-valued function defined on $[0, T]$. Let D^n be the collection of all such functions which are bounded and measurable. Let $D_+{}^n$ be the collection of all such nonnegative functions.

$f(\cdot)$ is an $m \times 1$ vector-valued function defined on D^n. $c(\cdot)$ is an $m \times 1$ vector-valued function defined on $[0, T]$. $g(\cdot, \cdot, \cdot)$ is an $m \times 1$ vector-valued function defined on $[0, t] \times [0, T] \times D^n$ for each $t \in [0, T]$. $\phi(\cdot)$ is a real-valued function defined on D^n.

Note. All vectors are considered column vectors and the integrals are in the Lebesgue sense. Let

$$S = \left\{ z(t) \in D^n : z(t) \geqslant 0 \ \forall t \in [0, T], f(z(t)) \leqslant c(t) + \int_0^t g(s, t, z(s))\, ds \right\}.$$

S is called the set of feasible solutions of MP.

If there exists a $\bar{z}(t) \in S$ such that $l(\bar{z}) = \max_{z(t) \in S} l(z)$, we say $\bar{z}(t)$ is a solution of the maximization problem (MP).

For our further study, we assume that D^n is suitably normed (say, $\| z(t) \| = \max_{t \in [0,T]} \sum_{i=1}^n | z_i(t) |$ for all $z(t) \in D^n$).

LOCAL MAXIMIZATION PROBLEM (LMP). Find a $\bar{z}(t)$ in S, if it exists such that for some open ball $B_\delta(\bar{z}(t)) = \{ z(t) \in D^n : \| \bar{z}(t) - z(t) \| < \delta \}$ around $\bar{z}(t)$ with radius $\delta > 0$, $z(t) \in B_\delta(\bar{z}(t)) \cap S$ implies $l(z) \leqslant l(\bar{z})$.

Let D^m be the collection of all $m \times 1$ vector-valued functions defined on $[0, T]$. Then we consider The Fritz–John saddle-point problem (FJSP).

Find $\bar{z}(t) \in D_n{}^+$, $\bar{r}(t) \in D^m$, $\bar{r}_0$ real, $(\bar{r}_0, \bar{r}(t)) \geqslant 0 \ \forall t \in [0, T]$, if they exist such that $G(\bar{z}(t), \bar{r}_0, r(t)) \leqslant G(\bar{z}(t), \bar{r}_0, \bar{r}(t)) \leqslant G(z(t), \bar{r}_0, \bar{r}(t))$ for all $r(t) \in D^m$, $r(t) \geqslant 0$ and for all $z(t) \in D_+{}^n$ and where $G(z(t), r_0, r(t)) = \int_0^T [-r_0 \phi(z(t)) + r'(t) \times \{ f(z(t)) - c(t) - \int_0^t g(s, t, z(s))\, ds \}]\, dt$. Throughout, a prime on a vector means the transpose of that vector.

THE KUHN–TUCKER SADDLE-POINT PROBLEM (KTSP). Find $\bar{z}(t) \in D_+{}^n$, $\bar{u}(t) \in D^m$, $\bar{u}(t) \geqslant 0 \ \forall t \in [0, T]$, if they exist such that $F(\bar{z}(t), u(t)) \leqslant F(\bar{z}(t), \bar{u}(t)) \leqslant F(z(t), \bar{u}(t)) \ \forall u(t) \in D^m$, $u(t) \geqslant 0 \ \forall t \in [0, T]$ and $\forall z(t) \in D_+{}^n$ where $F(z(t), u(t)) = \int_0^T [-\phi(z(t) + u'(t) \{ f(z(t)) - c(t) - \int_0^t g(s, t, z(s)\, ds \}]\, dt$.

Remark $*$. If $(\bar{z}(t), \bar{r}_0, \bar{r}(t))$ is a solution of FJSP and $r_0 > 0$, then $(\bar{z}(t), (1/\bar{r}_0)\bar{r}(t))$ is a solution of KTSP. Conversely, if $(\bar{z}(t), \bar{u}(t))$ is a solution of KTSP, then $(\bar{z}(t), 1, \bar{u}(t))$ is a solution of FJSP.

The following first four results are easy to establish. Our Theorems 1, 3, and 4 are analogous to Theorems 5.2.1, 5.2.2, and 5.2.4 in Mangasarian [9] and hence can be proven following Mangasarian's line of argument. We state them without proofs.

THEOREM 1. *If S is a convex set and ϕ is a concave function in $z(t)$, then the set of solutions of* MP *is convex.*

LEMMA 2. *If f is convex in $z(t)$ and g is concave in $z(t)$, then S is a convex set.*

THEOREM 3. *Let S be convex and $\bar{z}(t)$ be a solution of* MP. *If ϕ is strictly concave at $\bar{z}(t)$, then $\bar{z}(t)$ is the unique solution of* MP.

THEOREM 4. *If $\bar{z}(t)$ is a solution of* MP, *then it is also solution of* LMP. *The converse is true if S is convex and ϕ is concave at $\bar{z}(t)$.*

THEOREM 5 (Sufficiency). *If $(\bar{z}(t), \bar{u}(t))$ is a solution of* KTSP, *then $\bar{z}(t)$ is a solution of* MP *almost everywhere on* $[0, T]$. *If $(\bar{z}(t), \bar{r}_0, \bar{r}(t))$ is a solution of* FJSP *and $\bar{r}_0 > 0$ then $\bar{z}(t)$ is a solution of* MP *almost everywhere.*

Proof. The second statement is an easy consequence of the first statement and Remark $*$.

Let $(\bar{z}(t), \bar{u}(t))$ be a solution of KTSP. Then for all $u(t)$ in D^m, $u(t) \geqslant 0$ and for all $z(t)$ in S,

$$\int_0^T -\phi(\bar{z}(t))\,dt + \int_0^T \left\{u'(t)\,f(\bar{z}(t)) - u'(t)\,c(t) - \int_0^t u'(t)\,g(s, t, \bar{z}(s))\,ds\right\} dt$$
$$\leqslant \int_0^T -\phi(\bar{z}(t))\,dt + \int_0^T \left\{\bar{u}'(t)\,f(\bar{z}(t)) - \bar{u}'(t)\,c(t) - \int_0^t \bar{u}'(t)\,g(s, t, \bar{z}(s))\,ds\right\} dt$$
$$\leqslant \int_0^T -\phi(z(t))\,dt + \int_0^T \left\{\bar{u}'(t)\,f(z(t)) - \bar{u}'(t)\,c(t) - \int_0^t \bar{u}'(t)\,g(s, t, z(s))\,ds\right\} dt. \tag{2}$$

From the first inequality in (2), we have

$$\int_0^T (u'(t) - \bar{u}'(t)) \left\{f(\bar{z}(t)) - c(t) - \int_0^t g(s, t, \bar{z}(s))\,ds\right\} dt \leqslant 0. \tag{3}$$

Let

$$A_i = \left\{t \in [0, T]: f_i(\bar{z}(t)) - c_i(t) - \int_0^t g_i(s, t, \bar{z}(s))\,ds \leqslant 0\right\}$$

$$B_i = \left\{t \in [0, T]: f_i(\bar{z}(t)) - c_i(t) - \int_0^t g_i(s, t, \bar{z}(s))\,ds > 0\right\}$$

for $i = 1, 2,..., m$ where $f_i(\bar{z}(t))$ is the ith component of the vector function $f(\bar{z}(t)) = (f_1(\bar{z}(t)),..., f_m(\bar{z}(t)))'$. Similar interpretations apply to $c_i(t)$ and $g_i(s, t, \bar{z}(s))$. Note that $A_i \cap B_i = 0$ and $A_i \cup B_i = [0, T]$ for all $i = 1,..., m$. Since the vector function $u(t)$ is at our disposal, we can choose

$$u_i(t) = \begin{bmatrix} \bar{u}_i(t) & \forall t \in A_i \\ \bar{u}_i(t) + 1 & \forall t \in B_i \end{bmatrix}$$

for $i = 1, \ldots, m$. Now

$$0 \geqslant \int_0^T \{u_i'(t) - \bar{u}_i'(t)\} \left\{ f_i(\bar{z}(t) - c_i(t) - \int_0^t g_i(s, t, \bar{z}(s))\, ds \right\} dt$$

$$= \int_{A_i} 0 \left\{ f_i(\bar{z}(t)) - c_i(t) - \int_0^t g_i(s, t, \bar{z}(s))\, ds \right\} dt$$

$$+ \int_{B_i} l \left\{ f_i(\bar{z}(t) - c_i(t) - \int_0^t g_i(s, t, \bar{z}(s))\, ds \right\} dt$$

$$= \int_{B_i} \left\{ f_i(\bar{z}(t)) - c_i(t) - \int_0^t g_i(s, t, \bar{z}(s))\, ds \right\} dt.$$

But this is possible only (in view of the definition of B_i) if B_i has Lebesgue measure 0. Therefore, $f_i(\bar{z}(t)) - c_i(t) - \int_0^t g_i(s, t, \bar{z}(s))\, ds \leqslant 0$ a.e. on $[0, T]$, i.e.,

$$f_i(\bar{z}(t)) \leqslant c_i(t) + \int_0^t g_i(s, t, \bar{z}(s))\, ds \quad \text{a.e.} \quad \text{on} \quad [0, T].$$

Now $T = A_1 \cup B_1 = \cdots = A_m \cup B_m$ and $A_i \cap B_i = 0$ for $i = 1, \ldots, m$. We take $A = \bigcap_{i=1}^m A_i$, $B = \bigcup_{i=1}^m B_i$. Then $0 \leqslant \mu(B) \leqslant \sum_{i=1}^m \mu(B_i) = \sum_{i=1}^m 0 = 0 \Rightarrow \mu(B) = 0$, where μ is the Lebesgue measure on the sigma field of subsets of $[0, T]$. Also note that

$$A \cup B = \left(\bigcap_{i=1}^m A_i \right) \cup B$$

$$= \bigcap_{i=1}^m (A_i \cup B) = \bigcap_{i=1}^m T = T.$$

Hence

$$f(\bar{z}(t)) \leqslant c(t) + \int_0^t g(s, t, \bar{z}(s))\, ds \ \forall t \in A, \tag{3a}$$

i.e., $\bar{z}(t)$ is a feasible solution of MP a.e. on $[0, T]$.

Now $\bar{u}(t) \geqslant 0\ \forall t \in [0, T]$ and $f(\bar{z}(t)) \leqslant c(t) + \int_0^t g(s, t, \bar{z}(s))\, ds$ a.e. on $[0, T]$. Hence

$$\bar{u}(t) \left\{ f(\bar{z}(t)) - c(t) - \int_0^t g(s, t, \bar{z}(s))\, ds \right\} \leqslant 0 \quad \text{a.e.} \quad \text{on} \quad [0, T].$$

Therefore

$$\int_0^T \bar{u}'(t) \left\{ f(\bar{z}(t)) - c(t) - \int_0^t g(s, t, \bar{z}(s))\, ds \right\} dt \leqslant 0. \tag{4}$$

But taking $u(t) \equiv 0$ in (3), we have

$$\int_0^T \bar{u}'(t) \left\{ f(\bar{z}(t)) - c(t) - \int_0^t g(s, t, \bar{z}(s))\, ds \right\} dt \geqslant 0. \tag{5}$$

Combining (4) and (5) together, we get

$$\int_0^T \bar{u}'(t) \left\{ f(\bar{z}(t)) - c(t) - \int_0^t g(s, t, \bar{z}(s))\, ds \right\} dt = 0. \tag{6}$$

From the second inequality in (2) and by (6), we have

$$\int_0^T - \phi(\bar{z}(t))\, dt$$
$$\leqslant \int_0^T - \phi(z(t))\, dt + \int_0^T \bar{u}'(t) \left\{ f(z(t)) - c(t) - \int_0^t g(s, t, z(s))\, ds \right\} dt.$$

Hence

$$\int_0^T - \phi(\bar{z}(t))\, dt \leqslant \int_0^T - \phi(z(t))\, dt.$$

(This is because, $\bar{u}(t) \geqslant 0\ \forall t \in [0, T]$ and for all feasible $z(t)$, $f(z(t)) - c(t) - \int_0^t g(s, t, z(s))\, ds \leqslant 0$.) Hence $\int_0^T \phi(\bar{z}(t)) \geqslant \int_0^T \phi(z(t))$ for all $z(t) \in S$, i.e., $\bar{z}(t)$ is an optimal solution of MP a.e. on $[0, T]$ and this proves the theorem.

With some suitable restrictions on the functions f, c, and g, we can easily produce an optimal solution. We do this in the next theorem.

THEOREM 6. *Let*

(i) $(\bar{z}(t), \bar{u}(t))$ *be a solution of* KTSP;

(ii) $f(0) = 0$;

(iii) $c(t) \geqslant 0\ \forall t \in [0, T]$;

(iv) $g(s, t, z(s)) \geqslant 0\ \forall s \in [0, t]$, $t \in [0, T]$, $z(s) \geqslant 0\ \forall s \in [0, T]$.

Then there exists an optimal solution $z^*(t)$ *of* MP.

Proof. By Theorem 5, $\bar{z}(t)$ is an optimal solution of MP a.e. on $[0, T]$. Define

$$z^*(t) = \begin{bmatrix} \bar{z}(t) & \text{if} & t \in A \\ 0 & \text{if} & t \in B \end{bmatrix}.$$

Then by (3a) and hypotheses (ii), (iii), and (iv), $z^*(t)$ is a feasible solution of MP. Now

$$l(z^*) = \int_0^T \phi(z^*(t))\, dt = \int_A \phi(\bar{z}(t))\, dt + \int_B \phi(0)\, dt = \int_A \phi(\bar{z}(t))\, dt$$
$$= \int_0^T \phi(\bar{z}(t))\, dt = l(\bar{z})$$

(Since $\mu(B) = 0$, $\mu(A) = \mu([0, T])$.) Hence $z^*(t)$ is an optimal solution of MP.

3. Necessary Optimality Criteria

For the rest of the paper we assume that

A1. $\phi(z(t))$ is concave in $z(t)$.

A2. $f(z(t))$ is convex in $z(t)$.

A3. $g(s, t, z(s))$ is concave in $z(s)$ and hence S is convex.

A4. $H_i(z(t)) = f_i(z(t)) - c_i(t) - \int_0^t g_i(s, t, z(s))\, ds$ is in $L^1[0, T]$ for $i = 1,\dots, m$; $z(t) \in D^n$.

A5. $X = R \times L^\infty[0, T] \times \cdots \times L^\infty[0, T]$, where $L^\infty[0, T]$ repeats m times and R is the set of all reals.

We first prove the following important

Theorem 7. *If the system of inequalities*

$$l(\bar{z}) - l(z) < 0 \qquad \textit{and} \qquad H(z(t)) \leqslant 0$$

has no solution $z(t) \in D^n$ for a fixed $\bar{z}(t)$ S, then there exists

$$(\bar{r}_0, \bar{r}(t)) \in X, \qquad (\bar{r}_0, \bar{r}(t)) \geqslant 0 \qquad \forall t \in [0, T]$$

such that

$$\bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\, \bar{r}(t)\, dt \geqslant 0 \qquad \forall z(t) \in D^n,$$

where

$$H(z(t)) = (H(z(t)),\dots, H_m(z(t)))'.$$

Proof. Consider $W_{z(t)} = \{(r, y(t)) \in X: r > l(\bar{z}) - l(z), y(t) \geqslant H(z(t))\}$ for each $z(t) \in D^n$. Let $W = \bigcup_{z(t)\in D^n} W_{z(t)}$ and $V = \{(0, \bar{0})\}$ where $\bar{0}$ is the m-dimensional zero vector. Note that $V \cap W$ is empty by the hypothesis of the theorem. The set V is trivially convex. We show that W is also convex. Let $(r_1, y_1(t)) \in W$, $(r_2, y_2(t)) \in W$. Then there exist $z_1(t) \in D^n$, $z_2(t) \in D^n$ such that $\lambda z_1(t) + (l - \lambda)\, z_2(t) \in D^n$ for $0 < \lambda < 1$ and

$$\begin{bmatrix} r_1 > l(\bar{z}) - l(z_1),\ y_1(t) \geqslant H(z_1(t)) \\ \text{and} \\ r_2 > l(\bar{z}) - l(z_2),\ y_2(t) \geqslant H(z_2(t)) \end{bmatrix}$$

implies

$$\begin{bmatrix} \lambda r_1 > \lambda(l(\bar{z}) - l(z_1)),\ \lambda y_1(t) \geqslant \lambda H(z_1(t)) \\ \text{and} \\ (1 - \lambda)\, r_2 > (1 - \lambda)\, (l(\bar{z}) - l(z_2)),\ (1 - \lambda)\, y_2(t) \geqslant (1 - \lambda)\, H(z_2(t)) \end{bmatrix}$$

implies

$$\begin{bmatrix} \lambda r_1 + (1-\lambda) r_2 > l(\bar{z}) - \lambda l(z_1) - (1-\lambda) l(z_2) \\ \text{and} \\ \lambda y_1(t) + (1-\lambda) y_2(t) \geqslant \lambda H(z_1(t)) + (1-\lambda) H(z_2(t)) \end{bmatrix}.$$

Now

$$\begin{aligned} r_1 + (1-\lambda) r_2 &> l(\bar{z}) - [\lambda l(z_1) + (1-\lambda) l(z_2)] \\ &= l(\bar{z}) - \int_0^T [\lambda\phi(z_1(t)) + (1-\lambda)\phi(z_2(t))]\,dt \\ &> l(\bar{z}) - \int_0^T \phi(\lambda z_1(t) + (1-\lambda) Z_2(t))\,dt \qquad \text{(since } \phi \text{ is concave)} \\ &= l(\bar{z}) - l(\lambda z_1 + (1-\lambda) z_2). \end{aligned}$$

Also

$$\begin{aligned} \lambda y_1(t) + (1-\lambda) y_2(t) &\geqslant \lambda f(z_1(t)) + (1-\lambda) f(z_2(t)) - c(t) \\ &\quad - \int_0^t [\lambda g(s, t, z_1(s)) + (1-\lambda) g(s, t, z_2(s))]\,ds \\ &\geqslant f(\lambda z_1(t) + (1-\lambda) z_2(t)) - c(t) \\ &\quad - \int_0^t [\lambda g(s, t, z_1(s)) + (1-\lambda) g(s, t, z_2(s))]\,ds \\ &\qquad \text{(since } f \text{ is convex)} \\ &\geqslant f(\lambda z_1(t) + (1-\lambda) z_2(t)) - c(t) \\ &\quad - \int_0^t g(s, t, \lambda z_1(s) + 1 - \lambda) z_2(s))\,ds \\ &\qquad \text{(since } g \text{ is concave in } z(t)). \end{aligned}$$

Therefore $(\lambda r_1 + (1-\lambda) r_2, \lambda y_1(t) + (1-\lambda) y_2(t)) \in W$, i.e., W is convex. Because of the way r and $y(t)$ are chosen, W has at least one interior point. Hence by [11, Theorem 1, p. 219] there exists a continuous linear functional which separates W and V. This means there exists $\bar{r}_0 \in R$, $x^0(t)$ in $L^\infty[0, T] \times \cdots \times L^\infty[0, T]$ such that

$$0 \leqslant \bar{r}_0 r + \int_0^T y'(t)\, x^0(t)\,dt \qquad \forall (r, y(t)) \in W, \tag{7}$$

where $x^0(t) = (x_1{}^0(t),..., x_m{}^0(t))'$. Taking $y'(t) = (1, 0,..., 0)$ in (7) we have

$$0 \leqslant \bar{r}_0 r + \int_0^T x_1{}^0(t)\,dt. \tag{8}$$

Now r can be chosen as large as we wish. Therefore, for a fixed value of $\int_0^T x_1^0(t)\,dt$, $\bar{r}_0{}^r + \int_0^T x_1^0(t)\,dt$ can be made negative if we choose large r and assume $\bar{r}_0$ is negative. But this will violate (8). Hence

$$\bar{r}_0 \geqslant 0. \tag{9}$$

Next, since it is possible to have $y(t) = H(z(t))$, (7) implies $0 \leqslant \bar{r}_0 r + \int_0^T H'(z(t))\,x^0(t)\,dt$ which in turn implies

$$0 \leqslant \bar{r}_0(l(\bar{z}) - l(z)) + \epsilon\bar{r}_0 + \int_0^T H'(z(t))\,x^0(t)\,dt \tag{10}$$

where $r = l(\bar{z}) - l(z) + \epsilon$. (The choice of $\epsilon > 0$ depends upon z. But for any given z, ϵ can be made arbitrarily small since r can be brought arbitrarily close to $l(\bar{z}) - l(z)$.) From (10) it follows that

$$0 \leqslant \bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\,x^0(t)\,dt. \tag{11}$$

(From (10) we have $-\epsilon\bar{r}_0 \leqslant r_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\,x^0(t)\,dt$.) If

$$\inf_{z(t)\in D^n} \left\{\bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\,x^0(t)\,dt\right\} = -\delta < 0,$$

we can choose ϵ so small that $\epsilon\bar{r}_0 < \delta$. Hence

$$\inf_{z(t)\in D^n} \left\{\bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\,x^0(t)\,dt\right\} = -\delta < -\epsilon r_0$$

and this contradicts (10). Hence

$$\inf_{z(t)\in D^n} \left\{\bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\,x^0(t)\,dt\right\} \geqslant 0$$

which means

$$\bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\,x^0(t)\,dt \geqslant 0.$$

Next we show that $x^0(t) = (x_1{}^0(t),\ldots, x_m{}^0(t)) \geqslant 0$ a.e. on $[0, T]$. Suppose $x_i{}^0(t) < 0$ a.e. on $[0, T]$ for $i = 1,\ldots, m$. Then choose

$$z(t) = \bar{z}(t), \qquad r = -\int_0^T x_i{}^0(t)\,dt > 0$$

and

$$y(t) = (0,\ldots, 0, r_0 + 1, 0,\ldots, 0)'$$

and (7) becomes

$$0 \leqslant \bar{r}_0 r + \int_0^T y'(t)\, x^0(t)\, dt$$

$$= \bar{r}_0 r + \int_0^T (\bar{r}_0 + 1)\, x_i^0(t)\, dt$$

$$= \bar{r}_0 r - \bar{r}_0 r - r = -r < 0$$

which is nonsense.

Therefore $x_i^0(t) \geqslant 0$ a.e. on $[0, T]$ for $i = 1,..., m$. For $i = 1,..., m$, let

$$\bar{A}_i = \{t \in [0, T]: x_i^0(t) \geqslant 0\}$$
$$\bar{B}_i = \{t \in [0, T]: x_i^0(t) < 0\}.$$

Note that $\mu(\bar{B}_i) = 0$. Define

$$\bar{r}_i(t) = \begin{bmatrix} x_i^0(t) & \text{if} & t \in \bar{A}_i \\ -x_i^0(t) & \text{if} & t \in \bar{B}_i \end{bmatrix} \qquad \text{for } i = 1,..., m.$$

Therefore, $\bar{r}(t) = (r_1(t),..., \bar{r}_m(t))' \geqslant 0$, $t \in [0, T]$. From (11) we have

$$0 \leqslant \bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\, x^0(t)\, dt$$

$$= \bar{r}_0(l(\bar{z}) - l(z)) + \sum_{i=0}^m \int_0^T H_i(z(t))\, x_i^0(t)\, dt$$

$$= \bar{r}_0(l(\bar{z}) - l(z)) + \sum_{i=1}^m \left[\int_{\bar{A}_i} H_i(z(t))\, x_i^0(t)\, dt + \int_{\bar{B}_i} H_i(z(t))\, x_i^0(t)\, dt\right]$$

$$= \bar{r}_0(l(\bar{z}) - l(z)) + \sum_{i=1}^m \left[\int_{\bar{A}_i} H_i(z(t))\, x_i^0(t)\, dt + \int_{\bar{B}_i} H_i(z(t))\, (-x_i^0(t))\, dt\right].$$

(This is possible because Lebesgue measure of $\bar{B}_i$ is zero and so $\int_{\bar{B}_i} k(t)\, dt = 0$ for any bounded function $k(t)$.)

$$= \bar{r}_0(l(\bar{z}) - l(z)) + \sum_{i=1}^m \left[\int_{\bar{A}_i} H_i(z(t))\, \bar{r}_i(t)\, dt + \int_{\bar{B}_i} H_i(z(t))\, \bar{r}_i(t)\, dt\right]$$

$$= \bar{r}_0(l(\bar{z}) - l(z)) + \sum_{i=1}^m \left[\int_{\bar{A}_i \cup \bar{B}_i} H_i(z(t)\, \bar{r}_i(t)\, dt\right]$$

$$= \bar{r}_0(l(\bar{z}) - l(z)) + \sum_{i=1}^m \int_0^T H_i(z(t))\, \bar{r}_i(t)\, dt$$

$$= \bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\, \bar{r}(t)$$

and this proves the theorem.

We now prove the Fritz–John saddle-point necessary optimality theorem in our setup.

THEOREM 8. *Let* $\bar{z}(t)$ *be a solution of* MP. *Then there exists* $(\bar{r}_0, \bar{r}(t)) \in X$, $(\bar{r}_0, \bar{r}(t)) \geqslant 0$ *for all* $t \in [0, T]$ *such that*

(i) $\int_0^T H'(\bar{z}(t))\, \bar{r}(t)\, dt = 0$

(ii) $(\bar{z}(t), \bar{r}_0, \bar{r}(t))$ *solves* FJSP.

Proof. Existence of $(\bar{r}_0, \bar{r}(t)) \in X$, $(\bar{r}_0, \bar{r}(t)) \geqslant 0$ $\forall t \in [0, T]$ such that $\bar{r}_0(l(\bar{z}) - l(z)) + \int_0^T H'(z(t))\, \bar{r}(t) \geqslant 0$ is shown by Theorem 7. To prove (i), we observe that $\bar{r}(t) \geqslant 0$ and $H(\bar{z}(t)) = f(\bar{z}(t)) - c(t) - \int_0^t g(s, t, \bar{z}(s))\, ds \leqslant 0$. Hence $H'(\bar{z}(t))\, \bar{r}(t) \leqslant 0$ $\forall t \in [0, T]$. This implies that

$$\int_0^T H'(\bar{z}(t))\, \bar{r}(t) \leqslant 0 \tag{12}$$

But from (11), taking $z(t) = \bar{z}(t)$,

$$\int_0^T H'(\bar{z}(t))\, \bar{r}(t)\, dt \geqslant 0. \tag{13}$$

Hence by (12) and (13), we have (i).

To establish (ii), notice that by (11) $-\bar{r}_0(l(\bar{z})) \leqslant -\bar{r}_0(l(z)) + \int_0^T H'(z(t))\, \bar{r}(t)\, dt$. Therefore by (i),

$$-\bar{r}_0(l(\bar{z})) + \int_0^T H'(\bar{z}(t))\, \bar{r}(t)\, dt \leqslant -\bar{r}_0 l(z) + \int_0^T H'(z(t))\, \bar{r}(t)\, dt. \tag{14}$$

Again, since $H(\bar{z}(t)) = f(\bar{z}(t)) - c(t) - \int_0^t g(s, t, \bar{z}(s))\, ds \leqslant 0$, for $r(t) \geqslant 0$, $r(t)$ in $L^\infty[0, T] \times \cdots \times L^\infty[0, T]$, we have $H'(\bar{z}(t))\, r(t) \leqslant 0$. This implies that $\int_0^T H'(\bar{z}(t))\, r(t) \leqslant 0$. Therefore

$$-\bar{r}_0 l(\bar{z}) + \int_0^T H'(\bar{z}(t))\, r(t)\, dt \leqslant -\bar{r}_0 l(\bar{z}) + \int_0^T H'(\bar{z}(t))\, \bar{r}(t)\, dt. \tag{15}$$

Combining (14) and (15) we have (ii).

In order to establish Kuhn–Tucker saddle-point necessary optimality theorem, we introduce Karlin's constraint qualification for our problem.

KARLIN'S CONSTRAINT QUALIFICATION. We say $H(z(t))$ satisfy Karlin's constraint qualification on $[0, t] \times [0, T] \times D^n$ if and only if there does not exist $r'(t) = (r(t), \ldots, r_m(t)) \geqslant 0$, $r(t) \not\equiv 0$ such that $\int_0^T H'(z(t))\, r(t)\, dt \geqslant 0$ for all $z(t)$ in D^n.

THEOREM 9. *If* $H(z(t))$ *satisfies Karlin's constraint qualification on* $[0, t] \times [0, T] \times D^n$ *and* $\bar{z}(t)$ *is a solution of* MP, *then* $\bar{z}(t)$ *and some* $\bar{u}(t)$ *in* $L^\infty[0, T] \times \cdots \times L^\infty[0, T]$, $\bar{u}(t) \geqslant 0$ *for all* t *in* $[0, T]$ *solve* KTSP *and* $\int_0^T \bar{u}'(t)\, H(\bar{z}(t))\, dt = 0$.

Proof. If $\bar{r}_0$ of Theorem 8 is positive, then Theorem 9 follows from Remark $*$ and Theorem 8. So we only need to show that $\bar{r}_0 > 0$. Suppose $\bar{r}_0 = 0$, then $\bar{r}(t) \geqslant 0$ for all t in $[0, T]$ and from the second inequality of FJSP

$$0 = \int_0^T H'(\bar{z}(t))\, \bar{r}(t)\, dt$$

$$\leqslant \int_0^T H'(z(t))\, \bar{r}(t)\, dt \qquad \text{for all } z(t) \in D^n.$$

This violates Karlin's constraint qualification. Hence, $r_0 > 0$ and the theorem is proven.

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