Generic Fréchet Differentiability of Convex Functions Dominated by a Lower Semicontinuous Convex Function

Cheng Lixin, Shi Shuzhong, and Wang Bingwu

Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

and

E. S. Lee

Department of Industrial Engineering, Kansas State University, Manhattan, Kansas 66506-5101

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In this paper, an extended real-valued proper lower semicontinuous convex function \( f \) on a Banach space is said to have the Fréchet differentiability property (FDP) if every proper lower semicontinuous convex function \( g \) with \( g \leq f \) is Fréchet differentiable on a dense \( G_{\delta} \) subset of \( \text{int dom } g \), the interior of the effective domain of \( g \). We show that \( f \) has the FDP if and only if the \( \omega^* \)-closed convex hull of the image of the subdifferential map of \( f \) has the Radon–Nikodým property. This is a generalization of the main theorem in a paper by Lixin and Shuzhong (to appear). According to this result, it also gives several new criteria of Asplund spaces.

1. INTRODUCTION

Since E. Asplund’s pioneer work [1] in 1968, the study of Fréchet differentiability property of convex functions on an infinite-dimensional Banach space has continued for nearly 30 years. In 1975, I. Namioka and R. R. Phelps [8] gave a characterization of the spaces on which every
continuous convex function is generically Fréchet differentiable (i.e., Fréchet differentiable on a dense $G_δ$ subset), and called such spaces Asplund spaces. Now we have already known a great number of ideas that are equivalent to the notion of Asplund space, such as "the dual has the Radon–Nikodym property" (Stegall [12]), "the dual is separable on each separable subspace," "every equivalent norm has at least one Fréchet differentiability point" (see, for instance, [9]), and even "every locally Lipschitz function is densely Fréchet differentiable in the interior of its domain" (Preiss [10], [11]).

Asplund space is the only class of spaces on which every continuous convex function is generically Fréchet differentiable. However, in any Banach space, one can always find many nontrivial convex functions that are generically Fréchet differentiable. Recently, Tang Wee-Kee [13] and Giles and Sciffer [6] investigated the generic Fréchet differentiability property of convex functions whose domains are beyond Asplund spaces. They showed that if the image of the subdifferential map of a continuous convex function is separable on each separable subspace, then the function is generically Fréchet differentiable ([6]), and that on a separable Banach space, every continuous convex function dominated by a convex Lipschitz function $f$ is generically Fréchet differentiable if and only if the image of the subdifferential map $\partial f$ of $f$ is separable ([13]).

More recently, motivated by [13], Cheng Lixin and Shi Shuzhong [5] considered the differentiability of convex functions dominated by a continuous convex function and generalized as follows the results that we have just mentioned above.

**Theorem 1.1.** Suppose that $f$ is a proper lower semicontinuous (l.s.c.) convex function on a Banach space $E$ and that its effective domain $\text{dom } f$ is open. Then the following statements are equivalent:

(i) Every proper l.s.c. convex function $g$ on $E$ with $g \leq f$ is generically Fréchet differentiable in $\text{dom } g$, the effective domain of $g$.

(ii) The image of the subdifferential map $\partial f$ of $f$, $\partial f(E) := \{x^* \in E^* : x^* \in \partial f(x), x \in E\}$, is separable on each separable subspace of $E$.

(iii) $\partial f(E)$ has the Radon–Nikodym property (RNP).

(iv) The $\text{w}^*$-closed convex hull of $\partial f(E), \text{w}^*\text{-cl co}[\partial f(E)]$ has the RNP.

We say that a proper l.s.c. convex function $f$ satisfying (i) of Theorem 1.1 has the Fréchet differentiability property (FDP). This paper further shows that Theorem 1.1 is still valid for any proper l.s.c. convex function $f$ and gives more equivalent conditions for a proper l.s.c. convex function having the FDP. According to these results, it also presents some new criteria of Asplund spaces, which tell us that, in a non-Asplund space,
there never exists a proper l.s.c. convex function possessing the FDP with bounded effective domain, even with a bounded level set.

2. PRELIMINARY

We will always denote by $E$ a Banach space with its norm $\| \cdot \|$ and by $E^*$ its dual with the dual norm $\| \cdot \|^*$. For any set $A \subset E$, $\text{cl} \ A$ stands for the closed convex hull of $A$. If $A^* \subset E^*$, then $\text{w}^*\text{-cl} \ A^*$ means the $\text{w}^*$-closed convex hull of $A^*$. An extended real-valued function $f: E \to \mathbb{R} \cup \{ \pm \infty \}$ is said to be proper if it nowhere takes the value $-\infty$ and its effective domain $\text{dom} \ f := \{ x \in E: f(x) < \infty \}$ is nonempty. The epigraph of $f$, denoted by $\text{epi} \ f$, is defined by

$$\text{epi} \ f := \{ (x, r) \in E \times \mathbb{R}: f(x) \leq r, x \in \text{dom} \ f \}.$$ 

If $\text{epi} \ f$ is convex in $E \times \mathbb{R}$, then $f$ is convex on $E$; and if $\text{epi} \ f$ is closed, then $f$ is lower semicontinuous (l.s.c.) on $E$.

The subdifferential of $f$ at $x$, denoted by $\partial f(x)$, is defined by

$$\partial f(x) = \{ x^* \in E^*: \forall y \in E, f(y) - f(x) \geq \langle x^*, y-x \rangle \}.$$

If $f$ is continuous at $x$, then $\partial f(x)$ is nonempty, convex, and $\text{w}^*$-compact.

The conjugate function $f^*: E^* \to \mathbb{R} \cup \{ +\infty \}$ of $f$ is defined by

$$f^*(x^*) = \sup_{x \in E} \{ \langle x^*, x \rangle - f(x) \}. \quad (1)$$

It is easy to see

$$\partial f(x) = \{ x^* \in E^*: f(x) + f^*(x^*) = \langle x^*, x \rangle \}. \quad (2)$$

If $f$ is a proper l.s.c. convex function on $E$, then we have also that

$$f(x) = \sup_{x^* \in E^*} \{ \langle x^*, x \rangle - f^*(x^*) \}, \quad (3)$$

and in this case,

$$\forall x \in E, \quad f^{**}(x) = (f^*)^*(x) = f(x).$$

If $f_1$ and $f_2$ are two proper l.s.c. convex functions on $E$, then the inf-convolution of $f_1$ and $f_2$, denoted by $f_1 \Box f_2$, is defined by

$$\forall x \in E, \quad (f_1 \Box f_2)(x) := \inf_{y \in E} \{ f_1(y) + f_2(x-y) \}, \quad (4)$$

and we have that

$$\forall x^* \in E^*, \quad (f_1 \Box f_2)^*(x^*) = f_1^*(x^*) + f_2^*(x^*). \quad (5)$$
For a proper l.s.c. convex function $f$, if we define the $E$-subdifferential of $f^*$ at $x^* \in E^*$ by

$$
\partial_E f^*(x^*) = \partial f^*(x^*) \cap E
= \{ x \in E : \forall y^* \in E^*, f^*(y^*) - f^*(x^*) \geq \langle y^* - x^*, x \rangle \},
$$

then by (2) and (3), we have that

$$
x^* \in \partial f(x) \iff x \in \partial_E f^*(x^*);
$$
in particular,

$$
\partial f(E) := \{ x^* \in E^* : \exists y \in E, x^* \in \partial f(y) \}.
= \text{dom}(\partial_E f^*) := \{ x^* \in E^* : \partial_E f^*(x^*) \neq \emptyset \}.
$$

The following theorem is the Brøndsted–Rockafellar theorem ([4]; see also [9, p. 51]).

**Theorem 2.1.** Suppose that $f$ is a proper l.s.c. convex function on $E$. Then for any given point $x_0 \in \text{dom } f$, $\varepsilon > 0$, and any

$$
x_0^* \in \partial f(x_0)
:= \{ x^* \in E^* : \forall y \in E, f(y) - f(x_0) + \varepsilon \geq \langle x^*, y - x_0 \rangle \},
$$

there exist $x_{\varepsilon} \in \text{dom } f$ and $x_{\varepsilon}^* \in E^*$ such that

$$
x_{\varepsilon}^* \in \partial f(x_{\varepsilon}), \quad \| x_{\varepsilon} - x_0 \| \leq \sqrt{\varepsilon} \quad \text{and} \quad \| x_{\varepsilon}^* - x_0^* \| \leq \sqrt{\varepsilon}.
$$

It implies the following precise result, which is a key point in this paper.

**Proposition 2.1.** Let $f$ be a proper l.s.c. convex function on $E$. Then,

(i) $\partial f(E) = \text{dom}(\partial_E f^*)$ is dense in $\text{dom } f^* := \{ x^* \in E^* : f^*(x^*) < +\infty \}$.  

(ii) $f(x) = \sup_{x^* \in \partial f(E)} \{ \langle x^*, x \rangle - f^*(x^*) \}.$

**Proof.** If $x^* \in \text{dom } f^*$, then for any $\varepsilon > 0$, there exists $x_0 \in E$ such that

$$
\langle x^*, x_0 \rangle - f(x_0) + \varepsilon \geq f^*(x^*).
$$
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By (1) and (6), this leads to \( x^* \in \partial_E f(x_0) \). Hence, by Theorem 2.1, there exist \( x_\varepsilon \in \text{dom}\ f \) and \( x_\varepsilon ^* \in E^* \) such that \( x_\varepsilon ^* \in \partial f(x_\varepsilon ) \) and \( \| x_\varepsilon ^* - x_0 ^* \| \leq \sqrt{\varepsilon } \). Thus, (i) holds. (ii) is a consequence of (i).

This proposition also implies that for any proper l.s.c. convex function \( f \), \( \partial f(E) \neq \emptyset \) and \( \text{dom}(\partial f) := \{ x \in E : \partial f(x) \neq \emptyset \} \neq \emptyset \).

Every proper l.s.c. sublinear function \( p : E \to \mathbb{R} \cup \{ \infty \} \) is the support function of a \( w^* \)-closed convex subset \( C^* \) of \( E^* \), that is, for each such \( p \), there exists a (unique) \( w^* \)-closed convex set \( C^* \) in \( E^* \) such that
\[
\forall x \in E, \quad p(x) = \sigma_{C^*}(x) := \sup_{x^* \in C^*} \langle x^*, x \rangle.
\]

It is easy to show that
\[
C^* = \{ x^* \in E^* : \forall x \in E, \langle x^*, x \rangle \leq p(x) \}
\]  
(9)

and
\[
x^* \in \partial p(x) \iff x^* \in C^* \quad \text{with} \quad \langle x^*, x \rangle = p(x).
\]  
(10)

A closed convex set \( C \subset E \) with \( 0 \in C \) defines a nonnegative l.s.c.
sublinear function \( p \) by
\[
\forall x \in E, \quad p(x) = \inf\{ \lambda > 0 : \lambda^{-1} x \in C \},
\]  
(11)

and we also say that the function \( p \) is generated by \( C \). It is obvious that \( C = \{ x \in E : p(x) \leq 1 \} \). If, in addition, \( 0 \in \text{int} C \), then \( p \) is continuous on \( E \), and in this case, \( p \) is called the Minkowski functional generated by \( C \).

For such \( p \), \( C^* \) in (9) is the polar set of \( C \), defined by
\[
C^* = C^{\circ} := \{ x^* \in E^* : \forall x \in C, \langle x^*, x \rangle \leq 1 \},
\]
and (10) means that \( x^* \in \partial p(x) \) with \( p(x) = 1 \) is a support functional of \( C \); i.e., \( x^* \) attains its supremum on \( C \).

Let \( C \subset E \). The indicator function of \( C \), denoted by \( \delta_C \), is defined by
\[
\delta_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}
\]

If \( C \) is nonempty, closed, and convex, then \( \delta_C \) is proper, l.s.c., and convex. By definition,
\[
\forall x \in C, \quad \partial \delta_C(x) = \{ x^* \in E^* : \forall y \in C, \langle x^*, y \rangle \leq \langle x^*, x \rangle \}.
\]

It means that \( R \partial \delta_C(E) = \partial \delta_C(C) \) is just the set of the support functionals of \( C \). Let \( B^*(C) \) be the cone consisting of all functionals in \( E^* \) that are
bounded above on $C$. The following theorem is the Bishop–Phelps theorem ([2]; see also [9]):

**Theorem 2.2.** Let $C \subset E$ be a nonempty closed convex set. Then the set of support functionals of $C$ are dense in the cone consisting of those functionals in $E^*$ that are bounded above on $C$. That is, $\partial \delta_C(E)$ is dense in $B^*(C)$.

Suppose that $A^* \subset E^*$, $x \in E$, with $x \neq 0$ and $\alpha > 0$. We call

$$S(x, A^*, \alpha) = \{x^* \in A^*: \langle x^*, x \rangle > \sigma_A^*(x) - \alpha\}$$

a $w^*$-slice of $A^*$, where $\sigma_A^*$ is the support function of $A^*$. We say that $A^*$ is $w^*$-dentable if it admits $w^*$-slices of arbitrarily small diameter, that is, for any $\varepsilon > 0$, there exist $x \in E$ with $x \neq 0$ and $\alpha > 0$ such that

$$\text{diam } S(x, A^*, \alpha) < \varepsilon.$$

The set $A^*$ is said to have the Radon–Nikodym property (RNP) if every nonempty bounded subset of $A^*$ is $w^*$-dentable. It is well known (see [3] for more results about the RNP) that

**Proposition 2.2.** A $w^*$-closed convex set $C^* \subset E^*$ has the RNP if and only if it is separable on each separable subspace of $E$.

For a (set-valued) map $T: E \to 2^{E^*}$, the effective domain of $T$, denoted by $D(T)$, is defined by $D(T) = \{x \in E: T(x) \neq \emptyset\}$. $T$ is said to be monotone if $\langle x^* - y^*, x - y \rangle \geq 0$ whenever $x^* \in T(x)$, $y^* \in T(y)$, and to be maximal monotone if it is monotone and maximal in the family of the graphs of monotone maps ordered by inclusion. The subdifferential map $\partial f$ of a proper l.s.c. convex function is always maximal monotone (see, for instance, [9]).

The following theorem is a version of Kenderov’s result [7], which is presented in [5].

**Theorem 2.3.** Suppose that the $w^*$-closed convex set $C^* \subset E^*$ has the RNP. Then every maximal monotone map $T: E \to 2^{C^*}$ is single-valued and norm-to-norm upper semicontinuous on a dense $G_δ$ subset of $\text{int } D(T)$.

Recall that $f$ is Fréchet differentiable at $x$ provided that $\partial f(x) = \{x^*\}$ is a singleton and for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$0 \leq f(y) - f(x) - \langle x^*, y - x \rangle \leq \varepsilon \|y - x\|,$$

whenever $\|y - x\| < \delta$.

Since the subdifferential map $\partial f$ of a proper l.s.c. convex function $f$ is maximal monotone with $\text{int } \text{dom}(\partial f) = \text{int } \text{dom } f$, and since $f$ is Fréchet differentiable at $x \in \text{int } \text{dom } f$ if and only if $\partial f$ is single-valued and norm-to-norm upper semicontinuous at $x$ (see, for instance, [9]), we immediately obtain the following.

**Theorem 2.4.** Suppose that $f$ is a proper l.s.c. convex function on $E$ and $C^* \subset E^*$ is $w^*$-closed convex. If the subdifferential map $\partial f$ is from $E$ to $2^{C^*}$
and $C^*$ has the RNP, then $f$ is Fréchet differentiable on a dense $G_δ$ subset of int dom $f$.

We say that a convex function $f$ is generically Fréchet differentiable in an open set $D \subseteq E$ if it is Fréchet differentiable in a dense $G_δ$ subset of $D$.

We describe a proper l.s.c. convex function $f$ on $E$ as having the Frechet differentiability property (FDP) if every proper l.s.c. convex function $g$ dominated by $f$ (i.e., $g \leq f$) is generically Fréchet differentiable in int dom $g$.

3. EQUIVALENT CONDITIONS OF FDP

In this section, we will generalize Theorem 1.1 to any proper l.s.c. convex function. Before doing it, we need the following result, which is a generalization of Proposition 4.1 in [5].

Proposition 3.1. Suppose that $f$ is a proper l.s.c. convex function on a Banach space $E$. Let $C^* = \text{w}^*-\text{cl co} \partial f(E)$ and let $p = \sigma_{C^*}$. Then, for any $x_0 \in \text{dom } f$, 
\[
\forall x \in E, \quad f(x) - f(x_0) \leq p(x - x_0). \tag{12}
\]

Proof. Without loss of generality, we assume that $x_0 = 0$ with $f(0) = 0$. In this case, we have that for any $x^* \in E^*$, $f^*(x^*) \geq \langle x^*, 0 \rangle - f(0) = 0$. Hence, by Proposition 2.1,
\[
f(x) = \sup_{x^* \in \partial f(E)} \{ \langle x^*, x \rangle - f^*(x^*) \} \leq p(x),
\]
which leads to (12).

Proposition 3.2. Suppose that $f$ and $g$ are two proper l.s.c. convex functions on a Banach space $E$ with $g \leq f$ and $C^* = \text{w}^*-\text{cl co} \partial f(E)$. Then the subdifferential map $\partial g$ is from $E$ to $2^{C^*}$.

Proof. Let $p = \sigma_{C^*}$. Without loss of generality, we assume that $0 \in \text{dom } f$ and $f(0) = 0$. By Proposition 3.1, we have that
\[
\forall x \in E, \quad g(x) \leq f(x) \leq p(x). \tag{13}
\]
Suppose, to the contrary, that there exist points $z \in E$ and $z^* \in \partial g(z)$ such that $z^* \not\in C^*$. Applying the Separation Theorem to produce $z_0 \in E$ and $\delta > 0$ such that
\[
\langle z^*, z_0 \rangle > \delta + \sigma_{C^*}(z_0) = \delta + p(z_0),
\]
we have that, for all $t > 0$,
\[
g(tz_0) - g(z) \geq \langle z^*, tz_0 - z \rangle \geq t[\delta + p(z_0)] - \langle z^*, z \rangle.
\]
Take $\alpha = [g(z) - \langle z^*, z \rangle]/\delta$. Then we have
\[
\forall t > \max\{\alpha, 0\}, \quad g(tz_0) > p(tz_0),
\]
which contradicts (13).

Now we are ready to generalize Theorem 1.1 to any proper l.s.c. convex functions.

**Theorem 3.1.** Suppose that $f$ is a proper l.s.c. convex function on $E$. Then the following statements are equivalent:

(i) $f$ has the FDP.

(ii) $w^*-\text{cl co} [\partial f(E)]$ has the RNP.

(iii) For any proper l.s.c. convex function $g$ with $g \leq f$, $g$ has the FDP.

(iv) For any $n = 1, 2, \ldots$,
\[
f_n(x) := \inf_{y \in E} \{ f(y) + n\|x - y\| \}
\]
has the FDP.

(v) $\partial f(E)$ is separable on each separable subspace of $E$.

(vi) $\partial f(E)$ has RNP.

**Proof.** (i) $\Rightarrow$ (ii). The sufficiency part is immediately obtained by Theorem 2.3 and Proposition 3.2. The necessity direction is directly followed from the proof of the necessity of [5, Theorem 4.1] without any change.

(i) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Since $g, f_n \leq f, g$ and $f_n$ also have the FDP.

(iv) $\Rightarrow$ (i). Suppose, to the contrary, that there exists a $g$ with $g \leq f$ such that $g$ is not generically Fréchet differentiable in $\text{int dom } g$. Then there is a nonempty open set $U \subset \text{int dom } g$ such that $g$ is nowhere Fréchet differentiable in $U$.

Let
\[
g_n(x) = \inf_{y \in E} \{ g(y) + n\|y - x\| \}, \quad x \in E.
\]
Then we have that $g_n \leq f_n$, $n = 1, 2, \ldots$; and for each $x \in \text{int dom } g$, there is an open neighborhood $V$ of $x$ and $n \geq 1$ such that $g_n = g$ in $V$ (see [9, pp. 32–33]).

Choose any $x_0 \in U$ with a neighborhood of $x_0$, $V_0 \subset U$ and choose $n$ large enough such that $g_n = g$ in $V_0$. Since $f_n$ has the FDP, $g_n$ is
generically Fréchet differentiable in $E$. It is a contradiction to this that $g = g_n$ in $V_0$ is nowhere Fréchet differentiable.

(iv) $\iff$ (v) $\iff$ (vi). In fact, $f_n$ is the inf-convolution of $f$ and $N_n(\cdot) := n\|\cdot\|$ ([9, p. 33]). Since the conjugate function of $N_n$,$$
abla^*_n(x^*) = \sup_{x \in E} \{ \langle x^*, x \rangle - n\|x\| \} = \delta_{B_n^{*}}(x^*) := \left\{ \begin{array}{ll} 0, & \text{if } \|x^*\| \leq n, \\ +\infty, & \text{if } \|x^*\| > n, \end{array} \right.$$
where $B_n^{*} = \{ x^* \in E^* : \|x^*\| \leq n \}$, by (4) and (5), we have that $$f_n^*(x^*) = f^*(x^*) + \delta_{B_n^{*}}(x^*)$$ and $$f_n(x) = \sup_{\|x^*\| \leq n} \{ \langle x^*, x \rangle - f^*(x^*) \}.$$ Obviously, $$[\partial f(E)] \cap B_n^{*} \subset \partial f_n(E) \subset (\text{dom } f^*) \cap B_n^{*}.$$ By Proposition 2.1, $\partial f(E)$ is dense in $\text{dom } f^*$, so that $[\partial f(E)] \cap B_n^{*}$ is dense in $\partial f_n(E)$, $n = 1, 2, \ldots$. By Theorem 1.1, (iv) is equivalent to saying that for any $n \geq 1$, $\partial f_n(E)$ is separable on each subspace of $E$, which is equivalent to saying that $[\partial f(E)] \cap B_n^{*}$ is separable on each subspace of $E$; and (iv) is equivalent to saying that for any $n \geq 1$, $\partial f_n(E)$ has the RNP, which is equivalent to saying that $[\partial f(E)] \cap B_n^{*}$ has the RNP. Since $\partial f(E) = \bigcup_{n=1}^{\infty} [\partial f(E)] \cap B_n^{*}$, it leads to the conclusion. 

4. CONVEX FUNCTIONS HAVING FDP

Now we discuss properties of the class of convex functions having the FDP.

**Proposition 4.1.** Suppose that $f$ and $g$ are two proper l.s.c. convex functions on $E$. If $f + g$ has the FDP, then both $f$ and $g$ have FDP.

**Proof.** Since for any proper l.s.c. convex function $g$, $\partial g \neq \emptyset$, we can suppose that $x_0 \in \text{dom}(\partial g)$ and $x_0^* \in \partial g(x_0)$. Then $$\forall x \in E, \quad g(x) \geq g(x_0) + \langle x_0^*, x - x_0 \rangle.$$ If $h \leq f$, then, setting $$h_g(x) := h(x) + g(x_0) + \langle x_0^*, x - x_0 \rangle,$$
we have also $h_g \leq f + g$. But $h$ is generically Fréchet differentiable if and
only if $h_g$ is also. Hence, the FDP of $f + g$ implies the FDP of $f$. □

**Corollary 4.1.** Suppose that $f$ and $g$ are two proper l.s.c. convex functions on $E$. Then $f + g$ has the FDP if and only if $w^*\text{-cl}\co[\partial (f + g)(E)]$, $w^*\text{-cl}\co[\partial f(E)]$ and $w^*\text{-cl}\co[\partial g(E)]$ have the RNP.

**Corollary 4.2.** Suppose that $f$ is a proper l.s.c. convex function on $E$, and $C = \text{cl dom } f$. Then $f$ has the FDP if and only if both $w^*\text{-cl}\co[\partial f(E)]$ and $w^*\text{-cl}\co[\partial C(E)]$ have the RNP.

This is because $f = f + \delta_C$. And by Theorem 2.1, we have also

**Corollary 4.3.** Suppose that $f$ is a proper l.s.c. convex function on $E$ and $C = \text{cl dom } f$. Then "$\partial f(E)$ has the RNP" implies "$B^*(C)$ has the RNP," where $B^*(C)$ is the cone consisting of all functionals in $E^*$ that are bounded above on $C$.

**Corollary 4.4.** Suppose that $f$ is a proper l.s.c. convex function on $E$ and $C = \text{cl dom } f$. If $f$ has the FDP, then every proper l.s.c. convex function $h$ bounded above on $C$ is generically Fréchet differentiable.

**Proof.** This is because the FDP of $f$ implies the FDP of $\delta_C$, which leads to the conclusion. □

5. SOME CRITERIA OF ASPLOUND SPACES

**Proposition 5.1.** $E$ is an Asplund space if and only if there exists a proper l.s.c. convex function $f$ with a bounded effective domain such that $f$ has the FDP.

**Proof.** If $E$ is an Asplund space, then the indicator function of any bounded closed convex set is a required convex function. Conversely, if there exists such an $f$, then $C = \text{cl dom } f$ is bounded and $B^*(C) = E^*$ has the RNP, which implies that $E$ is an Asplund space. □

This is an improvement of Corollary 4.2 of [5]. More generally, we have

**Theorem 5.1.** $E$ is an Asplund space if and only if there exist a proper l.s.c. convex function $f$ having the FDP and $\inf \{ f(x) : x \in E \}$, such that the level set of $f$, $V(f, \alpha) := \{ x \in E : f(x) \leq \alpha \}$, is bounded.

**Proof.** We need only show the sufficiency. Without loss of generality, we assume that $\text{dom } f \neq V(f, \alpha)$; otherwise, the theorem is proved. Therefore, we can further assume that $\alpha = 1$, $0 \in V(f, 1)$ with $f(0) = 0$; otherwise, choose any $x_0 \in \text{dom } f$ such that $f(x_0) < \alpha$, let $g(x) = k(f(x + x_0) - f(x_0))$, and substitute $g$ for $f$, where $k = [\alpha - f(x_0)]^{-1}$. Thus, $V(f, 1)$ is closed and convex and contains 0.
We first show that $f$ is bounded below on $V(f, 1)$. Choose any $\bar{x} \in \text{dom} (\partial f)$ and $\bar{x}^* \in \partial f(\bar{x})$. We have
\begin{equation*}
\forall x \in E, \quad f(x) \geq \langle \bar{x}^*, x - \bar{x} \rangle + f(\bar{x}),
\end{equation*}
and
\begin{equation*}
\inf_{x \in V(f, 1)} f(x) \geq \inf_{x \in V(f, 1)} \langle \bar{x}^*, x - \bar{x} \rangle + f(\bar{x}) \\
\geq -\|\bar{x}^*\|\|\|V(f, 1)\| + \|\bar{x}\|] + f(\bar{x}),
\end{equation*}
where $\|V(f, 1)\| = \sup_{x \in V(f, 1)} \|x\| < \infty$. Therefore, $f$ is bounded below on $V(f, 1)$ and on $E$.

Now let $p_V$ be the nonnegative sublinear function, generated by $V(f, 1)$. Then $p_V$ is a proper l.s.c. convex function on $E$. We show that
\begin{equation}
\forall x \in E \setminus V(f, 1), \quad p_V(x) \leq f(x).
\end{equation}
Without loss of generality, we assume that $x \in E \setminus V(f, 1)$ satisfies $1 < f(x) < +\infty$. Since $f$ is l.s.c. and convex, $f$ is continuous on the interval $[0, x]$. Hence there exists $\lambda \in (0, 1)$ such that $f(\lambda x) = 1 = p_V(\lambda x)$, and then
\begin{equation*}
\lambda p_V(x) = p_V(\lambda x) = f(\lambda x) \leq \lambda f(x) + (1 - \lambda)f(0) = \lambda f(x).
\end{equation*}
Thus (14) holds. Joining up with the lower boundedness of $f$, we obtain that
\begin{equation*}
\forall x \in E, \quad p_V(x) \leq f(x) + (1 - \inf f(E));
\end{equation*}
and then $p_V$ also has the FDP.

By Theorem 3.1, $w^*\text{-cl co} [\partial p(E)]$ has the RNP. Since
\begin{equation*}
w^*\text{-cl co} [\partial p(E)] = \{ x^* \in E^* : \forall x \in E, \langle x^*, x \rangle \leq p(x) \}
\end{equation*}
\begin{equation*}
\sup_{x \in V(f, 1)} \langle x^*, x \rangle \leq 1 \} \supset \frac{1}{\|V(f, 1)\|} B^*,
\end{equation*}
where $B^*$ denotes the unit ball of $E^*$, this says that $B^*$ (and further, $E^*$) has the RNP.

Recall that a continuous function $f$ on $E$ is called coercive if $\|x\| \to +\infty$ implies $f(x) \to +\infty$.

**Corollary 5.1.** $E$ is an Asplund space if and only if there exists a coercive continuous convex function $f$ having the FDP.
6. FINAL REMARK

Theorem 3.1 tells us that, in any Banach space, there are many convex functions that are generically Fréchet differentiable, such as proper l.s.c. convex functions dominated by a support function of a w*-compact set in $E^*$, or, by an indicator function of a closed convex set $C \subset E$ whose support functionals are separable on each separable subspace of $E$. On the other hand, if $E$ is not an Asplund space, and if $f$ is a proper l.s.c. convex function and has the FDP on $E$, then dom $f$ must be unbounded, and even $f$ has no bounded level set (Theorem 5.1).

Both this note and [5] focus on the generic Fréchet differentiability of all proper l.s.c. convex functions dominated by a proper l.s.c. convex function. However, there are still continuous convex functions that are generically Fréchet differentiable going beyond our research field. For example, the natural norm on $l^\infty$ is Fréchet differentiable on a dense open set (see [14]), but the image of its subdifferential map does not have the RNP.

REFERENCES