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Group classes and mutually permutable products

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Abstract

Let $G = AB$ be the mutually permutable product of the nontrivial subgroups A and B of the group G. Then A or B contains a nontrivial normal subgroup of G. It is also established that $S(G) \cap A = S(A)$, where $S(U)$ is the solvable radical of U. These facts lead to some generalized results about mutually permutable products of SC-groups. It is also shown that if G is a PTS-group, then A and B are PST-groups.

1. Introduction

All groups considered are finite. Let $G = AB$ be the product of the subgroups A and B. This product is called mutually permutable (see [1,2,9]) (respectively totally permutable (see [2,4,6])) if A permutes with the subgroup of B and B permutes with the subgroups of A (respectively if every subgroup of A permutes with every subgroup of B). In [2] Asaad and Shaalan provide sufficient conditions for totally and mutually permutable products of two supersolvable subgroups to be supersolvable. The motivation for the seminal paper [2]

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is that G need not be supersolvable even when A and B are normal supersolvable subgroup of C.

Let $G = AB$ be a mutually permutable product of the subgroups A and B. In [1] it is shown that $G/(A \cap B)_G$ is supersolvable whenever A and B are supersolvable. Here $(A \cap B)_G$ is the core of $A \cap B$ in G. Moreover, in [9] $G/(A \cap B)_G$ is p-supersolvable, p a prime, if A and B are p-supersolvable. It is also shown in [9] that if A and B are Sylow tower groups with respect to a Sylow tower where all the primes q dividing $p - 1$ appear on top of p, then G is a Sylow tower of the same form.

The present paper is a continuation of [9] where we investigate further some of the basic properties of mutually permutable products. Our results are often motivated by corresponding results in totally permutable products.

2. Theorems and proofs

The authors showed in [6] that if $G = AB$ is a totally permutable product of the nontrivial subgroups A and B, then $A_G B_G \neq 1$. Using some of the results of [9] we prove

Theorem 1. *Assume that the group G is the mutually permutable product of its subgroups A and B. Then $A_G B_G \neq 1$.*

Proof. By Lemma 1 of [9] the theorem is true for simple groups. Assume that the theorem is false and that G is a counterexample of smallest order. By Corollary 3 of [9] the solvable residuals $D(A)$ of A and $D(B)$ of B are normal subgroups of G so that $D(A) \subseteq A_G = 1$ and $D(B) \subseteq B_G = 1$. Hence G is solvable by Corollary 2 of [9]. Note that all quotient groups of G satisfy the conditions of the theorem.

Let N be a minimal normal subgroup of G. By part (viii) of Lemma 1 in [9], $N \cap A = N \cap B = 1$. Moreover, by Lemma 2 of [9] the order of N is a prime, say p. Let M/N be a minimal normal subgroup of G/N. We may assume that M/N is a subgroup of AN/N. Thus $M = M \cap AN = N(M \cap A)$. By part (ii) of Lemma 1 in [9], $(M \cap A)(M \cap B)$ is a normal subgroup of G and since $M \cap A$ is not a normal subgroup of G, we have $M \cap B \neq 1$. This means $M/N \cap BN/N \neq 1$ so that B covers M/N by part (viii) of Lemma 1 in [9]. Hence also $M = N(M \cap B)$ and M/N is isomorphic to $M \cap A$ and $M \cap B$. Therefore, $|M|$ is a divisor of $|M \cap A|^2$ and M/N is a p-group. This means that M is also a p-group. By $M = (A \cap M)N = (A \cap M)Z(M)$ we find that M is elementary abelian.

Let us assume that M/N is noncyclic. Then $D = M \cap (A \cap B) = (A \cap M) \cap (B \cap M) \neq 1$. We have $1 = B_G = B_A = A \cap B_A = (A \cap B)_A$. Since $A \cap B$ is permutable in A, $A \cap B$ is contained in the hypercenter of A. This means that D is a p-group which is centralized by all p'-elements of A. Since $D_A = 1$, there is an element $a \in A$ such that $a^{-1}Da \neq D$ and the intersection has index p in D. By order relations we obtain $M \cap A = Da^{-1}Da$ and $M \cap A$ is contained in the hypercenter of A. In particular, $A/C_A(M \cap A)$ is a p-group. By symmetry, $B/C_B(M \cap B)$ is a p-group. Now $A \cap M \cong (A \cap M)N/N = M/N = (B \cap M)N/N \cong B \cap M$ and the two isomorphisms are operator isomorphisms to A and B respectively. We obtain that $(G/N)/(C_{G/N}(M/N))$ is a product of p-groups and hence a p-group. But M/N cannot be both noncyclic and a minimal normal subgroup of G/N. Thus we have derived that M/N must be cyclic and $|M| = p^2$.

Let $C = \mathcal{C}(M)$ and note that $G/C = (AC/C)(BC/C)$ is isomorphic to a subgroup of $C_{p-1} \times \mathrm{Hol}(C_p)$. Assume that there is an element $a \in A \setminus (A \cap C)$ such that $a^p \in A \cap C$. Then $N = [M, a] = [(A \cap M)N, a] = [A \cap M, a] = 1$, a contradiction. By symmetry we obtain the same contradiction if A is substituted with B. Now p does not divide $|G/C|$ and G/C is abelian of exponent dividing $p - 1$. Not all the elements induce power automorphisms on M, otherwise $M \cap A$ and $M \cap B$ would be normal subgroups of G, a contradiction. Consider an element $b \in B$ that does not induce a power automorphism on M, then b induces different power automorphisms on N and $M \cap B$. Let $g \in G$. Since G/C is abelian, b and $g^{-1}bg$ induce the same automorphism on M. Thus we have $(g^{-1}b^{-1}g)(M \cap B)(g^{-1}bg) = M \cap B$ and $b^{-1}(g(M \cap B)g^{-1})b = g(M \cap B)g^{-1}$ follows. This means that b induces the same power automorphism on $M \cap B$ and $g(M \cap B)g^{-1}$, and so $M \neq (M \cap B)(g(M \cap B)g^{-1})$. Therefore $M \cap B = g(M \cap B)g^{-1}$ for all $g \in G$. This is a contradiction and the theorem is true. □

A group G is called an SC-group (SNAC-group), if all chief factors (all nonabelian chief factors) of G are simple. D.J.S. Robinson [14] introduced these types of groups and characterized them.

We are now in the position to say something about products of these groups.

Theorem 2. *Assume that G is the mutually permutable product of the subgroups A and B. Then A and B are SNAC-groups if and only if G is a SNAC-group.*

Proof. Assume that A and B are SNAC-groups and also assume that G is not an SNAC-group and that it is of minimal order among all such groups. Then all proper quotient groups of G satisfy the theorem. Further G possesses just one minimal normal subgroup, M say, and M is nonabelian and not simple. By part (vi) of Lemma 1 in [9], M is contained in at least one of the two factors A and B. Assume now $M \subseteq A$ and $M \cap B = 1$. Then the elements of B induce in M power automorphisms by conjugation and A-invariant subgroups of M are also G-invariant. This means that M is simple, a contradiction. The same is true if $M \subseteq B$ and $M \cap A = 1$. Assume now $M \subseteq A \cap B$. Then M is at the same time a direct product of simple A-invariant and of simple B-invariant subgroups. Since the direct product of nonabelian simple groups is unique, we obtain a contradiction to the minimality of M. Thus G is an SNAC-group.

Conversely, assume that G is an SNAC-group. Let N be a minimal normal subgroup of G. By part (vii) of Lemma 1 in [9], A and B either cover or avoid N. Note that the SNAC-group G/N is the mutually permutable product of the subgroups AN/N and BN/N. Then AN/N and BN/N are SNAC-groups. If A avoids N, then $A \cong AN/N$ so that A is an SNAC-group. If A covers N, then A is also an SNAC-group since N is simple. This completes the proof of the theorem. □

For SC-groups the statement is less general.

Theorem 3. *Assume that G is the mutually permutable product of the subgroups A and B. Then*

(1) *If G is an SC-group, then A and B are SC-groups.*
(2) *If A and B are SC-groups, then $G/(A \cap B)_G$ is an SC-group.*

Proof. Statement (1) is established in the same way as the second part of the proof of Theorem 2. Assume now that A and B are SC-groups. By Theorem 2, G is an SNAC-group. Put $D = A \cap B$ and note by part (v) of Lemma 1 in [9] that D^G/D_G is nilpotent and G/D^G is a totally permutable product of AD^G/D^G and BD^G/D^G. Both factors are SC-groups so that G/D^G is an SC-group by Theorem A of [4]. It remains to show that every G-chief factor H/K with $D_G \subseteq H \subset K \subseteq D^G$ is cyclic. We know that H/K is a p-group for some prime p since D^G/D_G is nilpotent. By part (vi) of Lemma 1 in [9], A and B cover or avoid H/K. Suppose first that A and B avoid H/K. Then, by Lemma 2 of [9], $|H/K| = p$, and the chief factor is cyclic. Assume now that A covers H/K and B avoids H/K. Then B induces power automorphisms on H/K, and A-invariant subgroups of H/K are also B-invariant. Hence H/K is a minimal normal subgroup of AK/K and so it must be cyclic. The same is true if A and B are interchanged.

Assume finally that both A and B cover H/K. Consider $W/K = C_{(G/K)}(H/K)$. The group of automorphisms induced by AK/K on H/K is isomorphic to AW/W. Since A is an SC-group, there is a chief series of AK/K containing H/K and quotients of consecutive terms from 1 to H/K are simple of order p. This means that AW/W must be an extension of a p-group by an abelian group of exponent $p-1$. The same is true for BW/W. Note that D^G is contained in W and hence G/W is the totally permutable product of AW/W and BW/W. By Theorem 1 of [6], G/W is an extension of a p-group R/W by a totally permutable product of two abelian groups of exponent $p-1$ and therefore supersolvable by [2,6]. Since $[R,H]K/K$ is a normal subgroup of G/K different from H/K, we have $[R,H] \subseteq K$ and $R = W$ by definition of W. It now follows that AW/W and BW/W are abelian of exponent $p-1$. By Lemma 4 of [9] we also have $[A,B] \subseteq W$ so that G/W is abelian of exponent $p-1$. Now minimality of H/K yields that H/K is cyclic and so G/D_G is an SC-group. This completes the proof. □

Theorem 3 has also been established by Ballester-Bolinches, Cossey and Pedraza-Aquilera [3] using a completely different approach.

Let $G = AB$ be the mutually permutable product of the subgroups A and B and assume that A is an SC-group. What condition on B guarantees that G is an SC-group? In Theorem 7 of [9] a similar problem is considered: In that theorem A is a supersolvable group and the Frattini quotient group $B/\Phi(B)$ of B is a solvable T-group. In particular, B is supersolvable and G is also supersolvable. In [3] the authors consider the above question and show in Theorem 2 of [3] that G is an SC-group whenever B is a quasinilpotent group. A group G is called *quasinilpotent* if for every chief factor H/K the automorphisms induced by elements of G are already induced by elements of H. The class of quasinilpotent groups is considered in [10], it is quotient closed; by Theorem 13.6 of [10] a group G is quasinilpotent if and only if its quotient group $G/Z_*(G)$ modulo its hypercenter $Z_*(G)$ is semisimple. In particular, a quasinilpotent group is an SC-group.

The next result gives some more information about the above question.

Corollary 1. *Let the group G be a mutually permutable product of the subgroups A and B. If A is an SC-group and B is a T-group, then G is an SC-group.*

Proof. Note that B is an SC-group. Assume that the corollary is false and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G. By Theorems 2 and 3, N is an elementary abelian p-group for some prime p and $N \subseteq A \cap B$. Let x be a nontrivial element of N such that $\langle x \rangle$ is normal in A. Since B is a T-group, $\langle x \rangle$ is normalized by B. Thus $\langle x \rangle$ is normal in G, a contradiction; and the corollary follows. □

The class $\mathcal{F}$ of SC-groups is a formation, and from Corollary 1 we obtain the following result: if $G = AB$ is a normal product of the SC-group A and the T-group B, then G is an SC-group. Thus B is contained in the Fitting core of the formation $\mathcal{F}$. For details about the Fitting core of a formation see [7].

We will consider further the inner structure of the product of two mutually permutable subgroups in connection with its factors. We know by Corollary 3 of [9] that the solvable residuals of the factors are normal in the product G, so their product is the solvable residual of G by Corollary 2 of [9]. Somewhat dual is the concept of the solvable radical of G. We denote the solvable radical of the group X by $S(X)$. Here we obtain the following statement.

Theorem 4. *If G is the mutually permutable product of the subgroups A and B, then $S(A) = S(G) \cap A$.*

Proof. By construction, $S(G) \cap A$ is a solvable normal subgroup of A, so $S(G) \cap A \leqslant S(A)$ and likewise $S(G) \cap B \leqslant S(B)$. Let G be a minimal counterexample to the theorem. By Theorem 1 we can assume $A_G \neq 1$. Consider a minimal normal subgroup M of G with $M \leqslant A$. If M is abelian, then $S(G/M) = S(G)/M$, $S(A/M) = S(A)/M$ and further $S(BM/M) = S(B)M/M$. By minimality of G we obtain $S(G) \cap A = S(A)$ and $S(G) \cap BM = S(B)M$ with $S(G) \cap B = S(B)M \cap B = S(B)$. Assume now that M is nonabelian. Then $S(G) = S(C(M))$ and $S(A) = S(C(M) \cap A)$. If $M \leqslant B$, then $S(B) = S(C(M) \cap B)$; on the other hand if $M \nleqslant B$, then $M \leqslant C(B)$, by part (viii) of Lemma 1 of [9], and again $S(B) = S(C(M) \cap B)$. Now consider the product $(C(M) \cap A)(C(M) \cap B)$. This product is a proper normal subgroup of G which is also a mutually permutable product by parts (i) and (ii) of Lemma 1 of [9]. Thus

$$S\big(C(M)\big) \cap \big(C(M) \cap A\big) = S\big(C(M) \cap A\big) = S(A)$$

and

$$S\big(C(M)\big) \cap \big(C(M) \cap B\big) = S\big(C(M) \cap B\big) = S(B).$$

The theorem now follows from the nonexistence of a counterexample since $S(G) = S(C(M))$. □

In the light of Theorem 4 one might ask if $S(G) = S(A)S(B)$. That this is not case follows from

Example. Let $G = S_5 \times C_2$ where S_5 is the symmetric group on five letters with transposition (12). Put $C_2 = \langle x \rangle$. Now $A = S_5$ and $B = \langle A_5, x(12) \rangle$ are normal subgroups of $G = AB$, and $S(A) = S(B) = 1 \neq \langle x \rangle = S(G)$.

A group G is called a PT-group (T-group) if permutability (normality) is a transitive relation in G. By a result of Ore (see [13, 13.3.2]), PT-groups are exactly those groups where all the subnormal subgroups are permutable (see [2]). PST-groups are also defined via a transitivity property, namely with respect to S-permutability (see [4,8]): a subgroup of a group G is called S-permutable if it permutes with all the Sylow subgroup of G (see [13]). By a result of Kegel [12, Satz 1], every S-permutable subgroup is subnormal and hence PST-groups are exactly those groups in which all subnormal subgroups are permutable. In particular, PT-groups are PST-groups.

Let $\mathcal{T}, \mathcal{PT}, \mathcal{PST}$ denote the class of all finite T-groups, PT-groups, PST-groups respectively (see [1,5,8,12,14]). All three classes are closed with respect to quotient groups and normal subgroups. Let Θ be one of $\{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. A subnormal subgroup H of a finite group G is said to be *Θ-well embedded* in G if

(a) H is normal subgroup of G for $\Theta = \mathcal{T}$,
(b) H is permutable in G for $\Theta = \mathcal{PT}$,
(c) H is S-permutable in G for $\Theta = \mathcal{PST}$.

The following facts concerning Θ-well placed subnormal subgroups follow easily:

(1) If H is Θ-well placed in G and N is a normal subgroup of G then HN/N is Θ-well placed in G/N;
(2) if H is Θ-well placed in G if and only if H/H_G is Θ-well placed in G/H_G.

We are now able to formulate the next theorem and its proof.

Theorem 5. *Let $G = AB$ be a mutually permutable product of the subgroups A and B. If $G \in \Theta$, then $A \in \Theta$.*

Proof. We collect facts for a minimal counterexample.

(1) Every minimal normal subgroup of G is contained in A.

To show this we assume that the minimal normal subgroup N of G is not contained in A. Then by part (viii) of Lemma 1 in [9] we have $A \cap N = 1$ and so $A \cong A/(A \cap N) \cong AN/N$. Since G is a minimal counterexample and $G/N \in \Theta$, it follows that $A \cong AN/N \in \Theta$. This is a contradiction.

(2) Let H be a subnormal subgroup of A which is not Θ-well placed in A and assume that H is of minimal order. Then H is solvable.

Let $D(A), D(H)$ be the solvable residuals of A, H respectively. Then $D(H)$ is a subnormal subgroup of $D(A)$. By Corollary 3 of [9], $D(A)$ is normal in G and so $D(H)$ is subnormal in G. Because of Lemma 2 and Theorem 3 of [4] the perfect subnormal subgroup $D(H)$ of G is normal in G and so $D(H)$ is contained in H_G. Assume $H_G \neq 1$. Then H/H_G is Θ-well placed in G/H_G and hence H is Θ-well placed in G, a contradiction. This $D(H) \subseteq H_G = 1$ and H is solvable.

(3) Let $S(G), S(A)$ be the solvable radical of G, A respectively. Then H is contained in $S(G)$.

By Theorem 4 we have $H \subseteq S(A) = A \cap S(G)$, so $H \subseteq S(G)$.

Put $S = S(G)$ and let $S^{\mathcal{N}}$ denote the nilpotent residual of S. Then $S^{\mathcal{N}}$ is a normal subgroup of G and the elements of G act by conjugation on $S^{\mathcal{N}}$ as power automorphisms (see part (4) of Theorem 3 in [4]). In particular, every subgroup of $S^{\mathcal{N}}$ is normal in G.

(4) H is nilpotent, $H \cap S^{\mathcal{N}} = 1$ and $S^{\mathcal{N}} \neq 1$.

For assume $W = H \cap S^{\mathcal{N}} \neq 1$. Then W is a normal subgroup of G by (3) and $W \subseteq H_G = 1$. Now $H \cong HS^{\mathcal{N}}/S^{\mathcal{N}} \subseteq S/S^{\mathcal{N}}$ is nilpotent.

Assume that $S^{\mathcal{N}} = 1$. Then S is nilpotent and H is subnormal in G. But $G \in \Theta$ and so H is Θ-well placed A, a contradiction. So $S^{\mathcal{N}} \neq 1$.

(5) H is a p-group for some prime p.

Assume that H is not a p-group. Since H is nilpotent, it is a direct product of its Sylow subgroups each of which is Θ-well placed in A. Thus H is Θ-well placed in A, a contradiction. Therefore H is a p-group for some prime p.

(6) Let M be a minimal normal subgroup of G which is contained in $S^{\mathcal{N}}$. Then $(|H|, |M|) = 1$ and HM is Θ-well placed in A.

$M \subseteq A$ by (1). Now $S = S(G)$ is a solvable Θ-group and, by Theorem 3 of [8], $S^{\mathcal{N}}$ is a Hall subgroup of S. Since $H \cap S^{\mathcal{N}} = 1$ we obtain $(|H|, |M|) = 1$. Also by choice A/M is a Θ-group so that HM/M is Θ-well placed in A/M. Hence HM is Θ-well placed in A.

(7) $HM = H \times M$ and H is Θ-well placed in A.

By a result of Wielandt (see [13, 13.3.7]), M normalizes H and so $HM = H \times M$. If $\Theta = T$, then $H \times M$ is normal in A and H is normal in A. If $\Theta = PT$, then H is permutable in A by a result of Ito–Szep [11]. If $\Theta = PST$, then H is S-permutable in A by a result of P. Schmid [15]. Therefore, H is Θ-well placed in A in all cases, a final contradiction. □

References

[1] M.J. Alejandre, A. Ballester-Bolinches, J. Cossey, Permutable products of supersoluble groups, J. Algebra 276 (2004) 453–461.
[2] M. Assad, A. Shaalan, On the supersolubility of finite groups, Arch. Math. 53 (1989) 318–326.
[3] A. Ballester-Bolinches, J. Cossey, M.C. Pedraza-Aquilera, On mutually permutable products of finite groups, submitted for publication.
[4] J.C. Beidleman, P. Hauck, H. Heineken, Totally permutable products of certain classes of finite groups, J. Algebra 276 (2004) 826–835.
[5] J.C. Beidleman, H. Heineken, Pronormal and subnormal subgroups and permutability, Boll. Unione Mat. Ital. (8) 6 (2003) 605–615.
[6] J.C. Beidleman, H. Heineken, Totally permutable torsion groups, J. Group Theory 2 (1999) 377–392.
[7] J.C. Beidleman, H. Heineken, On the Fitting core of a formation, Bull. Austral. Math. Soc. 68 (2003) 107–112.
[8] J.C. Beidleman, H. Heineken, Finite soluble groups whose subnormal subgroups permute with certain classes of groups, J. Group Theory 6 (2003) 139–158.
[9] J.C. Beidleman, H. Heineken, Mutually permutable subgroups and group classes, Arch. Math. 85 (2005) 12–30.
[10] B. Huppert, N. Blackburn, Finite Groups III, Springer, Berlin, 1982.
[11] N. Ito, J. Szep, Über die Quasinormalteiler von endlichen Gruppen, Acta Sci. Math. (Szeged) 23 (1962) 168–170.
[12] O.H. Kegel, Sylowgruppen und Subnormalteiler endlicher Gruppen, Math. Z. 78 (1962) 205–221.
[13] D.J.S. Robinson, A Course in the Theory of Groups, Springer, Berlin, 1996.
[14] D.J.S. Robinson, The structure of finite groups in which permutability is a transitive relation, J. Aust. Math. Soc. 70 (2001) 143–159.
[15] P. Schmid, Subgroups permutable with all Sylow subgroups, J. Algebra 207 (1998) 285–293.