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Combinatorics of open covers (XI): Menger- and Rothberger-bounded groups ☆

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Abstract

We examine Menger-bounded (= o-bounded) and Rothberger-bounded groups. We give internal characterizations of groups having these properties in all finite powers (Theorems 6 and 7, and Theorem 15). In the metrizable case we also give characterizations in terms of measure-theoretic properties relative to left-invariant metrics (Theorems 12 and 19). Among metrizable σ-totally bounded groups we characterize the Rothberger-bounded groups by the corresponding game (Theorem 22).

Keywords: o-bounded group; Menger-bounded group; Rothberger-bounded group; Selection principle; Infinite game

According to [4] the topological group $(H, *)$ is *o-bounded* if there is for every sequence $(U_n\colon n \in \mathbb{N})$ of open neighborhoods of 1_H, a sequence $(F_n\colon n \in \mathbb{N})$ of finite subsets of H, such that $H = \bigcup_{n\in\mathbb{N}} F_n * U_n$. This notion was introduced by O. Okunev. In unpublished work [6] Kočinac defined the notion of a *Menger-bounded group* by the same definition. Additionally Kočinac defined the notions of a *Rothberger-bounded group* and a *Hurewicz-bounded group*. The group $(H, *)$ is a Rothberger-bounded group if there is for each sequence $(U_n\colon n \in \mathbb{N})$ of neighborhoods of 1_H a sequence $(x_n\colon n \in \mathbb{N})$ of elements of H such that $H = \bigcup_{n\in\mathbb{N}} x_n * U_n$. And $(H, *)$ is a Hurewicz-bounded group if there is for each sequence $(U_n\colon n \in \mathbb{N})$ of open neighborhoods of 1_H a sequence $(F_n\colon n \in \mathbb{N})$ of finite subsets of H such that each element of H belongs to all but finitely many of the sets $F_n * H_n$.

The following "OF" game on H, due to Tkačenko, is also described in [4]: Two players, ONE and TWO, play an inning per positive integer n. In the nth inning ONE first chooses a neighborhood U_n of 1_H, and then TWO responds with a finite subset F_n of H. A play $(U_1, F_1, U_2, F_2, \ldots, U_n, F_n, \ldots)$ is won by TWO if $H = \bigcup_{n\in\mathbb{N}} F_n * U_n$; else, ONE wins. According to [4] a group is said to be strictly o-bounded if TWO has a winning strategy in this game. It is clear how to define the corresponding games for the Rothberger-bounded and the Hurewicz-bounded groups, and this was also done by Kočinac in [6].

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In this paper we shall show that these o-bounded groups fit naturally into the arena of selection principles, and we shall analyse them as well as the Rothberger-bounded groups from this point of view. Strictly o-bounded groups and their connection with the Hurewicz-bounded groups is treated in [2].

Our paper is organized as follows: First we introduce the necessary background from the area of selection principles, and define relevant special types of open covers for topological groups. We treat Menger-bounded groups in the second section of the paper. In the third section we treat Rothberger-bounded groups.

The following classical fact about topological groups is useful and is often used without special mention:

Theorem 1 *(Van Dantzig). If $(H,*)$ is a topological group with identity element 1_H, then there is for each neighborhood U of 1_H a neighborhood V of 1_H such that $V = V^{-1} \subset U$.*

1. Selection principles, games and open covers

Let $\mathcal{A}$ and $\mathcal{B}$ be families of collections of subsets of the infinite set S. The symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the statement that for each sequence $(O_n\colon n\in\mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(T_n\colon n\in\mathbb{N})$ such that: For each n, $T_n\in O_n$ and $\{T_n\colon n\in\mathbb{N}\}\in\mathcal{B}$. A second selection principle studied also here is denoted by the symbol $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ denotes the statement that for each sequence $(O_n\colon n\in\mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(T_n\colon n\in\mathbb{N})$ such that: For each n, T_n is a finite subset of O_n and $\bigcup\{T_n\colon n\in\mathbb{N}\}\in\mathcal{B}$.

Corresponding to the selection principle $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ we have the following infinite game, denoted $\mathsf{G}_1(\mathcal{A},\mathcal{B})$. In this game two players, ONE and TWO, play an inning per positive integer. In the nth inning ONE first chooses $O_n\in\mathcal{A}$, and then TWO responds with a $T_n\in O_n$. In this way they construct a play

$$O_1, T_1, \ldots, O_n, T_n, \ldots.$$

Such a play is won by TWO if $\{T_n\colon n\in\mathbb{N}\}$ is an element of $\mathcal{B}$; else, ONE wins.

Corresponding to the selection principle $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ we have the following infinite game, denoted $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$. In this game two players, ONE and TWO, play an inning per positive integer. In the nth inning ONE first chooses $O_n\in\mathcal{A}$, and then TWO responds with a finite set $T_n\subseteq O_n$. In this way they construct a play

$$O_1, T_1, \ldots, O_n, T_n, \ldots.$$

Such a play is won by TWO if $\bigcup\{T_n\colon n\in\mathbb{N}\}$ is an element of $\mathcal{B}$; else, ONE wins.

We are particularly interested here in the case where $\mathcal{A}$ and $\mathcal{B}$ are open covers of topological spaces or topological groups. Specifically, let H and G be topological spaces with G a subspace of H. An open cover $\mathcal{U}$ of H is said to be an *ω-cover* if $H\notin\mathcal{U}$, and for each finite subset F of H there is a $V\in\mathcal{U}$ with $F\subseteq V$. We shall use the notations:

$\mathcal{O}_H$: The collection of open covers of H;
$\mathcal{O}_{HG}$: The collection of covers of G by sets open in H;
Ω_H: The collection of ω-covers of H;
Ω_{HG}: The collection of ω-covers of G by sets open in H.

The notion of a *weakly groupable* open cover was defined as follows in [3]: A countable open cover $\mathcal{U}$ for a topological space H is weakly groupable if there is a sequence $(\mathcal{U}_n\colon n\in\mathbb{N})$ such that: For each n, $\mathcal{U}_n$ is a finite subset of $\mathcal{U}$, for $m\neq n$, $\mathcal{U}_m\cap\mathcal{U}_n=\emptyset$, $\mathcal{U}=\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ and for each finite subset F of H there is an n with $F\subseteq\bigcup\mathcal{U}_n$. The symbol $\mathcal{O}_H^{\mathrm{wgp}}$ denotes the collection of weakly groupable open covers of a space H. And $\mathcal{O}_{HG}^{\mathrm{wgp}}$ denotes the collection of weakly groupable covers of G by sets open in H.

Now let $(H,*)$ be a topological group with identity element 1_H. If U is an open neighborhood of 1_H, then $x*U$ ($:=\{x*y\colon y\in U\}$) is an open neighborhood of x. Thus, $\{x*U\colon x\in H\}$ is an open cover of H and will be denoted by the symbol $\mathcal{O}_H(U)$. For F a finite subset of H and U an open neighborhood of 1_H, $F*U:=\{f*y\colon f\in F \text{ and } y\in U\}$ is an open set containing F. If U is an open neighborhood of 1_H such that there is no finite subset F of H with $F*U=H$, then the set $\Omega_H(U):=\{F*U\colon F \text{ a finite subset of } H\}$ does not have H as a member. Define:

$$\mathcal{O}_{\mathsf{nbd}}(H):=\big\{\mathcal{U}\in\mathcal{O}\colon (\exists \text{ open neighborhood } U \text{ of } 1_H)\ \big(\mathcal{U}=\{x*U\colon x\in H\}\big)\big\}.$$

Also define:

$$\Omega_{\mathsf{nbd}}(H):=\big\{\mathcal{U}\in\Omega\colon (\exists \text{ open neighborhood } U \text{ of } 1_H)\ \big(\mathcal{U}=\{F*U\colon F\subset H \text{ finite}\}\big)\big\}.$$

For each neighborhood U of 1_H, $\Omega_H(U) := \{F * U\colon F \subset H \text{ finite}\}$ is an ω-cover of H when H is not an element of this set.

2. Menger-bounded groups

In our notation the topological group $(H, *)$ is an o-bounded group if, and only if, it satisfies the selection principle $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_H)$. The well-known *Menger property* is $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_H, \mathcal{O}_H)$. Also, for G a sub*set* of H, the *relative Menger property* is $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_H, \mathcal{O}_{HG})$.

Theorem 2. *Let* $(G, *)$ *be a subgroup of topological group* $(H, *)$. *Then* $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_H, \mathcal{O}_{HG})$ *implies* $\mathsf{S}_{\mathrm{fin}}(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.

Proof. This is easy. □

Item (2) in Theorem 3 below explains the name Menger-bounded group.

Theorem 3. *Let* $(G, *)$ *be a subgroup of a topological group* $(H, *)$. *The following are equivalent*:

(1) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.
(2) $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.
(3) $\mathsf{S}_{\mathrm{fin}}(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.

Proof. (1) ⇒ (2): Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence from $\mathcal{O}_{\mathsf{nbd}}(H)$. For each n choose an open neighborhood U_n of 1_H such that $\mathcal{U}_n = \mathcal{O}_H(U_n)$. Then, for each n define $\mathcal{V}_n = \Omega_H(U_n)$. Apply $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$ to the sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ and select for each n a $W_n \in \mathcal{V}_n$, such that $\{W_n\colon n \in \mathbb{N}\}$ is a cover of G. For each n let F_n be a finite set with $W_n = F_n * U_n$. Define $\mathcal{W}_n := \{x * U_n\colon x \in F_n\}$ for each n. Then each $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a cover of G.

(2) ⇒ (3): This is clear.

(3) ⇒ (1): Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence in $\Omega_{\mathsf{nbd}}(H)$ and for each n let U_n be a neighborhood of 1_H such that $\mathcal{U}_n = \Omega_H(U_n)$. Apply $\mathsf{S}_{\mathrm{fin}}(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$ to the sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$: For each n let $\mathcal{V}_n$ be a finite subset of $\mathcal{U}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of G. Put

$$F_n = \bigcup\{F \subset H\colon F \text{ finite and there is a } V \in \mathcal{V}_n \text{ with } V = F * U_n\}.$$

Then each F_n is a finite subset of H. Put $V_n = F_n * U_n$. Then for each n we have $V_n \in \mathcal{U}_n$, and we have: $\{V_n\colon n \in \mathbb{N}\}$ is an open cover of G. □

Indeed, by writing $\mathbb{N}$ as a disjoint union of countably many infinite sets Y_n, $n \in \mathbb{N}$, and applying $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$ to each sequence $(\Omega(U_k)\colon k \in Y_n)$ independently, one finds a sequence $(F_n\colon n \in \mathbb{N})$ of finite subsets of H such that for each $x \in G$ there are infinitely many n with $x \in F_n * U_n$.

The relative versions of the selection principles considered in Theorem 3 hold if, and only if, the absolute versions hold for the subgroup G:

Theorem 4. *Let* $(G, *)$ *be a subgroup of the topological group* $(H, *)$. *Then the following are equivalent*:

(1) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.
(2) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G), \mathcal{O}_G)$.

Proof. The implication (2) ⇒ (1) is evidently true. We must show that (1) ⇒ (2). Thus, let $(\Omega(U_n)\colon n \in \mathbb{N})$ be a sequence in $\Omega_{\mathsf{nbd}}(G)$. Since each U_n is a neighborhood (in G) of the group identity we may choose for each n a neighborhood T_n in H of the identity, such that $U_n = T_n \cap G$. Next, choose for each n a neighborhood S_n in H of the identity, such that $S_n^{-1} * S_n \subseteq T_n$.

Apply (2) to the sequence $(\Omega(S_n)\colon n \in \mathbb{N})$ of elements of $\Omega(H)$: We find for each n a finite set $F_n \subset H$ such that $G \subseteq \bigcup_{n\in\mathbb{N}} F_n * S_n$.

For each n, and for each $f \in F_n$, choose a $p_f \in G$ as follows:

$$p_f \begin{cases} \in G \cap f * S_n & \text{if nonempty,} \\ = \text{ group identity} & \text{otherwise.} \end{cases}$$

Then put $G_n = \{p_f\colon f \in F_n\}$, a finite subset of G. Observe that for each n we have $G_n * U_n \in \Omega(U_n) \in \Omega_{\mathsf{nbd}}(G)$. We show that $G = \bigcup_{n\in\mathbb{N}} G_n * U_n$.

For let $g \in G_n$ be given. Choose n so that $g \in F_n * S_n$, and choose $f \in F_n$ so that $g \in f * S_n$. Then evidently $G \cap f * S_n$ is nonempty, and so $p_f \in G$ is defined as an element of this intersection. Since $p_f \in f * S_n$, we have $f \in p_f * S_n^{-1}$, and so $g \in p_f * S_n^{-1} * S_n \subseteq p_f * T_n$. Now $p_f^{-1} * g \in G \cap T_n = U_n$, and so we have $g \in p_f * U_n \subset G_n * U_n$. This completes the proof. □

As a consequence, we have the following general connection with the corresponding selection principle for general topological spaces:

Corollary 5. *For a subgroup $(G, *)$ of a topological group $(H, *)$: If $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_H, \mathcal{O}_{HG})$ holds then $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G), \mathcal{O}_G)$ holds.*

Proof. Theorems 2, 3 and 4. □

The converse of Theorem 2 is not true: In Theorem 8.2 of [7] the authors give an example of a σ-compact zero-dimensional metrizable group H which contains a subgroup G such that $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_G, \mathcal{O}_G)$ fails. Note that the group H, being σ-compact, has as topological space the property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_H, \mathcal{O}_H)$, and thus as topological group the property $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_H)$. Consequently the subgroup G also has the property $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G), \mathcal{O}_G)$.

2.1. Preservation under finite powers

For topological groups weak groupability properties of open covers capture the preservation of selection properties in finite powers. The following is our first step towards proving this.

Theorem 6. *Let $(G, *)$ be a subgroup of the topological group $(H, *)$. The following are equivalent*:

(1) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG}^{\mathrm{wgp}})$.
(2) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \Omega_{HG})$.

Proof. We must prove that (1) ⇒ (2). Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence from $\Omega_{\mathsf{nbd}}(H)$, and choose for each n a neighborhood U_n of 1_H with $\mathcal{U}_n = \Omega_H(U_n)$. Then define for each n: $V_n = \bigcap_{j\leqslant n} U_j$. Each V_n is an open neighborhood of 1_H, and each $\mathcal{V}_n = \Omega_H(V_n)$ is in $\Omega_{\mathsf{nbd}}(H)$.

Apply $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG}^{\mathrm{wgp}})$ to the sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$, and choose for each n a $W_n \in \mathcal{V}_n$ such that $\{V_n\colon n \in \mathbb{N}\}$ is a weakly groupable open cover of G. Choose an increasing sequence $m_1 < m_2 < \cdots < m_k < \cdots$ such that for each finite subset F of G there is a k with $F \subset \bigcup_{m_k\leqslant j<m_{k+1}} W_j$. Also, for each n choose a finite set $F_n \subset H$ with $W_n = F_n * V_n$.

For $i < m_1$ set $S_i = \bigcup_{j<m_1} F_j$, and for $m_k \leqslant i < m_{k+1}$ set $S_i = \bigcup_{m_k\leqslant j<m_{k+1}} F_j$. Then each S_i is a finite subset of H, and if we set $M_i = S_i * U_i$ for each i, then $M_i \in \mathcal{U}_i$ for each i, and $\{M_i\colon i \in \mathbb{N}\}$ is in Ω_{HG}. □

Theorem 7. *For a subgroup $(G, *)$ of a topological group $(H, *)$ the following are equivalent*:

(1) *For each n the selection principle $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H^n), \mathcal{O}_{H^nG^n})$ holds.*
(2) *$\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \Omega_{HG})$ holds.*

Proof. (1) ⇒ (2): Let $(\Omega(U_n)\colon n \in \mathbb{N})$ be a sequence from $\Omega_{\mathsf{nbd}}(H)$. Write $\mathbb{N} = \bigcup_{n\in\mathbb{N}} Y_n$ where for each n Y_n is infinite, and for $m \neq n$ we have $Y_m \cap Y_n = \emptyset$.

For each k, $(\Omega(U_n^k)\colon n \in Y_k)$ is in $\Omega_{\mathsf{nbd}}(H^k)$. By (1), choose for each k, for each $n \in Y_k$ a finite set $F_n \subset H$ such that $\{F_n^k * U_n^k\colon n \in Y_k\}$ covers G^k. We show now that $\{F_n * U_n\colon n \in \mathbb{N}\}$ is in Ω_{HG}. For let $F = \{f_1, \ldots, f_j\}$ be a finite subset of G. Then $f = (f_1, \ldots, f_j) \in G^j$. Pick an $n \in Y_j$ with $h \in F_n^j * U_n^j$. Then we have $F \subseteq F_n * U_n \in \Omega_H(U_n)$.

(2) ⇒ (1): Fix an $m \in \mathbb{N}$, and let $(\Omega_H(U_n)\colon n \in \mathbb{N})$ be a sequence in $\Omega_{\mathsf{nbd}}(H^m)$. For each k choose a neighborhood V_k of 1_H such that $V_k^m \subseteq U_k$.

Then $(\Omega_H(V_k)\colon k \in \mathbb{N})$ is a sequence from $\Omega_{\mathsf{nbd}}(H)$. Apply $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \Omega_{HG})$ and pick for each k a finite set $F_k \subset H$ with $\{F_k * V_k\colon k \in \mathbb{N}\} \in \Omega_{HG}$.

Then $\{F_k^m * V_k^m\colon k \in \mathbb{N}\}$ is an open cover of G^m. For each k choose $T_k \in \Omega_H(U_k)$ with $F_k^m * V_k^m \subset T_k$. Then $\{T_k\colon k \in \mathbb{N}\}$ is an open cover of G^m. □

As a result we have the following corollary:

Corollary 8. *Let $(G, *)$ be a subgroup of the topological group $(H, *)$. Then the following are equivalent*:

(1) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG}^{\mathrm{wgp}})$.
(2) *For each n,* $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G^n), \mathcal{O}_{G^n})$.
(3) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G), \Omega_G)$.
(4) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G), \mathcal{O}_G^{\mathrm{wgp}})$.

Proof. (1) ⇒ (2): (1) together with (2) ⇒ (1) of Theorem 7 gives that for each n, $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H^n), \mathcal{O}_{H^nG^n})$. And then (1) ⇒ (2) of Theorem 4 gives for each n that $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G^n), \mathcal{O}_{G^n})$ holds.

(2) ⇒ (3): This follows from (1) ⇒ (2) of Theorem 7, taking $H = G$.

(3) ⇒ (4): This is clear because any ω-cover is also a weakly groupable cover.

(4) ⇒ (1): Also this is evident. □

2.2. Metrizable groups

Proofs of the following two fundamental facts about metrizable groups can be found in [5]:

Theorem 9 *(Birkhoff–Kakutani). A topological group is metrizable if, and only if, it is* T_0 *and first-countable.*

Thus, when a group $(G, *)$ is metrizable, then there is a sequence $(V_k\colon k < \infty)$ of open neighborhoods of the identity element such that each V_k is symmetric (i.e., $V_k^{-1} = V_k$), and for each k, $V_{k+1}^2 \subseteq V_k$. A metric d on a topological group is *left-invariant* if for all g, x and y in G one has that $d(g * x, g * y) = d(x, y)$. The following theorem associates a left-invariant metric with neighborhood sequences $(V_k\colon k < \infty)$ as above:

Theorem 10 *(Kakutani). Let $(V_k\colon k < \infty)$ be a sequence of subsets of the topological group $(G, *)$ such that $\{V_k\colon k < \infty\}$ is a neighborhood basis of the identity element consisting of symmetric neighborhoods, and such that for each k also $V_{k+1}^2 \subseteq V_k$. Then there is a left-invariant metric d on G such that*

(1) *d is uniformly continuous in the left uniform structure on $G \times G$.*
(2) *If $y^{-1} * x \in V_k$ then $d(x, y) \leqslant (\frac{1}{2})^{k-2}$.*
(3) *If $d(x, y) < (\frac{1}{2})^k$ then $y^{-1} * x \in V_k$.*

In particular: A metrizable group is metrizable by a left-invariant metric d which has the additional properties stated in the theorem.

Measurelike properties of metrizable spaces are useful in analyzing selection properties in metrizable spaces. We shall show that similarly, for metrizable groups the measure-like properties with respect to left-invariant metrics characterize corresponding selection properties. We first review some of the basic facts regarding classical selection properties in metrizable spaces. Let $\mathcal{A}$ be a collection of sets consisting of subsets of H. Following [1] we shall say that a metric space (H, d) has *$\mathcal{A}$-measure zero* if there is for each sequence $(\epsilon_n\colon n \in \mathbb{N})$ of positive real numbers, a sequence ($\mathcal{I}_n$: $n \in \mathbb{N}$) such that: For each n, $\mathcal{I}_n$ is a finite set of subsets of H, each of diameter less than ϵ_n, and $\bigcup_{n\in\mathbb{N}} \mathcal{I}_n$ is a member of $\mathcal{A}$. The following fact is Theorem 4.2.9 of [1]:

Theorem 11. *Let H be a zero-dimensional metrizable space with no isolated points, and let G be a subspace of it. Then $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_H, \mathcal{O}_{HG})$ holds if, and only if, H has the $\mathcal{O}_{HG}$-measure zero property in all metrics of the space H.*

An analogous statement characterizes the metrizable o-bounded groups:

Theorem 12. *Let $(G, *)$ be a subgroup of the metrizable group $(H, *)$. The following are equivalent*:

(1) $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.
(2) *H has $\mathcal{O}_{HG}$-measure zero with respect to all left-invariant metrics of H.*

Proof. (1) ⇒ (2): By Theorem 4 $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G), \mathcal{O}_G)$ holds. Consider any left-invariant metric which generates the topology of H. Let (ϵ_n: $n \in \mathbb{N}$) be a sequence of positive real numbers. For each n choose an H-neighborhood U_n of 1_H, of diameter less than ϵ_n, and put $W_n = U_n \cap G$. Apply $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(G), \mathcal{O}_G)$ to the sequence ($\Omega(W_n)$: $n \in \mathbb{N}$), and choose for each n an element $G_n \in \Omega_G(W_n)$ such that $\{G_n\colon n \in \mathbb{N}\} \in \mathcal{O}_G$. For each n choose a finite set F_n with $G_n = F_n * W_n$, and put $\mathcal{I}_n = \{f * U_n\colon f \in F_n\}$. Observe that each $\mathcal{I}_n$ is finite, and each member of it is an open subset of H of diameter less than ϵ_n, and $\bigcup_{n\in\mathbb{N}} \mathcal{I}_n$ is a cover of G.

(2) ⇒ (1): Let a sequence ($\mathcal{O}(U_n)$: $n \in \mathbb{N}$) of elements of $\mathcal{O}_{\mathsf{nbd}}(H)$ be given. For each n choose a symmetric H-neighborhood V_n of the identity element 1_H such that

(1) For each n: $V_n \subseteq (\bigcap_{j<n}(U_j \cap V_j)) \cap U_n$;
(2) For each n: $V_{n+1}^2 \subset V_n$;
(3) $\{V_n\colon n \in \mathbb{N}\}$ is a neighborhood basis for the identity element.

For each n, put $\mathcal{V}_n = \mathcal{O}(V_n)$. Then each $\mathcal{V}_n$ is a member of $\mathcal{O}_{\mathsf{nbd}}(H)$. Let d be the left-invariant metric associated to the sequence (V_k: $k < \infty$) by Theorem 10. For each k put $\epsilon_k = (\frac{1}{2})^k$.

Claim. *If a subset S of H has diameter less than ϵ_k, then there is a $W \in \mathcal{V}_k$ with $S \subset W$.*

Proof. For consider such an S and choose an $x \in S$. Consider any $y \in S$. Then as $d(y, x) < \epsilon_k$ we have $x^{-1} * y \in V_k$, or equivalently, $y \in x * V_k$. But this implies $S \subset x * V_k$. Since $x * V_k$ is an element of $\mathcal{V}_k$, the Claim is proven. □

Now apply the hypothesis that H has $\mathcal{O}_{HG}$-measure zero with respect to the left-invariant metric d. For each k choose a finite set $\mathcal{I}_k$ of subsets of H, each such subset of diameter less than ϵ_k, such that $\bigcup_{k<\infty} \mathcal{I}_k$ covers G. Then for each k choose a finite subset $\mathcal{H}_k$ of $\mathcal{V}_k$ such that each element of $\mathcal{I}_k$ is contained in an element of $\mathcal{H}_k$ (this uses the Claim). Finally, for each element of $\mathcal{H}_k$, choose an element of $\mathcal{O}(U_k)$ containing it. This defines a finite subset $\mathcal{G}_k$ of $\mathcal{O}(U_k)$, and $\bigcup_{k<\infty} \mathcal{G}_k$ is a cover of G by sets open in H. □

Observe that whereas zero-dimensionality is a hypothesis in Theorem 11, it is not a hypothesis in Theorem 12.

3. Rothberger-bounded groups

Recall that a topological group $(H, *)$ is said to be a Rothberger-bounded group if it satisfies $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_H)$.

Theorem 13. *Let $(G, *)$ be a subgroup of the topological group $(H, *)$. Then the following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.
(2) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(G), \mathcal{O}_G)$.

Proof. The implication (2) $\Rightarrow$ (1) is evidently true. We must show that (1) $\Rightarrow$ (2). Thus, let $(\mathcal{O}(U_n)\colon n \in \mathbb{N})$ be a sequence in $\mathcal{O}_{\mathsf{nbd}}(G)$. Since each U_n is a neighborhood (in G) of the group identity we may choose for each n a neighborhood T_n in H of the identity, such that $U_n = T_n \cap G$. Next, choose for each n a neighborhood S_n in H of the identity, such that $S_n^{-1} * S_n \subseteq T_n$.

Apply (2) to the sequence $(\mathcal{O}(S_n)\colon n \in \mathbb{N})$ of elements of $\mathcal{O}_{\mathsf{nbd}}(H)$: We find for each n a point $x_n \in H$ such that $G \subseteq \bigcup_{n\in\mathbb{N}} x_n * S_n$.

For each n choose a $p_n \in G$ as follows:

$$p_n \begin{cases} \in G \cap x_n * S_n & \text{if nonempty,} \\ = \text{ group identity} & \text{otherwise.} \end{cases}$$

Observe that for each n we have $x_n * U_n \in \Omega(U_n) \in \Omega_{\mathsf{nbd}}(G)$. We show that $G = \bigcup_{n\in\mathbb{N}} p_n * U_n$.

For let $g \in G_n$ be given. Choose n so that $g \in x_n * S_n$. Then evidently $G \cap x_n * S_n$ is nonempty, and so $p_n \in G$ is defined as an element of this intersection. Since $p_n \in x_n * S_n$, we have $x_n \in p_n * S_n^{-1}$, and so $g \in p_n * S_n^{-1} * S_n \subseteq p_n * T_n$. Now $p_n^{-1} * g \in G \cap T_n = U_n$, and so we have $g \in p_n * U_n$. This completes the proof. □

The next few concepts are useful in analyzing the product properties for Rothberger-bounded groups. For an open neighborhood U of the identity element 1_H and for $n \in \mathbb{N}$,

$$\mathcal{O}_H^n(U) = \{F * U\colon F \subset H \text{ and } 1 \leqslant |F| \leqslant n\}.$$

The symbol $\mathcal{O}^n_{\mathsf{nbd}}(H)$ denotes the collection of all open covers of the form $\mathcal{O}_H^n(U)$ of the group H. According to [9]: For a positive integer n, an open cover $\mathcal{U}$ of a topological space is said to be an n-cover if there is for each n-element subset F of the space a $U \in \mathcal{U}$ with $F \subseteq U$. Thus, each $\mathcal{O}_H^n(U)$ is an n-cover of the group H.

Now let $(R_n\colon n \in \mathbb{N})$ be a sequence of natural numbers, diverging to ∞. According to notation introduced in [9], the symbol $\mathsf{S}_1(\{\mathcal{O}^{R_n}_{\mathsf{nbd}}(H)\colon n \in \mathbb{N}\}, \Omega_{HG})$ denotes the statement:

For each sequence $(\mathcal{O}_H^{R_n}(U_n)\colon n \in \mathbb{N})$, where each U_n is an open neighborhood of 1_H, there is a sequence $(V_n\colon n \in \mathbb{N})$ such that: For each n, $V_n \in \mathcal{O}^{R_n}(U_n)$, and $\{V_n\colon n \in \mathbb{N}\} \in \Omega_{HG}$.

The following theorem is reminiscent of the "quantifier elimination theorem", Theorem 2.2, of [9].

Theorem 14. *For a topological group* $(H, *)$ *and subgroup* $(G, *)$ *the following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG}^{\mathrm{wgp}})$.
(2) $\mathsf{S}_1(\{\mathcal{O}^n_{\mathsf{nbd}}(H)\colon n \in \mathbb{N}\}, \Omega_{HG})$.
(3) *For each sequence* $(R_n\colon n \in \mathbb{N})$ *of natural numbers diverging to* ∞, $\mathsf{S}_1(\{\mathcal{O}^{R_n}_{\mathsf{nbd}}(H)\colon n \in \mathbb{N}\}, \Omega_{HG})$.

Proof. (1) $\Rightarrow$ (2): For each n, put $\mathcal{U}_n = \mathcal{O}^n(U_n)$. For each n put $V_n = \bigcap_{j\leqslant n} U_j$ and for each n put $\mathcal{V}_n = \mathcal{O}(V_n)$. Apply $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG}^{\mathrm{wgp}})$ to the sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$. For each n choose an x_n such that $\{x_n * V_n\colon n \in \mathbb{N}\}$ is in $\mathcal{O}_{HG}^{\mathrm{wgp}}$. Choose a sequence $m_1 < m_2 < \cdots < m_k < m_{k+1} < \cdots$ such that: For each finite set $F \subset G$ there is an n with $F \subseteq \bigcup_{m_n \leqslant j < m_{n+1}} x_j * V_j$. For each such i define

$$F_i = \begin{cases} \{x_j\colon j \leqslant i\} & \text{if } i < m_1, \\ \{x_j\colon m_n \leqslant j \leqslant i\} & \text{if } m_n \leqslant i < m_{n+1}. \end{cases}$$

Then put $S_n = F_n * U_n$. For each n we have $S_n \in \mathcal{O}_H^n(U_n)$, and $\{S_n\colon n \in \mathbb{N}\}$ is an ω-cover of G.

(2) $\Rightarrow$ (3): Choose $1 < k_1 < k_2 < \cdots < k_n < \cdots$ such that $(\forall j \geqslant k_n)\ (R_j \geqslant n)$. Let $\mathcal{U}_n = \mathcal{O}_H^{R_n}(U_n)$, $n \in \mathbb{N}$ be given. Define $V_1 = \bigcap_{j\leqslant k_1} U_j$, and for each n put $V_{n+1} = \bigcap_{k_n \leqslant j < k_{n+1}} U_j$. Then apply (2) to $(\mathcal{O}_H^n(V_n)\colon n \in \mathbb{N})$. For each n choose $F_n \subset G$ with $|F_n| \leqslant n$. With $S_n = F_n * G_n$, $n \in \mathbb{N}$, the set $\{S_n\colon n \in \mathbb{N}\} \in \Omega_{HG}$.

(3) $\Rightarrow$ (1): Let $(R_n\colon n \in \mathbb{N})$ be given. Choose $1 \leqslant n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ such that $R_1 \leqslant n_1$ and for each k, $R_{k+1} \leqslant (n_{k+1} - n_k)$.

Now, let $\mathcal{U}_n = \mathcal{O}_H(U_n)$ be given for each n. Define $V_1 = \bigcap_{j\leqslant n_1} U_j$, and $V_{k+1} = \bigcap_{n_k < j \leqslant n_{k+1}} U_j$. Then put $\mathcal{V}_n = \mathcal{O}_H^{R_n}(V_n)$, $n \in \mathbb{N}$. Apply (3) to the sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ and choose for each n an $S_n \in \mathcal{V}_n$ so that $\{S_n\colon n \in \mathbb{N}\} \in \Omega_{HG}$. For each n choose a finite set $F_n \subseteq H$ such that $|F_n| \leqslant R_n$, and $S_n = F_n * V_n$.

For each m write $F_m = \{x_{n_m+1}, \dots, x_{n_{m+1}}\}$ with repetitions, if necessary. Then

$$(x_k * U_k\colon k < \infty)$$

is a sequence with $x_k * U_k \in \mathcal{U}_k$ for each k, the sequence of n_j's witness the weak groupability of $\{x_k * U_k\colon k < \infty\}$. □

Now we characterize being Rothberger bounded in all finite powers in terms of weak groupability:

Theorem 15. *For a subgroup $(G, *)$ of a topological group $(H, *)$ the following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}^{\mathrm{wgp}}_{HG})$.
(2) *For each n*, $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H^n), \mathcal{O}^{\mathrm{wgp}}_{H^nG^n})$.
(3) *For each n*, $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H^n), \mathcal{O}_{H^nG^n})$.

Proof. (1) ⇒ (2): Fix $n > 1$ and consider $H^n = H \times \cdots \times H$ (n copies). For each m let $\mathcal{U}_m = \mathcal{O}^{m^m}(U_{m,1} \times \cdots \times U_{m,n})$ be given and define $W_m = \bigcap_{j \leqslant n} U_{m,j}$, a neighborhood of 1_H. For each m choose a finite set $F_m \subset H$ such that $|F_m| \leqslant m$, and such that $\{F_m * W_m\colon m < \infty\} \in \Omega_{HG}$. This is possible since $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}^{\mathrm{wgp}}_{HG})$ implies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}^{\mathrm{wgp}}_{HG})$ which implies $\mathsf{S}_1(\Omega_{\mathsf{nbd}}(H), \Omega_{HG})$, as we saw in Theorem 6.

Then for each m put $G_m = F_m \times \cdots \times F_m$ (n copies), $m < \infty$. Then put $S_m = G_m * (U_{m,1} \times \cdots \times U_{m,1})$. For each m we have $S_m \in \mathcal{O}^{m^m}(U_{m,1} \times \cdots \times U_{m,n})$ and we have $\{S_m\colon m < \infty\} \in \Omega_{HG}$. By (3) ⇒ (1) of Theorem 14, $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H^n), \mathcal{O}^{\mathrm{wgp}}_{H^nG^n})$ holds.

(2) ⇒ (3): This implication is clear.

(3) ⇒ (1): For each n let $\mathcal{U}_n = \mathcal{O}^n(U_n)$ be given so that U_n is a neighborhood in H of 1_H. Write $\mathbb{N} = \bigcup_{k<\infty} Y_k$ where for each k, $k \leqslant \min(Y_k)$ and Y_k is infinite, and for $m \neq n$, $Y_m \cap Y_n = \emptyset$.

For each k: For $m \in Y_k$ put $\mathcal{V}_m = \mathcal{O}(U^k_m)$. Then $(\mathcal{V}_m\colon m \in Y_k)$ is a sequence from $\mathcal{O}_{\mathsf{nbd}}(H^k)$. Applying (3) choose for each $m \in Y_k$ an $x_m \in H^k$ such that $\{x_m * U^k_m\colon m \in Y_k\}$ is a cover of G^k. For each m in Y_k write $x_m = (x_m(1), \dots, x_m(k))$, and then put $\phi(x_m) = \{x_m(1), \dots, x_m(k)\}$. Observe that for each $m \in Y_k$ we have $|\phi(x_m)| \leqslant k \leqslant m$, and so $\phi(x_m) * U_m$ is in $\mathcal{O}^m(U_m)$. For each m put $V_m = \phi(x_m) * U_m$, a member of $\mathcal{O}^m(U_m) = \mathcal{V}_m$.

We claim that $\{V_m\colon m < \infty\}$ is in Ω_{HG}. For let $F \subset G$ be finite and put $k = |F|$. Write $F = \{f_1, \dots, f_k\}$. Consider $x = (f_1, \dots, f_k) \in G^k$. For some $m \in Y_k$ we have $x \in x_m * U^k_m$, and so $F \subset \phi(x_m) * U_m = V_m$. Now (2) ⇒ (1) of Theorem 14 implies that $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}^{\mathrm{wgp}}_{HG})$ holds. □

Using the techniques of this paper one also proves:

Theorem 16. *For a subgroup $(G, *)$ of a topological group $(H, *)$ the following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}^{\mathrm{wgp}}_{HG})$.
(2) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(G), \mathcal{O}^{\mathrm{wgp}}_{G})$.

3.1. The metrizable case

Recall that a metric space (X, d) has Borel *strong measure zero* if there is for each sequence $(\epsilon_n\colon n \in \mathbb{N})$ of positive reals a sequence $(X_n\colon n \in \mathbb{N})$ of subsets of X such that for each n we have $\mathrm{diam}(X_n) < \epsilon_n$, and $X = \bigcup_{n\in\mathbb{N}} X_n$.

We now show that metrizable Rothberger-bounded groups have a characterization in terms of strong measure zero which is quite analogous to the one for the Rothberger property of metrizable spaces. For metrizable spaces one has the following characterization:

Theorem 17 *(Fremlin, Miller). For a metrizable space X the following are equivalent*:

(1) *The space has the property* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.
(2) *The space has Borel strong measure zero with respect to each metrizing metric of the space.*

The relative version of this was proved in [8] and is as follows:

Theorem 18. *For a subspace G of σ-compact metrizable space H the following are equivalent*:

(1) *The space has the property* $\mathsf{S}_1(\mathcal{O}_H, \mathcal{O}_{HG})$.
(2) *The subspace G has Borel strong measure zero with respect to each metrizing metric of the space H.*

Here is the version for metrizable groups and Rothberger boundedness:

Theorem 19. *Let $(G, *)$ be a subgroup of the metrizable topological group $(H, *)$. The following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$ *holds.*
(2) *G has Borel strong measure zero in each left-invariant metrization of H.*

Proof. (1) $\Rightarrow$ (2): Let a sequence $(\epsilon_n\colon n \in \mathbb{N})$ of positive real numbers be given. Choose a left-invariant metrization d of H. For each n, choose an H-neighborhood U_n of 1_H which is of d-diameter less than ϵ_n. Then for each n put $\mathcal{U}_n = \mathcal{O}_H(U_n)$, an element of $\mathcal{O}_{\mathsf{nbd}}(H)$. Apply $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$ and select for each n a $T_n \in \mathcal{U}_n$ such that $\{T_n\colon n \in \mathbb{N}\}$ is a cover of G by sets open in H. Since the metric d is left-invariant, for each n we have $\mathrm{diam}_d(T_n) < \epsilon_n$. It follows that G has strong measure zero in the metric d.

(2) $\Rightarrow$ (1): Let a sequence $(\mathcal{O}(U_n)\colon n \in \mathbb{N})$ of elements of $\mathcal{O}_{\mathsf{nbd}}(H)$ be given. For each n choose a symmetric neighborhood V_n of the identity element such that

(1) For each n: $V_n \subseteq (\bigcap_{j<n}(U_j \cap V_j)) \cap U_n$;
(2) For each n: $V_{n+1}^2 \subset V_n$;
(3) $\{V_n\colon n \in \mathbb{N}\}$ is a neighborhood basis for the identity element.

For each n, put $\mathcal{V}_n = \mathcal{O}(V_n)$, a member of $\mathcal{O}_{\mathsf{nbd}}(H)$. Let d be a left-invariant metric associated to the sequence $(V_k\colon k < \infty)$ as per Theorem 10. For each k put $\epsilon_k = (\frac{1}{2})^k$. As in the proof of Theorem 12 we have: If a subset S of H has diameter less than ϵ_k, then there is a $W \in \mathcal{V}_k$ with $S \subset W$.

Now apply the hypothesis that G has strong Borel measure zero with respect to the left-invariant metric d. For each k choose a subset G_k of G of diameter less than ϵ_k, such that $\bigcup_{k<\infty} G_k = G$. Then for each k choose an element H_k of $\mathcal{V}_k$ such that $G_k \subset H_k$. Finally, for each H_k, choose an element V_k of $\mathcal{O}(U_k)$ containing it. Then $\{V_k\colon k < \infty\}$ is a cover of G by sets open in H. □

Using earlier results about finite powers, we have:

Theorem 20. *For a subgroup $(G, *)$ of a metrizable group $(H, *)$ the following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG}^{\mathrm{wgp}})$.
(2) *For each n, G^n has Borel strong measure zero in each left-invariant metric on H^n.*

Proof. By Theorems 15 and 19. □

Indeed, using the techniques of this paper one can prove that $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(G), \mathcal{O}_G^{\mathrm{wgp}})$ is, for metrizable groups G, equivalent to the following strengthened form of Borel strong measure zero: For each sequence $(\epsilon_n\colon n \in \mathbb{N})$ and left-invariant metric d, there is a sequence $(S_n\colon n \in \mathbb{N})$ of subsets of G such that for each n we have $\mathrm{diam}_d(S_n) < \epsilon_n$, and $\{S_n\colon n \in \mathbb{N}\}$ is a weakly groupable cover of G.

Lemma 21. *Let $(H, *)$ be a topological group, and let G be a dense subset (not necessarily a subgroup) of H. Let $V \subset H$ be an open neighborhood of 1_H. Then*:

(1) $\{F * V\colon F \subset G \text{ finite}\}$ *is an ω-cover of H.*
(2) $\{x * V\colon x \in G\}$ *is an open cover of H.*

Proof. Let U be a neighborhood of 1_H. By Theorem 1 pick a neighborhood W of 1_H such that $W^{-1} = W \subseteq U$.

(1) Consider any finite set $K \subset H$. Since G is dense in H, choose for $k \in K$ a $g_k \in G \cap k * W$. Then $F = \{g_k\colon k \in K\}$ is a finite subset of G and $K \subset F * U$.
(2) Consider any point $x \in H$. Since G is dense in H, choose a $g_x \in G \cap x * W$. Then $x \in g_x * W$. □

Lemma 21 is used in the following theorem:

Theorem 22. *Let $(G, *)$ be a subgroup of a σ-compact metrizable group $(H, *)$. The following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.
(2) *ONE has no winning strategy in* $\mathsf{G}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$.

Proof. (2) ⇒ (1): The argument for this direction is standard.

(1) ⇒ (2): We give the argument in the case when H is compact. A number of small but tedious modifications of the argument gives the proof for H σ-compact. See for example the proof of Theorem 9 in [8].

Let G be a subgroup of the metrizable compact group H. Since the closure of G in H is also a compact subgroup of H, we may assume that G is dense in H. Let d be any left-invariant metric of H.

Let F be a strategy for ONE in the game $\mathsf{G}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$ played on G. In the first inning ONE plays $F(\emptyset) = \mathcal{O}(U_\emptyset)$, an open cover of H. Since H is compact, fix a finite subset G_1 of G such that $\{x * U(H)_\emptyset\colon x \in G_\emptyset\}$ is a covering for H and a positive real number ϵ_1 which is a Lebesgue number for this finite cover relative to the metric d. Enumerate $G_\emptyset$ bijectively as $(x^0_j\colon j \leqslant n_\emptyset)$.

For $1 \leqslant j \leqslant n_\emptyset$ put $T_j = x^0_j * U_\emptyset$. For each T_j, a response by TWO, consider ONE's move $F(T_j) = \mathcal{O}(U_j)$. Then $\{x * U_j\colon x \in G\}$ is an open cover of H. Since H is compact, fix a finite set $G_j \subset G$ such that $\{x * U_j\colon x \in G_j\}$ is a cover for H, and fix a positive real number δ_j which is a Lebesgue number for this finite cover of H relative to the metric d. Put $\epsilon_2 = \min\{\delta_j\colon 1 \leqslant j \leqslant n_\emptyset\}$. Enumerate each G_j bijectively as $(x_{j,1}, \ldots, x_{j,n_j})$.

For $j_1 \leqslant n_\emptyset$ and $j_2 \leqslant n_{j_1}$ put $T_{j_1,j_2} = x_{j_1,j_2} * U_{j_1}$. Then ONE's response using the strategy F is $F(T_{j_1}, T_{j_1,j_2}) = \mathcal{O}(U_{j_1,j_2})$. Now $\{x * U_{j_1,j_2}\colon x \in G\}$ is an open cover of the compact space H. Choose a finite subset G_{j_1,j_2} of G and a positive real number δ_{j_1,j_2} such that $\{x * U_{j_1,j_2}\colon x \in G_{j_1,j_2}\}$ is a cover of H and has Lebesgue covering number δ_{j_1,j_2} relative to the metric d. Put $\epsilon_3 = \min(\{\delta_{j_1,j_2}\colon j_1 \leqslant n_\emptyset \text{ and } j_2 \leqslant n_{j_1}\} \cup \{\epsilon_2\})$. Enumerate each G_{j_1,j_2} bijectively as $\{x_{j_1,j_2,j}\colon j \leqslant n_{j_1,j_2}\}$.

For $j_1 \leqslant n_\emptyset$ and $j_2 \leqslant n_{j_1}$, and $j_3 \leqslant n_{j_1,j_2}$, put $T_{j_1,j_2,j_3} = x_{j_1,j_2,j_3} * U_{j_1,j_2}$. Then ONE's response using the strategy F is $\mathcal{O}(U_{j_1,j_2,j_3}) = F(T_{j_1}, T_{j_1,j_2}, T_{j_1,j_2,j_3})$. The set $\{x * U_{j_1,j_2,j_3}\colon x \in G\}$ is an open cover of H. Choose a finite subset G_{j_1,j_2,j_3} of G such that the finite subset $\{x * U_{j_1,j_2,j_3}\colon x \in G_{j_1,j_2,j_3}\}$ of the latter cover of H is a cover of the compact set H. Then choose a positive real number δ_{j_1,j_2,j_3} which is a Lebesgue number for this finite cover of H relative to the metric d. Finally, put $\epsilon_4 = \min(\{\delta_{j_1,j_2,j_3}\colon j_1 < n_\emptyset,\ j_2 < n_{j_1},\ j_3 < n_{j_2}\} \cup \{\epsilon_3\})$ and enumerate each G_{j_1,j_2,j_3} bijectively as $\{x_{j_1,j_2,j_3,j}\colon j \leqslant n_{j_1,j_2,j_3}\}$.

Continuing in this way we obtain the following families:

(1) $(n_\sigma\colon \sigma \in {}^{<\omega}\mathbb{N})$ of positive integers;
(2) $(U_\sigma\colon \sigma \in {}^{<\omega}\mathbb{N}$ and for $i \in \mathrm{dom}(\sigma)$, $\sigma(i) \leqslant n_{\sigma\lceil_i})$ of open neighborhoods in H of 1_H;
(3) $(x_\sigma\colon \sigma \in {}^{<\omega}\mathbb{N}$ and for $i \in \mathrm{dom}(\sigma)$, $\sigma(i) \leqslant n_{\sigma\lceil_i})$ of elements of G;
(4) $(\delta_\sigma\colon \sigma \in {}^{<\omega}\mathbb{N}$ and for $i \in \mathrm{dom}(\sigma)$, $\sigma(i) \leqslant n_{\sigma\lceil_i})$ of positive real numbers;
(5) $(\epsilon_k\colon k < \infty)$ of positive real numbers;
(6) $(G_\sigma\colon \sigma \in {}^{<\omega}\mathbb{N}$ and for $i \in \mathrm{dom}(\sigma)$, $\sigma(i) \leqslant n_{\sigma\lceil_i})$ of finite subsets of G; and
(7) $(T_\sigma\colon \sigma \in {}^{<\omega}\mathbb{N}$ and for $i \in \mathrm{dom}(\sigma)$, $\sigma(i) \leqslant n_{\sigma\lceil_i})$ of open subsets of H

such that:

- For each $\sigma \in {}^{<\omega}\mathbb{N}$ for which for $i \in \mathrm{dom}(\sigma)$, $\sigma(i) \leqslant n_{\sigma\lceil_i}$,

$$T_\sigma = x_\sigma * U_{\sigma\lceil_{\mathrm{dom}(\sigma)-1}};$$

- $\mathcal{O}(U_\sigma) = F(T_{\sigma(1)}, T_{\sigma(1),\sigma(2)}, \ldots, T_\sigma)$;
- $G_\sigma = \{x_{\sigma\frown j}\colon j \leqslant n_\sigma\}$;
- $\{x_{\sigma\frown j} * U_{\sigma\frown j}\colon j \leqslant n_\sigma\}$ is an open cover of H, with Lebesgue number δ_σ relative to the chosen metric d;
- $\epsilon_k = \min\{\delta_\sigma\colon \sigma \in {}^{\leqslant k}\mathbb{N}$ and for $i \in \mathrm{dom}(\sigma)$, $\sigma(i) \leqslant n_{\sigma\lceil_i}\}$.

Since G has Borel measure zero relative to the metric d, choose for the sequence $(\epsilon_k\colon k < \infty)$ of positive reals a partition $G = \bigcup_{k<\infty} X_k$ such that for each k we have $\mathrm{diam}_d(X_k) < \epsilon_k$.

Since $\mathrm{diam}_d(X_1) < \epsilon_1$, choose a $j_1 < n_\emptyset$ such that $X_1 \subset x_{j_1} * U_\emptyset$. Then also $X_1 \subseteq x_{j_1} * U_\emptyset = T_{j_1}$. Since $\mathrm{diam}_d(X_2) < \epsilon_2 < \delta_{j_1}$, choose a $j_2 < n_{j_1}$ such that $X_2 \subset x_{j_1,j_2} * U_{j_1}$. Then also $X_2 \subset x_{j_1,j_2} * U_{j_1} = T_{j_1,j_2}$. Continuing like this, using the fact that $\mathrm{diam}_d(X_k) < \epsilon_k$ and that $j_1, \ldots, j_k$ have been selected, we choose $j_{k+1} \leqslant n_{j_1,\ldots,j_k}$ such that $X_{k+1} \subseteq x_{j_1,\ldots,j_{k+1}} * U_{j_1,\ldots,j_k} = T_{j_1,\ldots,j_{k+1}}$. In this way we obtain a sequence $(j_1, j_2, \ldots, j_k, \ldots)$ where for each k we have $j_{k+1} \leqslant n_{j_1,\ldots,j_k}$ and $X_k \subseteq T_{j_1,\ldots,j_k}$.

But then the play

$$F(\emptyset), T_{j_1}, F(T_{j_1}), T_{j_1,j_2}, F(T_{j_1}, T_{j_1,j_2}), \ldots$$

is lost by ONE. Since F was an arbitrary strategy for ONE, it follows that ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}_{HG})$. □

Theorem 23. *For a subgroup $(G, *)$ of a σ-compact metrizable group $(H, *)$ the following are equivalent*:

(1) *ONE has no winning strategy in the game* $\mathsf{G}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}^{\mathrm{wgp}}_{HG})$.
(2) $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(H), \mathcal{O}^{\mathrm{wgp}}_{HG})$.
(3) *For each n,* $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}(G^n), \mathcal{O}_{G^n})$.

In Theorem 8.5 of [7] the authors show that under the appropriate set theoretic hypothesis (the Continuum Hypothesis is an example of one) there is a σ-compact zero-dimensional metrizable group H which contains a subgroup G which is Rothberger bounded in all finite powers, but as topological space G does not have the property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_G, \mathcal{O}_G)$ (and thus does not have property $\mathsf{S}_1(\mathcal{O}_G, \mathcal{O}_G)$).

4. A question about consistency

The group obtained in ZFC in Theorem 1 of [10] is a group of cardinality $\mathfrak{b}$, and has property $\mathsf{S}_{\mathrm{fin}}(\Omega_{\mathsf{nbd}}, \Omega)$. We have not found explicit examples of groups satisfying $\mathsf{S}_1(\Omega_{\mathsf{nbd}}, \mathcal{O})$, but not $\mathsf{S}_1(\Omega_{\mathsf{nbd}}, \Omega)$, in the literature. A few questions in this connection come to mind. We mention only the following two:

Problem 1. Is it consistent that there be a metrizable group having the property $\mathsf{S}_1(\Omega_{\mathsf{nbd}}, \mathcal{O})$, but which does not have this property in all finite powers?

Problem 2. Is it consistent that there be a metrizable group with property $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ which has $\mathsf{S}_1(\Omega_{\mathsf{nbd}}, \mathcal{O})$ in all finite powers, but does not have $\mathsf{S}_1(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ in all finite powers?

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Further reading

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