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Description of 2-integer continuous knapsack polyhedra

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Abstract

In this paper we discuss the polyhedral structure of several mixed integer sets involving two integer variables. We show that the number of the corresponding facet-defining inequalities is polynomial on the size of the input data and their coefficients can also be computed in polynomial time using a known algorithm [D. Hirschberg, C. Wong, A polynomial-time algorithm for the knapsack problem with two variables, Journal of the Association for Computing Machinery 23 (1) (1976) 147–154] for the two integer knapsack problem. These mixed integer sets may arise as substructures of more complex mixed integer sets that model the feasible solutions of real application problems.

Keywords: Mixed integer programming; Polyhedral characterization; Single node flow problem

1. Introduction

The description of the convex hull of elementary mixed integer sets has been useful in the generation of strong valid inequalities for general mixed integer problems. The so-called MIR inequalities are probably the most successful example in this area (see [2]). Recently, several elementary sets resulting from the aggregation of general MIP inequalities have been studied in order to generate strong cutting planes for those general models. Marchand and Wolsey [3] study the case of binary knapsack sets with a continuous variable and Günlük and Pochet [4] consider models with multiple constraints with a continuous variable and a single integer variable per inequality.

In this paper we describe the convex hull of some mixed integer sets involving only two integer variables. First we consider sets of the form

$$X = \{(y_1, y_2, s) : a_1 y_1 + a_2 y_2 \le D + \delta s,\ y_1, y_2, s \ge 0,\ y_1, y_2 \text{ integer}\},$$

where $a_1, a_2, D \in \mathbb{Z}$ and $\delta \in \{-1, 1\}$. Then we consider the integer single node flow set with two arcs:

$$Z = \{(x_1, x_2, y_1, y_2, s) : x_1 + x_2 \le D + s,\ 0 \le x_1 \le a_1 y_1,\ 0 \le x_2 \le a_2 y_2,\ s \ge 0,\ y_1, y_2 \text{ integer}\}$$

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and discuss the relationship between $\text{conv}(Z)$ and $\text{conv}(Z_{s=0})$ where

$$Z_{s=0} = \{(x_1, x_2, y_1, y_2) : x_1 + x_2 \le D, 0 \le x_1 \le a_1 y_1, 0 \le x_2 \le a_2 y_2, y_1, y_2 \text{ integer}\}.$$

Independently, Rajan [5] derived an equivalent description of $\text{conv}(X)$. That description is based on an algorithm that uses an algorithm from Kannan [6] and then solves a sequence of diophantine approximations.

Next we present the two main motivations to study the polyhedral structure of the convex hulls of these sets. The first motivation is based on the observation that most of the known valid inequalities for general mixed integer problems are generalizations of the basic MIR inequalities. These inequalities can be seen as the unique non-trivial facet-defining inequalities for the convex hull of sets involving one integer and one continuous variable. If we consider the set $\{(y, w) \in \mathbb{N}_0 \times \mathbb{R} : ay \le D + w\}$ the basic MIR inequality for this set is given by

$$y \le \lfloor D/a \rfloor + \frac{w}{a\lceil D/a \rceil - D} \tag{1}$$

where $D, a \in \mathbb{N}$ and a does not divide D. In order to obtain new families of valid inequalities we study a model which has two integer and one continuous variables. The introduction of an extra integer variable increases considerably the complexity of the model. The reason is that in the 1-integer case, if the continuous variable is fixed we obtain a set whose convex hull has only one non-trivial facet-defining inequality, while in the 2-integer case we obtain a knapsack set with two integer variables which requires an algorithm to compute all coefficients necessary to obtain the polyhedral description. Nevertheless, this model allows a polynomial description and the models studied here inherit this property.

A second motivation to study this kind of sets involving two integer and one continuous variable is the fact that they can be obtained from the relaxation and aggregation of more complex mixed integer sets that model real application problems. In particular, several such applications to lot-sizing models can be easily found. Sets of the form X and Z may arise when we consider machine capacity constraints in lot-sizing problems involving two types of batches, one small and one large. Other situations can occur from the aggregation of the well known flow conservation constraints and the elimination of the stock variables. For instance, Constantino [7] generates valid inequalities for the General Single Item (GSI) lot-sizing model with lower and upper bounds on production and start-up times by studying two 2-integer continuous sets obtained from relaxation and aggregation of the original model. Other applications also occur in telecommunication problems such as the capacitated expansion problem (see [8]). In this problem X occurs when we consider the flow capacity constraint on each arc. Typically, the flow capacity is equal to the installed capacity plus an integral multiple of several (in this case two) modularity sizes.

In Section 2 we describe the convex hull of the set $\{(y_1, y_2, s) \in \mathbb{N}_0^2 \times \mathbb{R} : a_1 y_1 + a_2 y_2 \le D + s, s \ge 0\}$ with a_1, a_2, D positive. This description is obtained using the following procedure:

(i) eliminate the continuous variable by considering the restriction of the polyhedra to one of the two faces $s = 0$ or $s = a_1 y_1 + a_2 y_2 - D$;

(ii) describe the convex hull of the resulting 2-integer knapsack sets using a polynomial algorithm from Hirschberg and Wong [1];

(iii) for each 2-integer facet-defining inequality find in polynomial time the lifting coefficient associated with the continuous variable.

In Section 3 we consider other 2-integer continuous sets of the form X which we classify accordingly to the polyhedral description and give some results on the polyhedral structure of those models. Finally, in Section 4, using the same approach as in Section 2, we describe the convex hull of the integer single node flow set with two arcs and of the set $Z_{s=0}$.

2. Polyhedral description of a 2-integer continuous set

In this section we consider the set $Y = \{(y_1, y_2, s) : a_1 y_1 + a_2 y_2 \le D + s,\ y_1, y_2, s \ge 0,\ y_1, y_2 \text{ integer}\}$ and assume that a_1, a_2, D are positive integers and $D > \max\{a_1, a_2\}$. Let Q denote $\text{conv}(Y)$. Consider the following sets (see Fig. 1):

$$Y_{\le} = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1 y_1 + a_2 y_2 \le D\},$$
$$Y_{\ge} = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1 y_1 + a_2 y_2 \ge D\}$$

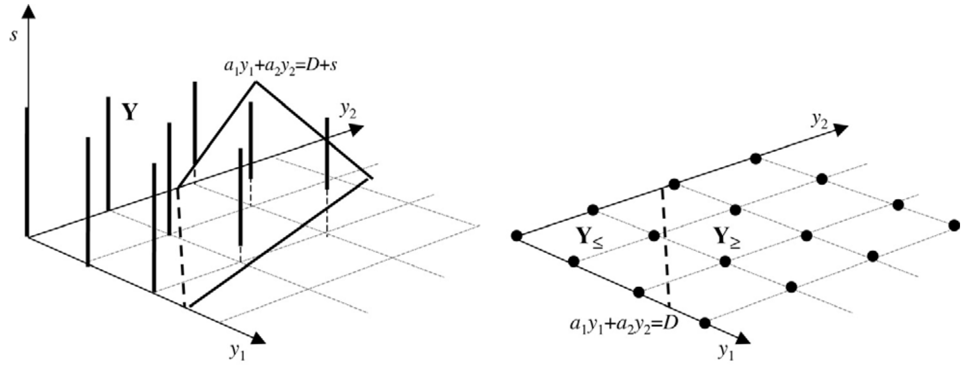

Fig. 1. Sets $Y, Y_{\leq}, Y_{\geq}$.

and the corresponding convex hulls $Q_{\leq} = \text{conv}(Y_{\leq})$ and $Q_{\geq} = \text{conv}(Y_{\geq})$. The sets $Y_{\leq}, Y_{\geq}$ are obtained when the continuous variable is eliminated by setting $s = 0$ and $s = a_1y_1 + a_2y_2 - D$, respectively.

Let $Q_1 = \{(y_1, y_2, s) \in Q : s = 0\}$ and $Q_2 = \{(y_1, y_2, s) \in Q : s = a_1y_1 + a_2y_2 - D\}$. We define the restriction of Q to $s = 0$ as the set $\text{Proj}_{(y_1,y_2)}(Q_1)$ and the restriction of Q to $s = a_1y_1 + a_2y_2 - D$ as the set $\text{Proj}_{(y_1,y_2)}(Q_2)$.

Remark 1. The convex hull of the restriction of a set S to the supporting hyperplane $\mathcal{H}$ of a valid inequality for S is equal to the restriction of the convex hull of S to $\mathcal{H}$.

This fact implies that the equalities $\text{Proj}_{(y_1,y_2)}(Q_1) = \text{conv}(Y_{\leq}) = Q_{\leq}$ and $\text{Proj}_{(y_1,y_2)}(Q_2) = \text{conv}(Y_{\geq}) = Q_{\geq}$ hold. For a given polyhedra P we denote by $V(P)$ its set of extreme points.

Lemma 2. *If* $(y_1^*, y_2^*, s^*) \in V(Q)$ *then either* $s^* = 0$ *or* $s^* = a_1y_1^* + a_2y_2^* - D$.

Proof. Suppose not. Considering $0 < \epsilon \leq \min\{s^*, s^* - a_1y_1^* - a_2y_2^* + D\}$ and $P^1 = (y_1^*, y_2^*, s^* - \epsilon)$, $P^2 = (y_1^*, y_2^*, s^* + \epsilon)$ then $P^1, P^2 \in Y$ and $(y_1^*, y_2^*, s^*) = 1/2P^1 + 1/2P^2$. □

Thus $V(Q) \subseteq V(Q_1) \cup V(Q_2)$. Since Q_1 and Q_2 are faces of Q the converse is also true, so $V(Q) = V(Q_1) \cup V(Q_2)$.

Lemma 3. $(y_1^*, y_2^*, s^*) \in V(Q)$ *if and only if one of the following conditions holds:*

(i) $s^* = 0$ *and* $(y_1^*, y_2^*) \in V(Q_{\leq})$.
(ii) $s^* = a_1y_1^* + a_2y_2^* - D$ *and* $(y_1^*, y_2^*) \in V(Q_{\geq})$.

Proof. Suppose $(y_1^*, y_2^*, s^*) \in V(Q)$. Lemma 2 implies $s^* = 0$ or $s^* = a_1y_1^* + a_2y_2^* - D$. Suppose $s^* = 0$ (the other case is similar). Thus $(y_1^*, y_2^*, 0) \in V(Q_1)$. Using Corollary 3.6 in [9] we have $(y_1^*, y_2^*) \in V(Q_{\leq})$.

To prove the converse suppose $(y_1^*, y_2^*) \in V(Q_{\leq})$. If $(y_1^*, y_2^*, 0) \notin V(Q)$ then there exists (y_1^1, y_2^1, s^1) and $(y_1^2, y_2^2, s^2) \in Q$ such that $(y_1^*, y_2^*, 0) = 1/2(y_1^1, y_2^1, s^1) + 1/2(y_1^2, y_2^2, s^2)$. As $s^1 \geq 0, s^2 \geq 0$ and $1/2s^1 + 1/2s^2 = 0$ then $s^1 = s^2 = 0$. Using Remark 1 we have $(y_1^1, y_2^1), (y_1^2, y_2^2) \in Q_{\leq}$, contradicting the hypothesis $(y_1^*, y_2^*) \in V(Q_{\leq})$. The proof for case (ii) is similar. □

Lemma 4. *Let* $C(Q) = \{(y_1, y_2, s) : a_1y_1 + a_2y_2 \leq s, y_1, y_2, s \geq 0\}$ *denote the characteristic cone of* Q. *Then* $C(Q) = \{r \in \mathbb{R}^3 : r = \sum_{j=1}^{3} \lambda_j r^j, \lambda_j \geq 0, j = 1, 2, 3\}$ *where* $r^1 = (0, 0, 1)$, $r^2 = (1, 0, a_1)$ *and* $r^3 = (0, 1, a_2)$.

In order to obtain $V(Q_{\leq})$ we present two algorithms given in [1]. Those algorithms compute the points in $V(Q_{\leq})$ that maximize a function $f_1y_1 + f_2y_2$ with $\frac{f_2}{f_1} \leq \frac{a_2}{a_1}$. In order to obtain the extreme points that maximize the function with $\frac{f_2}{f_1} > \frac{a_2}{a_1}$ it suffices to exchange y_1 with y_2. The first algorithm is a non-polynomial algorithm that computes coefficients with important properties (see Lemma 5 below) that are essential to prove Propositions 7 and 9. The second algorithm is a polynomial version of the first and it is the algorithm that we may use in practice to compute the necessary information to describe Q.

Algorithm HW

Step 0: $j \leftarrow 1, (y_1^j, y_2^j) \leftarrow \left(\left\lfloor \frac{D}{a_1} \right\rfloor, 0\right), k \leftarrow 1, (c^k, d^k) \leftarrow \left(\left\lfloor \frac{a_2}{a_1} \right\rfloor, 1\right), \ell \leftarrow 1,$

$(e^\ell, f^\ell) \leftarrow \left(\left\lceil \frac{a_2}{a_1} \right\rceil, 1\right), r \leftarrow 1.$

Step 1: While $y_1^j - c^k \geq 0$ do

Set $\gamma^j \leftarrow D - a_1 y_1^j - a_2 y_2^j,\ \rho_\leq^k \leftarrow -a_1 c^k + a_2 d^k,\ \rho_\geq^\ell \leftarrow a_1 e^\ell - a_2 f^\ell$

(i) if $\gamma^j \geq \rho_\leq^k$ set $j \leftarrow j+1, (y_1^j, y_2^j) \leftarrow (y_1^{j-1}, y_2^{j-1}) + r(-c^k, d^k)$;

(ii) if $\gamma^j < \rho_\leq^k$ and $\rho_\leq^k \geq \rho_\leq^\ell$ set
$k \leftarrow k+1, (c^k, d^k) \leftarrow (c^{k-1}, d^{k-1}) + r(e^\ell, f^\ell)$;

(iii) if $\gamma^j < \rho_\leq^k$ and $\rho_\leq^k < \rho_\leq^\ell$ set
$\ell \leftarrow \ell+1, (e^\ell, f^\ell) \leftarrow (e^{\ell-1}, f^{\ell-1}) + r(c^k, d^k)$.

The algorithm starts with a trivial extreme point $(y_1^1, y_2^1) = \left(\left\lfloor \frac{D}{a_1} \right\rfloor, 0\right)$ and, iteratively, computes rational approximations, $\frac{c}{d}$ and $\frac{e}{f}$, for $\frac{a_2}{a_1}$ ($\frac{c}{d} \leq \frac{a_2}{a_1} \leq \frac{e}{f}$). Once a fraction $\frac{c}{d}$ with a remainder $\rho_\leq = -a_1 c + a_2 d$ not greater than the gap γ^j is obtained another point (y_1^{j+1}, y_2^{j+1}) can be obtained replacing c units of a_1 with d units of a_2. Then $\gamma^{j+1} = \gamma^j - \rho_\leq$.

This algorithm is not polynomial and some points (y_1, y_2) generated may not be extreme. In order to obtain only the extreme points in polynomial time as a function of the input data instead of considering always $r = 1$ it suffices to compute r in Step 1 as follows:

(i) $r = \min\left\{\left\lfloor \dfrac{\gamma^j}{\rho_\leq^k} \right\rfloor, \left\lfloor \dfrac{y_1^j}{c^k} \right\rfloor\right\}$;

(ii) $r = \min\left\{\left\lfloor \dfrac{\rho_\leq^k}{\rho_\geq^\ell} \right\rfloor, \left\lceil \dfrac{\rho_\leq^k - \gamma^j}{\rho_\geq^\ell} \right\rceil\right\}$;

(iii) $r = \left\lfloor \dfrac{\rho_\geq^\ell}{\rho_\leq^k} \right\rfloor$.

Notice that these computations avoid the occurrence in consecutive iterations of the same case in Step 1. Although we are interested in obtaining the extreme points in polynomial time, the coefficients obtained using the non-polynomial Algorithm HW have some important properties we will use later. However, we notice here that all the information needed to describe Q can be obtained in $\mathcal{O}(\log(D/\min\{a_1, a_2\}))$ elementary operations using the polynomial version of Algorithm HW (see [1]).

Next we introduce the notation for the non-polynomial case. We denote by $k(\ell)$ the index of the pair (c, d) used, in (iii) of Step 1, to obtain (e^ℓ, f^ℓ), that is $(e^\ell, f^\ell) = (e^{\ell-1}, f^{\ell-1}) + (c^{k(\ell)}, d^{k(\ell)})$. Similarly, we use the notation $\ell(k)$ to denote the index of the pair (e, f) used, in (ii) of Step 1, to obtain (c, d). Let n_1 and n_2 denote the number of distinct pairs (c, d) and (e, f) generated, respectively. Now we summarize some of the properties of these coefficients.

Lemma 5. (i) $e^\ell d^k - f^\ell c^k = 1$ *if* $\ell = \ell(k)$ *or* $k = k(\ell)$. (ii) $\frac{c^1}{d^1} \leq \cdots \leq \frac{c^{n_1}}{d^{n_1}} \leq \frac{a_2}{a_1} \leq \frac{e^{n_2}}{f^{n_2}} \leq \cdots \leq \frac{e^1}{f^1}$. (iii) $\frac{c^k}{d^k}, k = 1, \ldots, n_1$ *(resp.* $\frac{e^\ell}{f^\ell}, \ell = 1, \ldots, n_2$*) are the best approximations from below (resp. from above) to* $\frac{a_2}{a_1}$ *for that size denominator.* (iv) (a) *The set* $\{(c^1, d^1), \ldots, (c^k, d^k)\}$ *is an integral Hilbert basis for* $\text{Cone}\{(c^1, d^1), (c^k, d^k)\}$. (b) *The set* $\{(e^1, f^1), \ldots, (e^\ell, f^\ell)\}$ *is an integral Hilbert basis for* $\text{Cone}\{(e^1, f^1), (e^\ell, f^\ell)\}$.

The first three properties are well known since the coefficients (c, d), (e, f) are the sequence of convergents of the reduced continuous fraction into which $\frac{a_2}{a_1}$ is expanded (see [10,1]). (iv) is essentially due to [11] and follows from the non-polynomial version of the algorithm.

These algorithms can be easily adapted to generate $V(Q_\geq)$.

We notice that as the number of extreme points is polynomial and these points can be obtained in polynomial time it is possible to give an extended formulation of polynomial size for Q using the relation $Q = \text{conv}(V(Q)) + C(Q)$.

In order to give an explicit linear description of Q we generate two families of inequalities. Restricting Y to $s = 0$ we obtain $Y_{\leq}$. Then generating a valid inequality to $Y_{\leq}$ and lifting s we obtain a valid inequality to Y. Restricting Y to $s = a_1 y_1 + a_2 y_2 - D$ we obtain $Y_{\geq}$. Generating a valid inequality to $Y_{\geq}$ and lifting s we obtain a valid inequality to Y.

Let us define $Y_= = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1 y_1 + a_2 y_2 = D\}$, $Y_> = Y_{\geq} \setminus Y_=$ and $Y_< = Y_{\leq} \setminus Y_=$.

Proposition 6. *If*

$$\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha \tag{2}$$

defines a non-trivial facet of $Q_{\leq}$*, then*

$$\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha + \beta s \tag{3}$$

defines a facet of Q*, where* $\beta = \max\left\{\frac{\alpha_1 y_1 + \alpha_2 y_2 - \alpha}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_>\right\}$.

Proof. Consider a point $(y_1^*, y_2^*, s^*) \in Y$ with $a_1 y_1^* + a_2 y_2^* \leq D \Rightarrow (y_1^*, y_2^*) \in Y_{\leq}$. Since (2) is valid for all $(y_1, y_2) \in Y_{\leq}$, $\beta > 0$ (in Proposition 7 below we determine this value, however by exhibiting points in $Y_>$ it is easy to verify that $\beta > 0$) and $s^* \geq 0$ it follows that $\alpha_1 y_1^* + \alpha_2 y_2^* \leq \alpha + \beta s^*$. If $a_1 y_1^* + a_2 y_2^* > D$ then $(y_1^*, y_2^*) \in Y_>$ and $s^* > 0$. Therefore,

$$\beta \geq \frac{\alpha_1 y_1^* + \alpha_2 y_2^* - \alpha}{a_1 y_1^* + a_2 y_2^* - D} \geq \frac{\alpha_1 y_1^* + \alpha_2 y_2^* - \alpha}{s^*} \Rightarrow \alpha_1 y_1^* + \alpha_2 y_2^* \leq \alpha + \beta s^*.$$

Notice that since (2) defines a facet of $Q_{\leq}$ and β is the smallest value in order for (3) to be valid for Q thus (3) must define a facet of Q. □

Now we address the question of how to compute β. We need to consider the coefficients computed by Algorithm HW. Notice that every facet of $Q_{\leq}$ contains two points $(y_1^j, y_2^j), (y_1^{j+1}, y_2^{j+1}) \in V(Q_{\leq})$ with $(y_1^{j+1}, y_2^{j+1}) = (y_1^j, y_2^j) + r(-c^k, d^k)$ for some $k \in \{1, \ldots, n_1\}$ and some positive integer r. Thus $\alpha_1 = d^k \times \kappa$, $\alpha_2 = c^k \times \kappa$ and $\alpha = \kappa(d^k y_1^j + c^k y_2^j)$ for some $\kappa > 0$. We assume the coefficients are normalized, that is, $\kappa = 1$. Let us also assume $\frac{\alpha_2}{\alpha_1} \leq \frac{a_2}{a_1}$ (the other case is similar, it suffices to exchange a_1 with a_2) and assume that D is not divided by a_1 (in that case there would be no facet with $\frac{\alpha_2}{\alpha_1} < \frac{a_2}{a_1}$ since there would be only one extreme point that maximizes every function $f_1 y_1 + f_2 y_2$ with $\frac{f_2}{f_1} \leq \frac{a_2}{a_1}$). We will use the notation $\gamma(y_1, y_2) = D - a_1 y_1 - a_2 y_2$. Notice that $\gamma^j = \gamma(y_1^j, y_2^j)$.

The following proposition states that β depends on the gap γ^j and on one of the remainders $\rho_{\geq}^{\ell(k)} = a_1 e^{\ell(k)} - a_2 f^{\ell(k)}$, $\rho_{\leq}^{k-1} = -a_1 c^{k-1} + a_2 d^{k-1}$ and, as we will see, it also depends explicitly on the position of the point (y_1^j, y_2^j) in the plane. Notice that the coefficients (y_1^j, y_2^j), $(e^{\ell(k)}, f^{\ell(k)})$ and (c^k, d^k) can be obtained using the polynomial version of Algorithm HW, and $(c^{k-1}, d^{k-1}) = (c^k, d^k) - (e^{\ell(k)}, f^{\ell(k)})$. Thus, all the coefficients necessary to obtain β can be computed in polynomial time.

Proposition 7. *Let* (2) *define a facet of* $Q_{\leq}$ *containing the points* (y_1^j, y_2^j), $(y_1^{j+1}, y_2^{j+1}) \in V(Q_{\leq})$ *with* $(y_1^{j+1}, y_2^{j+1}) = (y_1^j, y_2^j) + r(-c^k, d^k)$ *for some* $k \in \{1, \ldots, n_1\}$ *and some positive integer* r*. Assuming that* $\alpha_1 = d^k, \alpha_2 = c^k$*, then* $\beta = \frac{1}{\eta^j}$ *where* $\eta^j = a_1 \lceil (D/a_1) \rceil - D$ *if* $k = 1$ *and*

$$\eta^j = \begin{cases} \rho_{\geq}^{\ell(k)} - \gamma^j, & \text{if } y_2^j \geq f^{\ell(k)}, \\ \rho_{\leq}^{k-1} - \gamma^j, & \text{if } y_2^j < f^{\ell(k)}, \end{cases}$$

otherwise.

Proof. We will use properties from Lemma 5. First consider the case $k > 1$. Let

$$(y_1^*, y_2^*) = \begin{cases} (y_1^j, y_2^j) + (e^{\ell(k)}, -f^{\ell(k)}), & \text{if } y_2^j \geq f^{\ell(k)}, \\ (y_1^j, y_2^j) + (-c^{k-1}, d^{k-1}), & \text{if } y_2^j < f^{\ell(k)}. \end{cases}$$

Next we show that (y_1^*, y_2^*) is the optimal solution to the maximization problem associated with the computation of β. Let $\tau(y_1, y_2)$ denote $\tau(y_1, y_2) = \alpha_1 y_1 + \alpha_2 y_2 - \alpha$. So (2) can be written as $\tau(y_1, y_2) \le 0$. Notice that $\eta^j = -\gamma(y_1^*, y_2^*)$ and, from Lemma 5, $\tau(y_1^*, y_2^*) = 1$ which implies $(y_1^*, y_2^*) \in Y_>$. Using this notation we have $\beta = \max\left\{\frac{\tau(y_1,y_2)}{-\gamma(y_1,y_2)} : (y_1, y_2) \in Y_>\right\}$. Hence, we need to show that $\frac{\tau(y_1^*,y_2^*)}{-\gamma(y_1^*,y_2^*)} \ge \frac{\tau(y_1,y_2)}{-\gamma(y_1,y_2)}$ for all $(y_1, y_2) \in Y_>$. Since the left-hand side is positive and $(y_1, y_2) \in Y_> \Rightarrow \gamma(y_1, y_2) < 0$ we only need to consider those points such that $\tau(y_1, y_2) \ge 1$ (notice that $\alpha_1, \alpha_2, \alpha$ are integer).

Claim 1. (a) $a_1 - \alpha_1\eta^j \ge 0$; (b) $a_2 - \alpha_2\eta^j \ge 0$.

Proof of Claim 1: Let $w_1 = -c^{k-1}a_1 + d^{k-1}a_2$ and $w_2 = e^{\ell(k)}a_1 - f^{\ell(k)}a_2$. Using (a) in Lemma 5 we obtain $d^{k-1}w_2 + f^{\ell(k)}w_1 = a_1$. There is a positive integer r such that $\gamma(y_1^j, y_2^j) = \gamma(y_1^{j+1}, y_2^{j+1}) + r(-c^k a_1 + d^k a_2) = \gamma(y_1^{j+1}, y_2^{j+1}) + r(w_1 - w_2) \ge w_1 - w_2 \ge 0$. Since

$$\eta^j = \begin{cases} w_2 - \gamma^j, & \text{if } y_2^j \ge f^{\ell(k)}, \\ w_1 - \gamma^j, & \text{if } y_2^j < f^{\ell(k)}, \end{cases}$$

we have $w_2 \ge \eta^j$ and $w_1 \ge \eta^j$. Thus $a_1 = d^{k-1}w_2 + f^{\ell(k)}w_1 \ge (d^{k-1} + f^{\ell(k)})\eta^j = \alpha_1\eta^j$. The proof of (b) is similar.

Claim 2. (a) $a_1e^\ell - a_2 f^\ell \ge \eta^j \times (d^k e^\ell - c^k f^\ell)$ *for all* $\ell \le \ell(k)$; (b) $-a_1c^t + a_2d^t > \eta^j \times (-d^k c^t + c^k d^t)$ *for all* $t \le k$.

Proof of Claim 2: For $\ell = \ell(k)$ we have $d^k e^\ell - c^k f^\ell = 1$ and as we saw above $a_1e^\ell - a_2 f^\ell = w_2 \ge \eta^j$. For $t = k$ we have $-d^k c^t + c^k d^t = 0$ and $w_1 = -a_1c^t + a_2d^t > 0$. The proof for all $\ell < \ell(k)$ and for all $t < k$ can be made by induction using simultaneously (a) and (b) and noticing that $(c^{t-1}, d^{t-1}) = (c^t, d^t) - (e^{\ell(t)}, f^{\ell(t)})$ and $(e^{\ell-1}, f^{\ell-1}) = (e^\ell, f^\ell) - (c^{k(\ell)}, d^{k(\ell)})$.

Now consider $(y_1', y_2') \ne (y_1^*, y_2^*)$ such that $(y_1', y_2') \in Y_>$ and $\tau(y_1', y_2') \ge 1$. Three cases may occur:

Case 1. $y_1' \ge y_1^*, y_2' \ge y_2^*$. Let $(c, d) = (y_1' - y_1^*, y_2' - y_2^*)$. Thus

$$\frac{\tau(y_1', y_2')}{-\gamma(y_1', y_2')} = \frac{\tau(y_1^*, y_2^*) + \alpha_1 c + \alpha_2 d}{-\gamma(y_1^*, y_2^*) + a_1 c + a_2 d}.$$

From Claim 1 comes $c(a_1 - \alpha_1\eta^j) + d(a_2 - \alpha_2\eta^j) \ge 0 \Rightarrow \frac{\alpha_1 c+\alpha_2 d}{a_1 c+a_2 d} \le \frac{1}{\eta^j} \Rightarrow \frac{\tau(y_1',y_2')}{-\gamma(y_1',y_2')} \le \frac{1}{\eta^j}$.

Case 2. $y_1' \le y_1^*, y_2' \ge y_2^*$. Let $(c, d) = (y_1^* - y_1', y_2' - y_2^*)$. It cannot happen $\frac{c}{d} > \frac{c^k}{d^k}$ because that would imply $\tau(y_1', y_2') = \tau(y_1^*, y_2^*) - c^k c + d^k d < \tau(y_1^*, y_2^*) \Rightarrow \tau(y_1', y_2') \le 0$. So $\frac{c}{d} \le \frac{c^k}{d^k}$. If $\frac{c}{d} < \frac{c^1}{d^1}$ let $p = \left\lfloor d\frac{c^k}{d^k}\right\rfloor - c$, otherwise take $p = 0$. Thus $(c + p, d) \in \text{Cone}\{(c^1, d^1), (c^k, d^k)\}$. Therefore, using (iv) in Lemma 5 (c, d) can be written as $(c, d) = \sum_{t=1}^k \beta^t(c^t, d^t) - p(1, 0)$ where β^t are non-negative integers. Thus

$$\frac{\tau(y_1', y_2')}{-\gamma(y_1', y_2')} = \frac{\tau(y_1^*, y_2^*) + \sum_{t=1}^k \beta^t(-\alpha_1 c^t + \alpha_2 d^t) + p\alpha_1}{-\gamma(y_1^*, y_2^*) + \sum_{t=1}^k \beta^t(-a_1 c^t + a_2 d^t) + pa_1}.$$

From Claim 2, for $t = 1, \ldots, k$ we obtain $\frac{-\alpha_1 c^t+\alpha_2 d^t}{-a_1 c^t+a_2 d^t} \le \frac{1}{\eta^j}$ and, if $p \ne 0$ from Claim 1, $\frac{p\alpha_1}{pa_1} \le \frac{1}{\eta^j}$ thus $\frac{\tau(y_1',y_2')}{-\gamma(y_1',y_2')} \le \frac{1}{\eta^j}$.

Case 3. $y_1' \ge y_1^*, y_2' \le y_2^*$. First we prove that $(e, f) = (y_1' - y_1^*, y_2^* - y_2')$ satisfies $\frac{e}{f} \ge \frac{e^{\ell(k)}}{f^{\ell(k)}}$. Suppose $\frac{e}{f} < \frac{e^{\ell(k)}}{f^{\ell(k)}}$. Notice that $\tau(y_1', y_2') \ge 1$ implies $\frac{e}{f} \ge \frac{c^k}{d^k}$. If $\frac{a_2}{a_1} \ge \frac{e}{f} \ge \frac{c^k}{d^k}$, as $\frac{c^k}{d^k}$ is the best approximation from below to $\frac{a_2}{a_1}$ for that size denominator, we conclude $f \ge d^k$. Similarly, if $\frac{a_2}{a_1} < \frac{e}{f} < \frac{e^{\ell(k)}}{f^{\ell(k)}}$ then $f \ge f^{\ell(k)+1} \ge f^{\ell(k)} + d^k > d^k$. In both cases we have $y_2^j \ge y_2^* - d^{k-1} = y_2' + f - d^{k-1} \ge d^k - d^{k-1} = f^{\ell(k)}$. Thus $(y_1^*, y_2^*) = (y_1^j, y_2^j) + (e^{\ell(k)}, -f^{\ell(k)})$.

This implies $y_2^j = y_2^* + f^{\ell(k)} = y_2' + f + f^{\ell(k)} > d^k$. Now considering $(\overline{y}_1, \overline{y}_2) = (y_1^j + c^k, y_2^j - d^k)$ we have $\overline{y}_1, \overline{y}_2 \geq 0$ and $\gamma(\overline{y}_1, \overline{y}_2) \geq 0$. Thus $(\overline{y}_1, \overline{y}_2) \in Y_{\leq}$ and (y_1^j, y_2^j) can be obtained as a linear convex combination of $(\overline{y}_1, \overline{y}_2)$ and (y_1^{j+1}, y_2^{j+1}) which contradicts the hypotheses that $(y_1^j, y_2^j) \in V(Q_{\leq})$.

Let $p = \left\lceil f \frac{e^{\ell(k)}}{f^{\ell(k)}} \right\rceil - e$ if $\frac{e}{f} > \frac{e^1}{f^1}$ and $p = 0$ otherwise. Thus $(e - p, f) \in \text{Cone}\{(e^1, f^1), (e^{\ell(k)}, f^{\ell(k)})\}$. Now we can use (iv) in Lemma 5 and the proof is similar to the proof of Case 2.

If $k = 1$ set $(y_1^*, y_2^*) = (\lceil D/a_1 \rceil, 0)$. Notice that $k = 1$ implies $j = 1$, thus $(y_1^j, y_2^j) = (\lfloor D/a_1 \rfloor, 0)$ and so $\tau(y_1^*, y_2^*) = 1$. For all other $(y_1', y_2') \in Y_>$ such that $\tau(y_1', y_2') \geq 1$ only Case 1 or Case 2 considered above can occur. Claim 1 also holds because $c^1 = \lfloor a_2/a_1 \rfloor$, $d^1 = 1$ and $\eta^j < a_1$ imply $a_1 - d^1\eta^j \geq 0$ and $a_2 \geq \lfloor a_2/a_1 \rfloor a_1 \geq c^1\eta^j$. Thus, the proofs given for Case 1 and Case 2 are valid in this situation. Notice that in this case it is not necessary to use Claim 2 in the proof of Case 2. □

Considering the restriction of Y to $s = a_1y_1 + a_2y_2 - D$ we obtain similar results.

Proposition 8. *If*

$$\alpha_1 y_1 + \alpha_2 y_2 \geq \alpha \tag{4}$$

defines a non-trivial facet of $Q_{\geq}$ *then*

$$(a_1\beta - \alpha_1)y_1 + (a_2\beta - \alpha_2)y_2 \leq D\beta - \alpha + \beta s \tag{5}$$

defines a facet of Q, *where* $\beta = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in Y_<\right\}$.

Proof. Notice that (5) can be written as $\beta(s+D-a_1y_1-a_2y_2)+\alpha_1y_1+\alpha_2y_2 \geq \alpha$. The validity for $(y_1, y_2, s) \in Y$ with $(y_1, y_2) \in Y_{\geq}$ follows from the validity of (4) and $\beta \geq 0$, $s + D - a_1y_1 - a_2y_2 \geq 0$. If $(y_1, y_2, s) \in Y$ with $(y_1, y_2) \in Y_<$, then $\beta \geq \frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} \Rightarrow (D - a_1y_1 - a_2y_2)\beta \geq \alpha - \alpha_1y_1 - \alpha_2y_2 \Rightarrow \beta(s + D - a_1y_1 - a_2y_2) + \alpha_1y_1 + \alpha_2y_2 \geq \alpha$. Since (2) defines a facet of $Q_{\leq}$ and β is the smallest value in order for (5) to be valid for Q thus (5) must define a facet of Q. □

Now we obtain β considering $\frac{\alpha_2}{\alpha_1} \geq \frac{a_2}{a_1}$ and assuming that D is not divided by a_1. The case $\frac{\alpha_2}{\alpha_1} \leq \frac{a_2}{a_1}$ is similar. It suffices to exchange a_1 with a_2.

Proposition 9. *Let (4) define a facet of* $Q_{\geq}$ *containing the points* (y_1^j, y_2^j), $(y_1^{j+1}, y_2^{j+1}) \in V(Q_{\geq})$ *with* $(y_1^{j+1}, y_2^{j+1}) = (y_1^j, y_2^j) + r(-e^\ell, f^\ell)$ *for some* $\ell \in \{1, \ldots, n_2\}$ *and some positive integer* r. *Assuming that* $\alpha_1 = f^\ell$, $\alpha_2 = e^\ell$, *then* $\beta = \frac{1}{\eta^j}$ *where* $\eta^j = D - a_1\lfloor (D/a_1) \rfloor$ *if* $\ell = 1$ *and*

$$\eta^j = \begin{cases} \gamma^j + \rho_{\leq}^{k(\ell)}, & \text{if } y_2^j \geq d^{k(\ell)}, \\ \gamma^j + \rho_{\geq}^{\ell-1}, & \text{if } y_2^j < d^{k(\ell)}, \end{cases}$$

otherwise.

Since the proof of Proposition 9 is similar to the proof of Proposition 7 it will be omitted.

Theorem 10. *The families of inequality (3) and (5) with the trivial set of inequalities* $y_1 \geq 0$, $y_2 \geq 0$, $s \geq 0$, $a_1y_1 + a_2y_2 \leq D + s$ *suffice to describe* Q.

Proof. Considering a facet $\mathcal{F}$ defined by a non-trivial inequality

$$\mu_1 y_1 + \mu_2 y_2 \leq \mu + \nu s \qquad (f)$$

we show that this inequality either belongs to family (5) or it belongs to family (3).

First, we show that

(i) $\mu_1, \mu_2, \mu \geq 0$,
(ii) $\nu > 0$,
(iii) $\nu a_1 > \mu_1$, $\nu a_2 > \mu_2$.

Proof of (i). Suppose $\mu_1 < 0$. There must exist a point $(y_1', y_2', s') \in \mathcal{F}$ satisfying $y_1' > 0$ since otherwise $\mathcal{F} \subseteq \{(y_1, y_2, s) : y_1 = 0\}$. Thus $(0, y_2', s') \in Y$ and violates (f). Similarly we have $\mu_2 \geq 0$. As $(0, 0, 0) \in Y$ it follows that $\mu \geq 0$.

Proof of (ii). Since $(1, 0, a_1)$ and $(0, 1, a_2)$ are directions of Q and μ_1 and μ_2 cannot be simultaneously zero (otherwise $\mathcal{F} \subseteq \{(y_1, y_2, s) : s = 0\}$) we have $\nu > 0$.

Proof of (iii). We only prove $\nu a_1 > \mu_1$ since the other case is similar. The existence of the direction $(1, 0, a_1)$ implies $\nu a_1 \geq \mu_1$. Suppose $\mu_1 = \nu a_1$. Let $(y_1', y_2', s') \in \mathcal{F} \cap Y$ that satisfies $a_1 y_1' + a_2 y_2' < D + s'$ (there must exist such a point since, otherwise, $\mathcal{F} \in \{(y_1, y_2, s) : a_1 y_1 + a_2 y_2 = D + s\}$). From this inequality and from $\frac{\mu_1}{\nu} y_1' + \frac{\mu_2}{\nu} y_2' = \frac{\mu}{\nu} + s'$ we obtain $(a_2 - \frac{\mu_2}{\nu}) y_2' < (D - \frac{\mu}{\nu})$. Now, consider a point $(y_1'', y_2', s'') \in Y$ such that $s'' = a_1 y_1'' + a_2 y_2' - D$ (for instance, setting $y_1'' = \left\lceil \frac{D - a_2 y_2'}{a_1} \right\rceil$ if $D \geq a_2 y_2'$ or setting $y_1'' = y_1'$, otherwise). This point must satisfy $\frac{\mu_1}{\nu} y_1'' + \frac{\mu_2}{\nu} y_2' \leq \frac{\mu}{\nu} + s''$ because (f) is valid for Y. From this inequality and from $s'' = a_1 y_1' + a_2 y_2' - D$ we have $(a_2 - \frac{\mu_2}{\nu}) y_2' \geq (D - \frac{\mu}{\nu})$ contradicting the previous inequality.

Now we are ready to prove that (f) either belongs to (3) or to (5). As $\dim(\mathcal{F}) = 2$ there are three affinely independent points lying in $\mathcal{F}$. First we consider the case where $\mathcal{F}$ is bounded. Thus we may consider extreme points in Q.

From Lemma 3 every extreme point of Q lies in one of the planes defined by $s = 0$ and $a_1 y_1 + a_2 y_2 = D + s$. Noticing that the three extreme points cannot all lie in the same plane simultaneously, we first suppose two of them satisfy $s = 0$. If (y_1, y_2, s) is one of these two points, from Lemma 3, it must satisfy $(y_1, y_2) \in Q_{\leq}$ and, since $s = 0$, it also satisfies $\mu_1 y_1 + \mu_2 y_2 = \mu$. From the validity of (f) for Y it follows that $\mu_1 y_1 + \mu_2 y_2 \leq \mu$ is valid for $Y_{\leq}$ and therefore it defines a facet of $Q_{\leq}$. Since μ_1 and μ_2 are non-negative and they cannot be simultaneously zero we know that this inequality defines a non-trivial facet for $Q_{\leq}$. Now, consider a third extreme point (y_1^*, y_2^*, s^*) satisfying $a_1 y_1^* + a_2 y_2^* = D + s^*$ with $s^* > 0$ which implies $(y_1^*, y_2^*) \in Y_{>}$. Thus, every point in $\mathcal{F}$ satisfies $\mu_1 y_1 + \mu_2 y_2 = \mu + \nu s$ where $\nu = \frac{\mu_1 y_1^* + \mu_2 y_2^* - \mu}{a_1 y_1^* + a_2 y_2^* - D}$. In order for (f) to define a valid inequality ν must satisfy $\nu = \max\{\frac{\mu_1 y_1 + \mu_2 y_2 - \mu}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_{>}\}$. Therefore (f) belongs to family (3) where $\alpha_1 = \mu_1$, $\alpha_2 = \mu_2$, $\alpha = \mu$ and $\beta = \nu$. Now we consider the case where there are two extreme points in the facet satisfying $a_1 y_1 + a_2 y_2 = D + s$. Using Lemma 3, $(y_1, y_2) \in Q_{\geq}$. Such points must satisfy $\mu_1 y_1 + \mu_2 y_2 = \mu + \nu(a_1 y_1 + a_2 y_2 - D) \Rightarrow (\mu_1 - \nu a_1) y_1 + (\mu_2 - \nu a_2) y_2 = \mu - \nu D$. Notice that from (iii), $(\mu_1 - \nu a_1) < 0$ and $(\mu_2 - \nu a_2) < 0$ and the validity of (f) for Y implies that $(\nu a_1 - \mu_1) y_1 + (\nu a_2 - \mu_2) y_2 \geq \nu D - \mu$ is valid for $Y_{\geq}$ and therefore it defines a non-trivial facet of $Q_{\geq}$. Let $\alpha_1 = \nu a_1 - \mu_1$, $\alpha_2 = \nu a_1 - \mu_2$, $\alpha = \nu D - \mu$. Now, considering a third extreme point (y_1^*, y_2^*, s^*) with $s^* = 0$ and $(y_1^*, y_2^*) \in Y_{<}$ we have $\mu_1 y_1^* + \mu_2 y_2^* = \mu \Leftrightarrow (\nu a_1 - \alpha_1) y_1^* + (\nu a_2 - \alpha_2) y_2^* = \nu D - \alpha \Leftrightarrow \alpha - \alpha_1 y_1^* - \alpha_2 y_2^* = \nu(D - a_1 y_1^* - a_2 y_2^*)$. In order for (f) to define a valid inequality (y_1^*, y_2^*) must satisfy $(y_1^*, y_2^*) \in \arg\max\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in Y_{<}\}$. Therefore (f) belongs to family (5) with $\alpha_1 = \nu a_1 - \mu_1$, $\alpha_2 = \nu a_1 - \mu_2$, $\alpha = \nu D - \mu$, $\beta = \nu$.

It remains to consider the case in which $\mathcal{F}$ is not bounded. Let (v_1, v_2, v_3) be a direction of $\mathcal{F}$. Thus it must satisfy $\mu_1 v_1 + \mu_2 v_2 = \nu v_3$. From (iii) and noticing that $v_1, v_2 \geq 0$ and v_1 and v_2 cannot be simultaneously zero, we have $\nu a_1 v_1 + \nu a_2 v_2 > \nu v_3 \Leftrightarrow a_1 v_1 + a_2 v_2 > v_3$. But since (v_1, v_2, v_3) is also a direction of Q we obtain $a_1 v_1 + a_2 v_2 \leq v_3$ which contradicts the previous result. Thus there are no unbounded non-trivial facets. □

Example 11. Consider $Y = \{(y_1, y_2, s) : 21 y_1 + 76 y_2 \leq 1154 + s, y_1, y_2, s \geq 0, y_1, y_2 \text{ integer}\}$. Setting $s = 0$ we obtain $Y_{\leq} = \{(y_1, y_2) : 21 y_1 + 76 y_2 \leq 1154, y_1, y_2 \geq 0 \text{ and integer}\}$. Using Algorithm HW (see Table 1) we can obtain

$$Q_{\leq} = \{(y_1, y_2) : y_1 + 3y_2 \leq 54, 2y_1 + 7y_2 \leq 109, 5y_1 + 18y_2 \leq 274,$$
$$3y_1 + 11y_2 \leq 166, y_1 + 4y_2 \leq 60, y_1 \geq 0, y_2 \geq 0\}.$$

Based on the description of $Q_{\leq}$ we obtain the following set of facet-defining inequalities for Q:

$$y_1 + 3y_2 \leq 54 + s, 2y_1 + 7y_2 \leq 109 + s, 5y_1 + 18y_2 \leq 274 + s,$$
$$3y_1 + 11y_2 \leq 166 + \frac{s}{2}, y_1 + 4y_2 \leq 60 + \frac{s}{7}.$$

For instance, exchanging the coefficients of y_1 and y_2 (see the second table in Table 1) we obtain the inequality $11y_1 + 3y_2 \leq 166$ containing $(y_1^j, y_2^j) = (14, 4)$, $(y_1^{j+1}, y_2^{j+1}) = (8, 26)$. Thus $(c^k, d^k) = (3, 11)$,

Table 1
Coefficients obtained using the polynomial version of Algorithm HW

a	b	γ	c	d	$R_{\leq}$	e	f	$R_{\geq}$	a	b	γ	c	d	$R_{\leq}$	e	f	$R_{\geq}$
54	0	20	3	1	13	4	1	8	15	0	14	0	1	21	1	1	55
51	1	7	3	1	13	4	1	8	15	0	14	0	1	21	1	3	13
51	1	7	7	2	5	4	1	8	15	0	14	1	4	8	1	3	13
44	3	2	7	2	5	4	1	8	14	4	6	1	4	8	1	3	13
44	3	2	7	2	5	11	3	3	14	4	6	1	4	8	2	7	5
44	3	2	18	5	2	11	3	3	14	4	6	3	11	3	2	7	5
26	8	0	18	5	2	11	3	3	8	26	0	3	11	3	2	7	5
26	8	0	18	5	2	29	8	1	8	26	0	3	11	3	5	18	2
26	8	0	76	21	0	29	8	1	8	26	0	8	29	1	5	18	2
									8	26	0	8	29	1	13	47	1
									8	26	0	21	76	0	13	47	1

In the left table considering $a_1 = 21, a_2 = 76$ and in the right table exchanging a_1 with a_2 (corresponding to exchanging y_1 with y_2).

$(c^{k-1}, d^{k-1}) = (1, 4)$ and $(e^{\ell}, f^{\ell}) = (2, 7)$. As $b^j < f^{\ell}$, then consider $(14, 4) + (-1, 4) = (13, 8) \in Y_{\geq}$. So $\eta^j = -\gamma(y_1^j, y_2^j) - a_1 c^{k-1} + a_2 d^{k-1} = -6 - 76 \times 1 + 21 \times 4 = 2$. Hence $\beta = 1/2$. Exchanging again the coefficients of y_1 and y_2 we obtain the inequality $3y_1 + 11y_2 \leq 166 + \frac{s}{2}$.

Now, setting $s = 21y_1 + 76y_2 - 1154$, from $s \geq 0$ we have $Y_{\geq} = \{(y_1, y_2) : 21y_1 + 76y_2 \geq 1154,\ y_1, y_2 \geq 0 \text{ and integer}\}$. Again, computing the coefficients,

$$Q_{\geq} = \{(y_1, y_2) : 8y_1 + 29y_2 \geq 440, 5y_1 + 18y_2 \geq 274, 2y_1 + 7y_2 \geq 107, y_1 + y_2 \geq 16, y_1 \geq 0, y_2 \geq 0\}.$$

Based on the description of $Q_{\geq}$ we derive the following set of inequalities:

$$5y_1 + 18y_2 \leq 274 + s, 6y_1 + 22y_2 \leq 332 + s, 9y_1 + 34y_2 \leq 512 + s, 7y_1 + 62y_2 \leq 930 + s.$$

These two sets of inequalities with the trivial facet-defining inequalities $y_1 \geq 0, y_2 \geq 0, s \geq 0, 21y_1 + 76y_2 \leq 1154 + s$ suffice to describe Q.

3. Other 2-integer continuous sets

In this section we briefly discuss other 2-integer continuous sets that may have practical importance. Since the polyhedral description for those cases is similar to the description of Q we focus on the differences between Q and the other polyhedra.

First, consider $X = \{(y_1, y_2, s) \in \mathbb{N}_0^2 \times \mathbb{R}_+ : a_1 y_1 - a_2 y_2 \leq D + s\}$, where $a_1, a_2, D > 0$. Let $P = \text{conv}(X)$. The major differences between P and Q are the corresponding characteristic cones and the fact that P may have one non-trivial unbounded facet. Next we consider these two issues.

Lemma 12. *Let $C(P) = \{(v_1, v_2, v_3) : a_1 v_1 - a_2 v_2 \leq v_3, v_1, v_2, v_3 \geq 0\}$ be the characteristic cone of P. Then $C(P) = \{r : r = \sum_{i=1}^{4} \mu_i r^i, \mu_i \geq 0, i = 1, 2, 3, 4\}$ where $r^1 = (0, 0, 1), r^2 = (0, 1, 0), r^3 = (1, 0, a_1), r^4 = (a_2, a_1, 0)$.*

The bounded facets of P can be obtained in a similar fashion to the facets of Q, by considering the restriction of X to $s = 0$ and the restriction of X to $s = a_1 y_1 - a_2 y_2 - D$. However P may have one more unbounded facet.

Proposition 13. *The unbounded facets of P are defined by $y_1 \geq 0$, $y_2 \geq 0$, $s \geq 0$, $a_1 y_1 - a_2 y_2 \leq D + s$ and*

$$a_1 y_1 - a_2 y_2 \leq D - m + s \frac{g}{g - m} \tag{6}$$

where $g = g.c.d.\{a_1, a_2\}$ and $m = D - g\lfloor \frac{D}{g} \rfloor$.

Notice that if $m = 0$, i.e. g divides D (the equation $a_1 y_1 - a_2 y_2 = D$, has integer solutions), the last two inequalities become the same inequality. Thus we will assume $m \neq 0$.

Proof. First we show that (6) defines an unbounded facet. To prove validity we show that (6) can be seen as a basic MIR inequality. Considering $a = g$, $y = (a_1/g)y_1 - (a_2/g)y_2$ and $w = s$, then from (1) we obtain

$$(a_1/g)y_1 - (a_2/g)y_2 \leq \lfloor D/g \rfloor + \frac{s}{g\lceil D/g \rceil - D} \Leftrightarrow a_1 y_1 - a_2 y_2 \leq D - m + s\frac{g}{g-m}.$$

To prove that (6) defines a facet consider the following three affinely independent points: $(a, b, 0)$, $(a, b, 0) + (a_2, a_1, 0)$, $(a', b', g - m)$ where (a, b) is a solution to $a_1 y_1 - a_2 y_2 = D - m$ with $y_1, y_2 \in \mathbb{Z}_+$ and (a', b') is a solution to $a_1 y_1 - a_2 y_2 = D - m + g$ with $y_1, y_2 \in \mathbb{Z}_+$.

Next we show that if an inequality defines an unbounded non-trivial facet then it is defined up to the multiple of a constant by (6). Let

$$\alpha_1 y_1 - \alpha_2 y_2 \leq \alpha + \beta s \tag{7}$$

define such facet of P denoted by $\mathcal{F}$. As $(0, 0, 0) \in P$ we have $\alpha \geq 0$. If $\alpha_1 < 0$ then $\mathcal{F} \subseteq \{(y_1, y_2, s) : y_1 = 0\}$. So, $\alpha_1 \geq 0$. The existence of the directions $(a_2, a_1, 0)$ and $(1, 0, a_1)$ implies $\alpha_1 a_2 \leq \alpha_2 a_1$ and $\alpha_1 \leq a_1\beta$, respectively. Let (v_1, v_2, v_3) be a direction of $\mathcal{F}$. From $s \geq 0$ it follows that $v_3 \geq 0$. Suppose $v_3 > 0$. There exists one $(y_1', y_2', s') \in \mathcal{F}$ with $a_1 y_1' - a_2 y_2' < D + s'$. Thus, there exists a $\lambda > 0$ such that $(y_1^*, y_2^*, s^*) = (y_1', y_2', s') + \lambda(v_1, v_2, v_3)$ and it satisfies $a_1 y_1^* - a_2 y_2^* < D + s^*$ and $s^* > 0$. As $\beta > 0$ (the case $\beta = 0$ implies $\alpha_1 = 0$ and hence $\mathcal{F} \subseteq \{(y_1, y_2, s) : y_2 = 0\}$) then the point $(y_1^*, y_2^*, s^+) \in P$ with $s^+ = \max\{0, a_1 y_1^* - a_2 y_2^* - D\}$ violates (7). Thus $v_3 = 0$ and so $\alpha_1 v_1 = \alpha_2 v_2$. Using this equation and the inequality $\alpha_1 a_2 \leq \alpha_2 a_1$ we conclude that $a_2 v_2 \leq a_1 v_1$. To prove the equality $a_2 v_2 = a_1 v_1$ we suppose $a_2 v_2 > a_1 v_1$. Let $(y_1^0, y_2^0, s^0) \in \mathcal{F}$ with $a_1 y_1^0 - a_2 y_2^0 < D + s^0$. Setting $\lambda = \max\{0, (a_1 - (D + s^0 - a_1 y_1^0 - a_2 y_2^0))/(-a_1 v_1 + a_2 v_2)\}$, the point $(y_1^0 + \lambda v_1 + 1, y_2^0 + \lambda v_2, s^0) \in P$ violates (7). Since $a_2 v_2 = a_1 v_1$ and $\alpha_1 v_1 = \alpha_2 v_2$ the inequality (7) can be written as $a_1 y_1 - a_2 y_2 \leq \alpha' + \beta' s$ where $\alpha' = \frac{a_1}{\alpha_1}\alpha$ and $\beta' = \frac{a_1}{\alpha_1}\beta$. As $\beta' > 0$, every point lying in $\mathcal{F} \cap X$ either satisfies $s = 0$ or $s = a_1 y_1 - a_2 y_2 - D$ (otherwise we could decrease the value of s). Let us consider a point (y_1', y_2', s') satisfying $s' = 0$. Therefore $a_1 y_1' - a_2 y_2' = \alpha'$. Since (7) is a valid inequality we have $\alpha' = \max\{a_1 y_1 - a_2 y_2 : a_1 y_1 - a_2 y_2 \leq D, y_1, y_2 \in \mathbb{N}_0\} = D - m$. Therefore $\beta' > 1$ (otherwise every point $(y_1^*, y_2^*, s^*) \in X$ satisfying $a_1 y_1^* - a_2 y_2^* = D + s^*$ would violate (7)). Now consider $(y_1', y_2', s') \in \mathcal{F}$ satisfying $s' = a_1 y_1' - a_2 y_2' - D$. Thus $a_1 y_1' - a_2 y_2' = D - m + \beta'(a_1 y_1' - a_2 y_2' - D)$. So $\beta' = 1 + \frac{m}{a_1 y_1' - a_2 y_2' - D}$. Since (7) is a valid inequality we have $(y_1', y_2') = \arg\min\{a_1 y_1 - a_2 y_2 : a_1 y_1 - a_2 y_2 \geq D, y_1, y_2 \in \mathbb{N}_0\}$. Therefore $a_1 y_1' - a_2 y_2' - D = g - m \Rightarrow \beta' = \frac{g}{g-m}$. □

Now, we consider all mixed integer sets of the form:

$$X = \{(y_1, y_2, s) \in \mathbb{N}_0^2 \times \mathbb{R}_+ : a_1 y_1 + a_2 y_2 \leq D + \delta s\}$$

where $a_1, a_2, D \in \mathbb{Z}$, $\delta \in \{-1, 1\}$. The description of $\text{conv}(X)$ is trivial for sets X where a_1 and a_2 have the same sign and D has the opposite sign.

If $\delta < 0$ then all non-trivial facets of $\text{conv}(X)$ are obtained from the description of the convex hull of the restriction of X to $s = 0$. For these cases $\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha + \beta s$ defines a non-trivial facet of $\text{conv}(X)$ if and only if $\beta = 0$ and $\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha$ defines a non-trivial facet of $\text{conv}(X_{s=0})$ where $X_{s=0} = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1 y_1 + a_2 y_2 \leq D\}$. Notice that lifting each non-trivial facet-defining inequality for $\text{conv}(X_{s=0})$ we obtain $\beta = 0$ and, conversely, each non-trivial facet-defining inequality for $\text{conv}(X)$ satisfies $\beta = 0$.

It remains to consider four cases with $\delta > 0$. The case $a_1 > 0$, $a_2 > 0$ and $D > 0$ was considered in Section 2. The case $a_1 < 0$, $a_2 < 0$, $D < 0$ is similar to the previous one. The case where $a_1 > 0$ and $a_2 < 0$ was considered above and the case $a_1 < 0$ and $a_2 > 0$ is similar to that one.

Other important 2-integer continuous sets arise when upper and lower bounds on the integer variables are considered. In those cases a polyhedral description can be obtained in a way similar to that given in Section 2. However, for each particular situation it is necessary to adapt Algorithm HW to generate the extreme points of the pure 2-integer polyhedra. To compute the lifting coefficient β of the continuous variable, some minor modifications must be made in order to ensure that the point chosen to compute β (the point (y_1^*, y_2^*) considered in the proof of Proposition 7) satisfies these new constraints.

4. The integer single node flow set with two arcs

In this section we consider the mixed integer set Z defined by:

$$x_1 + x_2 \leq D + s, \tag{8}$$
$$x_1 \leq a_1 y_1, \tag{9}$$
$$x_2 \leq a_2 y_2, \tag{10}$$
$$x_1, x_2, s \geq 0, \tag{11}$$
$$y_1, y_2 \text{ integers.} \tag{12}$$

We assume that a_1, a_2, D are positive integers with $D > \max\{a_1, a_2\}$.

It is important to notice that there are only two integer variables involved in this model and so, for this particular structure, all the information needed to describe $\text{conv}(Z)$ will also be obtained from the 2-integer knapsack sets that result from the elimination of the continuous variables.

Since there are several similarities between the study of $\text{conv}(Z)$ and the study of Q in Section 2 we will omit some details.

First consider the characteristic cone of $\text{conv}(Z)$, denoted by $C(Z)$.

Lemma 14. *$C(Z) = \{(x_1, x_2, y_1, y_2, s) : x_1 + x_2 \leq s,\ x_1 \leq a_1 y_1,\ x_2 \leq a_2 y_2,\ x_1,\ x_2,\ s \geq 0\} = \{r \in \mathbb{R}^5 : r = \sum_{j=1}^{5} \lambda_j r^j, \lambda_j \geq 0, j = 1, \ldots, 5\}$ where $r^1 = (0, 0, 0, 0, 1), r^2 = (0, 0, 1, 0, 0), r^3 = (0, 0, 0, 1, 0), r^4 = (a_1, 0, 1, 0, a_1)$ and $r^5 = (0, a_2, 0, 1, a_2)$.*

All the extreme points of $\text{conv}(Z)$ lie in the intersection of three of the following four hyperplanes defined by $x_1 = a_1 y_1, x_2 = a_2 y_2, x_1 + x_2 = D + s$ and $s = 0$. Thus, every extreme point of $\text{conv}(Z)$ has to satisfy one of the following set of conditions: (i) $x_1 = a_1 y_1, x_2 = a_2 y_2, s = 0$, (ii) $x_1 = D - x_2 + s, x_2 = a_2 y_2, s = 0$, (iii) $x_1 = a_1 y_1, x_2 = D - x_1 + s, s = 0$, (iv) $x_1 = a_1 y_1, x_2 = a_2 y_2, s = x_1 + x_2 - D$.

In case (i) we have $(y_1, y_2) \in Y_{\leq}$. In case (ii), noticing that $0 \leq x_1 \leq a_1 y_1$ implies that $0 \leq D - a_2 y_2 \leq a_1 y_1$, we have $(y_1, y_2) \in Y_1$ where

$$Y_1 = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1 y_1 + a_2 y_2 \geq D, y_2 \leq D/a_2\}.$$

Note that Y_1 differs from $Y_{\geq}$ because it includes the additional constraint $y_2 \leq D/a_2$ that is implied by the non-negativity constraint $x_1 \geq 0$. Similarly, in case (iii) we have $(y_1, y_2) \in Y_2$ where

$$Y_2 = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1 y_1 + a_2 y_2 \geq D, y_1 \leq D/a_1\}.$$

Finally, in case (iv), we have $(y_1, y_2) \in Y_{\geq}$.

Let us define $Y_{1>} = Y_1 \setminus Y_{=}$ and $Y_{2>} = Y_2 \setminus Y_{=}$. First we consider the valid inequalities obtained from the lifting of facet-defining inequalities for $Q_{\leq} = \text{conv}(Y_{\leq})$ (corresponding to case (i)).

Proposition 15. *If*

$$\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha \tag{13}$$

is a valid facet-defining valid for $Q_{\leq}$ then the inequality

$$\beta_1(x_1 - a_1 y_1) + \beta_2(x_2 - a_2 y_2) + \alpha_1 y_1 + \alpha_2 y_2 \leq \alpha + \beta s \tag{14}$$

is a valid facet-defining inequality for $\text{conv}(Z)$, where

$$\beta = \max\left\{\frac{\alpha_1 y_1 + \alpha_2 - \alpha}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_{>}\right\},$$

$$\beta_1 = \max\left\{\frac{\alpha_1 y_1 + \alpha_2 y_2 - \alpha}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_{1>}\right\}$$

$$\text{and } \beta_2 = \max\left\{\frac{\alpha_1 y_1 + \alpha_2 y_2 - \alpha}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_{2>}\right\}.$$

Notice that the value of β can be computed as indicated in Proposition 7 and the computation of β_1 and β_2 is similar with small differences. If $k > 1$ it is necessary to check if the point $(y_1^j, y_2^j) + (e^{\ell(k)}, -f^{\ell(k)})$ satisfies the additional condition, $y_2 \le D/a_2$ or $y_1 \le D/a_1$, according to the coefficient we are computing, β_1 or β_2, respectively. If $k = 1$ with $\alpha_1 = 1, \alpha_2 = \lfloor a_2/a_1 \rfloor$ then $\beta_2 = 1/(a_1\lceil (D - a_2)/a_1\rceil + a_2 - D)$. If $\alpha_1 = \lfloor a_1/a_2 \rfloor, \alpha_2 = 1$ then $\beta_1 = 1/(a_2\lceil (D - a_1)/a_2\rceil + a_1 - D)$. Now we are ready to prove Proposition 15.

Proof. Consider $(x_1, x_2, y_1, y_2, s) \in Z$. If $(y_1, y_2) \in Y_{\le}$, from validity of (13) for $Y_{\le}$ and noticing that $\beta_1(x_1 - a_1y_1) \le 0$, $\beta_2(x_2 - a_2y_2) \le 0$ and $\beta s \ge 0$ we conclude that (x_1, x_2, y_1, y_2, s) satisfies (14).

Now suppose $(y_1, y_2) \in Y_{>}$. We may assume that only one of the three cases may occur: $s > 0$, $x_1 < a_1y_1$ or $x_2 < a_2y_2$. In fact we are checking all the extreme points. Note that this verification is enough since $\beta > 0$ and, as in Claim 1, we have $\alpha_j - \beta_j a_j \le 0$, $j \in \{1, 2\}$ and $\beta a_j \ge \alpha_j$, $j \in \{1, 2\}$ which ensures that moving along any direction of $C(Z)$ from a feasible point we obtain only feasible solutions. Assume $x_1 < a_1y_1$. If $(y_1, y_2) \in Y_{1>}$ then from the definition of β_1 we have

$$\beta_1(a_1y_1 + a_2y_2 - D) \ge \alpha_1y_1 + \alpha_2y_2 - \alpha \Rightarrow \beta_1(D - a_1y_1 - a_2y_2) + \alpha_1y_1 + \alpha_2y_2 \le \alpha.$$

Thus,

$$\beta_1(x_1 - a_1y_1) + \beta_2(x_2 - a_2y_2) + \alpha_1y_1 + \alpha_2y_2 \le \beta_1(D - a_1y_1 - a_2y_2) + \alpha_1y_1 + \alpha_2y_2 \le \alpha \le \alpha + \beta s.$$

The case $(y_1, y_2) \in Y_{>} \setminus Y_{1>}$ cannot occur because it would imply $y_2 > D/a_2$ and, as we are assuming $x_2 = a_2y_2,\ s = 0$ we would have $x_1 + x_2 > D + s$. The cases $x_2 < a_2y_2$ and $s > 0$ are similar. Since (13) defines a facet of $Q_{\le}$ and β, β_1, β_2 take the smallest values in order for (14) to be valid for $\text{conv}(Z)$ then (14) must define a facet of $\text{conv}(Z)$. □

Example 16. Consider the set, $Z = \{(x_1, x_2, y_1, y_2, s) : x_1 + x_2 \le 1154 + s, 0 \le x_1 \le 21y_1, 0 \le x_2 \le 76y_2, s \ge 0, y_1, y_2 \text{ integer}\}$. Based on the description of $Q_{\le}$ we obtain the following set of facet-defining inequalities for $\text{conv}(Z)$:

$$y_1 + 3y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{14}(x_2 - 76y_2) \le 54 + \frac{1}{1}s,$$
$$2y_1 + 7y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{6}(x_2 - 76y_2) \le 109 + \frac{1}{1}s,$$
$$5y_1 + 18y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{3}(x_2 - 76y_2) \le 274 + \frac{1}{1}s,$$
$$3y_1 + 11y_2 + \frac{1}{2}(x_1 - 21y_1) + \frac{1}{2}(x_2 - 76y_2) \le 166 + \frac{1}{2}s,$$
$$y_1 + 4y_2 + \frac{1}{7}(x_1 - 21y_1) + \frac{1}{7}(x_2 - 76y_2) \le 60 + \frac{1}{7}s.$$

For instance, considering the facet-defining inequality $y_1 + 3y_2 \le 54$ for $Q_{\le}$ we have $(c^1, d^1) = (1, 3)$, thus $k = 1$. So, $\beta = \beta_1 = 1/(21 \times \lceil 1154/21\rceil - 1154) = 1/1$. That is, β and β_1 are obtained for $(y_1, y_2) = (55, 0)$. However, $\beta_2 = 1/(21 \times \lceil (1154 - 76)/21\rceil + 76 - 1154) = 1/14$. We obtain the inequality $y_1 + 3y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{14}(x_2 - 76y_2) \le 54 + \frac{1}{1}s$.

Next we state similar results for the remaining cases without proof.

Proposition 17. *If*

$$\alpha_1y_1 + \alpha_2y_2 \ge \alpha \tag{15}$$

is a valid facet-defining inequality for $\text{conv}(Y_1)$ *containing only points in* $Y_{1>}$ *then the inequality*

$$\beta s + \alpha_1y_1 + \alpha_2y_2 \ge \alpha + \beta_1(x_1 + x_2 - D - s) + \beta_2(x_2 - a_2y_2) \tag{16}$$

is valid for Z *and defines a facet of* $\text{conv}(Z)$*, where*

$$\beta = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_>\right\},$$
$$\beta_1 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in Y_<\right\}$$
$$\textit{and } \beta_2 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_{2>}\right\}.$$

Note that β and β_2 are zero if (15) is valid for $Y_>$ and $Y_{2>}$, respectively, and they are strictly positive otherwise. In this last case it can be easily checked that β and β_2 are obtained for $(y_1, y_2) = (\lceil D/a_1\rceil, 0)$. So $\beta = \beta_2$. The restriction in Proposition 17 that the valid inequality (15) for Y_1 must contain only points in $Y_{1>}$ ensures that β and β_2 are non-negative. The facet-defining inequalities excluded by this restriction belong also to the family of valid inequalities introduced in Proposition 15.

In Example 16, $\text{conv}(Y_1) = \{(y_1, y_2) : 8y_1 + 29y_2 \geq 440,\ 5y_1 + 18y_2 \geq 274,\ 2y_1 + 7y_2 \geq 107,\ y_1 \geq 0,\ y_2 \geq 0\}$. Only $2y_1 + 7y_2 \geq 107$ defines a non-trivial facet that includes only points in $Y_{1>}$. Based on that inequality we obtain: $0s + 2y_1 + 7y_2 \geq 107 + \frac{1}{6}(x_1 + x_2 - D - s) + 0(x_2 - 76y_2)$.

Now we have a similar result considering Y_2.

Proposition 18. *If*

$$\alpha_1 y_1 + \alpha_2 y_2 \geq \alpha \tag{17}$$

is a valid facet-defining inequality for $\text{conv}(Y_2)$ *containing only points in* $Y_{2>}$ *the inequality*

$$\beta s + \alpha_1 y_1 + \alpha_2 y_2 \geq \alpha + \beta_1(x_1 - a_1 y_1) + \beta_2(x_2 + x_1 - D - s) \tag{18}$$

is valid for Z *and defines a facet of* $\text{conv}(Z)$*, where*

$$\beta = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_>\right\},$$
$$\beta_1 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_{1>}\right\}$$
$$\textit{and } \beta_2 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in Y_<\right\}.$$

Continuing Example 16, $\text{conv}(Y_2) = \{(y_1, y_2) : y_2 \geq 1,\ y_1 + 4y_2 \geq 56,\ 3y_1 + 11y_2 \geq 166,\ 5y_1 + 18y_2 \geq 274,\ 2y_1 + 7y_2 \geq 107,\ y_1 + y_2 \geq 16, y_1 \geq 0,\ y_2 \geq 0\}$. Based on these inequalities we derive the following set of facet-defining inequalities for $\text{conv}(Z)$:

$$\frac{1}{1}s + y_2 \geq 1 + \frac{1}{20}(x_1 + x_2 - 1154 - s) + \frac{1}{1}(x_1 - 21y_1),$$
$$\frac{1}{1}s + y_1 + 4y_2 \geq 56 + \frac{1}{7}(x_1 + x_2 - 1154 - s) + \frac{1}{1}(x_1 - 21y_1),$$
$$0s + 2y_1 + 7y_2 \geq 107 + \frac{1}{6}(x_1 + x_2 - 1154 - s) + 0(x_1 - 21y_1),$$
$$0s + y_1 + y_2 \geq 16 + \frac{1}{14}(x_1 + x_2 - 1154 - s) + 0(x_1 - 21y_1).$$

Finally, we consider the last set of inequalities.

Proposition 19. *If*

$$\alpha_1 y_1 + \alpha_2 y_2 \geq \alpha \tag{19}$$

is a valid facet-defining inequality for $\text{conv}(Y_\geq)$ *containing only points in* $Y_>$ *then the inequality*

$$\beta(s + D - x_1 - x_2) + \alpha_1 y_1 + \alpha_2 y_2 \geq \alpha + \beta_1(x_1 - a_1 y_1) + \beta_2(x_2 - a_2 y_2) \tag{20}$$

is valid for Z *and defines a facet of* $\text{conv}(Z)$, *where*

$$\beta = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in Y_<\right\},$$

$$\beta_1 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_{1>}\right\}$$

$$\text{and } \beta_2 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in Y_{2>}\right\}.$$

Continuing Example 16 it can be checked that every facet of the form (20) can also be obtained either from (16) or from (18). Note that the convex hulls of Y_1, Y_2 and $Y_\geq$ are very similar and, in general, they share a set of facets. Thus, several facet-defining inequalities of $\text{conv}(Z)$ may belong to more than one of the families (16), (18), (20). In fact, all but the facet in the border of $Q_\geq$ adjacent to the facet defined by $x_1 = 0$ (resp. $x_2 = 0$) are also facets of $\text{conv}(Y_1)$ (resp. $\text{conv}(Y_2)$). Hence, $\text{conv}(Y_\geq)$ can only contribute a new facet if it has only one non-trivial facet.

Now we consider two unbounded facet-defining inequalities that can be obtained by the MIR procedure.

Proposition 20. *The inequality*

$$x_i - \gamma_i y_i \leq (a_i - \gamma_i)\lfloor D/a_i \rfloor + s \tag{21}$$

where $\gamma_i = D - a_i \lfloor D/a_i \rfloor$, *and* $i \in \{1, 2\}$, *is valid for* Z.

Proof. Consider the basic MIR inequality (1) with $y = y_i$, $a = a_i$, $w = s + a_i y_i - x_i$. □

In Example 16 we have two unbounded non-trivial facet-defining inequalities: $x_1 - 20y_1 \leq 53 + s$, $x_2 - 69y_1 \leq 98 + s$. In this example the inequalities presented so far with the trivial facet-defining inequalities suffice to describe $\text{conv}(Z)$.

Proposition 21. *Inequalities (21) are the unique inequalities defining unbounded facets of* $\text{conv}(Z)$.

Proof. Suppose $\mathcal{F}$ is an unbounded facet of $\text{conv}(Z)$ defined by

$$\zeta_1 x_1 + \zeta_2 x_2 - \mu_1 y_1 - \mu_2 y_2 \leq \mu + \nu s. \tag{22}$$

First notice that $\zeta_1, \zeta_2, \mu_1, \mu_2, \mu, \nu \geq 0$ and $\nu \geq \max\{\zeta_1, \zeta_2\}$. The direction r^1 cannot belong to the characteristic cone of $\mathcal{F}$ (denoted by $C(\mathcal{F})$) because, otherwise, we would have $\nu = 0$ which implies $\zeta_1 = \zeta_2 = 0$ and, therefore, $\mu_1 = \mu_2 = 0$. Now we consider r^4. If $r^4 \in C(\mathcal{F})$ then $a_1\nu = a_1\zeta_1 - \mu_1$. Consider a point $(x_1, x_2, y_1, y_2, s) \in \mathcal{F} \cap Z$ satisfying $x_1 + x_2 < D + s$. As $\nu > 0$ and $\mu_1 \geq 0$ then $\zeta_1 > 0 \Rightarrow x_1 = a_1 y_1$. Thus $a_1 y_1 + x_2 < D + s$. On the other hand as $(x_1, x_2, y_1, y_2, s) \in \mathcal{F}$ we have $(\zeta_1 a_1 - \mu_1)y_1 + \zeta_2 x_2 - \mu_2 y_2 = \mu + \nu s$. Using $a_1\nu = a_1\zeta_1 - \mu_1$ we obtain $a_1 y_1 + (\zeta_2/\nu)x_2 - (\mu_2/\nu)y_2 = \mu/\nu + s$. From this equation and from the inequality $a_1 y_1 + x_2 < D + s$ we have $-(\zeta_2/\nu)x_2 + (\mu_2/\nu)y_2 + x_2 < D - \mu/\nu$. Now, considering the point $(x_1', x_2, y_1', y_2, s') \in Z$ with $y_1' = \lceil (D - x_2 - s)/a_1 \rceil$, $x_1' = a_1 y_1'$ and $s' = x_1' + x_2 - D$ we have $a_1 y_1' + x_2 = D + s'$. But, as (22) is valid for Z we also have $(\zeta_1 a_1 - \mu_1)y_1' + \zeta_2 x_2 - \mu_2 y_2 \leq \mu + \nu s'$. Again, since $a_1\nu = a_1\zeta_1 - \mu_1$ it follows that $a_1 y_1' + (\zeta_2/\nu)x_2 - (\mu_2/\nu)y_2 \leq \mu/\nu + s'$. Using the equation $a_1 y_1' + x_2 = D + s'$ we obtain $-(\zeta_2/\nu)x_2 + (\mu_2/\nu)y_2 + x_2 \geq D - \mu/\nu$, contradicting the previous strict inequality. Hence $r^4 \notin C(\mathcal{F})$. Similarly $r^5 \notin C(\mathcal{F})$.

Hence, only r^2 and r^3 may belong to $C(\mathcal{F})$. Suppose $r^2 \in C(\mathcal{F})$. Thus $\mu_1 = 0$ which implies $\zeta_1 = 0$ because, otherwise, we would have $x_1 + x_2 = D + s$ for every point in $\mathcal{F}$. Thus r^3 cannot belong to $C(\mathcal{F})$ because we would obtain a valid inequality dominated by $s \geq 0$. Since $\zeta_2 > 0$ we can write (22) as $x_2 - \mu_2' y_2 \leq \mu' + \nu' s$. We know that $\nu' = \nu/\zeta_2 \geq 1$. Suppose $\nu' > 1$ then $s = 0$ for every point in the facet because if $s > 0$ we could decrease x_2 and s in the same amount obtaining a point violating (22). Thus $\nu' = 1$. Now we prove that $\mu' = (a_2 - \mu_2')\lfloor D/a_2 \rfloor$. Considering the point $(x_1, x_2, y_1, y_2, s) = (0, a_2\lfloor D/a_2 \rfloor, 0, \lfloor D/a_2 \rfloor, 0) \in Z$ we have $\mu' \geq (a_2 - \mu_2')\lfloor D/a_2 \rfloor$. Suppose $\mu' > (a_2 - \mu_2')\lfloor D/a_2 \rfloor$. As $x_2 \leq a_2 y_2 \Rightarrow x_2 - \mu_2' y_2 \leq (a_2 - \mu_2')y_2$ we conclude that if $(x_1, x_2, y_1, y_2, s) \in \mathcal{F}$ then, as $\mu_2' < a_2$, we have $y_2 > \lfloor D/a_2 \rfloor$ and, therefore, $x_1 + x_2 = D + s$. Thus, $\mu' = (a_2 - \mu_2')\lfloor D/a_2 \rfloor$. Finally we show that $\mu_2' = (D - a_2\lfloor D/a_2 \rfloor)$. Considering the point $(0, D, 0, \lfloor D/a_2 \rfloor + 1, 0)$ it follows that $D - \mu_2'(\lfloor D/a_2 \rfloor + 1) \leq (a_2 - \mu_2')\lfloor D/a_2 \rfloor$. Thus $\mu_2' \geq D - a_2\lfloor D/a_2 \rfloor$. Suppose $\mu_2' > D - a_2\lfloor D/a_2 \rfloor$. From $x_2 \leq D + s$ we have $x_2 - \mu_2' y_2 \leq (D - a_2\lfloor D/a_2 \rfloor) + a_2\lfloor D/a_2 \rfloor - \mu_2' y_2 + s$. If $(x_1, x_2, y_1, y_2, s) \in \mathcal{F}$ then

$\mu'+s \le (D-a_2\lfloor D/a_2\rfloor)+a_2\lfloor D/a_2\rfloor-\mu_2'y_2+s \Rightarrow -\mu_2'\lfloor D/a_2\rfloor \le (D-a_2\lfloor D/a_2\rfloor)-\mu_2'y_2 \Rightarrow \mu_2'(y_2-\lfloor D/a_2\rfloor) \le (D-a_2\lfloor D/a_2\rfloor)$. As we are assuming $\mu_2' > D-a_2\lfloor D/a_2\rfloor$ we must have $y_2 \le \lfloor D/a_2\rfloor$ which implies $x_2 = a_2y_2$. Hence $\mu_2' = D-a_2\lfloor D/a_2\rfloor$. □

Theorem 22. conv(Z) *is completely described by the trivial facet-defining inequalities and the families* (14), (16), (18), (20) *and* (21).

Proof. Since the proof is similar to the proof of Theorem 10 we omit some technical details.

Consider a facet $\mathcal{F}$ defined by a non-trivial inequality

$$\zeta_1x_1+\zeta_2x_2-\mu_1y_1-\mu_2y_2 \le \mu+\nu s \qquad (f).$$

In the case where (f) defines an unbounded facet it was proven in Proposition 21 that (f) is of type (21). Thus, suppose (f) defines a bounded facet and therefore it must include five affinely independent extreme points. Notice that every extreme point satisfies one of the four types of conditions (i)–(iv). There must exist one extreme point satisfying each one of those conditions otherwise (f) would not define a facet. For instance, if there were no points satisfying conditions (i) then we would have $x_1+x_2 = D+s$ for every point in $\mathcal{F}$. At least one set of conditions must be satisfied by two extreme points. If that set is (i), (ii), (iii) or (iv) then, as in the proof of Theorem 10, we may conclude that (f) can be written as (14), (16), (18) or (20), respectively. Observe that those cases excluded in the hypothesis of Propositions 17–19 correspond to situations where (i) is satisfied by two extreme points and, therefore, the corresponding facets are also of type (14). □

Now we consider the set $Z_{s=0} = \{(x_1,x_2,y_1,y_2) : x_1+x_2 \le D,\ 0 \le x_1 \le a_1y_1,\ 0 \le x_2 \le a_2y_2,\ y_1, y_2 \text{ integer}\}$. Obviously, restricting conv(Z) to $s=0$ we obtain conv$(Z_{s=0})$. However the relation is stronger. In the description of the bounded facets of conv(Z) only the family of inequalities (20) was obtained from the lifting of 2-integer valid inequalities considering $s \ge 0$ (the sets of conditions (i), (ii), (iii) consider $s=0$). However, as is illustrated in Example 16, the set of facets of conv$(Y_\ge)$ is usually contained in the set of facets of conv(Y_1) and conv(Y_2). In that case the set of inequalities (20) is contained in the sets of inequalities (16) and (18). In fact, it can be checked that the unique exception is the case where conv$(Y_\ge)$ has only one non-trivial facet. Thus if conv$(Y_\ge)$ has more than one non-trivial facet then a valid inequality for Z,

$$\zeta_1x_1+\zeta_2x_2-\mu_1y_1-\mu_2y_2 \le \mu+\nu s$$

defines a non-trivial facet of conv(Z) if and only if $\nu = \max\{\zeta_1,\zeta_2\}$ and

$$\zeta_1x_1+\zeta_2x_2-\mu_1y_1-\mu_2y_2 \le \mu$$

is valid for $Z_{s=0}$ and defines a non-trivial facet of conv$(Z_{s=0})$.

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