Monotonicity of the Power Functions of Modified Likelihood Ratio Criterion for the Homogeneity of Variances and of the Sphericity Test*

E. M. CARTER

University of Guelph, Guelph, Ontario

AND

M. S. SRIVASTAVA

University of Toronto, Toronto, Canada

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The modified likelihood ratio criterion for testing the homogeneity of variances of $p$ univariate normal populations, and the sphericity test, are both shown in this paper to have a monotone nondecreasing power function.

1. INTRODUCTION

Let $s_j, j = 1, ..., p$ be independently distributed as $\sigma \chi^2_{n_j}$. The likelihood ratio criterion (hereafter referred to as LRC) for testing

$$H_0: \sigma_1 = \cdots = \sigma_p = \sigma$$

(say) vs $H_1: \sigma_i \neq \sigma_j$, for some $i \neq j$, $i, j = 1, ..., p$, where $\sigma$ is unknown, is given by:

Reject $H_0$ if

$$\left(\prod_{j=1}^{p} \frac{s_j}{\sigma_j^2}\right) \left(\sum_{j=1}^{p} \frac{s_j}{\sigma_j^2}\right)^{N(p)} \leq c,$$

where

$$N(r) = \sum_{j=1}^{r} n_j$$

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and \( c \) is so chosen that the error of the first kind (size of the test) is at a specified level. Without loss of generality we shall assume that the parameters \( \sigma_j \) are ordered as \( \sigma_1 \geq \cdots \geq \sigma_p \). The power of this test is shown in the next section to be a monotone nondecreasing function of \( \delta_j = (\sigma_j/\sigma_{j+1}) \), \( j = 1, 2, \ldots, p - 1 \), \( \delta_j \geq 1 \).

It is to be noted that the above test is the modified LRC [1] in which the \( N_i \)'s, the sample sizes from the \( i \)th population, are replaced by \( n_i \)'s, the degrees of freedom associated with \( s_i \)'s, for testing the homogeneity of variances of \( p \) normal populations. It may be mentioned that the unmodified LRC is not even unbiased unless the sample sizes are equal (see [3]). Similarly if \( x_1, x_2, \ldots, x_N \) are independent and identically distributed as \( N_p(\mu, \Sigma) \) then the LRC for testing

\[
\begin{align*}
H_0: \Sigma &= \sigma I, \quad \sigma > 0, \quad \mu \text{ unknown vs} \\
H_1: \Sigma &\neq \sigma I, \quad \mu \text{ unknown},
\end{align*}
\]

is given by

\[
|S| (\text{tr } S)^{-p} \leq c,
\]

where

\[
S = \sum_{j=1}^{N} (x_j - \bar{x})(x_j - \bar{x})' \sim W_p(\Sigma, n), \quad n = N - 1,
\]

and \( c \) is determined by the size of the test. The power of this test in (5) depends only on the characteristic roots of \( \Sigma \), say \( \sigma_1 \geq \cdots \geq \sigma_p \) and this power is a monotone nondecreasing function of \( \delta_j = \sigma_j/\sigma_{j+1} \), \( j = 1, 2, \ldots, p - 1 \), \( \delta_j \geq 1 \). This second problem can be reduced to a special case of the first problem.

2. Monotonicity of the Test for Homogeneity of Variances

This property is the content of the theorem below.

**Theorem 1.** Let \( s_i \) be independently distributed as \( \frac{1}{2} \sigma_i X_{n_j}^2 \), \( j = 1, \ldots, p \). Suppose also that \( \sigma_1 \geq \cdots \geq \sigma_p \). Let

\[
A = A(s_1, \ldots, s_p) = \left\{(s_1, \ldots, s_p): \left(\prod_{j=1}^{p} s_j^{n_j} \right)^{1/N(p)} \leq c\right\},
\]

where we define \( N(r) = \sum_{j=1}^{r} n_j \), and \( c \) is an arbitrary constant. Let \( P(A) = \) probability of \( A \). Then for any \( k, 1 \leq k \leq p - 1 \) \( P(A) \) is a nondecreasing function of \( \delta_k = \sigma_k/\sigma_{k+1} \), while the remaining \( (p - 2) \) parameters \( \delta_i = \sigma_i/\sigma_{i+1} \), \( i = 1, \ldots, p - 1, i \neq k \), are held fixed.
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Proof. As the region $A$ is invariant under scale transformations of the $s_i$'s, we consider the transformation

$$x_j = s_j/s_k, \quad j = 1, 2, ..., p, \quad j \neq k.$$ 

Then integrating over $s_k$ we obtain the joint density function of $x_1, ..., x_{k-1}, x_{k+1}, ..., x_p$ as

$$c_1 \left[ \prod_{j=1}^{p} \left( \sigma_k/s_j \right)^{f_j} x_j^{f_j-1} \right] \exp \left[ -\sum_{i=1}^{p} \sigma_k x_i/\sigma_i \right]^{f_0},$$

where $f_j = \frac{1}{2}n_j$, $j = 1, ..., p$, $f_0 = \sum_{j=1}^{p} f_j$, and $c_1 = (\prod_{i=1}^{p} \Gamma(f_i))/\Gamma(f_0)$. Now consider a transformation of the kind,

$$x_i = x_{k+1}u_i, \quad i = k + 2, ..., p,$$

$$x_{k+1} = u_{k+1} \left( 1 + \sum_{j=1}^{k-1} x_j \right) / \left( 1 + \sum_{j=k+2}^{p} u_j \right).$$

The joint p.d.f. of $x_1, ..., x_{k-1}, u_{k+1}, ..., u_p$ is given by

$$c_1 \left[ \prod_{j=1}^{k-1} \lambda_j \left( x_j^{f_j} \right)^{f_j-1} \prod_{i=k+2}^{p} \lambda_i \left( u_i^{f_i} \right)^{f_i-1} \right] \exp \left( \sum_{i=k+1}^{k} \delta_i y \right) u_{k+1}^{f_{k+1}-1} \left( 1 + \delta_i y u_{k+1} \right)^{-f_0}$$

$$\times \left( 1 + \sum_{j=1}^{k-1} \lambda_j x_j \right)^{-f(k+1)} \left( 1 + \sum_{j=1}^{k-1} \lambda_j x_j \right)^{-\sum_{i=k+2}^{p} \lambda_i f_i},$$

where $f_{k+1} = \sum_{i=k+1}^{p} f_i$, $\lambda_j = \sigma_k/\sigma_j$ for $1 \leq j \leq k$, $\lambda_i = \sigma_{k+1}/\sigma_i$ for $k + 2 \leq i \leq p$, $\delta_k = \sigma_k/\sigma_{k+1}$, and

$$\gamma = \left( 1 + \sum_{j=1}^{k-1} x_j \right) \left( 1 + \sum_{j=k+2}^{p} u_j \right) \left( 1 + \sum_{j=k+2}^{p} \lambda_j x_j \right)^{-1} \left( 1 + \sum_{j=k+2}^{p} u_j \right)^{-1}.$$ 

Hence the conditional p.d.f. of $u_{k+1}$ given $x_1, ..., x_{k-1}, u_{k+2}, ..., u_p$ is given by

$$\text{const} \left[ \delta_0 y \right]^{f_{k+1}} u_{k+1}^{f_{k+1}-1} \left[ 1 + \gamma \delta_2 u_{k+1} \right]^{-f_0},$$

and the conditional region for $u_{k+1}$ is

$$\omega_1: u_{k+1}^{f_{k+1}}(1 + u_{k+1})^{-f_0} \leq \frac{c(1 + \sum_{i=k+2}^{p} \lambda_i x_i^{f_i} \left( 1 + \sum_{j=k+1}^{k-1} x_j \right) \delta_i x_j f_j)}{(\prod_{j=1}^{k-1} x_j^{f_j})(\prod_{i=k+2}^{p} u_i^{f_i})}.$$ 

Since the parameter point is $\sigma_1 \geq \cdots \geq \sigma_k \geq \sigma_{k+1} \geq \cdots \geq \sigma_p$, we obtain that $\lambda_1 \leq \cdots \leq \lambda_{k+1} \leq 1 \leq \lambda_{k+2} \leq \cdots \leq \lambda_p$, and, hence, that $\gamma \geq 1$. Therefore from the properties of the $F$ test (see [2])

$$P(u_{k+1} \in \omega_1 | x_1, ..., x_{k-1}, u_{k+2}, ..., u_p)$$
increases as $\delta_k$ increases from one with $\lambda_i$, $i = 1, \ldots, p - 1, i \neq k$, kept fixed, or equivalently with $\delta_i = \sigma_i/\sigma_{i+1}$, $i = 1, \ldots, p - 1, i \neq k$, kept fixed. Hence on averaging with respect to the conditional variables we obtain the required result.

**Corollary 1.** Suppose that for any test the conditional acceptance region given $x_1, \ldots, x_{k-1}, u_{k+2}, \ldots, u_p$ is of the form $a < u_{k+1} < b$. If $a(1 + a)^{-\theta} \geq b'(1 + b)^{-\theta}$ then the power of the test is monotone increasing in $\delta_k = \sigma_k/\sigma_{k+1}$ for fixed $\delta_i = \sigma_i/\sigma_{i+1}$, $i = 1, \ldots, p, i \neq k$.

### 3. Monotonicity of the Sphericity Test

Let $S$ be distributed as $w_p(\Sigma, n)$. The density of $S$ is given by

$$
\pi^{-1}p(p-1) \prod_{j=1}^{p} \Gamma^{-1}(n - j + 1) | \Sigma |^{-1/2} | S |^{\frac{1}{2}(n-(p+1))} \exp - \text{tr} \Sigma^{-1}S.
$$

Define $P(B) = \text{probability of } B$ where $B = \{ S: | n(\text{tr } S)^{-1} < \epsilon \}$. As $B$ is invariant under orthogonal transformations we shall assume $\Sigma$ to be diagonal $(u_1, \ldots, u_p)$ with $u_1 \geq \cdots \geq u_p$. Therefore

$$
P(B) = \int_{B} \pi^{-1}p(p-1) \prod_{j=1}^{p} \Gamma^{-1}(\frac{1}{2}(n - j + 1)) | S |^{\frac{1}{2}(n-(p+1))} \prod_{j=1}^{p} \sigma_j^{-1/2} \left( \exp - \sum_{j=1}^{p} s_j \sigma_j^{-1} \right) dS.
$$

By making the transformation

$$
S = \text{diag}(s_{11}/\sqrt{s}, \ldots, s_{pp}/\sqrt{s})R \text{ diag}(s_{11}, \ldots, s_{pp})
$$

we obtain

$$
P(B) = \int_{B} \pi^{-1}p(p-1) \left( \prod_{j=1}^{p} \Gamma^{-1}(\frac{1}{2}(n - j + 1)) \right) \Gamma^{\frac{1}{2}(n)} | R |^{\frac{1}{2}(n-1)(p+1)}

\cdot \int_{B(s_{jj}, j-1, \ldots, p | R)} \prod_{j=1}^{p} \sigma_j^{-1/2} \exp - \sum_{j=1}^{p} s_j \sigma_j^{-1} \prod_{j=1}^{p} ds_j dR,
$$

$$
B(s_{jj}, j = 1, \ldots, p | R) = \left\{ (s_{11}, \ldots, s_{pp}) : \left( \prod_{j=1}^{p} s_{jj}^{n} \right) \left( \sum_{j=1}^{p} s_{jj} \right)^{-np} \leq C | R |^{-n} \right\}.
$$

By simply applying the theorem of Section 2 to the inner integral we obtain the result that $P(B)$ is a monotone nondecreasing function of each $\delta_k = \sigma_k/\sigma_{k+1}$.
REFERENCES

