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Homoclinic orbits for second order self-adjoint difference equations

Manjun Ma [a,*], Zhiming Guo [b]

[a] Department of Mathematics and Physics, Nanhua University, Hengyang, Hunan 421001, People's Republic of China
[b] School of Mathematics and Information Science, Guangzhou University, Guangzhou, Guangdong 510405, People's Republic of China

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Abstract

In this paper we discuss how to use variational methods to study the existence of nontrivial homoclinic orbits of the following nonlinear difference equations

$$\Delta\big[p(t)\Delta u(t-1)\big]+q(t)u(t)=f\big(t,u(t)\big),\quad t\in Z,$$

without any periodicity assumptions on $p(t)$, $q(t)$ and f, providing that $f(t,x)$ grows superlinearly both at origin and at infinity or is an odd function with respect to $x\in R$, and satisfies some additional assumptions.

Keywords: Discrete variational methods; Homoclinic orbits; Difference equations

1. Introduction

A lot of attention has been devoted in recent years to find periodic solutions of discrete dynamic models, for example, see [2,8–10,15]. It is our purpose in the present work to find other types of solutions, namely the doubly asymptotic solutions, first discovered by Poincaré [14] in continuous Hamiltonian systems. Hereafter this problem has always been an active subject, see [1,6,7,11,13,17,18] and the reference therein. Such doubly asymptotic solutions are called homoclinic solutions (orbits). Similarly we give the below definition: If $\bar{x}$ is a periodic solu-

* Corresponding author.
E-mail address: mjunm9@hotmail.com (M. Ma).

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tion of a discrete system, another solution z will be called homoclinic orbit emanating from $\bar{x}$ if $|z(t)-\bar{x}| \to 0$ when $t \to \pm\infty$.

In this paper, we consider the existence of nontrivial homoclinic orbits emanating from 0 of the equation

$$\Delta\big[p(t)\Delta u(t-1)\big]+q(t)u(t)=f\big(t,u(t)\big), \quad t \in Z, \tag{1.1}$$

where $p(t)$ and $q(t)$ are real valued on Z and $p(t)$ is nonzero for each $t \in Z$, $f(t,x): Z \times R \to R$ is continuous in x, the forward difference operator Δ is defined by $\Delta u(t)=u(t+1)-u(t)$. As usual, N, Z, and R denote the set of all natural numbers, integers and real numbers, respectively. For $a,b \in Z$, denote $Z(a)=\{a,a+1,\ldots\}$, $Z(a,b)=\{a,a+1,\ldots,b\}$ when $a \leqslant b$.

It is clear that Eq. (1.1) can be written as an equivalent first order nonlinear nonautonomous discrete Hamiltonian system

$$\Delta X(t)=J\nabla H_X\big(t,u(t+1),z(t)\big), \tag{1.2}$$

where $X(t)=(u(t),z(t))^T$; $z(t)$ is a discrete momentum variable defined by $z(t)=p(t)\times \Delta u(t-1)$; $H(t,X(t))=\frac{1}{2p(t)}z^2+\frac{1}{2}q(t)u^2(t)-F(t,u(t))$ is called the Hamiltonian function, $F(t,x)=\int_0^x f(t,s)\,ds$; $J=\left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right]$ is the normal symplectic matrix.

Moreover, Eq. (1.1) is a discretization of the following second order differential equation

$$\big(p(t)x'(t)\big)'+q(t)x(t)=f\big(t,x(t)\big) \tag{1.3}$$

which is also equivalent to a first order nonlinear Hamiltonian system. (1.1) and (1.2) are best approximations of (1.3) and its equivalent Hamiltonian system, respectively, when one lets the stepsize not equal to 1 but the variable stepsize go to zero. It is very verified that system (1.2) preserves the symplectic structure. So their solutions can give some desirable numerical features for the corresponding continuous Hamiltonian solutions. Equation (1.3) can be regarded as the more general form of the Emden–Fowler equation appearing in the study of astrophysics, gas dynamics, fluid mechanics, relativistic mechanics, nuclear physics and chemically reacting system in terms of various special forms of $f(t,x(t))$, for example, see [19] and the reference therein. In this survey paper, many well-known results concerning properties of solutions of (1.3) are collected. On the other hand, Eq. (1.1) do have its applicable setting as evidenced by the monograph [4,5], as mentioned in which when $f(t,x)\equiv 0$ for $(t,x)\in Z\times R$, Eq. (1.1) becomes the second order self-adjoint difference equation

$$Lu(t)\equiv \Delta\big[p(t)\Delta u(t-1)\big]+q(t)u(t)=0, \quad t\in Z, \tag{1.4}$$

which is in some ways a type of best expressive ways of the structure of the solution space for recurrence relations occurring in the study of series solutions of second order linear differential equations. So Eq. (1.4) arises with high frequency in various fields such as optimal control, filtering theory and discrete variational theory. Many authors have extensively studied its disconjugacy, disfocality, Boundary Value Problem, oscillation and asymptotic behavior. When $f(t,x(t))\equiv f(t)$, Eq. (1.1) was discussed on its Boundary Value Problem and uniqueness of solutions. Therefore, it is of practical importance and mathematical significance to study Eq. (1.1).

Yu and Guo recently dealt with the existence of periodic solutions for Eq. (1.1) by applying the Critical Point Theory in [20]. The variational methods has also been used to deal with the Boundary Value Problems of difference equations, for example, see [3]. In this paper we show that the Palais–Smale condition is satisfied on the unbounded domain and we use the usual Mountain Pass Theorem to prove the existence of a homoclinic orbit of (1.1). Moreover, if $f(t,.)$

is an odd function, we prove Eq. (1.1) possesses an unbounded sequence of homoclinic orbits emanating from 0 by invoking the "symmetric" Mountain Pass Theorem.

Throughout the paper, it will be assumed that f satisfies

(f_1) $\lim_{x\to 0}\frac{f(t,x)}{x}=0$ uniformly for $t\in Z$;

(f_2) there exist a constant $\beta>2$ such that

$$xf(t,x)\leqslant \beta\int_0^x f(t,s)\,ds<0 \tag{1.5}$$

for all $(t,x)\in Z\times R\setminus\{0\}$.

And further assume that

(p) $p(t)>0$ for all $t\in Z$;

(q) $q(t)<0$ for all $t\in Z$ and $\lim_{|t|\to\infty} q(t)=-\infty$.

Remarks.

(1) (f_1) implies that $u(t)\equiv 0$ is a trivial homoclinic solution of (1.1).

(2) (f_2) implies that for each $t\in Z$ there is a real function $\alpha(t)>0$ such that

$$\int_0^x f(t,s)\,ds\leqslant -\alpha(t)|x|^\beta \quad \text{for } |x|\geqslant 1. \tag{1.6}$$

(1.5) and (1.6) imply

$$\lim_{|x|\to\infty}\frac{f(t,x)}{x}=-\infty \quad \text{uniformly for } t\in Z. \tag{1.7}$$

This together with (f_1) shows that $f(t,x)$ grows superlinearly both at infinity and at origin.

2. Preliminaries

In this section, we will establish the corresponding variational framework for (1.1) and some technical lemmas which will play an important role in the proof of our main results.

Let S be the vector space of all real sequences of the form

$$u=\{u(t)\}_{t\in Z}=\big(\dots,u(-t),u(-t+1),\dots,u(-1),u(0),u(1),\dots,u(t),\dots\big),$$

namely

$$S=\big\{u=\{u(t)\}\ \big|\ u(t)\in R,\ t\in Z\big\}.$$

Define

$$E=\left\{u\in S\ \middle|\ \sum_{t=-\infty}^{+\infty}\big[p(t)\big(\Delta u(t-1)\big)^2-q(t)\big(u(t)\big)^2\big]<+\infty\right\}.$$

The space E is a Hilbert space with the inner product

$$\langle u,v\rangle=\sum_{t=-\infty}^{+\infty}\big[p(t)\Delta u(t-1)\Delta v(t-1)-q(t)u(t)v(t)\big]$$

for any $u, v \in E$, and the corresponding norm

$$\|u\|_E^2 = \langle u, u\rangle = \sum_{t=-\infty}^{+\infty} \left[p(t)\big(\Delta u(t-1)\big)^2 - q(t)\big(u(t)\big)^2\right], \quad \forall u \in E.$$

In what follows, l_I^2 and l^2 denote the space of functions whose second powers are summable on the interval I and $(-\infty, +\infty)$ equiped with the norm

$$\|u\|_I^2 = \sum_{t\in I}\left|u(t)\right|^2, \qquad \|u\|^2 = \sum_{t\in Z}\left|u(t)\right|^2,$$

respectively, $|.|$ is the usual absolute value in R. For $u \in E$, let

$$\begin{aligned}\varphi(u) &= \sum_{t=-\infty}^{+\infty}\left[\frac{1}{2}p(t)\big(\Delta u(t-1)\big)^2 - \frac{1}{2}q(t)\big(u(t)\big)^2 + F\big(t, u(t)\big)\right] \\ &= \frac{1}{2}\|u\|_E^2 + \sum_{t=-\infty}^{+\infty} F\big(t, u(t)\big), \end{aligned} \tag{2.1}$$

where $F(t,x) = \int_0^x f(t,s)\,ds$, and for any $u \in E$, we can compute the Fréchet derivative of (2.1) as

$$\frac{\partial I(u)}{\partial u(t)} = -\Delta\big[p(t)\Delta u(t-1)\big] - q(t)u(t) + f\big(t, u(t)\big), \quad t \in Z.$$

Thus, u is a critical point of φ on E if and only if u is a classical solution of Eq. (1.1) with $u(\pm\infty) = \Delta u(\pm\infty) = 0$. This means that the functional φ is just the variational framework of (1.1).

Now we prove two technical lemmas.

Lemma 2.1. *Assume that a sequence $\{u_k\} \subset E$ such that $u_k \rightharpoonup u$ in E, then the sequence u_k satisfies $u_k \to u$ in l^2.*

Proof. Suppose, without loss of generality, that $u_k \rightharpoonup 0$ in E. The Banach–Steinhaus theorem implies that

$$A = \sup_k \|u_k\|_E < +\infty.$$

By (q), for any $\varepsilon > 0$, there is $T_0 < 0$ such that $-\frac{1}{q(t)} \leqslant \varepsilon$ for all $t \in Z(-\infty, T_0)$. Similarly, there is $T_1 > 0$ such that $-\frac{1}{q(t)} \leqslant \varepsilon$ for all $t \in Z(T_1)$. By (p) and (q), clearly $u_k \rightharpoonup 0$ in E_I, where $E_I = \{u \in S_I \mid \sum_{t\in I}[p(t)(\Delta u(t-1))^2 - q(t)(u(t))^2] < +\infty\}$, $S_I = \{u = u(t) \mid u(t) \in R,\ t \in I\}$ and $I = Z(T_0, T_1)$. So $\{u_k\}$ is bounded in E_I, which implies $\{u_k\}$ is bounded in l_I^2. This together with the uniquence of the weak limit in l_I^2, we have $u_k \to 0$ on I, so there is a k_0 such that

$$\sum_{t\in I}\left|u_k(t)\right|^2 \leqslant \varepsilon \quad \text{for all } k \geqslant k_0. \tag{2.2}$$

Since $-\frac{1}{q(t)} \leqslant \varepsilon$ on $t \in Z(-\infty, T_0)$, we have

$$\sum_{t=-\infty}^{T_0}\left|u_k(t)\right|^2 \leqslant -\varepsilon \sum_{t=-\infty}^{T_0} q(t)\left|u_k(t)\right|^2 \leqslant \varepsilon A^2. \tag{2.3}$$

Similarly, since $-\frac{1}{q(t)} \leqslant \varepsilon$ on $t \in Z(T_1, +\infty)$, we have

$$\sum_{t=T_1}^{+\infty} |u_k(t)|^2 \leqslant \varepsilon A^2. \tag{2.4}$$

Note that ε is arbitrary, combining (2.2), (2.3) and (2.4), we get $u_k \to 0$ in l^2. □

Lemma 2.2. *Assume that (f_1), (f_2), (p) and (q) are satisfied. Then φ satisfies the Palais–Smale condition.*

Proof. Let $\{u_k\}$ be a sequence in E such that

$$\varphi(u_k) \to c, \quad \varphi'(u_k) \to 0 \quad \text{as } k \to \infty. \tag{2.5}$$

We show that $\{u_k\}$ possesses a convergent subsequence. By (2.5) and (f_2), there is a constant $d \geqslant 0$ such that

$$\begin{aligned} d + \|u_k\|_E &\geqslant \varphi(u_k) - \frac{1}{\beta}\varphi'(u_k)u_k \\ &= \left(\frac{1}{2} - \frac{1}{\beta}\right)\|u_k\|_E^2 + \sum_{t=-\infty}^{\infty}\left[F\big(t, u_k(t)\big) - \frac{1}{\beta} f(t, u_k)u_k\right] \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\beta}\right)\|u_k\|_E^2. \end{aligned}$$

Hence, $\{u_k\}$ is bounded in E. So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E and hence by Lemma 2.1 $u_k \to u_0$ in l^2. Moreover by (f_1) and $u_k, u_0 \in E$, for any $\varepsilon > 0$, we can choose $T \in N$ such that

$$|u_k| < \delta_1, \qquad |u_0| < \delta_2, \quad \text{and} \quad \|u_k - u_0\| < \varepsilon \quad \text{for all } |t| > T \tag{2.6}$$

and which can lead to

$$\big|f(t, u_k)\big| \leqslant \varepsilon |u_k|, \qquad \big|f(t, u_0)\big| \leqslant \varepsilon |u_0| \tag{2.7}$$

for all $|t| > T$, where δ_1, δ_2 are constants dependent on ε. So we have

$$\begin{aligned} &\sum_{t=-\infty}^{\infty} \big[F_u\big(t, u_k(t)\big) - F_u\big(t, u_0(t)\big)\big](u_k - u_0) \\ &\quad = \sum_{t=-\infty}^{\infty} \big[f\big(t, u_k(t)\big) - f\big(t, u_0(t)\big)\big](u_k - u_0) \\ &\quad = \sum_{t=-T}^{T} \big[f\big(t, u_k(t)\big) - f\big(t, u_0(t)\big)\big](u_k - u_0) \\ &\qquad + \sum_{|t|>T} \big[f\big(t, u_k(t)\big) - f\big(t, u_0(t)\big)\big](u_k - u_0). \end{aligned} \tag{2.8}$$

By the uniformly continuity of $f(t, x)$ in x and $u_k \to u_0$ in l^2_{loc}, the first term on the right-hand side of (2.6) approaches 0 as $k \to \infty$. In view of Lemma 2.1, (2.6) and (2.7) the second term can be estimated by

$$\begin{aligned}&\left|\sum_{|t|>T}\left[f\big(t,u_k(t)\big)-f\big(t,u_0(t)\big)\right](u_k-u_0)\right|\\&\quad\leqslant\left(\sum_{|t|>T}\left|f\big(t,u_k(t)\big)-f\big(t,u_0(t)\big)\right|^2\right)^{1/2}\left(\sum_{|t|>T}|u_k-u_0|^2\right)^{1/2}\\&\quad\leqslant\varepsilon^2\left(\sum_{|t|>T}\big(\left|u_k(t)\right|+|u_0|\big)^2\right)^{1/2}\leqslant\varepsilon^2 M,\end{aligned}$$

where the constant M dependents on the bounds for u_k and u_0 in E. So we have

$$\sum_{t=-\infty}^{\infty}\left[F_u\big(t,u_k(t)\big)-F_u\big(t,u_0(t)\big)\right](u_k-u_0)\to 0\quad\text{as } k\to\infty. \tag{2.9}$$

It follows from the definition of φ that

$$\begin{aligned}&\big(\varphi'(u_k)-\varphi'(u_0)\big)(u_k-u_0)\\&\quad=\|u_k-u_0\|_E^2-\sum_{t=-\infty}^{\infty}\left[F_u\big(t,u_k(t)\big)-F_u\big(t,u_0(t)\big)\right](u_k-u_0).\end{aligned}\tag{2.10}$$

By the continuity of φ', Lemma 2.1 and (2.9), (2.10) implies that $u_k\to u_0$ in E. So the proof is complete. □

Next we state two Lemmas extracted from the reference [12] and [16], respectively. Let $\beta_\rho(0)$ denote an open ball of radius ρ about 0.

Lemma 2.3 *(Mountain Pass Lemma [12]). Let E be a real Banach space and $\varphi\in C^1(E,R)$ satisfy the Palais–Smale condition. If further $\varphi(0)=0$,*

(A_1) there exist constants $\rho,\alpha>0$ such that

$$\varphi|_{\partial\beta_\rho(0)}\geqslant\alpha$$

and

(A_2) there exists $e\in E\setminus\bar\beta_\rho(0)$ such that $\varphi(e)\leqslant 0$, then I possesses a critical value $c\geqslant\alpha$ given by

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}\varphi\big(g(s)\big), \tag{2.11}$$

where

$$\Gamma=\left\{g\in C\big([0,1],E\big)\;\middle|\;g(0)=0,\ g(1)=e\right\}. \tag{2.12}$$

Lemma 2.4 *(Symmetric Mountain Pass Lemma [16]). Let E be a real Banach space, φ is even and $\varphi\in C^1(E,R)$ satisfies the Palais–Smale condition. If further $I(0)=0$,*

(A_3) there exist constants $\rho,\alpha>0$ such that

$$I|_{\partial\beta_\rho(0)}\geqslant\alpha$$

and

(A_4) for each finite-dimensional subspace $\tilde E\subset E$, there is $\gamma=\gamma(\tilde E)$ such that

$$\varphi\leqslant 0\quad\text{on } \tilde E\setminus\beta_\gamma.$$

Then I possesses an unbounded sequence of critical values.

3. Main results and their proofs

Theorem 3.1. *Suppose* (p), (q), (f_1) *and* (f_2) *are satisfied. Then there exists a homoclinic orbit* u *of Eq.* (1.1) *emanating from* 0 *such that*

$$0 < \sum_{t=-\infty}^{+\infty} \left[\frac{1}{2} p(t)\big(\Delta u(t-1)\big)^2 - \frac{1}{2} q(t)\big(u(t)\big)^2 + F\big(t, u(t)\big) \right] < +\infty.$$

Proof. We will prove the existence of a nontrivial critical point of φ. We have already known that $\varphi \in C^1(E, R)$, $\varphi(0) = 0$ and φ satisfies the Palais–Smale condition. Hence, it suffices to prove that φ satisfied the conditions (A_1) and (A_2). By Lemma 2.1, there is an $\alpha_0 > 0$ such that $\|u\| \leqslant \alpha_0 \|u\|_E$, $u \in E$. On the other hand, by (p) and (q), there is $\alpha_1 > 0$ such that $\|u\|_\infty \leqslant \alpha_1 \|u\|_E$, where $\|u\|_\infty = \max_{t\in Z} |u(t)|$. And by (f_1), for all $\varepsilon > 0$, there is $\delta > 0$ such that $|F(t,x)| \leqslant \varepsilon |x|^2$ whenever $|x| \leqslant \delta$. Let $\rho = \frac{\delta}{\alpha_1}$ and $\|u\|_E \leqslant \rho$, we have $\|u\|_\infty \leqslant \frac{\delta}{\alpha_1}\alpha_1 = \delta$. Hence, $|F(t, u(t))| \leqslant \varepsilon |u(t)|^2$ for all $t \in R$, then it follows that

$$F\big(t, u(t)\big) \geqslant -\varepsilon \big|u(t)\big|^2. \tag{3.1}$$

Summing (3.1) on Z, we get

$$\sum_{t=-\infty}^{+\infty} F\big(t, u(t)\big) \geqslant -\varepsilon \|u\|^2 \geqslant -\varepsilon \alpha_0^2 \|u\|_E^2.$$

So, if $\|u\|_E = \rho$, then

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|u\|_E^2 + \sum_{t=-\infty}^{+\infty} F\big(t, u(t)\big) \geqslant \frac{1}{2}\|u\|_E^2 - \varepsilon\alpha_0^2 \|u\|_E^2 \\ &= \left(\frac{1}{2} - \varepsilon\alpha_0^2\right)\|u\|_E^2 = \left(\frac{1}{2} - \varepsilon\alpha_0^2\right)\rho^2. \end{aligned}$$

It suffices to choose $\varepsilon = 1/4\alpha_0^2$ to get

$$\varphi(u) \geqslant \frac{1}{4}\rho^2 = \alpha > 0.$$

Consider

$$\varphi(\sigma u) = \frac{\sigma^2}{2}\|u\|_E^2 + \sum_{t=-\infty}^{+\infty} F(t, \sigma u)$$

for all $\sigma \in R$. Let $\bar{u} \in E$ be such that $|\bar{u}(t)| \geqslant 1$ on a nonempty integer interval $I \subset Z$. By (1.5), for any $\sigma \geqslant 1$, we have

$$\begin{aligned} \varphi(\sigma\bar{u}) &\leqslant \frac{\sigma^2}{2}\|\bar{u}\|_E^2 + \sum_I F(t, \sigma\bar{u}) \leqslant \frac{\sigma^2}{2}\|\bar{u}\|_E^2 - \sum_I |\sigma\bar{u}|^\beta \alpha_1(t) \\ &= \frac{\sigma^2}{2}\|\bar{u}\|_E^2 - \sigma^\beta \sum_I |\bar{u}|^\beta \alpha_1(t). \end{aligned}$$

Since $\beta > 2$, we can find a $\sigma \geqslant 1$ such that $\|\sigma\bar{u}\| > \rho$ and $\varphi(\sigma\bar{u}) \leqslant 0 = \varphi(0)$. So (A_1) and (A_2) are satisfied. The desired results follow. □

Theorem 3.2. *Suppose (p), (q), (f_1) and (f_2) are all true. If further*

(f_3) $\quad f(t,-x) = -f(t,x)$ *for all* $x \in R$ *and* $t \in Z$.

Then there exists an unbounded sequence in E of homoclinic orbits of Eq. (1.1) *emanating from* 0.

Proof. The condition (f_3) implies that φ is even. And we have already known that $\varphi \in C^1(E, R)$, $\varphi(0) = 0$ and φ satisfies the Palais–Smale condition. To apply the symmetric Mountain Pass Theorem, it suffices to prove that φ satisfies the conditions (A_3) and (A_4). (A_3) is identically the same as in Theorem 3.1, so it is already proved.

We prove (A_4). Let $\tilde{E} \subset E$ be a finite dimension subspace. Consider $u \in \tilde{E}$ with $u \neq 0$. By (1.5), there is a continuous function $\alpha_1(t) > 0$ such that

$$F(t, u(t)) \leqslant -\alpha_1(t)|u(t)|^\beta \quad \text{for } |u(t)| \geqslant 1.$$

Hence,

$$\sum_{|u(t)|>1} F(t, u(t)) \leqslant -\sum_{|u(t)|>1} \alpha_1(t)|u(t)|^\beta$$

and

$$\varphi(u) = \frac{1}{2}\|u\|_E^2 + \sum_{|u(t)|>1} F(t, u(t)) + \sum_{|u(t)|\leqslant 1} F(t, u(t)).$$

Now for all $u \in \tilde{E}$, we have $\|u\|_E^2 \leqslant c\|u\|_\infty^2$, where $c = c(\tilde{E})$. Let us define $m = \inf_{\|u\|_\infty = 2} \sum_{|u(t)|>1} \alpha_1(t)|u(t)|^2$. If $m = 0$, we will have $|u(t)| \equiv 0$ for all $t \in \{t \mid |u(t)| > 1\}$ which contradicts $\|u\|_\infty = 2$. Thus, $m > 0$ and we have obtain

$$\begin{aligned}
\varphi(u) &\leqslant \frac{1}{2}c\|u\|_\infty^2 + \sum_{|u(t)|>1} F(t, u(t)) + \sum_{|u(t)|\leqslant 1} F(t, u(t)) \leqslant \frac{1}{2}c\|u\|_\infty^2 + \sum_{|u(t)|>1} F(t, u(t)) \\
&\leqslant \frac{1}{2}c\|u\|_\infty^2 - \sum_{|u(t)|>1} \alpha_1(t)|u(t)|^\beta = \frac{1}{2}c\|u\|_\infty^2 - \frac{\|u\|_\infty^\beta}{2^\beta} \sum_{|u(t)|>1} \alpha_1(t) 2^\beta \frac{|u(t)|^\beta}{\|u\|_\infty^\beta} \\
&= \frac{1}{2}c\|u\|_\infty^2 - \frac{1}{2^\beta}\|u\|_\infty^\beta \sum_{|u(t)|>1} \alpha_1(t)\left(\frac{2|u(t)|}{\|u\|_\infty}\right)^\beta \leqslant \frac{1}{2}c\|u\|_\infty^2 - \frac{m}{2^\beta}\|u\|_\infty^\beta.
\end{aligned}$$

Since $\beta > 2$, we deduce that there is a $c \in c(\tilde{E})$ such that $\varphi(u) \leqslant 0$ whenever $\|u\|_\infty \geqslant c$. Hence, by Lemma 2.4, φ possesses an unbounded sequence of critical values (c_j) with $c_j = \varphi(u_j)$, when u_j is such that

$$0 = \varphi'(u_j)u_j = \|u_j\|_E^2 + \sum_{t=-\infty}^{\infty} f(t, u_j)u_j$$

so that

$$\sum_{t=-\infty}^{\infty} f(t, u_j)u_j = -\|u_j\|_E^2.$$

Thus, by (f_2), we have

$$c_j = \sum_{t=-\infty}^{+\infty}\left[-\frac{1}{2}f(t,u_j)u_j + F(t,u_j)\right] \leqslant -\frac{1}{2}\sum_{t=-\infty}^{+\infty} f(t,u_j)u_j = \frac{1}{2}\|u_j\|_E^2. \tag{3.2}$$

Since $c_j \to \infty$ as $j \to \infty$, (3.2) implies that $\{u_j\}$ is unbounded in E. Thus the proof is complete. □

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