Existence and numerical approximations of periodic solutions of semilinear fourth-order differential equations

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Abstract

A multiplicity result of existence of periodic solutions with prescribed wavelength for a class of fourth-order nonautonomous differential equations related either to the extended Fisher–Kolmogorov or to the Swift–Hohenberg equation is proved. Variational approach is used. Some numerical solutions are calculated via the finite element method.

Keywords: Periodic solutions; Extended Fisher–Kolmogorov equation; Swift–Hohenberg equation; Clark’s multiplicity theorem

1. Introduction

This paper is concerned with the existence of stationary periodic solutions of the equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4} - p \frac{\partial^2 u}{\partial x^2} - a(x)u + b(x)u^3, \]

known in the studies of phase transitions near a Lifschitz point [3,7,24]. Here the order parameter \( u \) depends only on one spatial direction. In our case the
coefficients $a$ and $b$ might be variables that corresponds to nonhomogeneous media. More precisely, in the field of relaxational dynamics near a Lifschitz point, Eq. (1) is prototype equation for phase transitions in condensed matter systems with randomly distributed impurities [6,12,23]. The case $p > 0$ is referred to as the extended Fisher–Kolmogorov equation while the case $p < 0$ yields the Swift–Hohenberg equation (cf. [5,20]).

We concentrate our attention to the scalar fourth-order equation

$$u^{iv} - pu'' - a(x)u + b(x)u^3 = 0$$

and suppose $p \in \mathbb{R}$, and the functions $a$ and $b$ are positive continuous even and $2L$ periodic. In this note we characterize the possible amount of $2L$ periodic solutions of Eq. (2) by the coefficient wavelength using variational techniques.

The autonomous case

$$u^{iv} - \beta u'' + F'(u) = 0, \quad \beta \in \mathbb{R},$$

with the double well potential $F$ with two nondegenerate global minima at $\pm 1$, has been extensively studied and different families of periodic, homoclinic and heteroclinic solutions have been found. A model potential is $F(u) = (u^2 - 1)^2/4$.

Note that Eq. (3) admits a first integral, usually called the energy

$$E(u) = u' u'''' - \frac{1}{2} u''^2 - \frac{\beta}{2} u'^2 + F(u),$$

also it is invariant both under the translations and reversibility (the transformation $x \to -x$). As it is remarked in [1,2,9–11,14–16], the behaviour of solutions of Eq. (3) is dramatically different for $\beta \geq 2 \sqrt{F''(\pm 1)}$, when the eigenvalues of the linearized equations around the equilibrium points $u = \pm 1$ are all real—two negative and two positive, and for $|\beta| < 2 \sqrt{F''(\pm 1)}$, when the eigenvalues are all complex—two with negative real part and two with positive real part.

In the case of real eigenvalues with $F(u) = (u^2 - 1)^2/4$ (i.e., $\beta \geq \sqrt{8}$), Peletier and Troy [14,16] show that Eq. (3) possesses a unique heteroclinic solution (kink) connecting $-1$ with $1$ which is odd and monotone, also there is a family of periodic solutions, parametrized by the energy $E \in (0, 1/4)$, or by the period $L \in (2 \pi \sqrt{(\beta + \sqrt{\beta^2 + 4})/(2\beta)}, \infty)$ (see also [1,17]), and there exist no periodic solutions with zero energy. In [1] Van den Berg has obtained the odd monotonically increasing kink as a transverse intersection of the unstable manifold of $-1$ and the stable manifold of $1$, and proved that it is asymptotically stable.

Due to the complex eigenvalues for $|\beta| < 2 \sqrt{F''(\pm 1)}$, the dynamics of the solutions of Eq. (3) becomes much richer. A numerous families of branches of single- and multi-bump periodic solutions bifurcating either from the kink at $\beta = \sqrt{8}$, or from the equilibrium $u = 1$ were constructed in [2]. All of these solutions have zero energy. In [15], using a topological shooting method, Peletier and Troy have found countably infinite number of odd heteroclinics from $-1$
to 1. In addition, homoclinic and heteroclinic orbits of different homotopy classes leading to ±1 do exist; the reader is referred to [8–11,21], etc.

In contrast to the autonomous case, Eq. (2) loses the translation invariance and the existence of a first integral. Also, it has only one equilibrium point \( u = 0 \). Nevertheless, Eq. (2) might be considered as a generalization of Eq. (3) with \( p = a(x) = b(x) = \beta^2 \) and after scaling \( u(x) = v(\sqrt{|\beta|}x) \). In our earlier paper [22] the existence result concerning Eq. (1) is for \( p > 0 \). Here we extend this result in both cases \( p > 0 \) and \( p < 0 \). We present some numerical experiments confirming the theoretical considerations using the finite element method.

In this note we are interested in 2\( L \) periodic solutions of Eq. (2) which are antisymmetric with respect to \( x = 0 \) and \( x = L \). Thus we consider the related two-point boundary value problem

\[
\begin{cases}
u'' - pu'' - a(x)u + b(x)u^3 = 0, & 0 < x < L, \\u(0) = u''(0) = u(L) = u''(L) = 0.
\end{cases}
\] (5)

Note that since the coefficients \( a \) and \( b \) are even and 2\( L \) periodic, if \( u(x) \) is classical solution of (5) and \( \tilde{u}(x) \) is its antisymmetric extension with respect to \( x = 0 \):

\[
\tilde{u}(x) = \begin{cases} u(x), & 0 \leq x \leq L, \\ -u(-x), & -L \leq x \leq 0, \end{cases}
\]

then the 2\( L \) periodic extension of \( \tilde{u}(x) \) over \( \mathbb{R} \) is classical 2\( L \) periodic solution of Eq. (2). The boundary value problem (5) could be considered as a nonlinear eigenvalue problem with respect to \( L \).

Consider the functional

\[
I(u) := \int_0^L \left( \frac{1}{2}(u''^2 + pu'^2 - a(x)u^2) + \frac{1}{4}b(x)u^4 \right) dx
\]

on the space \( X = H^2(0, L) \cap H_0^1(0, L) \). The functional \( I \) is Fréchet differentiable and its derivative is given by

\[
\langle I'(u), v \rangle = \int_0^L (u''v'' + pu'v' - a(x)uv + b(x)u^3v) \, dx, \quad v \in X.
\]

The weak solutions of (5) are critical points of \( I \).

Under suitable condition on \( L \) it can be proved that there exists a nontrivial solution of the problem (5) applying a general minimization theorem (cf. [13, Theorem 1.1]). However, a stronger result is obtained using Rabinowitz version of Clark’s multiplicity theorem (see [18, Theorem 9.1]) according to the reversibility of the functional \( I \).
Let us denote
\[ a_1 := \min_{x \in [0, L]} a(x), \quad a_2 := \max_{x \in [0, L]} a(x), \]
\[ L_{1,2} := \pi \sqrt{\frac{p + \sqrt{p^2 + 4a_{1,2}}}{2a_{1,2}}}. \] (6)

Note that if \( a_1 < a_2 \) then \( L_2 < L_1 \).

The following theorem contains our main result.

**Theorem 1.** Let \( p \neq 0 \). The problem (5) has only the trivial solution if \( 0 < L \leq L_2 \). The problem (5) has at least \( m \) distinct pairs of solutions if \( L > mL_1 \) for some \( m \in \mathbb{N} \).

Existence of nontrivial solutions of (5) in case \( L_2 < L < L_1 \) was considered as an open problem in [22]. We show that there is a number \( \overline{L} \), \( L_2 \leq \overline{L} \leq L_1 \), which exactly separates the nonexistence and existence cases. Exact multiplicity of solutions in case \( L > mL_1 \) is unknown. The numerical evidences show more than \( m \) nontrivial solutions in that case.

The paper is organized as follows. In Section 2 we prove the theoretical results. We construct a finite element method with cubic Hermite basis for the problem (5) in Section 3. Some numerical experiments are presented in Section 4.

2. Existence results

In this section we study the solvability of the problem (5)
\[
\begin{cases}
u^{iv} - pu'' - a(x)u + b(x)u^3 = 0, & 0 < x < L, \\
u(0) = u''(0) = u(L) = u''(L) = 0,
\end{cases}
\]
where \( p \neq 0 \), and the continuous positive functions \( a \) and \( b \) satisfy
\[ 0 < a_1 \leq a(x) \leq a_2, \quad 0 < b_1 \leq b(x) \leq b_2. \] (7)

For \( u, v \in X \) we define
\[
A(u, v) = \int_0^L \left( u''v'' + pu'v' - a(x)uv \right) dx,
\]
\[
B(u, v) = \int_0^L b(x)u^3v dx.
\]

A function \( u \in X \) is said to be a weak solution of the problem (5) if
\[ A(u, v) + B(u, v) = 0, \quad \forall v \in X. \]
It can be shown that weak solutions of (5) are also classical solutions (see [22]). We state Clark’s theorem in the form of Rabinowitz for reader convenience (cf. [18, Theorem 9.1]).

**Theorem** (Clark). Let $E$ be a real Banach space, $\Phi \in C^1(E, \mathbb{R})$ with $\Phi$ even, bounded below, and satisfying the Palais–Smale condition. Suppose $\Phi(0) = 0$, there is a set $K \subset E$ such that $K$ is homeomorphic to $S^{m-1}$, $m \in \mathbb{N}$, by an odd map, and $\sup_K \Phi < 0$. Then $\Phi$ possesses at least $m$ distinct pairs of critical points.

The proof of Theorem 1 is divided in a sequence of lemmata.

**Lemma 1.** Let $p \neq 0$. For every $\varepsilon \in (0, 1)$ there is a constant $a_3 > 0$ such that

$$p \int_0^L u^2 \, dx \geq -a_3 \int_0^L u^2 \, dx - (1 - \varepsilon) \int_0^L u''^2 \, dx, \quad \forall u \in X. \quad (8)$$

**Proof.** We have for every $u \in X$

$$p \int_0^L u^2 \, dx = -p \int_0^L uu'' \, dx \geq -a_3 \int_0^L u^2 \, dx - (1 - \varepsilon) \int_0^L u''^2 \, dx,$$

where $a_3 = p^2/(4(1 - \varepsilon))$. □

**Lemma 2.** The problem (5) has only the trivial solution if $L \leq L_2$.

**Proof.** Suppose that $u \in X$ is a nontrivial solution of (5). Multiplying Eq. (5) by $u$ and integrating by parts on $(0, L)$ we have

$$0 > -\int_0^L b(x)u^4 \, dx = \int_0^L ((u'')^2 + pu'^2 - a(x)u^2) \, dx. \quad (9)$$

For $u$ in $X$ we use the following Poincaré type inequalities (cf. [24]):

$$\int_0^L u^2 \, dx \leq \frac{L^2}{\pi^2} \int_0^L u'^2 \, dx, \quad \int_0^L u^2 \, dx \leq \frac{L^4}{\pi^4} \int_0^L u''^2 \, dx. \quad (10)$$

Let $p > 0$. We have

$$\left(\frac{\pi}{L}\right)^4 + p\left(\frac{\pi}{L}\right)^2 - a_2 \geq 0 \quad \text{if } 0 < L \leq L_2.$$
By (9) and (10) we obtain
\[
\int_0^L \left( u''^2 + pu'^2 - a(x)u^2 \right) dx \geq \int_0^L \left( \left( \frac{\pi}{L} \right)^4 + p \left( \frac{\pi}{L} \right)^2 - a_2 \right) u^2 dx \geq 0
\]
which is a contradiction.

Let \( p < 0 \). We have by Lemma 1 and (10) that for every \( \epsilon \in (0, 1) \)
\[
\int_0^L \left( u''^2 + pu'^2 - a(x)u^2 \right) dx \geq \int_0^L \left( \epsilon u''^2 - \left( \frac{p^2}{4(1 - \epsilon)} + a_2 \right) u^2 \right) dx
\]
\[
\geq \int_0^L \left( \epsilon \left( \frac{\pi}{L} \right)^4 - \left( \frac{p^2}{4(1 - \epsilon)} + a_2 \right) \right) u^2 dx. \quad (11)
\]
Observe that
\[
\epsilon \left( \frac{\pi}{L} \right)^4 - \left( \frac{p^2}{4(1 - \epsilon)} + a_2 \right) \geq 0 \quad \text{if } 0 < L \leq \pi \sqrt{\frac{4\epsilon(1 - \epsilon)}{p^2 + 4a_2(1 - \epsilon)}}.
\]
Note that the function
\[
f(\epsilon) = \pi \sqrt{\frac{4\epsilon(1 - \epsilon)}{p^2 + 4a_2(1 - \epsilon)}}
\]
is continuous on \([0, 1] \), \( f(0) = f(1) = 0 \) and \( f(\epsilon) > 0 \) in (0, 1). Let
\[
\epsilon_1 = \frac{p^2 + 4a_2 + p\sqrt{p^2 + 4a_2}}{4a_2} \in (0, 1).
\]
A simple computation shows that
\[
0 < \max_{\epsilon \in [0, 1]} f(\epsilon) = f(\epsilon_1) = \pi \sqrt{\frac{p + \sqrt{p^2 + 4a_2}}{2a_2}} = L_2.
\]
Taking \( \epsilon \) to be \( \epsilon_1 \) in (11) by
\[
\epsilon \left( \frac{\pi}{L} \right)^4 - \left( \frac{p^2}{4(1 - \epsilon)} + a_2 \right) \geq 0 \quad \text{if } 0 < L \leq L_2
\]
we reach to a contradiction again.
Therefore \( u = 0 \) if \( 0 < L \leq L_2 \) and \( p \neq 0 \). □

**Lemma 3.** The functional \( I \) is bounded below on \( X \).

**Proof.** Let \( \epsilon_0 \in (0, 1) \). We have by (7) and (8) for every \( u \in X \)}
\[ I(u) \geq \frac{1}{2} \int_0^L (u''^2 + pu'^2 - a_2u^2) \, dx + \frac{1}{4} \int_0^L b_1u^4 \, dx \]
\[ \geq \frac{1}{2} \int_0^L (\varepsilon_0 u'' - (a_2 + a_3)u^n) \, dx + \frac{1}{4} \int_0^L b_1u^4 \, dx \]
\[ \geq \int_0^L \left( -\frac{1}{2} (a_2 + a_3)u^2 + \frac{1}{4} b_1u^4 \right) \, dx \geq -\frac{(a_2 + a_3)^2}{4b_1} L. \]

**Lemma 4.** The functional \( I \) satisfies the Palais–Smale condition.

**Proof.** Let \((u_n)_n\) be a Palais–Smale sequence of \( I \), i.e.,
\[ (I(u_n))_n \text{ is bounded and } I'(u_n) \to 0. \] (12)

Let \( \varepsilon_0 \in (0, 1) \). By Lemmas 1, 3, and (12) there is a constant \( C_1 > 0 \) such that
\[ 0 \leq \int_0^L \left( \frac{\varepsilon_0}{2} u''^2 + \left( \frac{\sqrt{b_1}}{2} u_n^2 - \frac{a_2 + a_3}{2\sqrt{b_1}} \right)^2 \right) \, dx \]
\[ \leq I(u_k) + \frac{(a_2 + a_3)^2}{4b_1} L \leq C_1. \]

Then we have
\[ \int_0^L u_n''^2 \, dx \leq C. \]

Thus Lemma 1 and (10) implies that the sequence \((u_n)_n\) is bounded in \( H^2(0, L) \). Passing if necessary to a subsequence, by Sobolev embedding theorem we have
\[ u_n \to u \text{ in } H^2(0, L), \]
\[ u_n \to u \text{ in } C^1([0, L]), \]
\[ u_n \to u \text{ in } L^2(0, L), \]
\[ u_n \to u \text{ in } L^4(0, L). \] (13)

Letting \( n \to \infty \) in the equality
\[ \langle I'(u_n), u \rangle = \int_0^L (u_n''u'' + pu_n'u' - a(x)u_nu + b(x)u_n^3u) \, dx \]
from (12), (13), and Lebesgue convergence theorem we have

$$\int_0^L (u''^2 + pu'^2 - a(x)u^2 + b(x)u^4) \, dx = 0.$$  

From the boundedness of \((u_n)_n\) in \(X\) and (12) it follows \(\langle I'(u_n), u_n \rangle \to 0\). Thus

$$\int_0^L (u''^2_n + u'^2_n + u^2_n) \, dx$$

$$\to (1 - p) \int_0^L u'^2 \, dx + \int_0^L (1 + a(x))u^2_n \, dx - \int_0^L b(x)u^4 \, dx$$

$$= \int_0^L (u''^2 + u'^2 + u^2) \, dx.$$  

Hence \(\|u_n\| \to \|u\|\) and \(u_n \rightharpoonup u\) implies \(\|u_n - u\| \to 0\) in \(X\).  \(\square\)

**Lemma 5.** Let \(L > mL_1\) for some \(m \in \mathbb{N}\). Then there exists a set \(K \subset X\) which is homeomorphic to \(S^{m-1}\) by an odd map, and \(\sup_K I < 0\).

**Proof.** Let us consider the subset \(K\) of \(X\),

$$K = \left\{ \lambda_1 \sin \frac{\pi x}{L} + \cdots + \lambda_m \sin \frac{m\pi x}{L} : \lambda_1^2 + \cdots + \lambda_m^2 = \rho^2 \right\},$$

where \(\rho\) is a number to be chosen later. It is clear that the odd mapping \(H : K \to S^{m-1}\) defined by

$$H \left( \lambda_1 \sin \frac{\pi x}{L} + \cdots + \lambda_m \sin \frac{m\pi x}{L} \right) = \left( -\frac{\lambda_1}{\rho}, \ldots, -\frac{\lambda_m}{\rho} \right)$$

is a homeomorphism between \(K\) and \(S^{m-1}\). We have

$$\rho \sqrt{\frac{L}{2}} |H(w)|_{\mathbb{R}^m} \leq \|w\|_{H^2}$$

$$\leq \rho m \sqrt{\frac{L}{2} \left( 1 + \left( \frac{m\pi}{L} \right)^2 + \left( \frac{m\pi}{L} \right)^4 \right) |H(w)|_{\mathbb{R}^m}}.$$
for every \( w \in K \). \( K \) is a subset of the finite-dimensional space

\[
X_m := \text{sp}\left\{ \sin \frac{\pi x}{L}, \ldots, \sin \frac{m\pi x}{L} \right\}
\]

equipped with the norm

\[
\left\| \lambda_1 \sin \frac{\pi x}{L} + \cdots + \lambda_m \sin \frac{m\pi x}{L} \right\|_{X_m}^2 = \lambda_1^2 + \cdots + \lambda_m^2.
\]

There exist positive functions \( C_1(m) \) and \( C_2(m) \) such that

\[
C_1(m) \| w \|_{X_m} \leq \| w \|_{L^4} \leq C_2(m) \| w \|_{X_m}, \quad \forall w \in X_m,
\]

because the norms \( \| \cdot \|_{X_m} \) and \( \| \cdot \|_{L^4} \) are equivalent on \( X_m \). In particular,

\[
\| w \|_{L^4} \leq C_2(m) \rho, \quad \forall w \in K.
\] (14)

By (7) and (14) for \( w \in K \) we have

\[
I(w) \leq \frac{1}{2} \int_0^L \left( w''^2 + pw'^2 - a_1 w^2 \right) dx + \frac{1}{4} b_2 \int_0^L w^4 dx \\
\leq \frac{L}{4} \sum_{k=1}^m \lambda_k^2 \left( \left( \frac{k\pi}{L} \right)^4 + p \left( \frac{k\pi}{L} \right)^2 - a_1 \right) + \frac{1}{4} b_2 C_2(m) \rho^4.
\]

Observe that if \( L > mL_1 \) then for \( k = 1, 2, \ldots, m \)

\[
\left( \frac{k\pi}{L} \right)^4 + p \left( \frac{k\pi}{L} \right)^2 - a_1 < 0,
\]

and therefore

\[
A(m) := \max_{1 \leq k \leq m} \left( \left( \frac{k\pi}{L} \right)^4 + p \left( \frac{k\pi}{L} \right)^2 - a_1 \right) < 0.
\]

Hence

\[
I(w) \leq \frac{L}{4} A(m) \rho^2 + \frac{1}{4} b_2 C_2(m) \rho^4.
\]

For sufficiently small \( \rho > 0 \) we obtain that \( \sup_{K} I < 0 \) which completes the proof. \( \Box \)

These arguments show that if \( L > L_1 \) the functional \( I \) satisfies all hypotheses of Clark’s theorem. Hence \( I \) has at least \( m \) distinct pairs of critical points, and the problem (5) has at least \( m \) distinct pairs of solutions.
Remark. If \( m = 1 \), Clark’s theorem in the form of Rabinowitz is a special case of the minimization theorem [13, Theorem 1.1]. By the proof of Clark’s theorem in [18], the numbers \( c_k \) defined as
\[
c_k = \inf_{A \in \gamma_k} \sup_{u \in A} \Phi(u), \quad 1 \leq k \leq m,
\]
where
\[
\gamma_k = \{A \subset E \setminus \{0\}: A \text{ is closed, } x \in A \text{ iff } -x \in A, j(A) \geq k\},
\]
are critical values of \( \Phi \). Here \( j(A) \) is the genus of \( A \). In particular, if \( m = 1 \) the definition of \( c_k \) yields \( c_1 = \inf_E \Phi(u) \). Since \( \Phi \) is bounded below \( c_1 > -\infty \). Then there exists \( \bar{u} \in E \) such that \( \Phi(\bar{u}) = \inf_E \Phi(u) \).

We now turn to the problem of the existence of nontrivial solutions of (5) in case \( L_2 < L < L_1 \). We prove that there is a number \( \overline{L} \), \( L_2 \leq \overline{L} \leq L_1 \), which exactly separates the nonexistence and existence cases. Although \( \overline{L} \) is available the numbers \( L_1, L_2 \) are useful because of their easy computation. We apply the following result of the spectral theory.

**Lemma 6.** Let the symmetric quadratic forms \( a \) and \( b \) of the eigenvalue problem
\[
a(u, v) = \lambda b(u, v), \quad \forall v \in H,
\]
in the Hilbert space \( H \) satisfy the following conditions:

(i) \( a \) is coercive in \( H \), i.e., there is \( c > 0 \) such that \( a(u, u) \geq c\|u\|^2, \forall u \in H \),
(ii) if \( u_n \rightharpoonup u \) then \( b(u_n, u_n) \to b(u, u) \),
(iii) \( b(u, u) > 0 \) if \( u \in H \setminus \{0\} \).

If
\[
\lambda_1 := \min_{u \in H \setminus \{0\}} \frac{a(u, u)}{b(u, u)} > 1
\]
then \( a(u, u) - b(u, u) \) is coercive in \( H \).

**Lemma 7.** There exists a number \( \overline{L} \), \( L_2 \leq \overline{L} \leq L_1 \), such that if \( 0 < L \leq \overline{L} \) the problem (5) has only the trivial solution, and if \( L > \overline{L} \) the problem (5) admits a nontrivial solution.

**Proof.** Consider the eigenvalue problems
\[
(E_1) \quad \int_0^L u''''v'' + pu'v' = \lambda \int_0^L a(x)uv, \quad \forall v \in X,
\]
and
\[(E_2) \int_0^L u'' v'' = \lambda \int_0^L p u' v' + a(x) u v, \ \forall v \in X,\]
with \(p > 0\). For the first eigenvalue of \((E_1)\) and \((E_2)\) we have
\[\lambda_1(L) = \min_{u \in X \setminus \{0\}} \frac{\int_0^L u''^2 + pu'^2}{\int_0^L a(x) u^2}\]
and
\[\lambda_1(L) = \min_{u \in X \setminus \{0\}} \frac{\int_0^L u''^2}{\int_0^L pu'^2 + a(x) u^2},\]
respectively.

Let the minimum be attained in some eigenfunction \(u_L\). Then
\[\lambda_1(L) = \frac{\int_0^L u''^2 + pu'^2}{\int_0^L a(x) u^2} = \frac{1}{(\int_0^1 \bar{a}(x) v^2)} \left( \frac{1}{L^4} \left( \int_0^1 v''^2 \right) + \frac{1}{L^2} \left( \int_0^1 p v'^2 \right) \right),\]
\[\lambda_1(L) = \frac{\int_0^L u''^2}{\int_0^L pu'^2 + a(x) u^2} = \frac{(\int_0^1 v''^2)}{L^2 ((\int_0^1 p v'^2) + L^2 (\int_0^1 \bar{a}(x) v^2))},\]
where \(v(x) = u_L(Lx), \ \bar{a}(x) = a(Lx), \ x \in [0, 1]\. Since \(\lambda_1(L)\) is decreasing in \(L\), \(\lim_{L \to 0^+} \lambda_1(L) = +\infty, \ \lim_{L \to \infty} \lambda_1(L) = 0\),
there is some \(\bar{L}\) such that
\[\lambda_1(L) > 1 \ \text{if} \ 0 < L < \bar{L}\]
and
\[\lambda_1(L) < 1 \ \text{if} \ L > \bar{L}.\]

An easy computation using (10) shows that \(L_2 \leq \bar{L} \leq L_1\).

Since
\[\left( \int_0^L u''^2 + pu'^2 \right)^{1/2}, \ \left( \int_0^L u''^2 \right)^{1/2}\]
are equivalent norms in \(X\), by (13) and the previous lemma if \(0 < L < \bar{L}\) there exists \(c > 0\) such that
\[\int_0^L \left( \frac{1}{2} (u''^2 + pu'^2 - a(x) u^2) + \frac{1}{4} b(x) u^4 \right) \geq c \|u\|^2, \ \forall u \in X,\]
and
\[
\int_0^L \left( \frac{1}{2} u''^2 - pu'^2 - a(x)u^2 \right) + \frac{1}{4} b(x)u^4 \geq c \|u\|^2, \quad \forall u \in X.
\]

Hence the only critical point of the functional \( I \) is its global minimum at \( u = 0 \), and the problem (5) has only the trivial solution. ☐

3. Numerical treatment

For numerical solution of the problem (5) a finite element method with Hermite cubic functions is used [19].

We consider the uniform mesh
\[
0 = x_0 < x_1 < \cdots < x_N = L,
\]
where \( x_i = ih \), \( i = 0, \ldots, N \), \( h = L/N \). We denote by \( S^h \) the space of \( C^1 \) functions on \([0, L]\) which reduce to a cubic polynomial on every subinterval of the mesh. Let \( S^h_0 \) be the subspace consisting of those functions \( v^h \in S^h \) which satisfy the boundary conditions
\[
v^h(0) = 0, \quad v^h(L) = 0.
\]
Certainly \( S^h_0 \) is a finite-dimensional space, \( \dim S^h_0 = 2N \) and \( S^h_0 \subset X \). As usual, Galerkin approximation \( u^h \in S^h_0 \) is defined by
\[
A(u^h, v^h) + B(u^h, v^h) = 0, \quad v^h \in S^h_0.
\]

Let us denote by \( \Psi_k \), \( k = 1, \ldots, 2N \), the corresponding interpolatory basis in \( S^h_0 \). To compute the approximated solution
\[
u^h = u_1 \Psi_1(x) + \cdots + u_{2N} \Psi_{2N}(x), \quad x \in [0, L],
\]
where
\[
u_{2i-1} = \frac{du^h}{dx}(x_{i-1}), \quad i = 1, \ldots, N, \quad \nu_{2N} = \frac{du^h}{dx}(x_N),
\]
\[
u_{2i} = u^h(x_i), \quad i = 1, \ldots, N - 1,
\]
one has to solve the nonlinear system
\[
\sum_{i=1}^{2N} u_i A(\Psi_i, \Psi_j) + B \left( \sum_{i=1}^{2N} u_i \Psi_i, \Psi_j \right) = 0, \quad j = 1, \ldots, 2N. \tag{15}
\]

Every equation of (15) contains 20 cubic monomials of 6 nodal values. For evaluating the coefficients of (15) we use 20N numerical quadratures. We find solutions of the cubic system (15) applying the function FSOLVE2 of the MATLAB 5.2 libraries. This function implements the Algorithm 6.1.3 of [4] which is a globally convergent modification of Newton’s method, i.e., it is designed to converge to some solution of the system nonlinear equations from almost any initial solution.
4. Some numerical results

In this section we present some numerical results. We consider four cases. The first is used to test the accuracy of the algorithm on a slight modification of Eq. (5). The other three actually find distinct nontrivial solutions of Eq. (5).

To insure the reliability of the algorithm, we begin by solving

\[ \frac{u^{iv} - u''}{12} + \frac{u}{x^2 - x - 1} - \frac{u^3}{(x^2 - x - 1)^3} = 24 - x^3(x - 1)^3, \quad 0 < x < 1, \]

\[ u(0) = u''(0) = u(1) = u''(1) = 0. \]  

(16)

An exact solution of (16) is known to be \( x^4 - 2x^3 + x \). The solution \( u^h \) is computed with \( h = 0.1 \) by 4 iterations. The initial solution for the system (15) is chosen to be \( u_0 = 5(x^4 - 2x^3 + x) \). The error in the nodes is \( 10^{-8} \).

Satisfied with the accuracy of the algorithm we proceed with the problem (5). Hence, we solve

\[ u^{iv} - 1.5u'' - 50 \left( 1 + 12 \left( x - \frac{1}{2} \right)^2 \right) u + (1 + x)u^3 = 0, \quad 0 < x < L, \]

\[ u(0) = u''(0) = u(L) = u''(L) = 0 \]  

(17)

for three different values of \( L \). The initial solutions \( u_0 \) are chosen to be polynomials verifying the boundary value conditions. Since

\[
\begin{align*}
\min_{[0, \infty)} & \quad 50 \left( 1 + 12 \left( x - \frac{1}{2} \right)^2 \right) = 50, \\
\max_{[0, 1]} & \quad 50 \left( 1 + 12 \left( x - \frac{1}{2} \right)^2 \right) = 200,
\end{align*}
\]

we have \( L_1 \approx 1.2457 \) on \([0, \infty)\), \( L_2 \approx 0.8578 \) on \([0, 1]\) and decreases when increasing the interval.

First, let \( L = 2 \). From Theorem 1, at least one nonzero solution exists. We employed the algorithm with \( N = 20 \), \( u_0 = 4(x^4 - 4x^3 + 8x) \). By 16 iterations it found the solution shown in Fig. 1(1).

Second, let \( L = 3 \). In this case at least 2 nontrivial solutions exist according to Theorem 1. Having started with the initial solutions \( u_0^{(1)} \) and \( u_0^{(2)} \) such that

\[
\begin{align*}
\deg u_0^{(1)} & = 5, & u_0^{(1)}(2.5) & = 0, \\
\deg u_0^{(2)} & = 6, & u_0^{(2)}(1) & = 0, & u_0^{(2)}(1.5) & = 0,
\end{align*}
\]

by 13 and 22 iterations, respectively, we obtain the solution shown in Fig. 1(2). It has 3 zeros. Another nontrivial solution is found with \( u_0^{(3)} = 20(x^4 - 6x^3 + 27) \) by 36 iterations. Both solutions are given in Fig. 1(3).
Third, let $L = 4$. The solutions shown in Figs. 2(4), (5), and (6) are computed with $N = 40$ by 30, 21, and 93 iterations. The initial solutions are, respectively,

$$u_0^{(1)} = x^4 - 8x^3 + 64x,$$
$$u_0^{(2)} = x^5 - \frac{595}{57} x^4 + \frac{1720}{57} x^3 - \frac{1344}{19} x,$$

and $u_0^{(3)}$ is such that

$$\deg u_0^{(3)} = 9, \quad u_0^{(3)}(1) = 0, \quad u_0^{(3)}(1.5) = 0,$$
$$u_0^{(3)}(2) = 0, \quad u_0^{(3)}(2.5) = 0, \quad u_0^{(3)}(3) = 0.$$

The three approximated solutions are shown in Fig. 2(7).

**Remark.** The number $m$ is not the exact multiplicity of solutions of the problem (5) in the case $mL_1 < L < (m + 1)L_1$. For example, having $L = 2$ and $u_0 = 17(x^4 - 2x^3 + x)$ the algorithm reached by 15 iterations another nontrivial solution of the problem (17) which has one zero in $(0, 2)$. With $L = 3$ and $u_0 = x^4 - 6x^3 + 27x$ by 19 iterations we obtained a solution of (17) having one zero between 2 and 3.
Fig. 2. (4)–(6) Approximated and initial solutions of (17) with $L = 4$, $N = 40$. (7) Approximated solutions of (17) with $L = 4$, $N = 40$.

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**References**