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Properties of the limit shape for some last-passage growth models in random environments

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Abstract

We study directed last-passage percolation on the planar square lattice whose weights have general distributions, or equivalently, queues in series with general service distributions. Each row of the last-passage model has its own randomly chosen weight distribution. We investigate the limiting time constant close to the boundary of the quadrant. Close to the y-axis, where the number of random distributions averaged over stays large, the limiting time constant takes the same universal form as in the homogeneous model. But close to the x-axis we see the effect of the tail of the distribution of the random environment.

MSC: 60K35; 60K37; 60K25

Keywords: Corner growth model; Random environment; Limit shape; Last-passage percolation; Queues in series

1. Introduction

This paper studies the limit shapes of some last-passage percolation models in random environments, specifically, the corner growth model and two Bernoulli models with different rules for admissible paths.

We introduce the corner growth model through its queueing interpretation. Consider service stations in series, labeled $0, 1, 2, \ldots, \ell$, each with unbounded waiting room and first-in first-out (FIFO) service discipline. Initially customers $0, 1, 2, \ldots, k$ are queued up at server 0. At time $t = 0$ customer 0 begins service with server 0. Each customer moves through the system of

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servers in order, joining the queue at server $j+1$ as soon as service with server j is complete. After customer i departs server j, server j starts serving customer $i+1$ immediately if $i+1$ has been waiting in the queue, or then waits for customer $i+1$ to arrive from station $j-1$. Customers stay ordered throughout the process. Let $X(i, j)$ be the service time that customer i needs at station j, and $T(k, \ell)$ the time when customer k completes service with server ℓ.

Asymptotics for $T(k, \ell)$ as k and ℓ get large have been investigated a great deal in the past two decades. A seminal paper by Glynn and Whitt [6] studied the case of i.i.d. $\{X(i, j)\}$. They took advantage of the connection with directed last-passage percolation given by the identity

$$T(k, \ell) = \max_{\pi} \sum_{(i,j)\in\pi} X(i, j). \tag{1.1}$$

The maximum is taken over nondecreasing nearest-neighbor lattice paths $\pi \subseteq \mathbb{Z}_+^2$ from $(0, 0)$ to (k, ℓ) that are of the form $\pi = \{(0, 0) = (x_0, y_0), (x_1, y_1), \dots, (x_{k+\ell}, y_{k+\ell}) = (k, \ell)\}$ where $(x_i, y_i) - (x_{i-1}, y_{i-1}) = (1, 0)$ or $(0, 1)$. A quick inductive proof of (1.1) together with earlier references to this observation can be found in [6] (see Proposition 2.1). This particular last-passage model is also known as the *corner growth model*.

Next we add a random environment to this queueing model. The environment is a sequence $\{F_j : j \in \mathbb{Z}_+\}$ of probability distributions, generated by a probability measure-valued ergodic or i.i.d. process with distribution $\mathbb{P}$. Given the sequence $\{F_j\}$, we assume that the variables $\{X(i, j)\}$ are independent and $X(i, j)$ has distribution F_j. In the queueing picture this means that the service times $\{X(i, j) : i \in \mathbb{Z}_+\}$ at service station j have common distribution F_j, and at the outset the distributions $\{F_j : j \in \mathbb{Z}_+\}$ themselves are chosen randomly according to some given law $\mathbb{P}$. Obviously the labels "customer" and "server" are interchangeable because we can switch around the roles of the indices i and j.

The asymptotic regime that we consider for $T(k, \ell)$ is the *hydrodynamic* one where k and ℓ are both of order n and n is taken to ∞. Under some moment assumptions, standard subadditive considerations and approximations imply the existence of the deterministic limit

$$\Psi(x, y) = \lim_{n\to\infty} n^{-1} T(\lfloor nx\rfloor, \lfloor ny\rfloor) \quad \text{for all } (x, y) \in \mathbb{R}_+^2.$$

Only in the case where the distributions F_j are exponential or geometric has it been possible to describe explicitly the limit Ψ. This is the case of $\cdot/M/1$ queues in series, which in terms of interacting particle systems is the same as studying either the totally asymmetric simple exclusion process or the zero-range process with constant jump rate. For rate 1 i.i.d. exponential $\{X(i, j)\}$ the limit $\Psi(x, y) = (\sqrt{x} + \sqrt{y})^2$ was first derived by Rost [17] in a seminal paper on hydrodynamic limits of asymmetric exclusion processes. The random environment model with exponential F_j's was studied in [1,12,20].

Let us now set aside the queueing motivation and consider the last-passage model on the first quadrant $\mathbb{Z}_+^2$ of the planar integer lattice, defined by the nondecreasing lattice paths and the random weights $\{X(i, j)\}$. For the queueing application it is natural to assume the weights nonnegative, but in the general last-passage situation there is no reason to restrict to nonnegative weights.

The ideal limit shape result would have some degree of universality, that is, apply to a broad class of distributions. Such results have been obtained only close to the boundary: in [14] Martin showed that in the i.i.d. case, under suitable moment hypotheses and as $\alpha \searrow 0$,

$$\Psi(1, \alpha) = \mu + 2\sigma\sqrt{\alpha} + o(\sqrt{\alpha}), \tag{1.2}$$

where μ and σ^2 are the common mean and variance of the weights $X(i, j)$. Also, the $o(\sqrt{\alpha})$ term in the statement means that $\lim_{\alpha\searrow 0} \alpha^{-1/2}\big[\,\Psi(1, \alpha) - \mu - 2\sigma\sqrt{\alpha}\,\big] = 0$. In the i.i.d. case Ψ is symmetric so the same holds for $\Psi(\alpha, 1)$.

Our goal is to find the form Martin's result takes in the random environment setting. Ψ is no longer necessarily symmetric since the distribution of the array $\{X(i, j)\}$ is not invariant under transposition. So we must ask the question separately for $\Psi(1, \alpha)$ and $\Psi(\alpha, 1)$.

It turns out that for $\Psi(\alpha, 1)$, where the number of rows stays large relative to the number of columns, the fluctuations of the environment average out to the degree that our result in Theorem 2.2 is essentially identical to Martin's result in the homogeneous environment. We still have $\Psi(\alpha, 1) = \mu + 2\sigma\sqrt{\alpha} + o(\sqrt{\alpha})$ as $\alpha \searrow 0$, where now μ is the mean as before but σ^2 is the average of the "quenched" variance. That is, if we let $\mu_0 = \int x\, dF_0(x)$ and $\sigma_0^2 = \int (x - \mu_0)^2\, dF_0(x)$ denote the mean and variance of the random distribution F_0, and $\mathbb{E}$ expectation under $\mathbb{P}$, then $\mu = \mathbb{E}(\mu_0)$ and $\sigma^2 = \mathbb{E}(\sigma_0^2)$.

The case $\Psi(1, \alpha)$ does not possess a clean result such as the one above. Even though we are studying the deterministic limit obtained *after n* has been taken to infinity, we see an effect from the tail of the distribution of the quenched mean μ_0. We illustrate this with the case of exponential $\{F_j\}$. Now the number $n\alpha$ of distributions F_j is small compared to the number n of weights $X(i, j)$ in each row; hence the fluctuations among the F_j's become prominent. The effect comes in two forms: first, the leading term is no longer the averaged mean μ but the maximal mean. Second, if large values among the row means μ_j are rare, the order of the α-dependent correction is smaller than the $\sqrt{\alpha}$ seen above and this order of magnitude depends on the tail of the distribution of μ_0. As an exponent characterizing this tail changes, we can see a phase transition of sorts in the power of α, with a logarithmic correction at the transition point.

For general distributions we derive bounds on $\Psi(1, \alpha)$ that indicate that in the case of finitely many distributions the correction is of order $\sqrt{\alpha}$.

As auxiliary results we need bounds on the limits for last-passage models with Bernoulli weights under a random environment. However, with Bernoulli weights the standard corner growth model is *not* one of the explicitly solvable cases. The model with Bernoulli weights does become solvable when the path geometry is altered suitably. The model that we take up is the one where the paths are weakly increasing in one coordinate but strictly in the other. There are two cases, depending on which coordinate is required to increase strictly. If we require the x-coordinate to increase strictly then an admissible path $\{(x_0, y_0), (x_1, y_1), \ldots, (x_m, y_m)\}$ satisfies

$$x_{i+1} - x_i = 1 \quad \text{and} \quad y_0 \le y_1 \le \cdots \le y_m. \tag{1.3}$$

The other case interchanges x and y. These cases have to be addressed separately because the random environment attached to rows makes the model asymmetric. The sum of these two last-passage values gives a bound for the case where neither coordinate is required to increase strictly in each step.

We derive the exact limit constants for Bernoulli models with both kinds of strict/weak paths. For one of them this has been done before by Gravner et al. [9]. Their proof utilizes the fact that the distribution of $T(k, \ell)$ is a symmetric function of the environment (at least for the particular Bernoulli case that they study). Our proof is completely different. It is based on the idea in [19] where the limit for the homogeneous case was derived: the last-passage model is coupled with a particle system whose invariant distributions can be written down explicitly, and then through some convex analysis the speed of a tagged particle yields the explicit limit of the last-passage model.

Further remarks on the literature. The present paper does not address questions of fluctuations, but let us mention some highlights from the literature. For the last-passage model with i.i.d. exponential or geometric weights, the distributional limit with fluctuations of order $n^{1/3}$ and limit given by the Tracy–Widom GUE distribution was proved by Johansson [10]. As for the shape, universality has been achieved only close to the boundary, by Baik and Suidan [2] and Bodineau and Martin [3].

Fluctuations of the Bernoulli model with strict/weak paths and homogeneous weights were derived first in [11] and later also in [7]. For the model in a random environment, fluctuation limits appear in [9,8].

On the lattice $\mathbb{Z}_+^2$ we can imagine three kinds of nondecreasing paths: (i) weak–weak: both coordinates required to increase weakly, the type used in (1.1); (ii) strict–weak: one coordinate increases strictly, as above in (1.3); and (iii) strict–strict: both coordinates increase strictly, so an admissible path $\{(x_0, y_0), (x_1, y_1), \ldots, (x_m, y_m)\}$ satisfies $x_0 < \cdots < x_m$ and $y_0 < \cdots < y_m$. As mentioned, with Bernoulli weights the strict–weak case is solvable but the weak–weak case appears harder. The third case, strict–strict, is also solvable with Bernoulli weights. The shape was derived in [18] and recent work on this model appears in [5].

Organization of the paper. The main results on the shape close to the boundary are in Section 2 and the results for Bernoulli models in Section 3. Section 4 sketches the proof of the existence of the limiting shape, a result that we basically take for granted. The main proofs follow: in Section 5 for Theorem 2.2 on $\Psi(\alpha, 1)$, in Section 6 for Theorem 2.3 on $\Psi(1, \alpha)$, and in Section 7 for Theorem 2.4 for the exponential model.

Some frequently used notation. We write

$$\operatorname*{ess\,sup}_{\mathbb{P}} f = \inf\{s \in \mathbb{R} : \mathbb{P}(f > s) = 0\}$$

for the essential supremum of a function f under a measure $\mathbb{P}$. $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{R}_+ = [0, \infty)$. $I(A)$ is the indicator function of event A.

2. Main results

First a precise definition of the last-passage model in a random environment. Let $\mathbb{P}$ be a stationary, ergodic probability measure on the space $\mathcal{M}_1(\mathbb{R})^{\mathbb{Z}_+}$ of sequences of Borel probability distributions on $\mathbb{R}$. $\mathbb{E}$ denotes expectation under $\mathbb{P}$. For some of the main results, $\mathbb{P}$ will be assumed to be an i.i.d. product measure. A realization of the distribution-valued process under $\mathbb{P}$ is denoted by $\{F_j\}_{j\in\mathbb{Z}_+}$. This is the environment. Given $\{F_j\}$, the weights $\{X(z) : z \in \mathbb{Z}_+^2\}$ are independent real-valued random variables with marginal distributions $X(i, j) \sim F_j$ for $(i, j) \in \mathbb{Z}_+^2$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space on which all variables $\{F_j, X(i, j)\}$ are defined, and denote expectation under $\mathbf{P}$ by $\mathbf{E}$.

A (weakly) nondecreasing path is a sequence of points $z_0 = (x_0, y_0), z_1 = (x_1, y_1), \ldots, z_m = (x_m, y_m)$ in $\mathbb{Z}_+^2$ that satisfy $x_0 \le x_1 \le \cdots \le x_m$, $y_0 \le y_1 \le \cdots \le y_m$, and $|x_{i+1} - x_i| + |y_{i+1} - y_i| = 1$. For $z_1, z_2 \in \mathbb{Z}_+^2$ with $z_1 \le z_2$ (coordinatewise ordering), let $\Pi(z_1, z_2)$ be the set of nondecreasing paths from z_1 to z_2. Whether the endpoints z_1 and z_2 are included in the path makes no difference to the limit results below. The last-passage time $T(z_1, z_2)$ from z_1 to z_2 is defined by

$$T(z_1, z_2) = \max_{\pi\in\Pi(z_1,z_2)} \sum_{z\in\pi} X(z).$$

When $z_1 = 0$, use the abbreviations $\Pi(z) = \Pi(0, z)$ and $T(z) = T(0, z)$.

Impose these three assumptions on the model:

$$\mathbf{E}|X(z)| < \infty, \tag{2.1}$$

$$\int_0^\infty \Big\{1 - \mathbb{E}(F_0(x))\Big\}^{1/2} dx < \infty, \tag{2.2}$$

and

$$\int_0^\infty \operatorname*{ess\,sup}_{\mathbb{P}} (1 - F_0(x))\, dx < \infty. \tag{2.3}$$

We begin with this by now standard result that defines our object of study, namely the function Ψ. The proof is briefly commented on in Section 4.

Proposition 2.1. *Assume $\mathbb{P}$ is ergodic and satisfies (2.1)–(2.3). Then for all $(x, y) \in (0, \infty)^2$ the last-passage time constant*

$$\Psi(x, y) = \lim_{n\to\infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) \tag{2.4}$$

exists as a limit both $\mathbf{P}$-almost surely and in $L^1(\mathbf{P})$. Furthermore, $\Psi(x, y)$ is a homogeneous, concave and continuous function on $(0, \infty)^2$.

Assumption (2.2) is also used for the constant distribution case; see (2.5) in [14]. Some further control along the lines of assumption (2.3) is required for our case. For example, suppose $1 - F_j(x) = e^{-\xi_j x}$ for random $\xi_j \in (0, \infty)$. Then (2.3) holds iff $\operatorname{ess\,inf}_{\mathbb{P}}(\xi_0) > 0$. If the distribution of ξ_0 is not bounded away from zero, $n^{-1}T(n, n) \to \infty$ because we can simply collect all the weights from the row with minimal ξ_j among $\{\xi_0, \ldots, \xi_n\}$. However, assumption (2.2) can be satisfied without bounding ξ_0 away from zero.

We state the main results of the paper on the form of the limit shape at the boundary. As explained in the introduction, for $\Psi(\alpha, 1)$ we find a universal form as $\alpha \searrow 0$. In addition to the earlier assumptions, we need similar control of the left tail of the distributions:

$$\int_{-\infty}^0 \big(\mathbb{E}[F_0(x)]\big)^{1/2} dx < \infty \tag{2.5}$$

and

$$\int_{-\infty}^0 \operatorname*{ess\,sup}_{\mathbb{P}}\ F_0(x)\, dx < \infty. \tag{2.6}$$

Let us point out that (2.2) and (2.5) together guarantee $\mathbf{E}|X(z)|^2 < \infty$. Let $\mu_j = \mu(F_j)$ and $\sigma_j^2 = \sigma^2(F_j)$ denote the mean and variance of distribution F_j. These are random variables under $\mathbb{P}$ with expectations $\mu = \mathbb{E}(\mu_0)$ and $\sigma^2 = \mathbb{E}(\sigma_0^2)$.

Theorem 2.2. *Assume the process $\{F_j\}$ is i.i.d. under $\mathbb{P}$, and satisfies tail assumptions (2.2), (2.3), (2.5) and (2.6). Then, as $\alpha \downarrow 0$, $\Psi(\alpha, 1) = \mu + 2\sigma\sqrt{\alpha} + o(\sqrt{\alpha})$.*

Assumptions (2.2) and (2.5) are direct counterparts of what was used for Theorem 2.4 in [14]. Assumptions (2.3) and (2.6) are additional assumptions needed for handling the random environment. These assumptions are used to control estimates that come from bounding limits of Bernoulli models.

We turn to the case $\Psi(1,\alpha)$. The results will be qualitatively different from Theorem 2.2. The leading term will be the essential supremum of the mean instead of the averaged mean and we will see different orders for the first α-dependent correction term.

First a general result for which we restrict ourselves to the case of finitely many distributions, but we can relax the i.i.d. assumption of the random distributions.

Theorem 2.3. *Assume the process* $\{F_j\}$ *of probability distributions is stationary, ergodic, and has a state space of finitely many distributions* $H_1, \dots, H_L$ *each of which satisfies Martin's* [14] *hypothesis*

$$\int_0^\infty (1 - H_\ell(x))^{1/2}\,dx + \int_{-\infty}^0 H_\ell(x)^{1/2}\,dx < \infty. \tag{2.7}$$

Let $\mu^* = \max_\ell \mu(H_\ell)$ *be the maximal mean of the* H_ℓ*'s. Then there exist constants* $0 < c_1 < c_2 < \infty$ *such that, as* $\alpha \downarrow 0$,

$$\mu^* + c_1\sqrt{\alpha} + o(\sqrt{\alpha}) \le \Psi(1,\alpha) \le \mu^* + c_2\sqrt{\alpha} + o(\sqrt{\alpha}). \tag{2.8}$$

We would expect $c_1 = c_2$ but our proof does not give it.

Finally, we consider the case $\Psi(1,\alpha)$ for the exponential model where some (partially) explicit calculation is possible. Here we see how the tail of the random mean μ_0 creates different orders of magnitude for the α-dependent correction term. Let $\{\xi_j\}_{j\in\mathbb{Z}_+}$ be an i.i.d. sequence of random variables that satisfy $0 < c \le \xi_j < \infty$ with common distribution m. To distinguish the exponential model from the general one we write $G_j(x) = 1 - e^{-\xi_j x}$ for the distribution function of the exponential distribution with parameter ξ_j, and Ψ_G for the limiting time constant. We assume that c is the exact lower bound: $m[c, c+\varepsilon) > 0$ for each $\varepsilon > 0$. Then the essential supremum of the random mean is $\mu^* = c^{-1}$.

An implicit description of the limit shape was derived in [20] by way of studying an exclusion process with random jump rates attached to particles. We recall the result here. One explicit shape is needed for the proof of Theorem 2.2 also, so this result will serve there too.

Define first a critical value $u^* = \int_{[c,\infty)} \frac{c}{\xi - c}\, m(d\xi) \in (0,\infty]$. For $0 \le u < u^*$ define $a = a(u)$ implicitly by

$$u = \int_{[c,\infty)} \frac{a}{\xi - a}\, m(d\xi).$$

$a(u)$ is strictly increasing, strictly concave, continuously differentiable and one-to-one from $0 < u < u^*$ onto $0 < a < c$. We let $a(u) = c$ for $u \ge u^*$ if $u^* < \infty$. Then define $g : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$g(y) = \sup_{u\ge 0}\{-yu + a(u)\}, \qquad y \ge 0. \tag{2.9}$$

The function g is monotone decreasing, continuous, and $g(y) = 0$ for $y \ge a'(0+) = 1/\mu_G$. It is the level curve of the time constant. The equations connecting the two are $g(y) = \inf\{x > 0 : \Psi_G(x,y) \ge 1\}$ and

$$\Psi_G(x,y) = \inf\{t \ge 0 : t g(y/t) \ge x\}. \tag{2.10}$$

Qualitative properties of the limit shape depend on the tail of the distribution m at $c+$, and transitions occur where the integrals $\int_{[c,\infty)} (\xi - c)^{-2}\, m(d\xi)$ and $\int_{[c,\infty)} (\xi - c)^{-1}\, m(d\xi)$

blow up. (For details see [20].) These same regimes appear in our results below. For the case $\int_{[c,\infty)}(\xi - c)^{-2}\, m(d\xi) = \infty$ we make a precise assumption about the tail of the distribution of the random rate:

$$\exists \nu \in [-1, 1], \kappa > 0 \quad \text{such that } \lim_{\xi \searrow c} \frac{m[c, \xi)}{(\xi - c)^{\nu+1}} = \kappa. \tag{2.11}$$

The value $\nu = -1$ means that the bottom rate c has probability $m\{c\} = \kappa > 0$. Values $\nu < -1$ are of course not possible.

Theorem 2.4. *For the model with exponential distributions with i.i.d. random rates the limit Ψ_G has these asymptotics close to the x-axis.*

Case 1*: $\int_{[c,\infty)}(\xi - c)^{-2}\, m(d\xi) < \infty$. Then there exists $\alpha_0 > 0$ such that*

$$\Psi_G(1, \alpha) = c^{-1} + \alpha \int_{[c,\infty)} \frac{1}{\xi - c}\, m(d\xi) \quad \textit{for} \quad \alpha \in [0, \alpha_0]. \tag{2.12}$$

Case 2*: (2.11) holds and so, in particular, $\int_{[c,\infty)}(\xi - c)^{-2}\, m(d\xi) = \infty$. Then as $\alpha \searrow 0$,*

$$\textit{if } \nu \in (0, 1] \quad \textit{then } \Psi_G(1, \alpha) = c^{-1} + \alpha \int_{[c,\infty)} \frac{1}{\xi - c}\, m(d\xi) + o(\alpha); \tag{2.13}$$

$$\textit{if } \nu = 0 \quad \textit{then } \Psi_G(1, \alpha) = c^{-1} - \kappa\alpha \log \alpha + o(\alpha \log \alpha); \tag{2.14}$$

$$\textit{if } \nu \in [-1, 0) \quad \textit{then } \Psi_G(1, \alpha) = c^{-1} + B\alpha^{\frac{1}{1-\nu}} + o\big(\alpha^{\frac{1}{1-\nu}}\big). \tag{2.15}$$

In statement (2.15) above, $B = B(c, \kappa, \nu)$ is a constant whose explicit definition is in Eq. (7.7) in the proof section below. The extreme case $\nu = -1$ is the one that matches up with Theorem 2.3.

For some heuristic understanding of Theorem 2.4 we turn to the queueing interpretation discussed in the Introduction. Quantity $n\Psi(1, \alpha)$ represents the time when customer n departs from server $\lfloor n\alpha \rfloor$ (rigorously speaking in the $n \to \infty$ limit), when initially all customers are queued up at server 0. When $u^* < \infty$ (Case 1 and subcase $\nu > 0$ from Case 2) the results of [1] suggest that, at some distance but not too far from the first queue, the system should converge to an equilibrium where the queue length at server j (whose service rate is ξ_j) is geometric with mean $c/(\xi_j - c)$ and the departure process from each queue has rate c. A customer arriving at a queue with this geometric number of customers present spends on average time $1/(\xi_j - c)$ at that queue. In Case 1 this picture is precise enough that equation (2.12) can be naively understood in these terms: a single customer travels through $\lfloor n\alpha \rfloor$ servers in time $n\alpha \int_{[c,\infty)} \frac{1}{\xi - c}\, m(d\xi)$. After this it takes another n/c time to see n customers go through server $\lfloor n\alpha \rfloor$. Together these terms make up the right-hand side of (2.12). This argument requires a high density of customers because once the customer density drops below u^* the system chooses an equilibrium with flow rate below c. (Again, for details we refer the reader to [1].)

This point can be detected in the proofs for Case 2 where we find a parameter a_0 that in some sense represents a flow rate and replaces c in (2.12) (see Eq. (7.2)). The formulas in Case 2 are then obtained by estimating $c - a_0$.

3. Bernoulli models with strict–weak paths in a random environment

This section looks at last-passage models with Bernoulli-distributed weights. The environment is now an i.i.d. sequence $\{p_j\}_{j \in \mathbb{Z}_+}$ of numbers $p_j \in [0, 1]$, with distribution $\mathbb{P}$. Given $\{p_j\}$,

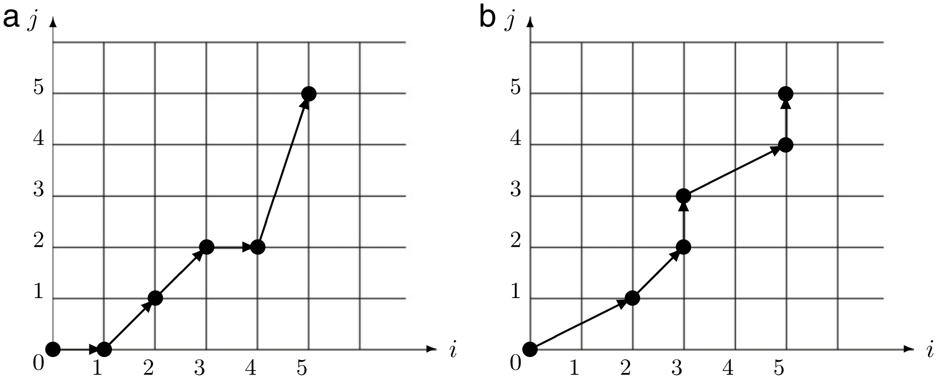

Fig. 1. Admissible paths in $\Pi_{\rightarrow}(z_1, z_2)$ and $\Pi_{\uparrow}(z_1, z_2)$.

the weights $\{X(i, j)\}$ are independent with marginal distributions $P(X(i, j) = 1) = p_j = 1 - P(X(i, j) = 0)$. We consider two last-passage times that differ by the type of admissible path: for $z_1, z_2 \in \mathbb{Z}_+^2$,

$$T_{\rightarrow}(z_1, z_2) = \max_{\pi \in \Pi_{\rightarrow}(z_1,z_2)} \sum_{z\in\pi} X(z) \quad \text{and} \quad T_{\uparrow}(z_1, z_2) = \max_{\pi \in \Pi_{\uparrow}(z_1,z_2)} \sum_{z\in\pi} X(z). \tag{3.1}$$

In terms of coordinates denote the endpoints by $z_k = (a_k, b_k)$, $k = 1, 2$. Then admissible paths $\pi \in \Pi_{\rightarrow}(z_1, z_2)$ are of the form $\pi = \{(a_1, y_0), (a_1 + 1, y_1), (a_1 + 2, y_2), \ldots, (a_2, y_{a_2-a_1})\}$ with $b_1 \le y_0 \le y_1 \le \cdots \le y_{a_2-a_1} \le b_2$, while paths $\pi \in \Pi_{\uparrow}(z_1, z_2)$ are of the form $\pi = \{(x_0, b_1), (x_1, b_1 + 1), \ldots, (x_{b_2-b_1}, b_2)\}$ with $a_1 \le x_0 \le x_1 \le \cdots \le x_{b_2-b_1} \le a_2$. Thus paths in $\Pi_{\rightarrow}(z_1, z_2)$ increase strictly in the x-direction while those in $\Pi_{\uparrow}(z_1, z_2)$ increase strictly in the y-direction. The last-passage times $T_{\rightarrow}(z_1, z_2)$ and $T_{\uparrow}(z_1, z_2)$ record the maximal weights of such paths in the lattice rectangle $([a_1, a_2] \times [b_1, b_2]) \cap \mathbb{Z}_+^2$. The diagrams in Fig. 1 illustrate the two kinds of admissible paths when we take $z_1 = (0, 0)$ and $z_2 = (5, 5)$.

As before we simplify notation with $T_{\rightarrow}(0, z) = T_{\rightarrow}(z)$. The almost sure limits are denoted by

$$\Psi_{\rightarrow}(x, y) = \lim_{n\to\infty} \frac{1}{n} T_{\rightarrow}(\lfloor nx \rfloor, \lfloor ny \rfloor) \quad \text{and} \quad \Psi_{\uparrow}(x, y) = \lim_{n\to\infty} \frac{1}{n} T_{\uparrow}(\lfloor nx \rfloor, \lfloor ny \rfloor) \tag{3.2}$$

for $(x, y) \in (0, \infty)^2$. The existence of the limits needs no further comment.

The next theorem gives the explicit limits. (3.3) is the same as in [9, Thm. 1]. Inside the $\mathbb{E}(\cdots)$ expectations below, p is the random Bernoulli probability. Let $b = \operatorname{ess\,sup}_{\mathbb{P}} p$ denote the maximal probability.

Theorem 3.1. *The limits in (3.2) are as follows for $x, y \in (0, \infty)$:*

$$\Psi_{\rightarrow}(x, y) = \begin{cases} bx + y(1-b)\mathbb{E}\left(\dfrac{p}{b-p}\right), & x/y \ge \mathbb{E}\left(\dfrac{p(1-p)}{(b-p)^2}\right) \\ yz_0^2\mathbb{E}\left(\dfrac{1-p}{(z_0-p)^2}\right) - y, & \mathbb{E}\left(\dfrac{p}{1-p}\right) < x/y < \mathbb{E}\left(\dfrac{p(1-p)}{(b-p)^2}\right) \\ x, & 0 < x/y \le \mathbb{E}\left(\dfrac{p}{1-p}\right) \end{cases} \tag{3.3}$$

and with $z_0 \in (b, 1)$ *uniquely defined by the equation*

$$x/y = \mathbb{E}\left[\frac{p(1-p)}{(z_0-p)^2}\right].$$

$$\Psi_{\uparrow}(x,y) = \begin{cases} y - yz_0^2\mathbb{E}\left(\dfrac{1-p}{(z_0+p)^2}\right), & 0 < x/y < \mathbb{E}\left(\dfrac{1-p}{p}\right) \\ y, & x/y \geq \mathbb{E}\left(\dfrac{1-p}{p}\right) \end{cases} \tag{3.4}$$

with $z_0 \in (0, \infty)$ *uniquely defined by the equation*

$$x/y = \mathbb{E}\left[\frac{p(1-p)}{(z_0+p)^2}\right].$$

Our second result gives simplified bounds that are useful for the proof of the main result, Theorem 2.2. Let $\bar{p} = \mathbb{E}(p)$ be the mean of the environment. $\Psi(x, y)$ is the limiting time constant with weakly increasing paths defined in Proposition 2.1.

Theorem 3.2. *The following three inequalities hold for the Bernoulli model:*

$$\Psi_{\rightarrow}(x,y) \leq bx + 2\sqrt{\bar{p}(1-b)xy}, \tag{3.5}$$

$$\Psi_{\uparrow}(x,y) \leq \bar{p}y + 2\sqrt{\bar{p}(1-\bar{p})xy} \tag{3.6}$$

and

$$\Psi(x,y) \leq \bar{p}y + 4\sqrt{\bar{p}(1-\bar{p})xy} + bx. \tag{3.7}$$

(3.7) follows from (3.5) and (3.6) because $\Psi(x, y) \leq \Psi_{\rightarrow}(x, y) + \Psi_{\uparrow}(x, y)$. Another loose estimate that we will use later following (3.7) is

$$\Psi(x,y) \leq \bar{p}y + 4\sqrt{\bar{p}(1-\bar{p})xy} + bx \leq (y + 4\sqrt{xy})\sqrt{\bar{p}} + bx. \tag{3.8}$$

We prove the formulas and inequalities first for $\Psi_{\rightarrow}$ and then for $\Psi_{\uparrow}$. For some parts of the proofs it is convenient to assume $b < 1$. Results for the case $b = 1$ follow by taking a limit.

Proof of (3.3) and (3.5). We adapt the proof from [19] to the random environment situation and sketch the main points.

Consider now the environment $\{p_j\}$ fixed, but the weights $X(i, j)$ random. For integers $0 \leq s < t$ and a, k, define an inverse to the last-passage time as

$$\Gamma((a,s),k,t) = \min\{l \in \mathbb{Z}_+ : T_{\rightarrow}((a+1,s+1),(a+l,t)) \geq k\}.$$

Note that $\Gamma((a, s), 0, t) = 0$ but $\Gamma((a, s), k, t) > 0$ for $k > 0$. Knowing the limits of the variables Γ is the same as knowing $\Psi_{\rightarrow}$. By the homogeneity of $\Psi_{\rightarrow}$ it is enough to find $h(x) = \Psi_{\rightarrow}(x, 1)$. By the homogeneity and superadditivity of $\Psi_{\rightarrow}$, h is concave and nondecreasing. Let g be the inverse function of h on $\mathbb{R}_+$. Then g is convex and nondecreasing, and

$$tg(x/t) = \lim_{n\to\infty} \frac{1}{n}\Gamma((0,0),\lfloor nx\rfloor,\lfloor nt\rfloor).$$

To find these functions we construct an exclusion-type process $z(t) = \{z_k(t) : k \in \mathbb{Z}\}$ of labeled, ordered particles $z_k(t) < z_{k+1}(t)$ that jump leftward on the lattice $\mathbb{Z}$, in discrete time $t \in \mathbb{Z}_+$. Given an initial configuration $\{z_i(0)\}$ that satisfies $z_{i-1}(0) \leq z_i(0) - 1$ and

$\liminf_{i\to-\infty} |i|^{-1} z_i(0) > -1/b$, the evolution is defined by

$$z_k(t) = \inf_{i:i\le k} \{z_i(0) + \Gamma((z_i(0), 0), k-i, t)\}, \quad k \in \mathbb{Z},\ t \in \mathbb{N}. \tag{3.9}$$

It can be checked that $z(t)$ is a well-defined Markov process, and in particular that $z_k(t) > -\infty$ almost surely.

Define the process $\{\eta_i(t)\}$ of interparticle distances by $\eta_i(t) = z_{i+1}(t) - z_i(t)$ for $i \in \mathbb{Z}$ and $t \in \mathbb{Z}_+$. By Prop. 1 in [19] process $\{\eta_i(t)\}$ has a family of i.i.d. geometric invariant distributions indexed by the mean $u \in [1, b^{-1})$ and defined by

$$P(\eta_i = n) = u^{-1}(1-u^{-1})^{n-1}, \quad n \in \mathbb{N}. \tag{3.10}$$

Let $x_k(t) = z_k(t-1) - z_k(t) \ge 0$ be the absolute size of the jump of the kth particle from time $t-1$ to t, and let $q_t = 1 - p_t$. From (6.5) in [19], in the stationary process,

$$P(x_k(t) = x) = \begin{cases} (1-up_t)q_t^{-1} & x = 0 \\ p_t(1-up_t)q_t^{-1}(u-1)^x(uq_t)^{-x} & x = 1, 2, 3, \ldots. \end{cases} \tag{3.11}$$

We track the motion of particle $z_0(t)$ in a stationary situation. The initial state is defined by setting $z_0(0) = 0$ and by letting $\{\eta_i(0)\}$ be i.i.d. with common distribution (3.10). With $k = 0$, divide by t in (3.9) and take $t \to \infty$. Apply laws of large numbers inside the braces in (3.9), with some simple estimation to pass to the limit through the infimum, to find the average speed of the tagged particle:

$$-\lim_{t\to\infty} \frac{1}{t} z_0(t) = \sup_{x\ge 0}\{ux - g(x)\} \equiv f(u). \tag{3.12}$$

The last equality defines the speed f as $f = g^+$, the *monotone conjugate* of g. It is natural to set $f(u) = 0$ for $u \in [0, 1)$, $f(b^{-1}) = f((b^{-1})-)$, and $f(u) = \infty$ for $u > b^{-1}$. By [16, Thm. 12.4],

$$g(x) = \sup_{u\ge 0}\{xu - g^+(u)\} = \sup_{1\le u\le 1/b}\{xu - f(u)\}. \tag{3.13}$$

Since $z_0(t)$ is a sum of jumps $x_0(k)$ with distribution (3.11), we have the second moment bound $\sup_{t\in\mathbb{N}} \mathbf{E}[(t^{-1}z_0(t))^2] < \infty$, and consequently the limit in (3.12) holds also in expectation. From this,

$$\begin{aligned} f(u) &= -\lim_{t\to\infty} \mathbf{E}[t^{-1}z_0(t)] = \lim_{t\to\infty} \mathbf{E}\left[t^{-1} \sum_{k=1}^{t} x_0(k) \right] = \mathbf{E}[x_0(0)] \\ &= \mathbb{E} \sum_{x=1}^{\infty} x(u-1)^x(uq)^{-x} p(1-up)(1-p)^{-1} = \mathbb{E}\left[\frac{pu(u-1)}{1-up} \right]. \end{aligned} \tag{3.14}$$

We find $g(x)$ from (3.13) and (3.14):

$$g(x) = \begin{cases} x/b - b^{-1}(1-b)\mathbb{E}\dfrac{p}{(b-p)} & x \ge b^2\mathbb{E}\dfrac{(1-p)}{(b-p)^2} - 1 \\ u_0^2\mathbb{E}\dfrac{p(1-p)}{(1-u_0p)^2} & \mathbb{E}\dfrac{p}{1-p} < x < b^2\mathbb{E}\dfrac{(1-p)}{(b-p)^2} - 1 \\ x & 0 < x < \mathbb{E}\dfrac{p}{1-p} \end{cases} \tag{3.15}$$

where $u_0 \in (1, b^{-1})$ is uniquely defined by the equation $x + 1 = \mathbb{E}(1 - p)(1 - u_0 p)^{-2}$. From this we find the inverse function $h(x) = g^{-1}(x)$ and then $\Psi_{\rightarrow}(x, y) = yh(x/y)$. We omit these details and consider (3.3) proved.

To prove (3.5) we return to the duality (3.13) and write

$$g(x) \geq \sup_{1 \leq u < 1/b} \{xu - \tilde{f}(u)\} \quad \text{for} \quad \tilde{f}(u) = \frac{u(u-1)}{1 - ub}\bar{p}. \tag{3.16}$$

$\tilde{f}'(u) = x$ is solved by $u^* = b^{-1}\left(1 - \sqrt{\frac{(1-b)\bar{p}}{bx + \bar{p}}}\right)$.

When $x \geq \frac{\bar{p}}{1-b}$, we have $u^* \in \left[1, \frac{1}{b}\right)$, and then

$$g(x) \geq xu^* - \tilde{f}(u^*) = \frac{1}{b^2}\left(\sqrt{(1-b)\bar{p}} - \sqrt{bx + \bar{p}}\right)^2.$$

Consequently

$$g^{-1}(x) \leq \frac{1}{b}\left(\sqrt{b^2 x} + \sqrt{(1-b)\bar{p}}\right)^2 - \frac{\bar{p}}{b} = bx - \bar{p} + 2\sqrt{(1-b)\bar{p}x}.$$

When $x < \frac{\bar{p}}{1-b}$, the supremum in (3.16) is attained at $u = 1$, and in this case

$$g^{-1}(x) \leq x \leq bx + 2\sqrt{(1-b)\bar{p}x}.$$

The bound (3.5) now follows from $\Psi_{\rightarrow}(x, y) = yg^{-1}(x/y)$. □

Proof of (3.4) and (3.6). The scheme is the same, so we omit some more details. The inverse of the last-passage time is now defined:

$$\Gamma((a, s), k, t) = \min\{l \in \mathbb{Z}_+ : T_{\uparrow}((a, s + 1), (a + l, t)) \geq k\}.$$

Vertical distance $t - s$ allows for at most $t - s$ marked points, so the above quantity must be set equal to ∞ for $k > t - s$. The particle process $\{z(t) : t \in \mathbb{Z}_+\}$ is defined by the same formula (3.9) as before but it is qualitatively different. The particles still jump to the left, but the ordering rule is now $z_k(t) \leq z_{k+1}(t)$ so particles are allowed to sit on top of each other. Well-definedness of the dynamics needs no further restrictions on admissible particle configurations because the minimum in (3.9) only considers $i \in \{k - t, \ldots, k\}$ so it is well-defined for all initial configurations $\{z_i(0) : i \in \mathbb{Z}\}$ such that $z_i(0) \leq z_{i+1}(0)$.

The following can be checked. Under a fixed environment $\{p_j\}$, the gap process $\{\eta_i(t) = z_{i+1}(t) - z_i(t) : i \in \mathbb{Z}\}$ has i.i.d. geometric invariant distributions $P(\eta_k = n) = \left(\frac{1}{1+u}\right)\left(\frac{u}{1+u}\right)^n$, $n \in \mathbb{Z}_+$, indexed by the mean $u \in \mathbb{R}_+$. In this stationary situation the successive jumps $x_k(t) = z_k(t - 1) - z_k(t)$ of a tagged particle have distribution

$$P(x_k(t) = y) = \begin{cases} \dfrac{1}{1 + up_t} & y = 0 \\ \left(\dfrac{u}{u+1}\right)^y \dfrac{p_t}{1 + up_t} & y \geq 1. \end{cases}$$

From here the analysis proceeds in the same way as for the other model. The speed function is defined by

$$f(u) = -\lim_{n\to\infty} \mathbf{E}\left[n^{-1} z_0(n)\right] = \mathbf{E}[x_0(0)] = u(u + 1)\mathbb{E}\left[\frac{p}{1 + up}\right]$$

and then convex analysis takes over. We omit the remaining details of the proof of (3.4).

To prove (3.6), note that

$$
\begin{aligned}
g(x) &= \sup_{u\geq 0}\{xu - f(u)\} \geq \sup_{u\geq 0}\left\{xu - \frac{\bar{p}u(u+1)}{1+u\bar{p}}\right\}\\
&= \begin{cases} \frac{1}{\bar{p}}(\sqrt{1-x}-\sqrt{1-\bar{p}})^2 & \bar{p}\leq x\leq 1\\ 0 & 0\leq x\leq \bar{p}.\end{cases}
\end{aligned}
$$

We used Jensen's inequality and concavity of $p \mapsto \frac{p}{1+up}$. From this

$$
g^{-1}(x) \leq \begin{cases} \bar{p}-\bar{p}x+2\sqrt{\bar{p}(1-\bar{p})x} & 0\leq x\leq \dfrac{1-\bar{p}}{\bar{p}}\\ 1 & x > \dfrac{1-\bar{p}}{\bar{p}}\end{cases}
$$

and (3.6) follows. □

4. Proof of Proposition 2.1

We comment briefly on the proof of Proposition 2.1. Further details can be found in [13]. The flow of arguments is standard. First one takes an integer point $(x, y) \in \mathbb{Z}_+^2$ and applies Liggett's version of the subadditive ergodic theorem to the process $Z_{m,n} = -T((mx, my), (nx, ny))$, $0 \leq m < n$, to prove that $\Psi(x, y)$ exists and is finite. Then rational (x, y) and real (x, y) are handled by approximations. Along the way, regularity properties of Ψ are established and used: superadditivity, homogeneity, concavity and continuity.

All this works easily for the Bernoulli case because last-passage times are uniformly bounded in terms of path length. Consequently we can assume that Proposition 2.1 has been proved for the Bernoulli case. For the general case we check that for integer points $(x, y) \in \mathbb{Z}_+^2$ the moment hypotheses of the subadditive ergodic theorem [4, p. 358] follow from our assumptions (2.1)–(2.3):

$$
\mathbf{E}Z_{0,1}^+ \leq \mathbf{E}|T((0,0),(x,y))| \leq \mathbf{E}\sum_{0\leq i\leq x, 0\leq j\leq y}|X(i,j)| = (x+1)(y+1)\mathbf{E}|X(0,0)| < \infty.
$$

Next,

$$
\begin{aligned}
\frac{1}{n}\mathbf{E}Z_{0,n} &\geq -\frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(nx,ny)}\sum_{z\in\pi}X(z)_+ = -\frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(nx,ny)}\sum_{z\in\pi}\int_0^\infty I(X(z)>u)\,du\\
&\geq -\frac{1}{n}\mathbf{E}\int_0^\infty \max_{\pi\in\Pi(nx,ny)}\sum_{z\in\pi}I(X(z)>u)\,du\\
&= -\frac{1}{n}\int_0^\infty \mathbf{E}\max_{\pi\in\Pi(nx,ny)}\sum_{z\in\pi}I(X(z)>u)\,du\\
&\geq -\int_0^\infty \sup_n \frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(nx,ny)}\sum_{z\in\pi}I(X(z)>u)\,du = -\int_0^\infty \Psi_{Ber[1-F(u)]}(x,y)\,du\\
&\geq -(y+4\sqrt{xy})\int_0^\infty\sqrt{1-\mathbb{E}F_0(u)}\,du - x\int_0^\infty\big(1-\operatorname*{ess\,inf}_{\mathbb{P}} F_0(u)\big)du.
\end{aligned}
$$

$I(A)$ is the indicator function of event A. $\Psi_{Ber[1-F(u)]}(x, y)$ is the limiting time constant for the Bernoulli model where the weights have distributions $P(X(i, j) = 1) = 1 - F_j(u) = 1 - P(X(i, j) = 0)$. On the last line above we used the Bernoulli estimate (3.8). By assumptions (2.2) and (2.3), $\mathbf{E}Z_{0,n} \geq n\gamma$ for a constant $\gamma > -\infty$. These estimates justify the application of the subadditive ergodic theorem. We omit the remaining details and consider Proposition 2.1 proved.

5. Proof of Theorem 2.2

For the first lemma, let $\{F_j\}$ and $\{G_j\}$ be ergodic sequences of distributions defined on a common probability space under probability measure $\mathbb{P}$. In a later step of the proof we need to assume $\{F_j\}$ i.i.d. Assume that both processes $\{F_j\}$ and $\{G_j\}$ satisfy the assumptions made in Theorem 2.2. With some abuse of notation we label the time constants, means, and even random weights associated with the processes $\{F_j\}$ and $\{G_j\}$ with subscripts F and G. So for example $\mu_F = \mathbb{E}\left(\int x\, dF_0(x)\right)$. The symbolic subscripts F and G should not be confused with the random distributions F_j and G_j assigned to the rows of the lattice. We write $\Psi_{Ber([G(x)-F(x)]_+)}$ for the limit of a Bernoulli model with weight distributions $P(X(i,j)=1) = (G_j(x) - F_j(x))_+ = 1 - P(X(i,j)=0)$ where x is a fixed parameter. An analogous convention will be used for other Bernoulli models along the way.

Lemma 5.1. *Assume $\{F_j\}$ and $\{G_j\}$ satisfy (2.2), (2.3), (2.5) and (2.6). Then for $\alpha > 0$,*

$$\begin{aligned}
&|\Psi_F(\alpha,1) - \Psi_G(\alpha,1) - (\mu_F - \mu_G)| \\
&\quad \le 8\sqrt{\alpha}\int_{-\infty}^{+\infty}\Big(\mathbb{E}|G_0(x) - F_0(x)|\Big)^{1/2} dx + \alpha\int_{-\infty}^{+\infty}\operatorname*{ess\,sup}_{\mathbb{P}}|F_0(x) - G_0(x)|\,dx.
\end{aligned} \tag{5.1}$$

Proof. The right-hand side of (5.1) is finite under the assumptions on $\{F_j\}$ and $\{G_j\}$. Couple the F_j- and G_j-distributed weights in a standard way. Let $\{u(z) : z = (i,j) \in \mathbb{Z}_+^2\}$ be i.i.d. Uniform(0, 1) random variables. Set $X_F(z) = F_j^{-1}(u(z))$, where $F_j^{-1}(u) = \sup\{x : F_j(x) < u\}$, and similarly $X_G(z) = G_j^{-1}(u(z))$. Write $\mathbf{E}$ for expectation over the entire probability space of distributions and weights.

$$\begin{aligned}
&\Psi_F(\alpha,1) - \Psi_G(\alpha,1) \\
&= \lim_{n\to\infty}\frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)}\sum_{z\in\pi}X_F(z) - \lim_{n\to\infty}\frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)}\sum_{z\in\pi}X_G(z) \\
&\le \varlimsup_{n\to\infty}\frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)}\sum_{z\in\pi}\big(X_F(z) - X_G(z)\big) \\
&= \varlimsup_{n\to\infty}\frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)}\sum_{z\in\pi}\int_{-\infty}^{+\infty}\Big\{I\big(X_G(z)\le x < X_F(z)\big) - I\big(X_F(z)\le x < X_G(z)\big)\Big\}dx \\
&\le \varlimsup_{n\to\infty}\frac{1}{n}\mathbf{E}\int_{-\infty}^{+\infty}\max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)}\sum_{z\in\pi}\Big\{I\big(X_G(z)\le x < X_F(z)\big) - I\big(X_F(z)\le x < X_G(z)\big)\Big\}dx.
\end{aligned}$$

We check that Fubini's theorem allows us to interchange the integral and the expectation. Since F and G are interchangeable it is enough to consider the first indicator function from above. Let a be an integer $\ge \alpha$.

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)}\sum_{z\in\pi}I\big(X_G(z)\le x < X_F(z)\big)dx \\
&\le \int_{-\infty}^{+\infty}\sup_n\frac{1}{n}\mathbf{E}\max_{\pi\in\Pi(an,n)}\sum_{z\in\pi}I\big(X_G(z)\le x < X_F(z)\big)\,dx \\
&= \int_{-\infty}^{+\infty}\Psi_{Ber([G(x)-F(x)]_+)}(a,1)\,dx \\
&\le \int_{-\infty}^{+\infty}\Big(\mathbb{E}|G_0(x) - F_0(x)| + 4\sqrt{a}\big(\mathbb{E}|G_0(x) - F_0(x)|\big)^{1/2} + a\operatorname*{ess\,sup}_{\mathbb{P}}|G_0(x) - F_0(x)|\Big)\,dx \\
&< \infty
\end{aligned}$$

by estimate (3.7) and the finiteness of the right-hand side of (5.1). Continue from the limit above by applying Fubini's theorem. Then take the limit inside the dx-integral by dominated convergence, justified by the n-uniformity in the bound above. Finally apply again the Bernoulli estimate (3.7).

$$\begin{aligned}
&\Psi_F(\alpha,1) - \Psi_G(\alpha,1) \\
&\le \overline{\lim_{n\to\infty}} \int_{-\infty}^{+\infty} \frac{1}{n}\mathbf{E} \max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)} \sum_{z\in\pi} \Big\{ I\big(X_G(z)\le x < X_F(z)\big) - I\big(X_F(z)\le x < X_G(z)\big)\Big\}\, dx \\
&\le \int_{-\infty}^{+\infty} \lim_{n\to\infty} \frac{1}{n}\Big\{\mathbf{E} \max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)} \sum_{z\in\pi} I\big(X_G(z)\le x < X_F(z)\big) \\
&\quad + \mathbf{E} \max_{\pi\in\Pi(\lfloor\alpha n\rfloor,n)} \sum_{z\in\pi}\big(1 - I\big(X_F(z)\le x < X_G(z)\big)\big) - \sum_{z\in\pi} 1\Big\}\, dx \\
&= \int_{-\infty}^{+\infty} \big\{ \Psi_{Ber([G(x)-F(x)]_+)}(\alpha,1) + \Psi_{Ber(1-[F(x)-G(x)]_+)}(\alpha,1) - (1+\alpha)\big\} dx \\
&\le \int_{-\infty}^{+\infty} \Big\{ \mathbb{E}\big(G_0(x)-F_0(x)\big)_+ + 1 - \mathbb{E}\big(F_0(x)-G_0(x)\big)_+ \\
&\quad + 4\sqrt{\alpha}\Big(\sqrt{\mathbb{E}\big(G_0(x)-F_0(x)\big)_+} + \sqrt{\mathbb{E}\big(F_0(x)-G_0(x)\big)_+}\Big) \\
&\quad + \alpha\Big(\operatorname*{ess\,sup}_{\mathbb{P}}[G_0(x)-F_0(x)]_+ + 1 - \operatorname*{ess\,inf}_{\mathbb{P}}[F_0(x)-G_0(x)]_+\Big) - (1+\alpha)\Big\} dx \\
&\le (\mu_F - \mu_G) + 8\sqrt{\alpha}\int_{-\infty}^{+\infty} \sqrt{\mathbb{E}|F_0(x)-G_0(x)|}\, dx + \alpha \int_{-\infty}^{+\infty} \operatorname*{ess\,sup}_{\mathbb{P}} |G_0(x)-F_0(x)|\, dx.
\end{aligned}$$

Interchanging F and G gives the bound from the other direction and concludes the proof. □

For a while we make two convenient assumptions: that the weights are uniformly bounded, so for a constant $M < \infty$,

$$\mathbb{P}\{F_0(-M) = 0 \text{ and } F_0(M) = 1\} = 1, \tag{5.2}$$

and that variances are uniformly bounded away from zero, so for a constant $0 < c_0 < \infty$,

$$\mathbb{P}\{\sigma^2(F_0) \ge c_0\} = 1. \tag{5.3}$$

Note that then

$$c_0 \le \sigma^2(F_0) \le M^2 \quad \mathbb{P}\text{-a.s.} \tag{5.4}$$

and the conditions assumed for Theorem 2.2 are trivially satisfied by the uniform boundedness.

Henceforth $r = r(\alpha)$ denotes a positive integer-valued function such that $r(\alpha) \nearrow \infty$ as $\alpha \searrow 0$. Tile the lattice with $1\times r$ blocks $B_r(x,y) = \{(x, ry+k) : k = 0,1,\ldots,r-1\}$ for $(x,y)\in\mathbb{Z}_+^2$. A coarse-grained last-passage model is defined by adding up the weights in each block:

$$X_r(z) = \sum_{v\in B_r(z)} X(v).$$

The distribution of the new weight $X_r(i,j)$ on row $j\in\mathbb{Z}_+$ of the rescaled lattice is the convolution $F_{r,j} = F_{rj} * F_{rj+1} * \cdots * F_{rj+r-1}$.

We repeat Lemma 4.4 from [14] with a sketch of the argument.

Lemma 5.2. *Let $\Psi_F(x,y)$ and $\Psi_{F_r}(x,y)$ be the last-passage time functions obtained by using F_j and $F_{r,j}$ as the distributions on the jth row, respectively. If $r\to\infty$ and $r\sqrt{\alpha}\to 0$ as $\alpha\downarrow 0$, then*

$$\lim_{\alpha\downarrow 0} \frac{1}{\sqrt{\alpha}} \left| \Psi_F(\alpha,1) - \frac{1}{r}\Psi_{F_r}(\alpha r,1)\right| = 0.$$

Proof. Given a path $\pi \in \Pi(m, nr-1)$, consider all the blocks that it intersects; this gives a path $\tilde{\pi} \in \Pi(m, n-1)$ in the rescaled lattice such that $\left|(\cup_{z\in\tilde{\pi}} B_r(z))\Delta\pi\right| \le mr$. Then by (5.2)

$$\left| \max_{\pi\in\Pi(m,nr)} \sum_{z\in\pi} X(z) - \max_{\tilde{\pi}\in\Pi(m,n)} \sum_{z\in\tilde{\pi}} X_r(z) \right| \le mrM.$$

Take $m = \lfloor \alpha nr \rfloor$, divide through by nr, and the conclusion follows. □

Let $\mu_{r,y}$ and $V_{r,y}$ be the mean and variance of $F_{r,y}$:

$$\mu_{r,y} = \sum_{i=0}^{r-1} \mu_{ry+i}, \quad \text{and} \quad V_{r,y} = \sum_{i=0}^{r-1} \sigma^2_{ry+i}.$$

Let $\Phi_{r,y}$ be the distribution function of the normal $\mathcal{N}(\mu_{r,y}, V_{r,y})$ distribution, and $\widetilde{\Phi}_{r,y}$ the distribution function of $\mathcal{N}(r\mu_F, V_{r,y})$. The difference between $\Phi_{r,y}$ and $\widetilde{\Phi}_{r,y}$ is that the latter has a nonrandom mean. We shall also find it convenient to use $\{X_j\}$ as a sequence of independent variables with (random) distributions $X_j \sim F_j$. For the next lemma we need to assume $\{F_j\}$ an i.i.d. sequence under $\mathbb{P}$.

As in [14], a key step in the proof is the replacement of the rescaled weights with Gaussian weights, which is undertaken in the next lemma.

Lemma 5.3. *Assume $\{F_j\}$ i.i.d. under $\mathbb{P}$. If $r \to \infty$ and $r\sqrt{\alpha} \to 0$ as $\alpha \downarrow 0$, then*

$$\lim_{\alpha\downarrow 0} \frac{1}{r\sqrt{\alpha}} |\Psi_{F_r}(\alpha r, 1) - \Psi_{\Phi_r}(\alpha r, 1)| = 0. \tag{5.5}$$

Proof. According to Theorem 5.17 of [15], independent mean 0 random variables $X_1, X_2, X_3, \ldots$ satisfy the estimate

$$\left| P\left\{ B_r^{-1/2} \sum_{i=1}^{r} X_i \le x \right\} - \Phi(x) \right| \le A \frac{\sum_{i=1}^{r} E|X_i|^3}{B_r^{3/2}} (1+|x|)^{-3}, \quad x \in \mathbb{R},$$

where $B_r = \sum_{i=1}^{r} \mathrm{Var}(X_i)$, Φ is the standard normal distribution function, and A is a constant that is independent of the distribution functions of $X_1, X_2, \ldots, X_r$. Then,

$$\begin{aligned} |F_{r,y}(x) - \Phi_{r,y}(x)| &\le A \frac{\sum_{i=0}^{r-1} E|X_{ry+i} - \mu_{ry+i}|^3}{\left(\sum_{i=0}^{r-1} \sigma^2_{ry+i}\right)^{3/2}} \left(1 + V_{r,y}^{-1/2}|x-\mu_{r,y}|\right)^{-3} \\ &\le \frac{C}{\sqrt{r}} \left(1 + M^{-1} r^{-1/2} |x - \mu_{r,y}|\right)^{-3} \end{aligned} \tag{5.6}$$

where the second inequality used the assumptions $P(|X_i| \le M) = 1$ and $\sigma_i^2 \ge c_0^2 > 0$.

Armed with (5.6) we now estimate the right-hand side of (5.1) for the processes $\{F_{r,y}\}_{y\in\mathbb{Z}_+}$ and $\{\Phi_{r,y}\}_{y\in\mathbb{Z}_+}$ and with α replaced by αr.

For the first term on the right in (5.1), note this Schwarz trick: for a probability density f on $\mathbb{R}$ and a function $H \ge 0$,

$$\int \sqrt{H}\, dx = \int f^{1/2} \sqrt{f^{-1} H}\, dx \le \left(\int f^{-1} H\, dx \right)^{1/2}.$$

For the calculation below take $\delta > 0$ and $f(x) = c_1(1+|x - r\mu_F|^{1+\delta})^{-1}$ for the right constant $c_1 = c_1(\delta)$. Factors that depend on M and δ are subsumed in a constant C. Then

$$
\begin{aligned}
&\sqrt{\alpha r}\int_{-\infty}^{+\infty}\Big(\mathbb{E}|F_{r,0}(x)-\Phi_{r,0}(x)|\Big)^{1/2}dx\\
&\le C\alpha^{1/2}r^{1/4}\int_{-\infty}^{+\infty}\Big\{\mathbb{E}\Big[\big(1+M^{-1}r^{-1/2}|x-\mu_{r,0}|\big)^{-3}\Big]\Big\}^{1/2}dx\\
&\le C\alpha^{1/2}r^{1/4}\left\{\mathbb{E}\int_{-\infty}^{+\infty}\big(1+|x-r\mu_F|^{1+\delta}\big)\left(1+\frac{|x-\mu_{r,0}|}{M\sqrt{r}}\right)^{-3}dx\right\}^{1/2}
\end{aligned}
$$

by a change of variables $x=\mu_{r,0}+yM\sqrt{r}$

$$
\begin{aligned}
&=C\alpha^{1/2}r^{1/2}\left\{\mathbb{E}\int_{-\infty}^{+\infty}\frac{1+|\mu_{r,0}-r\mu_F+yM\sqrt{r}|^{1+\delta}}{(1+|y|)^3}dy\right\}^{1/2}\\
&\le C\alpha^{1/2}r^{1/2}\Big\{\mathbb{E}|\mu_{r,0}-r\mu_F|^{1+\delta}+r^{(1+\delta)/2}\Big\}^{1/2}\le C\alpha^{1/2}r^{(3+\delta)/4}.
\end{aligned}
$$

In the last step we used $\mathbb{E}|\mu_{r,0}-r\mu_F|^{1+\delta}\le Cr^{(1+\delta)/2}$ which follows because $\mu_{r,0}-r\mu_F$ is a sum of r bounded mean zero i.i.d. random variables.

For the second term on the right in (5.1),

$$
\begin{aligned}
&\alpha r\int_{-\infty}^{+\infty}\operatorname*{ess\,sup}_{\mathbb{P}}|F_{r,0}(x)-\Phi_{r,0}(x)|\,dx\le C\alpha r^{1/2}\int_{-\infty}^{+\infty}\operatorname*{ess\,sup}_{\mathbb{P}}\left(1+\frac{|x-\mu_{r,y}|}{M\sqrt{r}}\right)^{-3}dx\\
&\le C\alpha r^{1/2}\left\{\int_{-\infty}^{-rM}\left(1+\frac{-rM-x}{M\sqrt{r}}\right)^{-3}dx+\int_{-rM}^{rM}dx+\int_{rM}^{+\infty}\left(1+\frac{x-rM}{M\sqrt{r}}\right)^{-3}dx\right\}\\
&\le C\alpha r^{3/2}.
\end{aligned}
$$

To summarize, with these estimates and (5.1) we have

$$
\frac{1}{r\sqrt{\alpha}}|\Psi_{F_r}(\alpha r,1)-\Psi_{\Phi_r}(\alpha r,1)|\le\frac{C}{r\sqrt{\alpha}}(\alpha^{1/2}r^{(3+\delta)/4}+\alpha r^{3/2}).
$$

If δ is fixed small enough, assumptions $r\to\infty$ and $r\sqrt{\alpha}\to 0$ make this vanish as $\alpha\to 0$. □

The next lemma makes a further approximation that puts us in the situation where all sites have normal variables with the same mean.

Lemma 5.4. *Let Ψ_{Φ_r} and $\Psi_{\widetilde{\Phi}_r}$ be defined as before, and again $r\sqrt{\alpha}\to 0$ as $\alpha\to 0$. Then*

$$
\lim_{\alpha\downarrow 0}\frac{1}{r\sqrt{\alpha}}|\Psi_{\Phi_r}(\alpha r,1)-\Psi_{\widetilde{\Phi}_r}(\alpha r,1)|=0.
$$

Proof. For $z=(i,j)\in\mathbb{Z}_+^2$, let $X^{(r)}(z)$ have distribution $\Phi_{r,j}$ so that $\widetilde{X}^{(r)}(z)=X^{(r)}(z)-\mu_{r,j}+r\mu_F$ has distribution $\widetilde{\Phi}_{r,j}$. Now estimate

$$
\begin{aligned}
\Psi_{\widetilde{\Phi}_r}(\alpha r,1)&=\lim_{n\to\infty}\frac{1}{n}\max_{\pi\in\Pi(\lfloor\alpha nr\rfloor,n)}\sum_{z\in\pi}\widetilde{X}^{(r)}(z)\\
&\le\lim_{n\to\infty}\frac{1}{n}\max_{\pi\in\Pi(\lfloor\alpha nr\rfloor,n)}\sum_{z\in\pi}X^{(r)}(z)+\lim_{n\to\infty}\frac{1}{n}\max_{\pi\in\Pi(\lfloor\alpha nr\rfloor,n)}\sum_{z\in\pi}\big(-\mu_{r,j}+r\mu_F\big)\\
&\le\Psi_{\Phi_r}(\alpha r,1)+\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\big(-\mu_{r,j}+r\mu_F\big)+\lim_{n\to\infty}\frac{1}{n}2Mr\cdot\lfloor\alpha nr\rfloor\\
&=\Psi_{\Phi_r}(\alpha r,1)+2M\alpha r^2.
\end{aligned}
$$

The opposite bound $\Psi_{\widetilde{\Phi}_r}(\alpha r,1)\ge\Psi_{\Phi_r}(\alpha r,1)-2M\alpha r^2$ comes similarly, and the lemma follows. □

Let us separate the mean by letting $\overline{\Phi}_{r,y}$ denote the $N\left(0, \sum_{i=0}^{r-1}\sigma^2_{ry+i}\right)$ distribution function. Since the last-passage functions of the normal distributions satisfy $\Psi_{\widetilde{\Phi}_r}(\alpha r, 1) = r\mu_F(1+\alpha r) + \Psi_{\overline{\Phi}^{(r)}}(\alpha r, 1)$, we can summarize the effect of the last three lemmas as follows.

Lemma 5.5. *Assume $\{F_j\}$ i.i.d. under $\mathbb{P}$, and assume that $r = r(\alpha)$ satisfies $r \to \infty$ and $r\sqrt{\alpha} \to 0$ as $\alpha \downarrow 0$. Under assumptions* (5.2) and (5.3)

$$\lim_{\alpha\downarrow 0} \frac{1}{\sqrt{\alpha}} \left| \Psi_F(\alpha, 1) - \mu_F - \frac{1}{r}\Psi_{\overline{\Phi}^{(r)}}(\alpha r, 1)\right| = 0. \tag{5.7}$$

In order to deduce a limit from (5.7) we utilize the explicitly computable case of exponential distributions from [20]. We need to match up the random variances of the exponentials with the variances σ_j^2 of the sequence $\{F_j\}$. Thus, given the i.i.d. sequence of quenched variances $\sigma_j^2 = \sigma^2(F_j)$ that we have worked with up to now under condition (5.4), let $\xi_j = 1/\sigma_j$ and $G_j(x) = 1 - e^{-\xi_j x}$, the rate ξ_j exponential distribution. Then $\{\xi_j\}_{j\in\mathbb{Z}_+}$ is an i.i.d. sequence of bounded random variables $0 < c \le \xi_j \le b$ with distribution m. We can assume that c is the exact lower bound: $m[c, c+\varepsilon) > 0$ for each $\varepsilon > 0$. G_j has mean and variance $\mu(G_j) = \xi_j^{-1}$ and $\sigma^2(G_j) = \xi_j^{-2} = \sigma_j^2$.

Assumptions (2.2) and (2.3) are easily checked, and so the last-passage function Ψ_G is well-defined. We would like to apply Lemma 5.5 to this exponential model, but obviously assumption (5.2) is not satisfied. To get around this difficulty we make the following approximation which leaves the quenched means and variances intact. We learned this trick from [14].

Let Y_j denote a G_j-distributed random variable. For a fixed $\tau > 0$, let

$$m_j = E(Y_j|Y_j > \tau) \quad \text{and} \quad w_j = E(Y_j^2|Y_j > \tau).$$

The quantities

$$s_j = \frac{(m_j-\tau)^2}{(m_j-\tau)^2 + w_j - m_j^2} \quad \text{and} \quad u_j = \frac{w_j - \tau^2}{m_j - \tau} - \tau$$

satisfy the equations

$$(1-s_j)\tau + s_j u_j = m_j \quad \text{and} \quad (1-s_j)\tau^2 + s_j u_j^2 = w_j.$$

Then $0 \le s_j \le 1$, $u_j \ge \tau$ and $w_j \ge \tau^2$. Define distribution functions

$$\widetilde{G}_j(x) = \begin{cases} G_j(x) & 0 \le x < \tau \\ 1 - s_j[1 - G_j(\tau)] & \tau \le x < u_j \\ 1 & x \ge u_j. \end{cases} \tag{5.8}$$

$\widetilde{Y}_j \sim \widetilde{G}_j$ satisfies $EY_j = E\widetilde{Y}_j$ and $EY_j^2 = E\widetilde{Y}_j^2$. Moreover, for any fixed $\tau > 0$,

$$u_j = \frac{E(Y_j^2|Y_j > \tau) - \tau^2}{E(Y_j|Y_j > \tau) - \tau} - \tau = \frac{2}{p_j} + \tau \le \frac{2}{c} + \tau,$$

so the distributions $\{\widetilde{G}_j\}$ are all supported on the nonrandom bounded interval $[0, 2/c + \tau]$. Consequently Lemma 5.5 applies to $\widetilde{G}$. We can draw the same conclusion for G once we have the next estimate:

Lemma 5.6. *Given $\varepsilon > 0$, we can select τ large enough and define $\widetilde{G}_j$ as in* (5.8) *so that*

$$\overline{\lim_{\alpha\downarrow 0}} \frac{1}{\sqrt{\alpha}} |\Psi_G(\alpha, 1) - \Psi_{\widetilde{G}}(\alpha, 1)| < \varepsilon.$$

Proof. This comes from an application of Lemma 5.1. $G_j = \widetilde{G}_j$ on $(-\infty, \tau)$ and $1 - \widetilde{G}_j \le 1 - G_j$ on all of $\mathbb{R}$. The integrals on the right-hand side of (5.1) are finite and can be made arbitrarily small by choosing τ large. □

Currently we have shown that

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} \left| \Psi_G(\alpha, 1) - \mathbb{E}\sigma_0 - \frac{1}{r} \Psi_{\overline{\Phi}_r}(\alpha r, 1) \right| = 0. \tag{5.9}$$

It remains to perform an explicit calculation on $\Psi_G(\alpha, 1)$. As before, utilize the notation $\mu_G = \mathbb{E}\xi_0^{-1}$ and $\sigma_G^2 = \mathbb{E}\xi_0^{-2}$.

Lemma 5.7. *For random exponential distributions with rates bounded away from zero,*

$$\Psi_G(\alpha, 1) = \mu_G - 2\sigma_G\sqrt{\alpha} + O(\alpha).$$

Proof. Recall the definition of the limit shape $\Psi_G(\alpha, 1)$ from (2.10). From (2.9) one can read that $tg(1/t)$ is nondecreasing in t. Thus by (2.10) $\Psi_G(\alpha, 1) = t = t(\alpha)$ such that $tg(1/t) = \alpha$.

Next we argue that when α is close enough to 0, $g(1/t) = -u_0/t + a(u_0)$ for some $0 < u_0 < u^*$ with $a'(u_0) = 1/t$. Since $a(0) = 0$ and $a(u^*-) = c$, strict concavity gives for $0 < u < u^*$

$$\left\{ \int_{[c,\infty)} \frac{\xi}{(\xi - c)^2} m(d\xi) \right\}^{-1} = a'(u^*-) < a'(u) = \left\{ \int_{[c,\infty)} \frac{\xi}{(\xi - a(u))^2} m(d\xi) \right\}^{-1}$$
$$< a'(0+) = \left\{ \int_{[c,\infty)} \xi^{-1} m(d\xi) \right\}^{-1} = \frac{1}{\mu_G}.$$

On the other hand, $0 < \Psi_G(\alpha, 1) - \mu_G \le C\sqrt{\alpha} + C\alpha$ where the second inequality comes from comparing $\{G_j\}$ in (5.1) with identically zero weights. Thus when α is small enough, $1/t$ is in the range of a'. Consequently there exists $u_0 \in (0, u^*)$ such that $a'(u_0) = 1/t$, or equivalently,

$$\int_{[c,\infty)} \frac{\xi}{(\xi - a(u_0))^2} m(d\xi) = t. \tag{5.10}$$

From the choice of t, $\alpha = tg(1/t) = t\big(-u_0/t + a(u_0)\big) = -u_0 + ta(u_0)$ and so

$$\Psi_G(\alpha, 1) = t = \frac{\alpha}{a(u_0)} + \frac{u_0}{a(u_0)} = \frac{\alpha}{a(u_0)} + \int_{[c,\infty)} \frac{1}{\xi - a(u_0)} dm(\xi). \tag{5.11}$$

Combining (5.10) and (5.11) gives

$$\alpha = a(u_0)^2 \int_{[c,\infty)} \frac{1}{(\xi - a(u_0))^2} m(d\xi). \tag{5.12}$$

From this,

$$a(u_0)^2 \sigma_G^2 = a(u_0)^2 \int_{[c,\infty)} \frac{1}{\xi^2} m(d\xi) \le \alpha.$$

Hence we have $0 \le a(u_0) \le \sqrt{\alpha}/\sigma_G$.

When α and, hence, $a(u_0)$ are small, (5.12) and the last bound on $a(u_0)$ yield

$$0 \le \frac{\alpha}{a(u_0)^2} - \sigma_G^2 = \int_{[c,\infty)} \left[\frac{1}{(\xi - a(u_0))^2} - \frac{1}{\xi^2} \right] m(d\xi)$$
$$= \int_{[c,\infty)} \frac{2\xi a(u_0) - a(u_0)^2}{\xi^2 (\xi - a(u_0))^2} m(d\xi)$$
$$\le 2 \int_{[c,\infty)} \frac{a(u_0)}{\xi (c - a(u_0))^2} m(d\xi) = O(\sqrt{\alpha}). \tag{5.13}$$

Consequently

$$\frac{\sqrt{\alpha}}{a(u_0)} - \sigma_G = \frac{\frac{\alpha}{a(u_0)^2} - \sigma_G^2}{\frac{\sqrt{\alpha}}{a(u_0)} + \sigma_G} = O(\sqrt{\alpha}).$$

Now we put all the above together to prove the lemma.

$$\begin{aligned}
&\Psi_G(\alpha, 1) - \mu_G - 2\sigma_G\sqrt{\alpha} \\
&= \frac{\alpha}{a(u_0)} + \int_{[c,\infty)} \frac{1}{\xi - a(u_0)} m(d\xi) - \mu_G - 2\sigma_G\sqrt{\alpha} \\
&= \frac{\alpha}{a(u_0)} + \int_{[c,\infty)} \left[\frac{1}{\xi} + \frac{1}{\xi^2} a(u_0) + O\big(a(u_0)^2\big)\right] m(d\xi) - \mu_G - 2\sigma_G\sqrt{\alpha} \\
&= \sqrt{\alpha}\left(\frac{\sqrt{\alpha}}{a(u_0)} - \sigma_G\right) + \sigma_G a(u_0)\left(\sigma_G - \frac{\sqrt{\alpha}}{a(u_0)}\right) + \alpha \cdot O\left(\frac{a(u_0)^2}{\alpha}\right) \\
&= O(\alpha) \quad \text{as } \alpha \downarrow 0. \quad \square
\end{aligned}$$

Combining Lemma 5.7 and (5.9) gives

$$\lim_{\alpha\downarrow 0} \frac{1}{\sqrt{\alpha}} \left| \frac{1}{r} \Psi_{\overline{\Phi}_r}(\alpha r, 1) - 2\sigma_G\sqrt{\alpha} \right| = 0.$$

Substitute this back into (5.7) and recall that $\sigma_F = \sigma_G$. The conclusion that we get is

$$\lim_{\alpha\downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi_F(\alpha, 1) - \mu_F - 2\sigma_F\sqrt{\alpha}| = 0. \tag{5.14}$$

We have proved Theorem 2.2 under assumptions (5.2) and (5.3). We now lift (5.3). For $\varepsilon > 0$, let $\{W(z)\}$ be i.i.d. weights with distribution H defined by $P(W(z) = \pm\varepsilon) = 1/2$. Let $\widetilde{F}_j = F_j * H$ be the distribution of the weight $\widetilde{X}(i, j) = X(i, j) + W(i, j)$. Let Ψ_H and $\Psi_{\widetilde{F}}$ be the time constants of the last-passage models with weights $\{W(z)\}$ and $\{\widetilde{X}(z)\}$, respectively. The Bernoulli bound (3.7) gives the estimate $\Psi_H(x, y) \le 4\varepsilon\sqrt{xy}$. The corresponding last-passage times satisfy

$$T_{\widetilde{F}}(z) - T_H(z) \le T_F(z) \le T_{\widetilde{F}}(z) + \hat{T}_H(z)$$

where $\hat{T}_H(z)$ uses the weights $-W(z)$. In the limit,

$$\Psi_{\widetilde{F}}(\alpha, 1) - 4\varepsilon\sqrt{\alpha} \le \Psi_F(\alpha, 1) \le \Psi_{\widetilde{F}}(\alpha, 1) + 4\varepsilon\sqrt{\alpha}. \tag{5.15}$$

Since $\sigma^2(\widetilde{F}_j) = \sigma^2(F_j) + \varepsilon^2$ while $\mu_{\widetilde{F}} = \mu_F$, and $\varepsilon > 0$ can be arbitrarily small, this estimate suffices for limit (5.14).

As the last item of the proof of Theorem 2.2 we remove the uniform boundedness assumption (5.2). Suppose $\{F_j\}$ satisfy the conditions required for Theorem 2.2, but there is no common bounded support. For a fixed $M > 0$ define the truncated distributions

$$F_{j,M}(x) = \begin{cases} 1 & x \ge M \\ F_j(x) & -M \le x < M \\ 0 & x < -M. \end{cases}$$

Let μ_M, σ_M^2 and $\Psi_{F_M}(x, y)$ be quantities associated with $\{F_{j,M}\}$.

From (5.1) and the conditions assumed in Theorem 2.2,

$$\frac{1}{\sqrt{\alpha}}|\Psi_F(\alpha,1)-\Psi_{F_M}(\alpha,1)-(\mu-\mu_M)|$$
$$\le 8\int_{-\infty}^{+\infty}\Big(\mathbb{E}|F_0(x)-F_{0,M}(x)|\Big)^{1/2}dx+\sqrt{\alpha}\int_{-\infty}^{+\infty}\operatorname*{ess\,sup}_{\mathbb{P}}|F_0(x)-F_{0,M}(x)|\,dx$$
$$=8\left[\int_{-\infty}^{-M}\Big(\mathbb{E}|F_0(x)|\Big)^{1/2}dx+\int_M^{\infty}\Big(\mathbb{E}|1-F_0(x)|\Big)^{1/2}dx\right]$$
$$+\sqrt{\alpha}\left[\int_{-\infty}^{-M}\operatorname*{ess\,sup}_{\mathbb{P}}|F_0(x)|\,dx+\int_M^{+\infty}\operatorname*{ess\,sup}_{\mathbb{P}}|1-F_0(x)|\,dx\right]\le\varepsilon.$$

The last inequality comes from choosing M large enough, and is valid for all $\alpha\le 1$. Since $\mathbb{E}(EX^2(0,0))<\infty$, dominated convergence gives $\sigma_M\to\sigma$ and so we can pick M such that $|\sigma-\sigma_M|<\varepsilon$. Now

$$\frac{1}{\sqrt{\alpha}}|\Psi_F(\alpha,1)-\mu-2\sigma\sqrt{\alpha}|\le\frac{1}{\sqrt{\alpha}}|\Psi_{F_M}(\alpha,1)-\mu_M-2\sigma_M\sqrt{\alpha}|+2\varepsilon.$$

Since ε is arbitrary and limit (5.14) holds for $\{F_{j,M}\}$, we get the conclusion for the sequence $\{F_j\}$. This concludes the proof of Theorem 2.2.

6. Proof of Theorem 2.3

Proof of Theorem 2.3. The lower bound in (2.8) can be proved by applying Martin's result (1.2) to the homogeneous problem where a maximal path is constructed by using only those rows j where $F_j=H_{i^*}$, the distribution with the maximal mean $\mu^*=\mu(H_{i^*})$. This is fairly straightforward and we leave the details to the reader.

To prove the upper bound in (2.8), we start by increasing all the weights $X(z)$ by moving their means to μ^*. Then we subtract the common mean μ^* from the weights, so that for the proof we can assume that all distributions $H_1,\dots,H_L$ have *mean zero*.

Create the following coupling. Independently of the process $\{F_j\}$, let $\{X_\ell(z):1\le\ell\le L,\ z\in\mathbb{Z}_+^2\}$ be a collection of independent weights such that $X_\ell(z)$ has distribution H_ℓ. Then define the weights used for computing $\Psi(1,\alpha)$ by

$$X(z)=\sum_{\ell=1}^{L}X_\ell(z)I_{\{F_j=H_\ell\}}\quad\text{for}\quad z=(i,j)\in\mathbb{Z}_+^2.$$

Begin with this elementary bound:

$$\Psi(1,\alpha)=\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\left[\max_{\pi\in\Pi(n,\lfloor\alpha n\rfloor)}\sum_{z\in\pi}X(z)\right]$$
$$\le\sum_{\ell=1}^{L}\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\left[\max_{\pi\in\Pi(n,\lfloor\alpha n\rfloor)}\sum_{z\in\pi}X_\ell(z)I_{\{F_j=H_\ell\}}\right].\tag{6.1}$$

The next lemma contains a convexity argument that will remove the indicators from the last-passage values above.

Lemma 6.1. *Let $\mathcal{D}$ be a sub-σ-field on a probability space $(\Omega,\mathcal{F},P)$, D an event in $\mathcal{D}$, and ξ and η two integrable random variables. Assume that $E\eta=0$, η is independent of $\mathcal{D}$, and ξ and η are independent conditionally on $\mathcal{D}$. Then $E[\xi\vee(\eta I_D)]\le E[\xi\vee\eta]$.*

Proof. By Jensen's inequality, for any fixed $x\in\mathbb{R}$,

$$x\vee E(\eta\,|\,\mathcal{D})\le E(x\vee\eta\,|\,\mathcal{D}).$$

Since η is independent of $\mathcal{D}$ and mean zero,

$$x \vee 0 \le E(x \vee \eta \,|\, \mathcal{D}).$$

Integrate this against the conditional distribution $P(\xi \in dx \,|\, \mathcal{D})$ of ξ, given $\mathcal{D}$, and use the conditional independence of ξ and η:

$$E(\xi \vee 0 \,|\, \mathcal{D}) \le E(\xi \vee \eta \,|\, \mathcal{D}).$$

Next integrate this over the event D^c:

$$E\big[I_{D^c} \cdot \xi \vee (\eta I_D)\big] = E\big[I_{D^c} \cdot \xi \vee 0\big] \le E\big[I_{D^c} \cdot \xi \vee \eta\big].$$

The corresponding integral over the event D needs no argument. □

Fix a lattice point $z_0 = (i_0, j_0)$ for the moment. We split the maximum in (6.1) according to whether the path π goes through z_0 or not, and if it does we also separate the weight at z_0:

$$\begin{aligned}\max_{\pi \in \Pi(n, \lfloor n\alpha \rfloor)} \sum_{z \in \pi} X_\ell(z) I_{\{F_j = H_\ell\}} &= A \vee \big(B + X_\ell(z_0) I_{\{F_{j_0} = H_\ell\}}\big) \\ &= B + \big(A - B\big) \vee \big(X_\ell(z_0) I_{\{F_{j_0} = H_\ell\}}\big)\end{aligned}$$

where

$$A = \max_{\pi \not\ni z_0} \sum_{z \in \pi} X_\ell(z) I_{\{F_j = H_\ell\}} \quad \text{and} \quad B = \max_{\pi \ni z_0} \sum_{z \in \pi \setminus \{z_0\}} X_\ell(z) I_{\{F_j = H_\ell\}}.$$

Now apply Lemma 6.1 with $\xi = A - B$, $\eta = X_\ell(z_0)$, and $D = \{F_{j_0} = H_\ell\}$. Given F_{j_0}, $A - B$ does not look at $X_\ell(z_0)$, so the independence assumed in Lemma 6.1 is satisfied. The outcome from that lemma is the inequality

$$\mathbf{E}\left[\max_{\pi \in \Pi(n, \lfloor \alpha n \rfloor)} \sum_{z \in \pi} X_\ell(z) I_{\{F_j = H_\ell\}}\right] \le \mathbf{E}\big[A \vee (B + X_\ell(z_0))\big].$$

This is tantamount to replacing the weight $X_\ell(z_0) I_{\{F_{j_0} = H_\ell\}}$ at z_0 with $X_\ell(z_0)$.

We can repeat this at all lattice points z_0 in (6.1). In the end we have an upper bound in terms of homogeneous last-passage values, to which we can apply Martin's result (1.2):

$$\begin{aligned}\Psi(1, \alpha) &\le \sum_{\ell=1}^{L} \lim_{n \to \infty} \frac{1}{n} \mathbf{E}\left[\max_{\pi \in \Pi(n, \lfloor \alpha n \rfloor)} \sum_{z \in \pi} X_\ell(z)\right] = \sum_{\ell=1}^{L} \Psi_{H_\ell}(1, \alpha) \\ &= 2\sqrt{\alpha} \sum_{\ell=1}^{L} \sigma(H_\ell) + o(\sqrt{\alpha}).\end{aligned}$$

This completes the proof of Theorem 2.3. □

7. Proofs for the exponential model

Proof of Theorem 2.4. Eq. (2.10) gives

$$\Psi_G(1, \alpha) = \inf\{t \ge 0 : t g(\alpha/t) \ge 1\} = t(\alpha) = t. \tag{7.1}$$

That the infimum is achieved can be seen from (2.9).

Under Case 1 the critical value $u^* = \int_{[c,\infty)} c(\xi - c)^{-1} \, m(d\xi) < \infty$, and also

$$a'(u^*-) = \left\{\int_{[c,\infty)} \frac{\xi}{(\xi - c)^2} \, m(d\xi)\right\}^{-1} > 0.$$

By the concavity of a and (2.9), for $0 \le y \le a'(u^*-)$ we have $g(y) = -yu^* + c$. Consequently for small enough α,

$$1 = tg(\alpha/t) = -\alpha c \int_{[c,\infty)} \frac{1}{\xi - c} m(d\xi) + ct$$

and Eq. (2.12) follows.

In Case 2, $a'(0+) > a'(u^*-) = 0$ and hence for small enough $\alpha > 0$, there exists a unique $u_0 \in (0, u^*)$ such that $a'(u_0) = \alpha/t$. Set $a_0 = a(u_0) \in (0, c)$. As $\alpha \searrow 0$, both $u_0 \nearrow u^*$ and $a_0 \nearrow c$. We have the equations

$$a'(u_0)^{-1} = \int_{[c,\infty)} \frac{\xi}{(\xi - a_0)^2} m(d\xi) = \frac{t}{\alpha}, \quad 1 = tg(\alpha/t) = -\alpha u_0 + t a_0,$$

$$\Psi_G(1, \alpha) = t = \frac{1}{a} + \frac{\alpha u_0}{a_0} = \frac{1}{a_0} + \alpha \int_{[c,\infty)} \frac{1}{\xi - a_0} m(d\xi) \tag{7.2}$$

and

$$\frac{1}{a_0^2} = \alpha \int_{[c,\infty)} \frac{1}{(\xi - a_0)^2} m(d\xi). \tag{7.3}$$

Assuming (2.11), start with $\nu \in (-1, 0) \cup (0, 1)$. For a small enough $\varepsilon > 0$, there are constants $0 < \kappa_1 < \kappa_2$ such that

$$\kappa_1(\xi - c)^{\nu+1} \le m[c, \xi) \le \kappa_2(\xi - c)^{\nu+1} \quad \text{for} \quad \xi \in [c, c + \varepsilon] \tag{7.4}$$

and as $\varepsilon \searrow 0$ we can take $\kappa_1, \kappa_2 \to \kappa$. First we estimate $c - a_0$. Fix $\varepsilon > 0$.

$$\begin{aligned} \frac{1}{\alpha} &= a_0^2 \int_{[c,\infty)} \frac{1}{(\xi - a_0)^2} m(d\xi) = 2a_0^2 \int_c^\infty \frac{m[c, \xi)}{(\xi - a_0)^3} d\xi \\ &= 2a_0^2 \int_c^{c+\varepsilon} \frac{m[c, \xi)}{(\xi - a_0)^3} d\xi + C_1(\varepsilon) \end{aligned}$$

for a quantity $C_1(\varepsilon) = O(\varepsilon^{-2})$. The first term above can be bounded above and below by (7.4), and we develop both bounds together for κ_i, $i = 1, 2$, as

$$\begin{aligned} &2\kappa_i a_0^2 \int_c^{c+\varepsilon} \frac{(\xi - c)^{\nu+1}}{(\xi - a_0)^3} d\xi + C_1(\varepsilon) \\ &= 2\kappa_i a_0^2 \int_c^{c+\varepsilon} \frac{\left[(\xi - a_0) - (c - a_0)\right]^{\nu+1}}{(\xi - a_0)^3} d\xi + C_1(\varepsilon) \\ &= 2\kappa_i a_0^2 \sum_{k=0}^\infty \binom{\nu+1}{k} (-1)^k (c - a_0)^k \int_c^{c+\varepsilon} (\xi - a_0)^{\nu-k-2} d\xi + C_1(\varepsilon) \\ &= 2\kappa_i a_0^2 \sum_{k=0}^\infty \binom{\nu+1}{k} (-1)^k (c - a_0)^k \frac{(c - a_0)^{\nu-k-1} - (c + \varepsilon - a_0)^{\nu-k-1}}{k - \nu + 1} + C_1(\varepsilon) \\ &= 2\kappa_i a_0^2 A_\nu (c - a_0)^{\nu-1} - 2\kappa_i a_0^2 \sum_{k=0}^\infty \binom{\nu+1}{k} \frac{(-1)^k}{k - \nu + 1} (c - a_0)^k (c + \varepsilon - a_0)^{\nu-k-1} + C_1(\varepsilon) \\ &= 2\kappa_i a_0^2 A_\nu (c - a_0)^{\nu-1} + C_1(\varepsilon). \end{aligned} \tag{7.5}$$

$C_1(\varepsilon)$ changed of course in the last equality. In the next to last equality above we defined

$$A_\nu = \sum_{k=0}^\infty \binom{\nu+1}{k} \frac{(-1)^k}{k - \nu + 1}.$$

Rewrite the above development in the form

$$(c-a_0)^{1-\nu} = 2\kappa c^2 A_\nu \alpha + \alpha[2A_\nu(\kappa_i a_0^2 - \kappa c^2) + C_1(\varepsilon)(c-a_0)^{1-\nu}].$$

Now choose $\varepsilon = \varepsilon(\alpha) \searrow 0$ as $\alpha \searrow 0$ but slowly enough that $C_1(\varepsilon)(c-a_0)^{1-\nu} \to 0$ as $\alpha \searrow 0$. Then also $\kappa_i a_0^2 \to \kappa c^2$ and we can write

$$c - a_0 = B_0 \alpha^{\frac{1}{1-\nu}} + o\big(\alpha^{\frac{1}{1-\nu}}\big) \tag{7.6}$$

with a new constant $B_0 = (2\kappa c^2 A_\nu)^{\frac{1}{1-\nu}}$.

Now consider the case $\nu \in (0,1)$ which also guarantees $\int_{[c,\infty)}(\xi - c)^{-1}\, m(d\xi) < \infty$. From (7.2) and (7.6) as $\alpha \searrow 0$

$$\begin{aligned}\Psi_G(1,\alpha) &= \frac{1}{a_0} + \alpha \int_{[c,\infty)} \frac{1}{\xi - a_0}\, m(d\xi)\\ &= \frac{1}{c} + \alpha \int_{[c,\infty)} \frac{1}{\xi - c} m(d\xi) + O\big(\alpha^{\frac{1}{1-\nu}}\big) + \alpha \left(\int_{[c,\infty)} \frac{1}{\xi - a_0}\, m(d\xi) - \int_{[c,\infty)} \frac{1}{\xi - c}\, m(d\xi) \right)\\ &= \frac{1}{c} + \alpha \int_{[c,\infty)} \frac{1}{\xi - c}\, m(d\xi) + o(\alpha).\end{aligned}$$

Next the case $\nu \in (-1,0)$. The steps are similar to those above so we can afford to be sketchy.

$$\begin{aligned}\Psi_G(1,\alpha) &= \frac{1}{a_0} + \alpha \int_{[c,\infty)} \frac{1}{\xi - a_0}\, m(d\xi)\\ &= \frac{1}{c} + \frac{c-a_0}{c^2} + \frac{(c-a_0)^2}{c^2 a_0} + \alpha \int_c^{c+\varepsilon} \frac{m[c,\xi)}{(\xi - a_0)^2}\, d\xi + \alpha C_1(\varepsilon).\end{aligned}$$

Again, using (7.4) and proceeding as in (7.5), we develop an upper and a lower bound for the quantity above with distinct constants κ_i, $i = 1, 2$. After bounding $m[c,\xi)$ above and below with $\kappa_i(\xi - c)^{\nu+1}$ in the integral, write $(\xi - c)^{\nu+1} = ((\xi - a_0) - (c - a_0))^{\nu+1}$ and expand in power series.

$$\begin{aligned}&\frac{1}{c} + B_0 c^{-2} \alpha^{\frac{1}{1-\nu}} + o\big(\alpha^{\frac{1}{1-\nu}}\big) + \alpha\kappa_i \int_c^{c+\varepsilon} \frac{(\xi - c)^{\nu+1}}{(\xi - a_0)^2}\, d\xi + \alpha C_1(\varepsilon)\\ &= \frac{1}{c} + B_0 c^{-2} \alpha^{\frac{1}{1-\nu}} + o\big(\alpha^{\frac{1}{1-\nu}}\big) + \alpha\kappa_i (c-a_0)^\nu \sum_{k=0}^\infty \binom{\nu+1}{k} \frac{(-1)^k}{k-\nu}\\ &\quad + \alpha\kappa_i (c - a_0 + \varepsilon)^\nu \sum_{k=0}^\infty \binom{\nu+1}{k} \frac{(-1)^k}{\nu - k} \left(\frac{c-a_0}{c-a_0+\varepsilon} \right)^k + \alpha C_1(\varepsilon)\\ &= \frac{1}{c} + B\alpha^{\frac{1}{1-\nu}} + o\big(\alpha^{\frac{1}{1-\nu}}\big) + A_{\nu,2}\alpha(\kappa_i - \kappa)(c-a_0)^\nu + \alpha C_1(\varepsilon).\end{aligned}$$

In the last equality the next to last term with the $\sum_{k=0}^\infty$ sum was subsumed in the $\alpha C_1(\varepsilon)$ term. Then we introduced new constants

$$A_{\nu,2} = \sum_{k=0}^\infty \binom{\nu+1}{k} \frac{(-1)^k}{k-\nu} \quad \text{and} \quad B = B_0 c^{-2} + \kappa B_0^\nu A_{\nu,2}. \tag{7.7}$$

As before, by letting $\varepsilon = \varepsilon(\alpha) \searrow 0$ slowly enough as $\alpha \searrow 0$ we can extract $\Psi_G(1,\alpha) = c^{-1} + B\alpha^{\frac{1}{1-\nu}} + o\big(\alpha^{\frac{1}{1-\nu}}\big)$ from the above bounds.

It remains to treat the cases $\nu = -1, 0, 1$ where integration of the type done in (7.5) is elementary. We omit the details. □

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