Radiative nonrecoil nuclear finite size corrections of order $\alpha(Z\alpha)^5$ to the hyperfine splitting of S-states in muonic hydrogen

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A B S T R A C T

On the basis of quasipotential method in quantum electrodynamics we calculate nuclear finite size radiative corrections of order $\alpha(Z\alpha)^5$ to the hyperfine structure of S-wave energy levels in muonic hydrogen and muonic deuterium. For the construction of the particle interaction operator we employ the projection operators on the particle bound states with definite spins. The calculation is performed in the infrared safe Fried–Yennie gauge. Modern experimental data on the electromagnetic form factors of the proton and deuteron are used.

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In last years a significant theoretical interest in the investigation of fine and hyperfine energy structure of simple atoms is related with light muonic atoms: muonic hydrogen, muonic deuterium and ions of muonic helium. This is generated by essential progress achieved by experimental collaboration CREMA (Charge Radius Experiment with Muonic Atoms) in studies of such simple atoms [1]. The measurement of the transition frequency $2S_{1/2}^{f=1} - 2P_{3/2}^{f=2}$ in muonic hydrogen leads to a new more precise value of the proton charge radius. For the first time the hyperfine splitting of $2S$ state in muonic hydrogen was measured. Analogous measurements in muonic deuterium are also carried out and planned for the publication. It is important to point out that the CREMA experiments set a task to improve by an order of the magnitude numerical values of charge radii of simplest nuclei (proton, deuteron, helion and α-particle). Successful realization of such program is based on precise theoretical calculations of different corrections to the energy intervals of fine and hyperfine structure of muonic atoms [2–6]. Nuclear structure corrections play in this investigation a special role and, possibly, can solve the proton charge radius puzzle [1]. There exist a number of attempts to reconsider a calculation of nuclear structure corrections in [7] (see also other references in [1,3]) accounting among other things the off-shell effects in two-photon exchange amplitudes. In this work we study the corrections of special kind of order $\alpha(Z\alpha)^5$ related to the finite size of the proton and deuteron in the hyperfine structure of muonic hydrogen. Preliminary estimate of possible value of

such contribution to hyperfine splitting for muonic hydrogen, as an example, gives the numerical value $\alpha^2 E_F(\mu p) \approx 0.011$ meV. This means that present corrections can be important in order to obtain hyperfine splittings of S-states with high accuracy. For precise determination of order $\alpha(Z\alpha)^5$ contribution we should take into account that the distributions of the charge and magnetic moment of nuclei are described by electromagnetic form factors.

Our calculation is performed on the basis of quasipotential method in quantum electrodynamics (QED) as applied to particle bound states, which was used previously for the solution of different problems [8]. In terms of perturbation theory in QED the contribution to the scattering amplitude and quasipotential is determined by the Feynman diagrams presented in Fig. 1. To evaluate corrections of order $\alpha(Z\alpha)^5$ we neglect relative momenta of particles in initial and final states and construct separate hyperfine potentials corresponding to muon self-energy, vertex and spanning photon diagrams.

Basic contribution to hyperfine splitting of S-states in muonic deuterium (below we present general equations for muonic deuterium) is given by the following one-photon potential:

$$\Delta V^{hfs} = \frac{4\pi\alpha}{3m_1 m_p}\mu_d(\mathbf{s}_1 \cdot \mathbf{s}_2)\delta(\mathbf{x}), \tag{1}$$

where $\mathbf{s}_{1,2}$ are the spin operators of the muon and deuteron, $\mu_d = 0.8574382308$ is magnetic moment of the deuteron in nuclear magnetons, m_1 and m_p are the masses of the muon and deuteron correspondingly. Averaging (1) over the bound state wave functions we obtain the leading order contribution to hyperfine
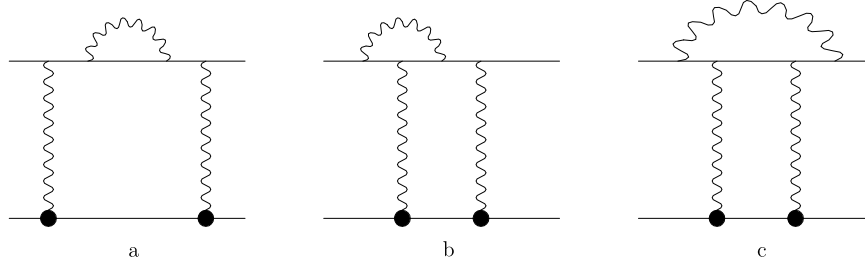
Fig. 1. Direct two-photon exchange amplitudes with radiative corrections to muon line giving contributions of order $\alpha(Z\alpha)^5$ to the hyperfine structure. Wave line on the diagram denotes the photon. Bold point on the diagram denotes the vertex operator of the proton or deuteron.

splitting (the Fermi energy) in muonic deuterium:

$$\Delta E_F(S) = \frac{2\mu^3\alpha^4\mu_d}{m_1 m_p n^3} = \begin{cases} 49.0875 \text{ meV}, & n = 1, \\ 6.1359 \text{ meV}, & n = 2. \end{cases} \tag{2}$$

The contribution of two-photon exchange diagrams to hyperfine structure of order $(Z\alpha)^5$ was investigated earlier by many authors [2]. The lepton line radiative corrections to two-photon exchange amplitudes were studied in detail in [9] in the case of muonium hyperfine splitting. Total integral expression for all radiative corrections in Fig. 1 to hyperfine splitting of order $\alpha(Z\alpha)^5$ including recoil effects was obtained in [10] in the Fried–Yennie gauge for radiative photon [11]. The advantage of the Fried–Yennie gauge in the analysis of corrections in Fig. 1 consists in the fact that it leads to infrared-finite renormalizable integral expressions for muon self-energy operator, vertex function and lepton tensor describing the "jellyfish-type" diagram (with spanning photon) [12]. Using such expressions we can perform analytical calculation of order $\alpha(Z\alpha)^5$ corrections to hyperfine structure in the point-like nucleus approximation. If the approximation of point-like nucleus is inappropriate then these expressions allow to obtain numerical values of diagrams (a, b, c) in Fig. 1 separately. In this work we perform independent construction of all enumerated above muon radiative corrections in the Fried–Yennie gauge and obtain new integral contributions for the muon self-energy, vertex and spanning photon amplitudes separately to hyperfine structure in the case of finite size nucleus. The muon–deuteron scattering amplitude can be presented in the form (direct two-photon exchange diagrams with radiative corrections to the muon line):

$$\mathcal{M} = \frac{-i(Z\alpha)^2}{\pi^2} \int d^4k \left[\bar{u}(q_1) L_{\mu\nu} u(p_1)\right] D_{\mu\omega}(k) D_{\nu\lambda}(k)$$
$$\times \left[\epsilon_\rho^*(q_2) \Gamma_{\omega,\rho\beta}(q_2, p_2+k) \mathcal{D}_{\beta\tau}(p_2+k) \Gamma_{\lambda,\tau\alpha}(p_2+k, p_2) \right.$$
$$\left. \times \epsilon_\alpha(p_2)\right] \tag{3}$$

where $\epsilon_\rho^*(q_2)$ ($\epsilon_\alpha(p_2)$) denote the polarization vector of the final (initial) deuteron, $p_{1,2}$ and $q_{1,2}$ are four-momenta of the muon and deuteron in initial and final states: $p_{1,2} \approx q_{1,2}$. k stands for the four-momentum of the exchange photon. The vertex operator describing the photon–deuteron interaction is determined by three electromagnetic form factors in the form

$$\Gamma_{\omega,\rho\beta}(q_2, p_2+k)$$
$$= \frac{(2p_2+k)_\omega}{2m_2} g_{\rho\beta} \cdot F_1(k) - \frac{(2p_2+k)_\omega}{2m_2} \frac{k_\rho k_\beta}{2m_2^2} \cdot F_2(k)$$
$$- (g_{\rho\gamma} g_{\beta\omega} - g_{\rho\omega} g_{\beta\gamma}) \frac{k_\gamma}{2m_2} \cdot F_3(k). \tag{4}$$

The form factors $F_{1,2,3}(k^2)$ are related to the charge, magnetic and quadrupole deuteron form factors as ($\eta = k^2/4m_2^2$)

$$F_C = F_1 + \frac{2}{3}\eta[f_1 + (1+\eta)F_2 - F_3],$$

$$F_M = F_3, \qquad F_Q = F_1 + (1+\eta)F_2 - F_3. \tag{5}$$

The propagators of the deuteron and photon in the Coulomb gauge are the following ones:

$$\mathcal{D}_{\alpha\beta}(p) = \frac{-g_{\alpha\beta} + \frac{p_\alpha p_\beta}{m_2^2}}{(p^2 - m_2^2 + i0)},$$

$$D_{\lambda\sigma}(k) = \frac{1}{k^2}\left[g_{\lambda\sigma} + \frac{k_\lambda k_\sigma - k_0 k_\lambda g_{\sigma 0} - k_0 k_\sigma g_{\lambda 0}}{\mathbf{k}^2}\right]. \tag{6}$$

The lepton tensor $L_{\mu\nu}$ has a completely definite form for each amplitude in Fig. 1. It is equal to the sum of the self-energy (Σ), vertex (Λ), and spanning photon (Ξ) insertions in the muon line:

$$L_{\mu\nu} = L_{\mu\nu}^\Sigma + 2L_{\mu\nu}^\Lambda + L_{\mu\nu}^\Xi. \tag{7}$$

Using the FeynCalc package [13] we construct the renormalized muon self-energy operator and vertex function as in [10,12] and obtain the following expressions for leptonic tensors corresponding to muon self-energy, vertex contributions and the diagram with spanning photon in the Fried–Yennie gauge:

$$L_{\mu\nu}^\Sigma = -\frac{3\alpha}{4\pi}\gamma_\mu(\hat{p}_1 - \hat{k})\gamma_\nu \int_0^1 \frac{(1-x)dx}{(1-x)m_1^2 + x\mathbf{k}^2}, \tag{8}$$

$$L_{\mu\nu}^\Lambda = \frac{\alpha}{4\pi} \int_0^1 dz \int_0^1 dx\gamma_\mu \frac{\hat{p}_1 - \hat{k} + m_1}{(p_1-k)^2 - m_1^2 + i0}$$
$$\times \left[F_\nu^{(1)} + \frac{F_\nu^{(2)}}{\Delta} + \frac{F_\nu^{(3)}}{\Delta^2}\right], \tag{9}$$

$$F_\nu^{(1)} = -6x\gamma_\nu \ln \frac{m_1^2 x + \mathbf{k}^2 z(1-xz)}{m_1^2 x},$$

$$F_\nu^{(3)} = 2x^3(1-x)\hat{Q}(\hat{p}_1 - \hat{k} + m_1)\gamma_\nu(\hat{p}_1 + m_1)\hat{Q}, \tag{10}$$

$$F_\nu^{(2)} = -x^3(2\gamma_\nu Q^2 - 2\hat{Q}\gamma_\nu\hat{Q}) - x^2[\gamma_\alpha\hat{Q}\gamma_\nu(\hat{p}_1 + m_1)\gamma_\alpha$$
$$+ \gamma_\alpha(\hat{p}_1 - \hat{k} + m_1)\gamma_\nu\hat{Q}\gamma_\alpha + 2\gamma_\nu(\hat{p}_1 + m_1)\hat{Q}$$
$$+ 2\hat{Q}(\hat{p}_1 - \hat{k} + m_1)\gamma_\nu]$$
$$- x(2-x)\gamma_\alpha(\hat{p}_1 - \hat{k} + m_1)\gamma_\nu(\hat{p}_1 + m_1)\gamma_\alpha, \tag{11}$$

$$Q = -p_1 + kz,$$

$$\Delta = x^2 m_1^2 - xz(1-xz)k^2 + 2kp_1 xz(1-x),$$

$$L_{\mu\nu}^\Xi = \frac{\alpha}{4\pi} \int_0^1 dz \int_0^1 dx\left(\frac{F_{\mu\nu}^{(1)}}{\Delta} + \frac{F_{\mu\nu}^{(2)}}{\Delta^2} + \frac{F_{\mu\nu}^{(3)}}{\Delta^3}\right), \tag{12}$$

$$
\begin{aligned}
F^{(1)}_{\mu\nu} =\ & 12\hat{p}_1 g_{\mu\nu} - 15\hat{p}_1\gamma_\mu\gamma_\nu + 9\hat{k}\gamma_\mu\gamma_\nu - 24x\hat{Q}\gamma_\mu\gamma_\nu \\
& + 16x\hat{Q}\,g_{\mu\nu} - 8x\gamma_\mu Q_\nu \\
& + \gamma_\nu(30p_{1,\mu} - 18k_\mu + 40Q_\mu x),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
F^{(2)}_{\mu\nu} =\ & \hat{p}_1\hat{Q}\gamma_\mu\big(-2p_{1,\nu}x - 8Q_\nu x^2\big) + \hat{p}_1\hat{Q}\gamma_\nu\big(2p_{1,\mu}x + 8Q_\mu x^2\big) \\
& + \hat{p}_1\gamma_\mu\gamma_\nu\big(-8m_1^2 x + 5Q^2x^2\big) + \hat{p}_1\gamma_\mu(2Q_\nu m_1 x) \\
& + \hat{p}_1\gamma_\nu(-2Q_\mu m_1 x) + \hat{p}_1\big(12p_{1,\mu}p_{1,\nu} - 12p_{1,\mu}k_\nu \\
& + 20p_{1,\mu}Q_\nu x + 24p_{1,\nu}Q_\mu x - 12k_\mu Q_\nu x - 12k_\nu Q_\mu x \\
& + 32Q_\mu Q_\nu x^2 + 16g_{\mu\nu}m_1^2 x + 12g_{\mu\nu}kQx - 12g_{\mu\nu}Q^2x^2\big) \\
& + 6\hat{k}\hat{p}_1\gamma_\mu p_{1,\nu} - 6\hat{k}\hat{p}_1\gamma_\nu p_{1,\mu} + \hat{k}\hat{Q}\gamma_\mu\big(12p_{1,\nu}x + 8Q_\nu x^2\big) \\
& + \hat{k}\hat{Q}\gamma_\nu\big(-12p_{1,\mu}x - 8Q_\mu x^2\big) - 3\hat{k}\gamma_\mu\gamma_\nu Q^2x^2 \\
& - 6\hat{k}\gamma_\mu Q_\nu m_1 x + \hat{k}\gamma_\nu(6Q_\mu m_1 x) + \hat{k}\big(12p_{1,\mu}Q_\nu x \\
& - 12p_{1,\nu}Q_\mu x + 12g_{\mu\nu}m_1^2 x - 8g_{\mu\nu}Q^2x^2\big) \\
& + \hat{Q}\gamma_\mu\gamma_\nu\big(2m_1^2 x - 24m_1^2 x^2 - 8kQx^2 + 16Q^2x^3\big) \\
& + \hat{Q}\gamma_\mu\big(2p_{1,\nu}m_1 x + 8Q_\nu m_1 x^2\big) \\
& + \hat{Q}\gamma_\nu\big(-2p_{1,\mu}m_1 x - 8Q_\mu m_1 x^2\big) \\
& + \hat{Q}\big(-4p_{1,\mu}p_{1,\nu}x - 12p_{1,\mu}k_\nu x + 12p_{1,\nu}k_\mu x \\
& + 16p_{1,\nu}Q_\mu x^2 - 16k_\nu Q_\mu x^2 + 20Q_\mu Q_\nu x^3 + 16g_{\mu\nu}m_1^2 x^2 \\
& + 16g_{\mu\nu}kQx^2 - 14g_{\mu\nu}Q^2x^3\big) \\
& + g_{\mu\nu}\big(-2m_1^3 x - 6kQm_1 x + 4Q^2m_1 x^2\big) + \gamma_\mu\big(4p_{1,\nu}m_1^2 x \\
& - 12p_{1,\nu}kQx - 8p_{1,\nu}Q^2x^2 - 12k_\nu m_1^2 x + 8k_\nu Q^2x^2 \\
& + 2Q_\nu m_1^2 x - 16Q_\nu m_1^2 x^2 - 16Q_\nu kQx^2 + 2Q_\nu Q^2x^3\big) \\
& + \gamma_\nu\big(24p_{1,\mu}m_1^2 x + 12p_{1,\mu}kQx - 18p_{1,\mu}Q^2x^2 \\
& - 12k_\mu m_1^2 x + 14k_\mu Q^2x^2 - 2Q_\mu m_1^2 x + 48Q_\mu m_1^2 x^2 \\
& + 16Q_\mu kQx^2 - 30Q_\mu Q^2x^3\big) - 24p_{1,\mu}p_{1,\nu}m_1 \\
& + 12p_{1,\mu}k_\nu m_1 - 36p_{1,\mu}Q_\nu m_1 x + 12p_{1,\nu}k_\mu m_1 \\
& - 36p_{1,\nu}Q_\mu m_1 x + 24k_\mu Q_\nu m_1 x + 12k_\nu Q_\mu m_1 x \\
& - 48Q_\mu Q_\nu m_1 x^2 - 16g_{\mu\nu}m_1^3 x - 12g_{\mu\nu}kQm_1 x \\
& + 8g_{\mu\nu}Q^2m_1 x^2,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
F^{(3)}_{\mu\nu} =\ & \hat{p}_1\hat{Q}\gamma_\mu\big(8p_{1,\nu}kQx^2 - 8Q_\nu m_1^2 x^3 + 4Q_\nu Q^2x^4\big) \\
& + \hat{p}_1\hat{Q}\gamma_\nu\big(-8p_{1,\mu}kQx^2 + 8Q_\mu m_1^2 x^3 - 4Q_\mu Q^2x^4\big) \\
& + \hat{p}_1\big(-16p_{1,\mu}p_{1,\nu}Q^2x^2 + 8p_{1,\mu}k_\nu Q^2x^2 \\
& + 16p_{1,\mu}Q_\nu kQx^2 - 16p_{1,\mu}Q_\nu Q^2x^3 + 8p_{1,\nu}k_\mu Q^2x^2 \\
& - 16p_{1,\nu}Q_\mu kQx^2 - 16p_{1,\nu}Q_\mu Q^2x^3 + 8k_\mu Q_\nu Q^2x^3 \\
& + 8k_\nu Q_\mu Q^2x^3 - 16Q_\mu Q_\nu Q^2x^4 - 8g_{\mu\nu}kQQ^2x^3 \\
& - 8g_{\mu\nu}Q^2m_1^2 x^3 + 4g_{\mu\nu}Q^4x^4\big) \\
& + \hat{k}\hat{p}_1\hat{Q}\big(8p_{1,\mu}Q_\nu x^2 - 8p_{1,\nu}Q_\mu x^2\big) \\
& + \hat{k}\hat{Q}\gamma_\mu\big(8p_{1,\nu}m_1^2 x^2 - 8p_{1,\nu}Q^2x^3 - 4Q_\nu Q^2x^4\big) \\
& + \hat{k}\hat{Q}\gamma_\nu\big(-8p_{1,\mu}m_1^2 x^2 + 8p_{1,\mu}Q^2x^3 + 4Q_\mu Q^2x^4\big) \\
& + \hat{k}\hat{Q}\big(8p_{1,\mu}Q_\nu m_1 x^2 - 8p_{1,\nu}Q_\mu m_1 x^2\big) \\
& + \hat{k}\gamma_\mu\big(4p_{1,\nu}Q^2m_1 x^2 + 4Q_\nu Q^2m_1 x^3\big) \\
& + \hat{k}\gamma_\nu\big(-4p_{1,\mu}Q^2m_1 x^2 - 4Q_\mu Q^2m_1 x^3\big)
\end{aligned}
$$

$$
\begin{aligned}
& + \hat{k}\big(16p_{1,\mu}Q_\nu m_1^2 x^2 - 8p_{1,\mu}Q_\nu Q^2x^3 - 16p_{1,\nu}Q_\mu m_1^2 x^2 \\
& + 8p_{1,\nu}Q_\mu Q^2x^3 - 8g_{\mu\nu}Q^2m_1^2 x^3 + 4g_{\mu\nu}Q^4x^4\big) \\
& + \hat{Q}\gamma_\mu\gamma_\nu\big(-8m_1^4 x^3 + 4kQQ^2x^4 + 12Q^2m_1^2 x^4 - 4Q^4x^5\big) \\
& + \hat{Q}\gamma_\mu\big(-8p_{1,\nu}kQm_1 x^2 + 8Q_\nu m_1^3 x^3 - 4Q_\nu Q^2m_1 x^4\big) \\
& + \hat{Q}\gamma_\nu\big(8p_{1,\mu}kQm_1 x^2 - 8Q_\mu m_1^3 x^3 + 4Q_\mu Q^2m_1 x^4\big) \\
& + \hat{Q}\big(-16p_{1,\mu}p_{1,\nu}m_1^2 x^2 + 8p_{1,\mu}k_\nu Q^2x^3 \\
& - 16p_{1,\mu}Q_\nu m_1^2 x^3 + 16p_{1,\nu}k_\mu m_1^2 x^2 - 8p_{1,\nu}k_\mu Q^2x^3 \\
& - 8p_{1,\nu}Q_\mu Q^2x^4 + 8k_\nu Q_\mu Q^2x^4 - 8Q_\mu Q_\nu Q^2x^5 \\
& - 8g_{\mu\nu}kQQ^2x^4 - 8g_{\mu\nu}Q^2m_1^2 x^4 + 4g_{\mu\nu}Q^4x^5\big) \\
& + g_{\mu\nu}\big(4kQQ^2m_1 x^3 + 4Q^2m_1^3 x^3 - 2Q^4m_1 x^4\big) \\
& + \gamma_\mu\big(8p_{1,\nu}kQQ^2x^3 - 8p_{1,\nu}Q^2m_1^2 x^3 + 4p_{1,\nu}Q^4x^4 \\
& + 8k_\nu Q^2m_1^2 x^3 - 4k_\nu Q^4x^4 - 16Q_\nu m_1^4 x^3 + 8Q_\nu kQQ^2x^4 \\
& + 8Q_\nu Q^2m_1^2 x^4\big) + \gamma_\nu\big(-8p_{1,\mu}kQQ^2x^3 - 8p_{1,\mu}Q^2m_1^2 x^3 \\
& + 4p_{1,\mu}Q^4x^4 + 8k_\mu Q^2m_1^2 x^3 - 4k_\mu Q^4x^4 + 16Q_\mu m_1^4 x^3 \\
& - 8Q_\mu kQQ^2x^4 - 24Q_\mu Q^2m_1^2 x^4 + 8Q_\mu Q^4x^5\big) \\
& + 24p_{1,\mu}p_{1,\nu}Q^2m_1 x^2 - 8p_{1,\mu}k_\nu Q^2m_1 x^2 \\
& - 16p_{1,\mu}Q_\nu kQm_1 x^2 + 24p_{1,\mu}Q_\nu Q^2m_1 x^3 \\
& - 16p_{1,\nu}k_\mu Q^2m_1 x^2 + 16p_{1,\nu}Q_\mu kQm_1 x^2 \\
& + 24p_{1,\nu}Q_\mu Q^2m_1 x^3 - 16k_\mu Q_\nu Q^2m_1 x^3 \\
& - 8k_\nu Q_\mu Q^2m_1 x^3 + 24Q_\mu Q_\nu Q^2m_1 x^4 \\
& + 8g_{\mu\nu}kQQ^2m_1 x^3 + 8g_{\mu\nu}Q^2m_1^3 x^3 \\
& - 4g_{\mu\nu}Q^4m_1 x^4.
\end{aligned}
\tag{15}
$$

For the further construction of hyperfine splitting potentials corresponding to the amplitude (3) we introduce the projection operators on the states of muon–deuteron pair with the spin 3/2 and 1/2:

$$
\hat{\Pi}_{\mu,3/2} = \big[u(p_1)\epsilon_\mu(p_2)\big]_{3/2} = \Psi_\mu(P),
$$

$$
\hat{\Pi}_{\mu,1/2} = \frac{i}{\sqrt{3}}\gamma_5(\gamma_\mu - v_{1,\mu})\Psi(P),
\tag{16}
$$

$$
\begin{aligned}
& \sum_\lambda \Psi_\mu^\lambda(P)\bar{\Psi}_\nu^\lambda(P) \\
& = -\frac{\hat{v}_1 + 1}{2}\bigg(g_{\mu\nu} - \frac{1}{3}\gamma_\mu\gamma_\nu - \frac{2}{3}v_{1,\mu}v_{1,\nu} \\
& \quad + \frac{1}{3}(v_{1,\mu}\gamma_\nu - v_{1,\nu}\gamma_\mu)\bigg),
\end{aligned}
\tag{17}
$$

where the spin-vector $\Psi_\mu(P)$ and spinor $\Psi(P)$ describe the muon–deuteron bound states with spins 3/2 and 1/2, $v_{1,\mu} = P_\mu/M$, $P = p_1 + p_2$, $M = m_1 + m_2$. The insertion (16) into (3) allows us to pass to the trace calculation and contractions over the Lorentz indices by means of the system Form [14]. A general structure of potentials contributing to the energy shifts for states with the angular momenta 1/2 and 3/2 is the following one:

$$
\begin{aligned}
N_{1/2} = \frac{1}{6}\,\mathrm{Tr}\bigg\{ & \sum_\sigma \Psi^\sigma(P)\bar{\Psi}^\sigma(P)(\gamma_\rho - v_{1,\rho})\gamma_5(1 + \hat{v}_1) \\
& \times L_{\mu\nu}(1 + \hat{v}_1)\gamma_5(\gamma_\alpha - v_{1,\alpha})\bigg\} \\
& \times \Gamma_{\omega,\rho\beta}(q_2, p_2 + k)\mathcal{D}_{\beta\tau}(p_2 + k)\Gamma_{\lambda,\tau\alpha}(p_2 + k, p_2) \\
& \times D_{\mu\omega}(k)D_{\nu\lambda}(k),
\end{aligned}
\tag{18}
$$

Table 1
Radiative nuclear finite size corrections of order $\alpha(Z\alpha)^5$, to hyperfine structure of S-states in muonic hydrogen. Numerical results for the ground state are presented. The contribution to the hyperfine structure for the point nucleus is indicated in round brackets.

Bound state	SE correction, meV	Vertex correction, meV	Spanning photon contribution, meV	Summary contribution, meV
Point-like nucleus	$E_F\alpha(Z\alpha)\frac{3}{2}$	$-E_F\alpha(Z\alpha)(3\ln 2 + \frac{9}{4})$	$E_F\alpha(Z\alpha)(4\ln 2 - \frac{5}{2})$	$E_F\alpha(Z\alpha)(\ln 2 - \frac{13}{4})$
Muonic hydrogen	0.0083 (0.0146)	−0.0915 (−0.0421)	−0.0028 (0.0026)	−0.0860 (−0.0249)
Muonic deuterium	0.0014 (0.0039)	−0.0042 (−0.0113)	−0.0011 (0.0007)	−0.0039 (−0.0067)

$$N_{3/2} = \frac{1}{4}\text{Tr}\left\{\sum_\sigma \Psi_\alpha^\sigma(P)\bar{\Psi}_\rho^\sigma(P)(1+\hat{v}_1)L_{\mu\nu}(1+\hat{v}_1)\right\}$$
$$\times \Gamma_{\omega,\rho\beta}(q_2, p_2+k)\mathcal{D}_{\beta\tau}(p_2+k)\Gamma_{\lambda,\tau\alpha}(p_2+k, p_2)$$
$$\times D_{\mu\omega}(k)D_{\nu\lambda}(k). \tag{19}$$

The expressions (18) and (19) contain both recoil and nonrecoil corrections of order $\alpha(Z\alpha)^5$. Since we neglect the recoil effects the denominator of the deuteron propagator is simplified as follows: $1/[(p_2+k)^2 - m_2^2 + i0] \approx 1/(k^2 + 2kp_2 + i0) \approx 1/(2k_0 m_2 + i0)$. The crossed two-photon amplitudes give in this case a similar contribution to hyperfine splitting which is determined also by relations (3)–(12) with the replacement $k \to -k$ in the deuteron propagator. As a result the summary contribution is proportional to the $\delta(k_0)$:

$$\frac{1}{2m_2k_0 + i0} + \frac{1}{-2m_2k_0 + i0} = -\frac{i\pi}{m_2}\delta(k_0). \tag{20}$$

In the case of muonic hydrogen the transformation of the scattering amplitude and a construction of muon–proton potential can be done in much the same way. The main difference is related to the structure of proton–photon vertex functions which are parameterized by two electromagnetic form factors. Another difference appears in the projection operators on the states with spin 1 and 0 which have the form:

$$\hat{\Pi}_{0,1} = \frac{\hat{v}_1 + 1}{2\sqrt{2}}\gamma_5(\hat{\epsilon}), \tag{21}$$

where ϵ_μ is the polarization vector of muon–proton state with spin 1. The energy shift caused by interactions shown in Fig. 1 is given by

$$\Delta E_{1/2,3/2} = \mathcal{M}_{1/2,3/2}|\psi_n(0)|^2, \tag{22}$$

where $|\psi_n(0)|^2 = (\mu Z\alpha)^3/\pi n^3$ is the squared modulus of the bound state wave function at the origin. The lower subscript denotes total angular momentum for the muon–deuteron state. Then the hyperfine splitting (hfs) is determined as follows:

$$\Delta E^{hfs} = \Delta E_{3/2} - \Delta E_{1/2}. \tag{23}$$

As a result three types of corrections of order $\alpha(Z\alpha)^5$ to hyperfine structure in both cases of muonic hydrogen are presented in the integral form over the loop momentum $\mathbf{k}$ and the Feynman parameters x and z:

$$\Delta E_\Sigma^{hfs} = E_F 6\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 x\, dx \int_0^\infty \frac{F_1(k^2)F_3(k^2)dk}{x + (1-x)k^2}, \tag{24}$$

$$\Delta E_{\Lambda 1}^{hfs} = -E_F 24\frac{\alpha(Z\alpha)}{\pi^2}$$
$$\times \int_0^1 dz \int_0^1 x\, dx \int_0^\infty \frac{F_1(k^2)F_3(k^2)\ln[\frac{x+k^2z(1-xz)}{x}]dk}{k^2}, \tag{25}$$

$$\Delta E_{\Lambda 2}^{hfs} = E_F 8\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 dz \int_0^1 dx \int_0^\infty \frac{dk}{k^2}\left\{\frac{F_1(k^2)F_3(k^2)}{[x + k^2z(1-xz)]^2}\right.$$
$$\times \left[-2xz^2(1-xz)k^4 + zk^2(3x^3z - x^2(9z+1)\right.$$
$$\left.+ x(4z+7) - 4) + x^2(5-x)\right] - \frac{1}{2}\Big\}, \tag{26}$$

$$\Delta E_\Xi^{hfs} = E_F 4\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 (1-z)dz \int_0^1 (1-x)dx \int_0^\infty \frac{F_1(k^2)F_3(k^2)dk}{[x+(1-x)k^2]^3}$$
$$\times \left[6x + 6x^2 - 6x^2z + 2x^3 - 12x^3z - 12x^4z\right.$$
$$+ k^2(-6z + 18xz + 4xz^2 + 7x^2z - 30x^2z^2$$
$$- 2x^2z^3 - 36x^3z^2 + 12x^3z^3 + 24x^4z^3)$$
$$\left.+ k^4(9xz^2 - 31x^2z^3 + 34x^3z^4 - 12x^4z^5)\right], \tag{27}$$

where we extracted the value of the deuteron magnetic moment from $F_3(k^2)$ so that $F_3(0) = 1$ and $F_1(0) = 1$. The dimensionless variable k is introduced in (24)–(27). The contribution of the form factor $F_2(k^2)$ to (24)–(27) is omitted because the terms $F_2(k^2)F_3(k^2)$ are suppressed by powers of the mass m_2. The term $1/2$ in figure brackets (26) is related to the subtraction term of the quasipotential. All corrections (24), (25), (26) and (27) are expressed through the convergent integrals. In the case of point-like deuteron (proton) all integrations can be done analytically. Firstly, the integration over the parameter x is performed and after that the integration over k and z. The diagrams of the seagull type for point-like deuteron doesn't contribute to hyperfine splitting. In Table 1 we present separate results for muon self-energy, vertex and spanning photon contributions in the Fried–Yennie gauge. Total analytical result equal to $E_F\alpha(Z\alpha)(\ln 2 - \frac{13}{4})$ was obtained for the first time in [15]. In [12] the expressions for the lepton tensors of the vertex and spanning photon diagrams were constructed in a slightly different form but they lead to the same contributions (24)–(27) to hyperfine splitting of S-states in the case of point-like nucleus. In numerical calculations (24)–(27) with finite size nucleus we employ the known parameterizations [16,17] for electromagnetic form factors of the deuteron and proton used also in our previous papers [18].

It follows from obtained results in Table 1 that the account of proton and deuteron form factors essentially changes the results for point-like nuclei. In a number of cases there is the change of the correction sign. This follows from the fact that for muonic atoms the integral over k in (24)–(27) is specified by the interval of order of muon mass and a sign-alternating integrand. We perform independent calculation of radiative nonrecoil corrections of order $\alpha(Z\alpha)^5$ to hyperfine structure of S-states in muonic hydrogen using the Fried–Yennie gauge for radiative photon. In the case of muonic hydrogen these corrections decrease the theoretical value of hyperfine splitting of 2S-state approximately on

0.01 meV. There exists also another contribution of order $\alpha(Z\alpha)^5$ to hyperfine splitting which is determined by the electron vacuum polarization effect and nuclear structure in two-photon exchange amplitudes. Thus, for example, in the case of 2S-state in muonic deuterium this contribution is equal to -0.0019 meV. In order of magnitude it is close to radiative corrections of order $\alpha(Z\alpha)^5$ presented in Table 1. This correction is generally considered together with the main nuclear structure contribution of order $(Z\alpha)^5$ from two-photon exchange amplitudes. Nuclear structure corrections of order $(Z\alpha)^5$ (the Zemach contribution and recoil correction) were studied by many authors. Let us present the value of the Zemach correction for muonic hydrogen and deuterium as given in [3] in the case of 2S-state:

$$\Delta E_{Zem} = \begin{cases} \mu p: & -0.1676(6) \text{ meV}, \\ \mu d: & -0.1177(7) \text{ meV}, \end{cases} \tag{28}$$

where uncertainties of the calculation equal approximately to 0.5 per cent are presented in round brackets. Other estimate of the uncertainty related with a parametrization of the nuclear form factors is quoted in [5] for muonic hydrogen 2S-state hyperfine splitting: ± 0.0041 meV. Another large theoretical error connected with the polarizability contribution $\Delta E_{pol}(\mu p) = 0.0105$ meV is equal to (± 0.0026) meV [5]. For an estimate of similar contribution to hyperfine splitting in muonic deuterium the result for muonic hydrogen is commonly used while it can be calculated more precisely (this calculation is in progress). Uncertainties near 0.5 per cent at small values of Q^2 in the fits for the proton form factors G_{E_p} and G_{M_p}/μ_p are presented in Fig. 1 of [19] (see also Table II from [20]). Then theoretical error in the calculation of basic nuclear structure contribution of order $(Z\alpha)^5$ which is determined by a product of two electromagnetic form factors cannot be less than one per cent or ± 0.0017 meV. As it follows from Table 1 the contribution of order $\alpha(Z\alpha)^5$ calculated in present work is important to obtain a high accuracy result with the account of basic theoretical uncertainties. To construct the quasipotential corresponding to amplitudes in Fig. 1 we develop the method of projection operators on the bound states with definite spins. It allows to employ different systems of analytical calculations [14,13]. In this approach more complicated corrections, for example, radiative recoil corrections to hyperfine structure of order $\alpha(Z\alpha)^5 m_1/m_2$ can be evaluated if an increase of the accuracy will be needed. The results from Table 1 should be taken into account to obtain total value of hyperfine splittings in muonic hydrogen for a comparison with experimental data [1].

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