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Canonical matrices of bilinear and sesquilinear forms

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Abstract

Canonical matrices are given for

(i) bilinear forms over an algebraically closed or real closed field;
(ii) sesquilinear forms over an algebraically closed field and over real quaternions with any nonidentity involution; and
(iii) sesquilinear forms over a field $\mathbb{F}$ of characteristic different from 2 with involution (possibly, the identity) up to classification of Hermitian forms over finite extensions of $\mathbb{F}$; the canonical matrices are based on any given set of canonical matrices for similarity over $\mathbb{F}$.

A method for reducing the problem of classifying systems of forms and linear mappings to the problem of classifying systems of linear mappings is used to construct the canonical matrices. This method has its origins in representation theory and was devised in [V.V. Sergeichuk, Classification problems for systems of forms and linear mappings, Math. USSR-Izv. 31 (1988) 481–501].

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1. Introduction

We give canonical matrices of bilinear forms over an algebraically closed or real closed field (familiar examples are $\mathbb{C}$ and $\mathbb{R}$), and of sesquilinear forms over an algebraically closed field and over $\mathbb{P}$-quaternions ($\mathbb{P}$ is a real closed field) with respect to any nonidentity involution. We also give canonical matrices of sesquilinear forms over a field $\mathbb{F}$ of characteristic different from 2 with involution (possibly, the identity) up to classification of Hermitian forms over finite extensions of $\mathbb{F}$; the canonical matrices are based on any given set of canonical matrices for similarity.

Bilinear and sesquilinear forms over a field $\mathbb{F}$ of characteristic different from 2 have been classified by Gabriel, Riehm, and Shrader-Frechette. Gabriel [7] reduced the problem of classifying bilinear forms to the nondegenerate case. Riehm [20] assigned to each nondegenerate bilinear form $\mathscr{A}: \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ a linear mapping $A: V \to V$ and a finite sequence $\varphi_1^{\mathscr{A}}, \varphi_2^{\mathscr{A}}, \ldots$ consisting of ε_i-Hermitian forms $\varphi_i^{\mathscr{A}}$ over finite extensions of $\mathbb{F}$ and proved that two nondegenerate bilinear forms $\mathscr{A}$ and $\mathscr{B}$ are equivalent if and only if the corresponding mappings A and B are similar and each form $\varphi_i^{\mathscr{A}}$ is equivalent to $\varphi_i^{\mathscr{B}}$ (results of this kind were earlier obtained by Williamson [38]). This reduction was studied in [25] and was improved and extended to sesquilinear forms by Riehm and Shrader-Frechette [21]. But this classification of forms was not expressed in terms of canonical matrices, so it is difficult to use.

Using Riehm's reduction, Corbas and Williams [2] obtained canonical forms of nonsingular matrices under congruence over an algebraically closed field of characteristic different from 2 (their list of nonsingular canonical matrices contains an inaccuracy, which can be easily fixed; see [12, p. 1013]). Thompson [36] gave canonical pairs of symmetric or skew-symmetric matrices over $\mathbb{C}$ and $\mathbb{R}$ under simultaneous congruence. Since any square complex or real matrix can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix, Thompson's canonical pairs lead to canonical matrices for congruence; they are studied in [17]. We construct canonical matrices that are much simpler than the ones in [2,17].

We construct canonical matrices of bilinear and sesquilinear forms by using the technique for reducing the problem of classifying systems of forms and linear mappings to the problem of classifying systems of linear mappings that was devised by Roiter [24] and the second author [27,28,31]. A system of forms and linear mappings satisfying some relations is given as a representation of a partially ordered graph P with relations: each vertex corresponds to a vector space, each arrow or nonoriented edge corresponds to a linear mapping or a bilinear/sesquilinear form (see Section 3). The problem of classifying such representations over a field or skew field $\mathbb{F}$ of characteristic different from 2 reduces to the problems of classifying

- representations of some quiver $\underline{P}$ with relations and involution (in fact, representations of a finite dimensional algebra with involution) over $\mathbb{F}$, and
- Hermitian forms over fields or skew fields that are finite extensions of the center of $\mathbb{F}$.

The corresponding reduction theorem was extended in [31] to the problem of classifying selfadjoint representations of a linear category with involution and in [33] to the problem of classifying symmetric representations of an algebra with involution. Similar theorems were proved by Quebbermann, Scharlau, and Schulte [18,26] for additive categories with quadratic or Hermitian forms on objects, and by Derksen, Shmelkin, and Weyman [3,35] for generalizations of quivers involving linear groups.

Canonical matrices of

(i) bilinear and sesquilinear forms,
(ii) pairs of symmetric or skew-symmetric forms, and pairs of Hermitian forms, and
(iii) isometric or selfadjoint operators on a space with scalar product given by a nondegenerate symmetric, skew-symmetric, or Hermitian form

were constructed in [28,31] by this technique over a field $\mathbb{F}$ of characteristic different from 2 up to classification of Hermitian forms over fields that are finite extensions of $\mathbb{F}$. Thus, the canonical matrices of (i)–(iii) over $\mathbb{C}$ and $\mathbb{R}$ follow from the construction in [28,31] since classifications of Hermitian forms over these fields are known.

The canonical matrices of bilinear and sesquilinear forms over an algebraically closed field of characteristic different from 2 and over a real closed field given in [31, Theorem 3], and the canonical matrices of bilinear forms over an algebraically closed field of characteristic 2 given in [30] are based on the Frobenius canonical form for similarity. In this article we simplify them by using the Jordan canonical form. Such a simplification was given by the authors in [10] for canonical matrices of bilinear and sesquilinear forms over $\mathbb{C}$; a direct proof that the matrices from [10] are canonical is given in [11,12]; applications of these canonical matrices were obtained in [4–6,12,13]. We also construct canonical matrices of sesquilinear forms over quaternions; they were given in [32] with incorrect signs for the indecomposable direct summands; see Remark 3.1. Analogous results for canonical matrices of isometric operators have been obtained in [34].

The paper is organized as follows. In Section 2 we formulate our main results: Theorem 2.1 about canonical matrices of bilinear and sesquilinear forms over an algebraically or real closed field and over quaternions, and Theorem 2.2 about canonical matrices of bilinear and sesquilinear forms over any field $\mathbb{F}$ of characteristic not 2 with an involution, up to classification of Hermitian forms. In Section 3 we give a brief exposition of the technique for reducing the problem of classifying systems of forms and linear mappings to the problem of classifying systems of linear mappings. We use it in Sections 4 and 5, in which we prove Theorems 2.1 and 2.2.

2. Canonical matrices for congruence and *congruence

Let $\mathbb{F}$ be a field or skew field with involution $a \mapsto \bar{a}$, i.e., a bijection $\mathbb{F} \to \mathbb{F}$ satisfying $\overline{a+b} = \bar{a} + \bar{b}$, $\overline{ab} = \bar{b}\bar{a}$, and $\bar{\bar{a}} = a$. Thus, the involution may be the identity only if $\mathbb{F}$ is a field.

For any matrix $A = [a_{ij}]$ over $\mathbb{F}$, we write $A^* := \bar{A}^{\mathrm{T}} = [\bar{a}_{ji}]$. Matrices $A, B \in \mathbb{F}^{n\times n}$ are said to be **congruent* over $\mathbb{F}$ if there is a nonsingular $S \in \mathbb{F}^{n\times n}$ such that $S^*AS = B$. If $S^{\mathrm{T}}AS = B$, then the matrices A and B are called *congruent*. The transformations of congruence ($A \mapsto S^{\mathrm{T}}AS$) and *congruence ($A \mapsto S^*AS$) are associated with the bilinear form $x^{\mathrm{T}}Ay$ and the sesquilinear form x^*Ay, respectively.

2.1. Canonical matrices over an algebraically or real closed field and over quaternions

In this section we give canonical matrices for congruence over:

- an algebraically closed field, and
- a *real closed field*—i.e., a field $\mathbb{P}$ whose algebraic closure $\mathbb{K}$ has a finite degree $\neq 1$ (that is, $1 < \dim_{\mathbb{P}} \mathbb{K} < \infty$).

We also give canonical matrices for *congruence over:

- an algebraically closed field with nonidentity involution, and
- the *skew field of* $\mathbb{P}$*-quaternions*

$$\mathbb{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{P}\},$$

in which $\mathbb{P}$ is a real closed field, $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$.

We consider only two involutions on $\mathbb{H}$: *quaternionic conjugation*

$$a + bi + cj + dk \longmapsto a - bi - cj - dk, \quad a, b, c, d \in \mathbb{P}, \tag{1}$$

and *quaternionic semiconjugation*

$$a + bi + cj + dk \longmapsto a - bi + cj + dk, \quad a, b, c, d \in \mathbb{P}, \tag{2}$$

because if an involution on $\mathbb{H}$ is not quaternionic conjugation, then it becomes quaternionic semiconjugation after a suitable reselection of the imaginary units i, j, k; see [19].

There is a natural one-to-one correspondence

$$\begin{Bmatrix} \text{algebraically closed fields} \\ \text{with nonidentity involution} \end{Bmatrix} \quad \longleftrightarrow \quad \{\text{real closed fields}\}$$

sending an algebraically closed field with nonidentity involution to its fixed field. This follows from our next lemma, in which we collect known results about such fields.

Lemma 2.1. (a) *Let* $\mathbb{P}$ *be a real closed field and let* $\mathbb{K}$ *be its algebraic closure. Then* $\operatorname{char}\mathbb{P} = 0$ *and*

$$\mathbb{K} = \mathbb{P} + \mathbb{P}i, \quad i^2 = -1. \tag{3}$$

The field $\mathbb{P}$ *has a unique linear ordering* $\leqslant$ *such that*

$$a > 0 \text{ and } b > 0 \implies a + b > 0 \text{ and } ab > 0.$$

The positive elements of $\mathbb{P}$ *with respect to this ordering are the squares of nonzero elements.*

(b) *Let* $\mathbb{K}$ *be an algebraically closed field with nonidentity involution. Then* $\operatorname{char}\mathbb{K} = 0$,

$$\mathbb{P} := \{k \in \mathbb{K} | \bar{k} = k\} \tag{4}$$

is a real closed field,

$$\mathbb{K} = \mathbb{P} + \mathbb{P}i, \quad i^2 = -1, \tag{5}$$

and the involution is "complex conjugation":

$$\overline{a + bi} = a - bi, \quad a, b \in \mathbb{P}. \tag{6}$$

(c) *Every algebraically closed field* $\mathbb{F}$ *of characteristic* 0 *contains at least one real closed subfield. Hence,* $\mathbb{F}$ *can be represented in the form* (5) *and possesses the involution* (6).

Proof. (a) Let $\mathbb{K}$ be the algebraic closure of $\mathbb{F}$ and suppose $1 < \dim_{\mathbb{P}} \mathbb{K} < \infty$. By Corollary 2 in [16, Chapter VIII, §9], we have $\operatorname{char}\mathbb{P} = 0$ and (3). The other statements of part (a) follow from Proposition 3 and Theorem 1 in [16, Chapter XI, §2].

(b) If $\mathbb{K}$ is an algebraically closed field with nonidentity involution $a \mapsto \bar{a}$, then this involution is an automorphism of order 2. Hence $\mathbb{K}$ has degree 2 over its *fixed field* $\mathbb{P}$ defined in (4). Thus, $\mathbb{P}$ is a real closed field. Let $i \in \mathbb{K}$ be such that $i^2 = -1$. By (a), every element of $\mathbb{K}$ is uniquely represented in the form $k = a + bi$ with $a, b \in \mathbb{P}$. The involution is an automorphism of $\mathbb{K}$, so $\bar{i}^2 = -1$. Thus, $\bar{i} = -i$ and the involution has the form (6).

(c) This statement is proved in [37, §82, Theorem 7c]. $\square$

For notational convenience, write

$$A^{-\mathrm{T}} := (A^{-1})^{\mathrm{T}} \quad \text{and} \quad A^{-*} := (A^{-1})^{*}.$$

The *cosquare* of a nonsingular matrix A is $A^{-\mathrm{T}}A$. If two nonsingular matrices are congruent then their cosquares are similar because

$$(S^{\mathrm{T}}AS)^{-\mathrm{T}}(S^{\mathrm{T}}AS) = S^{-1}A^{-\mathrm{T}}AS.$$

If Φ is a cosquare, every matrix C such that $C^{-\mathrm{T}}C = \Phi$ is called a *cosquare root* of Φ; we choose any cosquare root and denote it by $\sqrt[T]{\Phi}$.

Analogously, $A^{-*}A$ is the **cosquare* of A. If two nonsingular matrices are *congruent then their *cosquares are similar. If Φ is a *cosquare, every matrix C such that $C^{-*}C = \Phi$ is called a **cosquare root* of Φ; we choose any *cosquare root and denote it by $\sqrt[*]{\Phi}$.

For each real closed field, we denote by $\leqslant$ the ordering from Lemma 2.1(a). Let $\mathbb{K} = \mathbb{P} + \mathbb{P}i$ be an algebraically closed field with nonidentity involution represented in the form (5). By the *absolute value* of $k = a + bi \in \mathbb{K}$ ($a, b \in \mathbb{P}$) we mean a unique nonnegative "real" root of $a^2 + b^2$, which we write as

$$|k| := \sqrt{a^2 + b^2} \tag{7}$$

(this definition is unambiguous since $\mathbb{K}$ is represented in the form (5) uniquely up to replacement of i by $-i$). For each $M \in \mathbb{K}^{m\times n}$, its *realification* $M^{\mathbb{P}} \in \mathbb{P}^{2m\times 2n}$ is obtained by replacing every entry $a + bi$ of M by the 2×2 block

$$\begin{matrix} a & -b \\ b & a \end{matrix} \tag{8}$$

Define the n-by-n matrices

$$\Delta_n(\lambda) := \begin{bmatrix} 0 & & & \lambda \\ & & \cdot^{\cdot^{\cdot}} & i \\ & \lambda & \cdot^{\cdot^{\cdot}} & \\ \lambda & i & & 0 \end{bmatrix}, \quad J_n(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix},$$

$$\Gamma_n := \begin{bmatrix} 0 & & & & & \cdot^{\cdot^{\cdot}} \\ & & & & 1 & \cdot^{\cdot^{\cdot}} \\ & & & -1 & -1 & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix},$$

and

$$\Gamma'_n := \begin{bmatrix} 0 & & & & & -1 \\ & & & & \iddots & 1 \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & \iddots & \iddots & & & \\ 1 & 1 & & & & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} m \end{matrix} \quad \text{if } n = 2m,$$

$$\Gamma'_n := \begin{bmatrix} 0 & & & & & & 1 \\ & & & & & \iddots & 0 \\ & & & & 1 & \iddots & \\ & & & 1 & 0 & & \\ & & 1 & 1 & & & \\ & \iddots & \iddots & & & & \\ 1 & 1 & & & & & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} m \\ \}1 \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} m \end{matrix} \quad \text{if } n = 2m,$$

The *skew sum* of two matrices A and B is

$$[A \backslash B] := \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}.$$

The main result of this article is the following theorem, which is proved in Section 5. It was obtained for complex matrices in [10,12].

Theorem 2.1. (a) *Over an algebraically closed field of characteristic different from* 2, *every square matrix is congruent to a direct sum*, *determined uniquely up to permutation of summands*, *of matrices of the form*:

(i) $J_n(0)$;
(ii) $[J_n(\lambda)\backslash I_n]$, *in which* $\lambda \neq (-1)^{n+1}$, $\lambda \neq 0$, *and* λ *is determined up to replacement by* λ^{-1};
(iii) $\sqrt[T]{J_n((-1)^{n+1})}$.

Instead of the matrix (iii), *one may use* Γ_n, *or* Γ'_n, *or any other nonsingular matrix whose cosquare is similar to* $J_n((-1)^{n+1})$; *these matrices are congruent to* (iii).

(b) *Over an algebraically closed field of characteristic* 2, *every square matrix is congruent to a direct sum*, *determined uniquely up to permutation of summands*, *of matrices of the form*:

(i) $J_n(0)$;
(ii) $[J_n(\lambda)\backslash I_n]$, *in which* λ *is nonzero and is determined up to replacement by* λ^{-1};
(iii) $\sqrt[T]{J_n(1)}$ *with odd* n; *no blocks of the form* $[J_n(1)\backslash I_n]$ *are permitted for any odd* n *for which a block* $\sqrt[T]{J_n(1)}$ *occurs in the direct sum.*[2]

Instead of the matrix (iii), *one may use* Γ'_n *or any other nonsingular matrix whose cosquare is similar to* $J_n(1)$, *these matrices are congruent to* (iii).

[2] If the direct sum would otherwise contain both $\sqrt[T]{J_n(1)}$ and $[J_n(1)\backslash I_n]$ for the same odd n, then this pair of blocks must be replaced by three blocks $\sqrt[T]{J_n(1)}$. This restriction is imposed to ensure uniqueness of the canonical direct sum because $\sqrt[T]{J_n(1)} \oplus [J_n(1)\backslash I_n]$ is congruent to $\sqrt[T]{J_n(1)} \oplus \sqrt[T]{J_n(1)} \oplus \sqrt[T]{J_n(1)}$; see [30] and Remark 2.1.

(c) *Over an algebraically closed field with nonidentity involution, every square matrix is *congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the form*:

(i) $J_n(0)$;
(ii) $[J_n(\lambda)\backslash I_n]$, *in which* $|\lambda| \neq 1$ (*see* (7)), $\lambda \neq 0$, *and* λ *is determined up to replacement by* $\bar{\lambda}^{-1}$ (*alternatively, in which* $|\lambda| > 1$);
(iii) $\pm\sqrt[*]{J_n(\lambda)}$, *in which* $|\lambda| = 1$.

Instead of the matrices (iii), *one may use any of the matrices*

$$\mu\sqrt[*]{J_n(1)}, \quad \mu\Gamma_n, \quad \mu\Gamma'_n, \quad \mu\Delta_n(1), \quad \mu A \tag{9}$$

with $|\mu| = 1$, *where* A *is any* $n \times n$ *matrix whose *cosquare is similar to a Jordan block.*

(d) *Over a real closed field* $\mathbb{P}$ *whose algebraic closure is represented in the form* (3), *every square matrix is congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the form*:

(i) $J_n(0)$;
(ii) $[J_n(a)\backslash I_n]$, *in which* $0 \neq a \in \mathbb{P}$, $a \neq (-1)^{n+1}$, *and* a *is determined up to replacement by* a^{-1} (*alternatively*, $a \in \mathbb{P}$ *and* $|a| > 1$ *or* $a = (-1)^n$);
(iii) $\pm\sqrt[T]{J_n((-1)^{n+1})}$;
(ii′) $[J_n(\lambda)^{\mathbb{P}}\backslash I_{2n}]$, *in which* $\lambda \in (\mathbb{P} + \mathbb{P}i) \setminus \mathbb{P}$, $|\lambda| \neq 1$, *and* λ *is determined up to replacement by* $\bar{\lambda}$, λ^{-1}, *or* $\bar{\lambda}^{-1}$ (*alternatively*, $\lambda = a + bi$ *with* $a, b \in \mathbb{P}$, $b > 0$, *and* $a^2 + b^2 > 1$);
(iii′) $\pm\sqrt[T]{J_n(\lambda)^{\mathbb{P}}}$, *in which* $\lambda \in (\mathbb{P} + \mathbb{P}i) \setminus \mathbb{P}$, $|\lambda| = 1$, *and* λ *is determined up to replacement by* $\bar{\lambda}$ (*alternatively*, $\lambda = a + bi$ *with* $a, b \in \mathbb{P}$, $b > 0$, *and* $a^2 + b^2 = 1$).

Instead of (iii), *one may use* $\pm\Gamma_n$ *or* $\pm\Gamma'_n$.
Instead of (iii′), *one may use* $\pm(\sqrt[*]{J_n(\lambda)})^{\mathbb{P}}$ *with the same* λ, *or any of the matrices*

$$((c + i)\Gamma_n)^{\mathbb{P}}, \quad ((c + i)\Gamma'_n)^{\mathbb{P}}, \quad \Delta_n(c + i)^{\mathbb{P}} \tag{10}$$

with $0 \neq c \in \mathbb{P}$.

(e) *Over a skew field of* $\mathbb{P}$*-quaternions* ($\mathbb{P}$ *is real closed*) *with quaternionic conjugation* (1) *or quaternionic semiconjugation* (2), *every square matrix is *congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the form*:

(i) $J_n(0)$;
(ii) $[J_n(\lambda)\backslash I_n]$, *in which* $0 \neq \lambda \in \mathbb{P} + \mathbb{P}i$, $|\lambda| \neq 1$, *and* λ *is determined up to replacement by* $\bar{\lambda}$, λ^{-1}, *or* $\bar{\lambda}^{-1}$ (*alternatively*, $\lambda = a + bi$ *with* $a, b \in \mathbb{P}$, $b \geqslant 0$, *and* $a^2 + b^2 > 1$);
(iii) $\varepsilon\sqrt[*]{J_n(\lambda)}$, *in which* $\lambda \in \mathbb{P} + \mathbb{P}i$, $|\lambda| = 1$, λ *is determined up to replacement by* $\bar{\lambda}$, *and*

$$\varepsilon := \begin{cases} 1, & \text{if the involution is (1), } \lambda = (-1)^n, \\ & \text{and if the involution is (2), } \lambda = (-1)^{n+1}, \\ \pm 1, & \text{otherwise.} \end{cases} \tag{11}$$

Instead of (iii), *one may use*

$$(a + bi)\Gamma_n \quad \text{or} \quad (a + bi)\Gamma'_n, \tag{12}$$

in which $a, b \in \mathbb{P}$, $a^2 + b^2 = 1$, *and*

$$\begin{cases} b \geqslant 0 & \text{if the involution is (1)}, \\ a \geqslant 0 & \text{if the involution is (2)}. \end{cases}$$

Instead of (iii), *one may also use*

$$(a+bi)\Delta_n(1), \tag{13}$$

in which $a, b \in \mathbb{P}$, $a^2+b^2=1$, *and*

$$\begin{cases} a \geqslant 0, & \text{if the involution is (1), } n \text{ is even,} \\ & \text{and if the involution is (2), } n \text{ is odd,} \\ b \geqslant 0, & \text{otherwise.} \end{cases}$$

In this theorem "determined up to replacement by" means that a block is congruent or *congruent to the block obtained by making the indicated replacements.

Remark 2.1. Theorem 3.2 in Section 3 ensures that each system of linear mappings and bilinear forms on vector spaces over an algebraically closed field of characteristic not two or real closed field as well as each system of linear mappings and sesquilinear forms on vector spaces over an algebraically closed field with nonidentity involution, or real quaternions with nonidentity involution decomposes into a direct sum of indecomposable systems that is unique up to isomorphisms of summands. Over any field of characteristic not 2, two decompositions into indecomposables may have nonisomorphic direct summands, but Theorem 3.1 tells us that the number of indecomposable direct summands does not depend on the decomposition.

However, over an algebraically closed field $\mathbb{F}$ of characteristic 2, not even the number of indecomposable direct summands is invariant. For example, the matrices

$$[1] \oplus [1] \oplus [1], \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [1] \tag{14}$$

are congruent over $\mathbb{F}$ since

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

but each of the direct summands in (14) is indecomposable by Theorem 2.1(b). The cancellation theorem does not hold for bilinear forms over $\mathbb{F}$: the matrices (14) are congruent but the matrices

$$[1] \oplus [1], \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are not congruent because they are canonical.

2.2. *Canonical matrices for *congruence over a field of characteristic different from 2*

Canonical matrices for congruence and *congruence over a field of characteristic different from 2 were obtained in [31, Theorem 3] up to classification of Hermitian forms. They were based on the Frobenius canonical matrices for similarity. In this section we rephrase [31, Theorem 3] in terms of an *arbitrary* set of canonical matrices for similarity. This flexibility is used in the proof of Theorem 2.1. The same flexibility is used in [10] to construct simple canonical matrices for congruence or *congruence over $\mathbb{C}$, and in [34] to construct simple canonical matrices of pairs (A, B) in which B is a nondegenerate Hermitian or skew-Hermitian form and A is an isometric operator over an algebraically or real closed field or over real quaternions.

In this section $\mathbb{F}$ denotes a field of characteristic different from 2 with involution $a \mapsto \bar{a}$, which can be the identity. Thus, congruence is a special case of *congruence.

For each polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{F}[x],$$

we define the polynomials

$$\bar{f}(x) := \bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \cdots + \bar{a}_n,$$
$$f^{\vee}(x) := \bar{a}_n^{-1}(\bar{a}_n x^n + \cdots + \bar{a}_1 x + \bar{a}_0) \quad \text{if } a_n \neq 0.$$

The following lemma was proved in [31, Lemma 6] (or see [34, Lemma 2.3]).

Lemma 2.2. *Let $\mathbb{F}$ be a field with involution $a \mapsto \bar{a}$, let $p(x) = p^{\vee}(x)$ be an irreducible polynomial over $\mathbb{F}$, and let r be the integer part of* $(\deg p(x))/2$. *Consider the field*

$$\mathbb{F}(\kappa) = \mathbb{F}[x]/p(x)\mathbb{F}[x], \quad \kappa := x + p(x)\mathbb{F}[x], \tag{15}$$

with involution

$$f(\kappa)^{\circ} := \bar{f}(\kappa^{-1}). \tag{16}$$

Then each element of $\mathbb{F}(\kappa)$ on which the involution acts identically is uniquely representable in the form $q(\kappa)$, in which

$$q(x) = a_r x^r + \cdots + a_1 x + a_0 + \bar{a}_1 x^{-1} + \cdots + \bar{a}_r x^{-r}, \quad a_0 = \bar{a}_0, \tag{17}$$

$a_0, \ldots a_r \in \mathbb{F}$; if $\deg p(x) = 2r$ *is even, then*

$$a_r = \begin{cases} 0 & \textit{if the involution on } \mathbb{F} \textit{ is the identity}, \\ \bar{a}_r & \textit{if the involution on } \mathbb{F} \textit{ is not the identity and } p(0) \neq 1, \\ -\bar{a}_r & \textit{if the involution on } \mathbb{F} \textit{ is not the identity and } p(0) = 1. \end{cases}$$

We say that a square matrix is *indecomposable for similarity* if it is not similar to a direct sum of square matrices of smaller sizes. Denote by $\mathcal{O}_{\mathbb{F}}$ any maximal set of nonsingular indecomposable canonical matrices for similarity; this means that each nonsingular indecomposable matrix is similar to exactly one matrix from $\mathcal{O}_{\mathbb{F}}$.

For example, $\mathcal{O}_{\mathbb{F}}$ may consist of all nonsingular *Frobenius blocks*, i.e., the matrices

$$\Phi = \begin{bmatrix} 0 & & 0 & -c_n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -c_2 \\ 0 & & 1 & -c_1 \end{bmatrix} \tag{18}$$

whose characteristic polynomials $\chi_\Phi(x)$ are powers of irreducible monic polynomials $p_\Phi(x) \neq x$:

$$\chi_\Phi(x) = p_\Phi(x)^s = x^n + c_1 x^{n-1} + \cdots + c_n. \tag{19}$$

If $\mathbb{F}$ is an algebraically closed field, then we may take $\mathcal{O}_{\mathbb{F}}$ to be all nonsingular Jordan blocks.

It suffices to construct *cosquare roots $\sqrt[*]{\Phi}$ (see page 197) only for $\Phi \in \mathcal{O}_{\mathbb{F}}$: then we can take

$$\sqrt[*]{\Psi} = S^* \sqrt[*]{\Phi} S \quad \text{if } \Psi = S^{-1}\Phi S \text{ and } \sqrt[*]{\Phi} \text{ exists} \tag{20}$$

since $\Phi = A^{-*}A$ implies $S^{-1}\Phi S = (S^*AS)^{-*}(S^*AS)$.

Existence conditions and an explicit form of $\sqrt[*]{\Phi}$ for Frobenius blocks Φ over a field of characteristic not 2 were established in [31, Theorem 7]; this result is presented in Lemma 2.3. In the proof of Theorem 2.1, we take another set $\mathcal{O}_{\mathbb{F}}$ and construct simpler *cosquare roots over an algebraically or real closed field $\mathbb{F}$.

The version of the following theorem given in [31, Theorem 3] considers the case in which $\mathcal{O}_{\mathbb{F}}$ consists of all nonsingular Frobenius blocks.

Theorem 2.2. (a) *Let $\mathbb{F}$ be a field of characteristic different from* 2 *with involution* (*which can be the identity*). *Let $\mathcal{O}_{\mathbb{F}}$ be a maximal set of nonsingular indecomposable canonical matrices for similarity over $\mathbb{F}$. Every square matrix A over $\mathbb{F}$ is *congruent to a direct sum of matrices of the following types*:

(i) $J_n(0)$;
(ii) $[\Phi \backslash I_n]$, *in which $\Phi \in \mathcal{O}_{\mathbb{F}}$ is an $n \times n$ matrix such that $\sqrt[*]{\Phi}$ does not exist* (*see Lemma* 2.3); *and*
(iii) $\sqrt[*]{\Phi}q(\Phi)$, *in which $\Phi \in \mathcal{O}_{\mathbb{F}}$ is such that $\sqrt[*]{\Phi}$ exists and $q(x) \neq 0$ has the form* (17) *in which r is the integer part of* $(\deg p_\Phi(x))/2$ *and $p_\Phi(x)$ is the irreducible divisor of the characteristic polynomial of Φ.*

The summands are determined to the following extent:

Type (i) *uniquely.*
Type (ii) *up to replacement of Φ by the matrix $\Psi \in \mathcal{O}_{\mathbb{F}}$ that is similar to Φ^{-*}* (*i.e., whose characteristic polynomial is $\chi_\Phi^\vee(x)$*).
Type (iii) *up to replacement of the whole group of summands*

$$\sqrt[*]{\Phi}q_1(\Phi) \oplus \cdots \oplus \sqrt[*]{\Phi}q_s(\Phi)$$

with the same Φ by a direct sum

$$\sqrt[*]{\Phi}q_1'(\Phi) \oplus \cdots \oplus \sqrt[*]{\Phi}q_s'(\Phi)$$

such that each $q_i'(x)$ is a nonzero function of the form (17) *and the Hermitian forms*

$$q_1(\kappa)x_1^\circ x_1 + \cdots + q_s(\kappa)x_s^\circ x_s,$$
$$q_1'(\kappa)x_1^\circ x_1 + \cdots + q_s'(\kappa)x_s^\circ x_s$$

are equivalent over the field (15) *with involution* (16).

(b) *In particular, if $\mathbb{F}$ is an algebraically closed field of characteristic different from* 2 *with the identity involution, then the summands of type* (iii) *can be taken equal to $\sqrt[*]{\Phi}$. If $\mathbb{F}$ is an algebraically closed field with nonidentity involution, or a real closed field, then the summands of type* (iii) *can be taken equal to $\pm\sqrt[*]{\Phi}$. In these cases the summands are uniquely determined by the matrix A.*

Let

$$f(x) = \gamma_0 x^m + \gamma_1 x^{m-1} + \cdots + \gamma_m \in \mathbb{F}[x], \quad \gamma_0 \neq 0 \neq \gamma_m.$$

A vector $(a_1, a_2, \ldots, a_n)$ over $\mathbb{F}$ is said to be f-*recurrent* if $n \leqslant m$, or if

$$\gamma_0 a_l + \gamma_1 a_{l+1} + \cdots + \gamma_m a_{l+m} = 0, \quad l = 1, 2, \ldots, n - m$$

(by definition, it is not f-recurrent if $m = 0$). Thus, this vector is completely determined by any fragment of length m.

The following lemma was proved in [31, Theorem 7]; we give a more detailed proof.

Lemma 2.3. *Let $\mathbb{F}$ be a field of characteristic not 2 with involution $a \mapsto \bar{a}$ (possibly, the identity). Let $\Phi \in \mathbb{F}^{n\times n}$ be nonsingular and indecomposable for similarity; thus, its characteristic polynomial is a power of some irreducible polynomial $p_\Phi(x)$.*

(a) *$\sqrt[*]{\Phi}$ exists if and only if*

$$p_\Phi(x) = p_\Phi^\vee(x), \quad \textit{and} \tag{21}$$

$$\textit{if the involution on } \mathbb{F} \textit{ is the identity, also } p_\Phi(x) \neq x + (-1)^{n+1}. \tag{22}$$

(b) *If (21) and (22) are satisfied and Φ is a nonsingular Frobenius block (18) with characteristic polynomial*

$$\chi_\Phi(x) = p_\Phi(x)^s = x^n + c_1x^{n-1} + \cdots + c_n, \tag{23}$$

then for $\sqrt[*]{\Phi}$ one can take the Toeplitz matrix

$$\sqrt[*]{\Phi} := [a_{i-j}] = \begin{bmatrix} a_0 & a_{-1} & \ddots & a_{1-n} \\ a_1 & a_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \ddots & a_1 & a_0 \end{bmatrix}, \tag{24}$$

whose vector of entries $(a_{1-n}, a_{2-n}, \dots, a_{n-1})$ is the χ_Φ-recurrent extension of the vector

$$v = (a_{1-m}, \dots, a_m) = (a, 0, \dots, 0, \bar{a}) \tag{25}$$

of length

$$2m = \begin{cases} n & \textit{if } n \textit{ is even,} \\ n+1 & \textit{if } n \textit{ is odd,} \end{cases} \tag{26}$$

in which

$$a := \begin{cases} 1 & \textit{if } n \textit{ is even, except for the case} \\ & \quad p_\Phi(x) = x + c \textit{ with } c^{n-1} = -1, \\ \chi_\Phi(-1) & \textit{if } n \textit{ is odd and } p_\Phi(x) \neq x+1, \\ e - \bar{e} & \textit{otherwise, with any fixed } \bar{e} \neq e \in \mathbb{F}. \end{cases} \tag{27}$$

Proof. (a) Let $\Phi \in \mathbb{F}^{n\times n}$ be nonsingular and indecomposable for similarity. We prove here that if $\sqrt[*]{\Phi}$ exists then the conditions (21) and (22) are satisfied; we prove the converse statement in (b).

Suppose $A := \sqrt[*]{\Phi}$ exists. Since

$$A = A^*\Phi = \Phi^*A\Phi, \tag{28}$$

we have $A\Phi A^{-1} = \Phi^{-*}$ and

$$\begin{aligned}\chi_\Phi(x) &= \det(xI - \Phi^{-*}) = \det(xI - \bar{\Phi}^{-1}) = \det((-\bar{\Phi}^{-1})(I - x\bar{\Phi})) \\ &= \det(-\bar{\Phi}^{-1}) \cdot x^n \cdot \det(x^{-1}I - \bar{\Phi}) = \chi_\Phi^\vee(x).\end{aligned}$$

In the notation (19), $p_\Phi(x)^s = p_\Phi^\vee(x)^s$, which verifies (21).

It remains to prove (22). Because of (20), we may assume that Φ is a nonsingular Frobenius block (18) with characteristic polynomial (23). If a_{ij} are the entries of A, then we define $a_{i,n+1}$ by

$A\Phi = [a_{ij}]\Phi = [a_{i,j+1}]$, $a_{n+1,j}$ by $\Phi^*A\Phi = \Phi^*[a_{i,j+1}] = [a_{i+1,j+1}]$; and we then use (28) to obtain $[a_{ij}] = [a_{i+1,j+1}]$. Hence the matrix entries depend only on the difference of the indices and A has the form (24) with $a_{i-j} := a_{ij}$. That $(a_{1-n}, a_{2-n}, \dots, a_{n-1})$ is χ_Φ-recurrent follows from:

$$[a_{i-j}]\Phi = [a_{i-j-1}]. \tag{29}$$

In view of

$$\begin{aligned}\chi_\Phi(x) &= x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n \\ &= \chi_\Phi^\vee(x) = \bar{c}_n^{-1}(\bar{c}_nx^n + \bar{c}_{n-1}x^{n-1} + \cdots + \bar{c}_1x + 1),\end{aligned} \tag{30}$$

the vector $(\bar{a}_{n-1}, \dots, \bar{a}_{1-n})$ is χ_Φ-recurrent, so $[a_{i-j}] = A = A^*\Phi = [\bar{a}_{j-i+1}]$, and we have

$$(a_{1-n}, \dots, a_{n-1}) = (a_{1-n}, \dots, a_0, \bar{a}_0, \dots, \bar{a}_{2-n}). \tag{31}$$

Since this vector is χ_Φ-recurrent, it is completely determined by the fragment

$$(a_{1-m}, \dots, a_0, \bar{a}_0, \dots, \bar{a}_{1-m}) \tag{32}$$

of length $2m$ defined in (26).

Write

$$\mu_\Phi(x) := p_\Phi(x)^{s-1} = x^t + b_1x^{t-1} + \cdots + b_t, \quad b_0 := 1. \tag{33}$$

Suppose that (22) is not satisfied; i.e., the involution is the identity and $p_\Phi(x) = x + (-1)^{n-1}$. Let us prove that

$$\text{the vector (32) is } \mu_\Phi(x)\text{-recurrent.} \tag{34}$$

If $n = 2m$ then $\mu_\Phi(x) = (x-1)^{2m-1}$ and (34) is obvious.

Let $n = 2m - 1$. Then the coefficients of $\chi_\Phi(x) = (x+1)^n$ in (23) and $\mu_\Phi(x) = (x+1)^{n-1}$ in (33) are binomial coefficients:

$$c_i = \binom{n}{i}, \qquad b_i = \binom{n-1}{i}.$$

Standard identities for binomial coefficients ensure that

$$c_i = b_i + b_{i-1} = b_i + b_{n-i}, \quad 0 < i < n.$$

Thus (34) follows since

$$\begin{aligned}&2[b_0a_{1-m} + b_1a_{2-m} + \cdots + b_{n-2}a_{3-m} + b_{n-1}a_{2-m}] \\ &\quad = (b_0 + 0)a_{1-m} + (b_1 + b_{n-1})a_{2-m} + (b_2 + b_{n-2})a_{3-m} \\ &\qquad + \cdots + (b_{n-1} + b_1)a_{2-m} + (0 + b_0)a_{1-m} \\ &\quad = c_0a_{1-m} + c_1a_{2-m} + \cdots + c_na_{1-m} = 0\end{aligned}$$

in view of the χ_Φ-recurrence of (32). But then the μ_Φ-recurrent extension of (32) coincides with (31) and we have

$$(0, \dots, 0, b_0, \dots, b_t)A = 0$$

(see (33)), which contradicts our assumption that A is nonsingular.

(b) Let Φ be a nonsingular Frobenius block (18) with characteristic polynomial (23) satisfying (21) and (22).

We first prove the nonsingularity of every Toeplitz matrix $A := [a_{i-j}]$ whose vector of entries

$$(a_{1-n}, a_{2-n}, \dots, a_{n-1}) \tag{35}$$

is χ_Φ-recurrent (and so (29) holds) but is not μ_Φ-recurrent. If $w := (a_{n-1}, \dots, a_0)$ is the last row of A, then

$$w\Phi^{n-1}, w\Phi^{n-2}, \dots, w \tag{36}$$

are all the rows of A by (29). If they are linearly dependent, then $wf(\Phi) = 0$ for some nonzero polynomial $f(x)$ of degree less than n. If $p_\Phi(x)^r$ is the greatest common divisor of $f(x)$ and $\chi_\Phi(x) = p_\Phi(x)^s$, then $r < s$ and

$$p_\Phi(x)^r = f(x)g(x) + \chi_\Phi(x)h(x) \quad \text{for some } g(x), h(x) \in \mathbb{F}[x].$$

Since $wf(\Phi) = 0$ and $w\chi_\Phi(\Phi) = 0$, we have $wp_\Phi(\Phi)^r = 0$. Thus, $w\mu_\Phi(\Phi) = 0$. Because (36) are the rows of A,

$$\begin{aligned}&(0, \dots, 0, b_0, \dots, b_t, \underbrace{0, \dots, 0}_{i})A\\ &\quad = b_0 w\Phi^{i+t} + b_1 w\Phi^{i+t-1} + \cdots + b_t w\Phi^i = w\Phi^i \mu_\Phi(\Phi) = 0\end{aligned}$$

for each $i = 0, 1, \dots, n-t-1$. Hence, (35) is μ_Φ-recurrent, a contradiction.

Finally, we must show that (25) is χ_Φ-recurrent but not μ_Φ-recurrent (and so in view of (30) its χ_Φ-recurrent extension has the form (31), which ensures that $A = [a_{j-i}] = A^*\Phi$ is nonsingular and can be taken for $\sqrt[*]{\Phi}$).

Suppose first that $n = 2m$. Since (25) has length n, it suffices to verify that it is not μ_Φ-recurrent. This is obvious if $\deg \mu_\Phi(x) < n-1$. Let $\deg \mu_\Phi(x) = n-1$. Then $\mu_\Phi(x) = (x+c)^{n-1}$ for some c and we need to show only that

$$a + b_{n-1}\bar{a} = a + c^{n-1}\bar{a} \neq 0. \tag{37}$$

If $c^{n-1} \neq -1$ then by (27) $a = 1$ and so (37) holds. Let $c^{n-1} = -1$. If the involution on $\mathbb{F}$ is the identity then by (21) $c = \pm 1$ and so $c = -1$, contrary to (22). Hence the involution is not the identity, $a = e - \bar{e}$, and (37) is satisfied.

Now suppose that $n = 2m-1$. Since (25) has length $n+1$, it suffices to verify that it is χ_Φ-recurrent, i.e., that

$$a + c_n\bar{a} = 0. \tag{38}$$

By (30), $c_n = \bar{c}_n^{-1}$. Because $\chi_\Phi(x) = \chi_\Phi^\vee(x) = \bar{c}_n^{-1}x^n\bar{\chi}_\Phi(x^{-1})$, we have

$$\chi_\Phi(-1) = -c_n\overline{\chi_\Phi(-1)}.$$

If $p_\Phi(x) \neq x+1$ then $a = \chi_\Phi(-1) \neq 0$ and (38) holds. If $p_\Phi(x) = x+1$ then the involution on $\mathbb{F}$ is not the identity by (22). Hence $a = e - \bar{e}$ and (38) is satisfied. $\square$

3. Reduction theorems for systems of forms and linear mappings

Classification problems for systems of forms and linear mappings can be formulated in terms of representations of graphs with nonoriented, oriented, and doubly oriented ($\longleftrightarrow$) edges; the notion of quiver representations was extended to such representations in [27]. In this section we give a brief summary of definitions and theorems about such representations; for the proofs and a more detailed exposition we refer the reader to [31] and [34]. For simplicity, we consider representations of graphs without doubly oriented edges. All vector spaces that we consider are right vector spaces.

Let $\mathbb{F}$ be a field or skew field with involution $a \mapsto \bar{a}$ (possibly, the identity). A *sesquilinear form* on vector spaces U and V over $\mathbb{F}$ is a mapping $B\colon U \times V \to \mathbb{F}$ satisfying

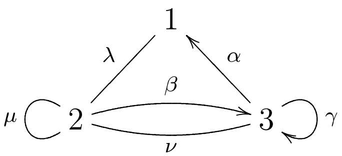
$$B(ua + u'a', v) = \bar{a}B(u, v) + \bar{a'}B(u', v)$$

and

$$B(u, va + v'a') = B(u, v)a + B(u, v')a'$$

for all $u, u' \in U$, $v, v' \in V$, and $a, a' \in \mathbb{F}$. This form is *bilinear* if the involution $a \mapsto \bar{a}$ is the identity. If $e_1, \dots, e_m$ and $f_1, \dots, f_n$ are bases of U and V, then $B(u, v) = [u]_e^* B_{ef}[v]_f$ for all $u \in U$ and $v \in V$, in which $[u]_e$ and $[v]_f$ are the coordinate vectors and $B_{ef} := [B(e_i, f_j)]$ is the matrix of B. Its matrix in other bases is $R^* B_{ef} S$, in which R and S are the transition matrices.

A *pograph* (partially ordered graph) is a graph in which every edge is nonoriented or oriented; for example,



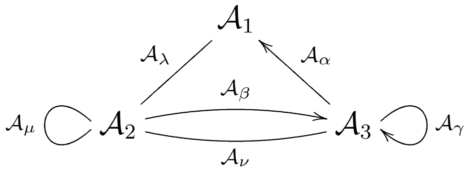
(39)

We suppose that the vertices are $1, 2, \dots, n$, and that there can be any number of edges between any two vertices.

A *representation* $\mathscr{A}$ of a pograph P over $\mathbb{F}$ is given by assigning to each vertex i a vector space $\mathscr{A}_i$ over $\mathbb{F}$, to each arrow $\alpha\colon i \to j$ a linear mapping $\mathscr{A}_\alpha\colon \mathscr{A}_i \to \mathscr{A}_j$, and to each nonoriented edge $\lambda\colon i — j$ $(i \leqslant j)$ a sesquilinear form $\mathscr{A}_\lambda\colon \mathscr{A}_i \times \mathscr{A}_j \to \mathbb{F}$.

For example, each representation of the pograph (39) is a system

$\mathcal{A}$:



of vector spaces $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ over $\mathbb{F}$, linear mappings $\mathscr{A}_\alpha, \mathscr{A}_\beta, \mathscr{A}_\gamma$, and forms $\mathscr{A}_\lambda\colon \mathscr{A}_1 \times \mathscr{A}_2 \to \mathbb{F}$, $\mathscr{A}_\mu\colon \mathscr{A}_2 \times \mathscr{A}_2 \to \mathbb{F}$, $\mathscr{A}_\nu\colon \mathscr{A}_2 \times \mathscr{A}_3 \to \mathbb{F}$.

A *morphism* $f = (f_1, \dots, f_n)\colon \mathscr{A} \to \mathscr{A}'$ of representations $\mathscr{A}$ and $\mathscr{A}'$ of P is a set of linear mappings $f_i\colon \mathscr{A}_i \to \mathscr{A}'_i$ that transform $\mathscr{A}$ to $\mathscr{A}'$; this means that

$$f_j \mathscr{A}_\alpha = \mathscr{A}'_\alpha f_i, \qquad \mathscr{A}_\lambda(x, y) = \mathscr{A}'_\lambda(f_i x, f_j y)$$

for all arrows $\alpha\colon i \longrightarrow j$ and nonoriented edges $\lambda\colon i — j (i \leqslant j)$. The composition of two morphisms is a morphism. A morphism $f\colon \mathscr{A} \to \mathscr{A}'$ is called an *isomorphism* and is denoted by $f\colon \mathscr{A} \xrightarrow{\sim} \mathscr{A}'$ if all f_i are bijections. We write $\mathscr{A} \simeq \mathscr{A}'$ if $\mathscr{A}$ and $\mathscr{A}'$ are isomorphic. If $\mathscr{A} = \mathscr{A}'$, then a morphism or isomorphism $f\colon \mathscr{A} \to \mathscr{A}'$ is called an *endomorphism* or *automorphism*, respectively.

The *direct sum* $\mathscr{A} \oplus \mathscr{A}'$ of representations $\mathscr{A}$ and $\mathscr{A}'$ of P is the representation consisting of the vector spaces $\mathscr{A}_i \oplus \mathscr{A}'_i$, the linear mappings $\mathscr{A}_\alpha \oplus \mathscr{A}'_\alpha$, and the forms $\mathscr{A}_\lambda \oplus \mathscr{A}'_\lambda$ for all vertices

i, arrows α, and nonoriented edges λ. A representation $\mathscr{A}$ is *indecomposable* if $\mathscr{A} \simeq \mathscr{B} \oplus \mathscr{C}$ implies $\mathscr{B} = 0$ or $\mathscr{C} = 0$, where 0 is the representation in which all vector spaces are 0.

The **dual space* to a vector space V is the vector space V^* of all mappings $\varphi : V \to \mathbb{F}$ that are *semilinear*, this means that

$$\varphi(va + v'a') = \bar{a}(\varphi v) + \bar{a'}(\varphi v'), \quad v, v' \in V, a, a' \in \mathbb{F}.$$

We identify V with V^{**} by identifying $v \in V$ with $\varphi \mapsto \overline{\varphi v}$. For every linear mapping $A : U \to V$, we define the **adjoint mapping* $A^*\colon V^* \to U^*$ setting

$$A^*\varphi := \varphi A \text{ for all } \varphi \in V^*.$$

For every pograph P, we construct the *quiver $\underline{P}$ with an involution on the set of vertices and an involution on the set of arrows* as follows: we replace

- each vertex i of P by two vertices i and i^*,
- each oriented edge $\alpha\colon i \to j$ by two arrows $\alpha\colon i \to j$ and $\alpha^*\colon j^* \to i^*$,
- each nonoriented edge $\lambda\colon k — l$ $(k \leqslant l)$ by two arrows $\alpha\colon l \to k^*$ and $\alpha^*\colon k \to l^*$,

and set $u^{**} := u$ and $\alpha^{**} := \alpha$ for all vertices and arrows of the quiver $\underline{P}$. For example,

$$P:\quad \begin{tikzcd} 2 \arrow[d, "\alpha"'] \arrow[d, dash, bend left, "\lambda"] \\ 1 \arrow[loop right, dash, "\mu"] \end{tikzcd} \qquad\qquad \underline{P}:\quad \begin{tikzcd} 2 \arrow[d, "\alpha"'] \arrow[dr, near start, "\lambda"] & 2^* \\ 1 \arrow[ur, near end, "\lambda^*"] \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mu^*"'] & 1^* \arrow[u, "\alpha^*"'] \end{tikzcd} \tag{40}$$

Respectively, for each representation $\mathscr{M}$ of P over $\mathbb{F}$, we define the representation $\underline{\mathscr{M}}$ of $\underline{P}$ by replacing

- each vector space V in $\mathscr{M}$ by the pair of spaces V and V^*,
- each linear mapping $A\colon U \to V$ by the pair of mutually *adjoint mappings $A\colon U \to V$ and $A^*\colon V^* \to U^*$,
- each sesquilinear form $B\colon V \times U \to \mathbb{F}$ by the pair of mutually *adjoint mappings

$$B\colon u \in U \mapsto B(?, u) \in V^*, \quad B^*\colon v \in V \mapsto \overline{B(v, ?)} \in U^*.$$

For example, the following are representations of (40):

$$\mathcal{A}:\quad \begin{tikzcd} U \arrow[d, "A"'] \arrow[d, dash, bend left, "B"] \\ V \arrow[loop right, dash, "C"] \end{tikzcd} \qquad\qquad \underline{\mathcal{A}}:\quad \begin{tikzcd} U \arrow[d, "A"'] \arrow[dr, near start, "B"] & U^\star \\ V \arrow[ur, near end, "B^\star"] \arrow[r, shift left, "C"] \arrow[r, shift right, "C^\star"'] & V^\star \arrow[u, "A^\star"'] \end{tikzcd} \tag{41}$$

For each representation $\mathscr{M}$ of $\underline{P}$ we define an *adjoint representation* $\mathscr{M}^\circ$ of $\underline{P}$ consisting of the vector spaces $\mathscr{M}_v^\circ := \mathscr{M}_{v^*}^*$ and the linear mappings $\mathscr{M}_\alpha^\circ := \mathscr{M}_{\alpha^*}^*$ for all vertices v and arrows α of $\underline{P}$. For example, the following are representations of the quiver $\underline{P}$ defined in (40):

$$\mathcal{M}:\quad \begin{tikzcd} U_1 \arrow[d, "A_1"'] \arrow[dr, near start, "B_1"] & U_2 \\ V_1 \arrow[ur, near end, "B_2"] \arrow[r, shift left, "C_1"] \arrow[r, shift right, "C_2"'] & V_2 \arrow[u, "A_2"'] \end{tikzcd} \qquad\qquad \mathcal{M}^\circ:\quad \begin{tikzcd} U_2^\star \arrow[d, "A_2^\star"'] \arrow[dr, near start, "B_2^\star"] & U_1^\star \\ V_2^\star \arrow[ur, near end, "B_1^\star"] \arrow[r, shift left, "C_2^\star"] \arrow[r, shift right, "C_1^\star"'] & V_1^\star \arrow[u, "A_1^\star"'] \end{tikzcd}$$

The second representation in (41) is *selfadjoint*: $\underline{\mathscr{A}}^\circ = \underline{\mathscr{A}}$.

In a similar way, for each morphism $f: \mathscr{M} \to \mathscr{N}$ of representations of $\underline{P}$ we construct the *adjoint morphism*

$$f^\circ: \mathscr{N}^\circ \to \mathscr{M}^\circ, \quad \text{in which } f_i^\circ := f_{i^*}^* \tag{42}$$

for all vertices i of $\underline{P}$. An isomorphism $f: \mathscr{M} \xrightarrow{\sim} \mathscr{N}$ of selfadjoint representations $\mathscr{M}$ and $\mathscr{N}$ is called a *congruence* if $f^\circ = f^{-1}$.

There is a natural one-to-one correspondence between isomorphisms of representations of a pograph P and congruences of the corresponding selfadjoint representations of $\underline{P}$: each isomorphism $f: \mathscr{A} \xrightarrow{\sim} \mathscr{B}$ of representations of P corresponds to the congruence $\underline{f}: \underline{\mathscr{A}} \xrightarrow{\sim} \underline{\mathscr{B}}$, in which

$$\underline{f}_i := f_i, \quad \underline{f}_{i^*} := f_i^{-*} \quad \text{for each vertex } i \text{ of } P.$$

Thus, *the problem of classifying representations of a pograph P up to isomorphism reduces to the problem of classifying selfadjoint representations of the quiver $\underline{P}$ up to congruence.*

Let us show how to solve the latter problem if we know a maximal set $\mathrm{ind}(\underline{P})$ of nonisomorphic indecomposable representations of the quiver $\underline{P}$ (this means that every indecomposable representation of $\underline{P}$ is isomorphic to exactly one representation from $\mathrm{ind}(\underline{P})$). We first replace each representation in $\mathrm{ind}(\underline{P})$ that is *isomorphic* to a selfadjoint representation by one that is *actually* selfadjoint—i.e., has the form $\underline{\mathscr{A}}$, and denote the set of these $\underline{\mathscr{A}}$ by $\mathrm{ind}_0(\underline{P})$. Then in the set $\mathrm{ind}(\underline{P}) \smallsetminus \mathrm{ind}_0(\underline{P})$ we delete one representation from each pair $\{\mathscr{M}, \mathscr{L}\}$, $\mathscr{M}^\circ \simeq \mathscr{L} \neq \mathscr{M}$, and denote the set of remaining representations by $\mathrm{ind}_1(\underline{P})$.

We obtain a new set $\mathrm{ind}(\underline{P})$ that we partition into 3 subsets:

$$\mathrm{ind}(\underline{P}) = \begin{array}{|c|}\hline \mathscr{M} \mid \mathscr{M}^\circ (\text{if } \mathscr{M}^\circ \not\simeq \mathscr{M}) \\ \hline \underline{\mathscr{A}} \\ \hline \end{array}, \quad \begin{array}{l} \mathscr{M} \in \mathrm{ind}_1(\underline{P}), \\ \underline{\mathscr{A}} \in \mathrm{ind}_0(\underline{P}). \end{array} \tag{43}$$

For each representation $\mathscr{M}$ of $\underline{P}$, we define a representation $\mathscr{M}^+$ of P by setting $\mathscr{M}_i^+ := \mathscr{M}_i \oplus \mathscr{M}_{i^*}^*$ for all vertices i in P and

$$\mathscr{M}_\alpha^+ := \begin{bmatrix} \mathscr{M}_\alpha & 0 \\ 0 & \mathscr{M}_{\alpha^*}^* \end{bmatrix}, \quad \mathscr{M}_\beta^+ := \begin{bmatrix} 0 & \mathscr{M}_{\beta^*}^* \\ \mathscr{M}_\beta & 0 \end{bmatrix} \tag{44}$$

for all edges $\alpha: i \longrightarrow j$ and $\beta: i — j$ $(i \leqslant j)$. Note that the corresponding selfadjoint representation $\underline{\mathscr{M}}^+$ of $\underline{P}$ is isomorphic to $\mathscr{M} \oplus \mathscr{M}^\circ$.

For every representation $\mathscr{A}$ of P and for every selfadjoint automorphism $f = f^\circ: \underline{\mathscr{A}} \xrightarrow{\sim} \underline{\mathscr{A}}$, we denote by $\mathscr{A}^f$ the representation of P that is obtained from $\mathscr{A}$ by replacing each form $\mathscr{A}_\beta$ $(\beta: i — j, i \leqslant j)$ by $\mathscr{A}_\beta^f := \mathscr{A}_\beta f_j$.

Let $\mathrm{ind}(\underline{P})$ be partitioned as in (43), and let $\underline{\mathscr{A}} \in \mathrm{ind}_0(\underline{P})$. By [31, Lemma 1], the set R of noninvertible elements of the endomorphism ring $\mathrm{End}(\underline{\mathscr{A}})$ is an ideal, which coincides with the radical of $\mathrm{End}(\underline{\mathscr{A}})$. Clearly, each nonzero element of the quotient ring $\mathbb{T}(\mathscr{A}) := \mathrm{End}(\underline{\mathscr{A}})/R$ is invertible; that is, $\mathbb{T}(\mathscr{A})$ is a field or skew field. We define the involution

$$(f + R)^\circ := f^\circ + R \tag{45}$$

on $\mathbb{T}(\mathscr{A})$.

For each nonzero $a = a^\circ \in \mathbb{T}(\mathscr{A})$, we fix a selfadjoint automorphism

$$f_a = f_a^\circ \in a, \quad \text{and define } \mathscr{A}^a := \mathscr{A}^{f_a} \tag{46}$$

(we can take $f_a := (f + f^\circ)/2$ for any $f \in a$).

For each Hermitian form

$$\varphi(x) = x_1^\circ a_1 x_1 + \cdots + x_r^\circ a_r x_r, \quad 0 \neq a_i = a_i^\circ \in \mathbb{T}(\mathscr{A}),$$

we write

$$\mathscr{A}^{\varphi(x)} := \mathscr{A}^{a_1} \oplus \cdots \oplus \mathscr{A}^{a_r}.$$

The following two theorems are special cases of [31, Theorem 1] (or [34, Theorem 3.1]) and [34, Theorem 3.2].

Theorem 3.1. *Over a field or skew field $\mathbb{F}$ of characteristic different from 2 with involution $a \mapsto \bar{a}$ (possibly, the identity), every representation of a pograph P is isomorphic to a direct sum*

$$\mathscr{M}_1^+ \oplus \cdots \oplus \mathscr{M}_p^+ \oplus \mathscr{A}_1^{\varphi_1(x)} \oplus \cdots \oplus \mathscr{A}_q^{\varphi_q(x)},$$

in which

$$\mathscr{M}_i \in \operatorname{ind}_1(\underline{P}), \qquad \underline{\mathscr{A}}_j \in \operatorname{ind}_0(\underline{P}),$$

and $\mathscr{A}_j \neq \mathscr{A}_{j'}$ if $j \neq j'$. This sum is determined by the original representation uniquely up to permutation of summands and replacement of $\mathscr{A}_j^{\varphi_j(x)}$ by $\mathscr{A}_j^{\psi_j(x)}$, in which $\varphi_j(x)$ and $\psi_j(x)$ are equivalent Hermitian forms over $\mathbb{T}(\mathscr{A}_j)$ with involution (45).

Theorem 3.1 implies the following generalization of the law of inertia for quadratic forms; its proof is the same as the proof of [34, Theorem 3.2].

Theorem 3.2. *Let $\mathbb{F}$ be either*

(i) *an algebraically closed field of characteristic different from 2 with the identity involution, or*
(ii) *an algebraically closed field with nonidentity involution, or*
(iii) *a real closed field, or the skew field of quaternions over a real closed field.*

Then every representation of a pograph P over $\mathbb{F}$ is isomorphic to a direct sum, uniquely determined up to permutation of summands, of representations of the types:

$$\mathscr{M}^+, \begin{cases} \mathscr{A} & \text{if } \mathscr{A}^- \simeq \mathscr{A}, \\ \mathscr{A}, \mathscr{A}^- & \text{if } \mathscr{A}^- \not\simeq \mathscr{A}, \end{cases} \tag{47}$$

in which $\mathscr{M} \in \operatorname{ind}_1(\underline{P})$ and $\underline{\mathscr{A}} \in \operatorname{ind}_0(\underline{P})$. In the respective cases (i)–(iii), *the representations* (47) *have the form*

(i) $\mathscr{M}^+$, $\mathscr{A}$,
(ii) $\mathscr{M}^+$, $\mathscr{A}$, $\mathscr{A}^-$,
(iii) $\mathscr{M}^+, \begin{cases} \mathscr{A}, & \text{if } \mathbb{T}(\mathscr{A}) \text{ is an algebraically closed field with the identity involution or a skew field of quaternions with involution different from quaternionic conjugation,} \\ \mathscr{A}, \mathscr{A}^-, & \text{otherwise.} \end{cases}$

Remark 3.1. Theorem 3.2 is a special case of Theorem 2 in [31], which was formulated incorrectly in the case of quaternions. To correct it, remove "or the algebra of quaternions …" in (a) and (b) and add "or the algebra of quaternions over a maximal ordered field" in c). The paper [32] is based on the incorrect Theorem 2 in [31] and so the signs $\pm$ of the sesquilinear forms in the indecomposable direct summands in [32, Theorems 1–4] are incorrect. Correct canonical forms are given for bilinear/sesquilinear forms in Theorem 2.1, for pairs of symmetric/skew-symmetric matrices in [22,23], for selfadjoint operators in [15], and for isometries in [34].

4. Proof of Theorem 2.2

Each sesquilinear form defines a representation of the pograph

$$P: \qquad \begin{tikzcd} 1 \arrow[loop right, dash, "\alpha"] \end{tikzcd} \tag{48}$$

Its quiver is

$$\underline{P}: \quad \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\alpha^*"'] & 1^* \end{tikzcd}$$

We prove Theorem 2.2 using Theorem 3.1; to do this, we first identify in Lemma 4.1 the sets $\mathrm{ind}_1(\underline{P})$ and $\mathrm{ind}_0(\underline{P})$, and the orbit of $\mathscr{A}$ for each $\underline{\mathscr{A}} \in \mathrm{ind}_0(\underline{P})$.

Every representation of P or $\underline{P}$ over $\mathbb{F}$ is isomorphic to a representation in which all vector spaces are $\mathbb{F} \oplus \cdots \oplus \mathbb{F}$. From now on, we consider only such representations of P and $\underline{P}$; they can be given by a square matrix A:

$$\mathscr{A}: \begin{tikzcd} \bullet \arrow[loop right, dash, "A"] \end{tikzcd} \quad (\text{we write } \mathscr{A} = A) \tag{49}$$

and, respectively, by rectangular matrices A and B of the same size:

$$\mathscr{M}: \begin{tikzcd} \bullet \arrow[r, shift left, "A"] \arrow[r, shift right, "B"'] & \bullet \end{tikzcd} \quad (\text{we write } \mathscr{M} = (A, B)),$$

we omit the spaces $\mathbb{F} \oplus \cdots \oplus \mathbb{F}$ since they are completely determined by the sizes of the matrices.

The adjoint representation

$$\mathscr{M}^\circ: \begin{tikzcd} \bullet \arrow[r, shift left, "B^*"] \arrow[r, shift right, "A^*"'] & \bullet \end{tikzcd}$$

is given by the matrix pair

$$(A, B)^\circ = (B^*, A^*). \tag{50}$$

A morphism of representations

$$\begin{tikzcd} \mathcal{M}: \arrow[d, "f"'] & \bullet \arrow[r, shift left, "A"] \arrow[r, shift right, "B"'] \arrow[d, "F_1"'] & \bullet \arrow[d, "F_2"] \\ \mathcal{M}': & \bullet \arrow[r, shift left, "A'"] \arrow[r, shift right, "B'"'] & \bullet \end{tikzcd}$$

is given by the matrix pair $f = [F_1, F_2]: \mathscr{M} \to \mathscr{M}'$ (for morphisms we use square brackets) satisfying

$$F_2A = A'F_1, \qquad F_2B = B'F_1. \tag{51}$$

Denote by 0_{m0} and 0_{0n} the $m \times 0$ and $0 \times n$ matrices representing the linear mappings $0 \to \mathbb{F}^m$ and $\mathbb{F}^n \to 0$. Thus, $0_{m0} \oplus 0_{0n}$ is the $m \times n$ zero matrix.

Lemma 4.1. *Let $\mathbb{F}$ be a field or skew field of characteristic different from 2. Let $\mathscr{O}_{\mathbb{F}}$ be a maximal set of nonsingular indecomposable canonical matrices over $\mathbb{F}$ for similarity. Let P be the pograph* (48). *Then*:

(a) *The set* $\operatorname{ind}(\underline{P})$ *can be taken to be the set of all representations*

$$(\Phi, I_n),\ (J_n(0), I_n),\ (I_n, J_n(0)),\ (M_n, N_n),\ (N_n^{\mathrm{T}}, M_n^{\mathrm{T}}) \tag{52}$$

in which $\Phi \in \mathscr{O}_{\mathbb{F}}$ is n-by-n and

$$M_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \quad N_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}$$

are $(n-1)$-by-n for each natural number n.

(b) *The set* $\operatorname{ind}_1(\underline{P})$ *can be taken to be the set of all representations*

$$(\Phi, I_n), \quad (J_n(0), I_n), \quad (M_n, N_n)$$

in which $\Phi \in \mathscr{O}_{\mathbb{F}}$ is an $n \times n$ matrix such that $\sqrt[*]{\Phi}$ does not exist, and

$$\begin{aligned} &\Phi \text{ is determined up to replacement by} \\ &\quad \text{the unique } \Psi \in \mathscr{O}_{\mathbb{F}} \text{ that is similar to } \Phi^{-*}. \end{aligned} \tag{53}$$

The corresponding representations of P are

$$(\Phi, I_n)^+ = [\Phi \backslash I_n], \tag{54}$$

$$(M_n, N_n)^+ \simeq J_{2n-1}(0), \quad (J_n(0), I_n)^+ \simeq J_{2n}(0). \tag{55}$$

(c) *The set* $\operatorname{ind}_0(\underline{P})$ *can be taken to be the set of all representations*

$$\underline{\mathscr{A}}_{\Phi} := (\sqrt[*]{\Phi}, (\sqrt[*]{\Phi})^*) \tag{56}$$

in which $\Phi \in \mathscr{O}_{\mathbb{F}}$ is such that $\sqrt[*]{\Phi}$ exists. The corresponding representations of P are

$$\mathscr{A}_{\Phi} = \sqrt[*]{\Phi}, \quad \mathscr{A}_{\Phi}^{-} = -\sqrt[*]{\Phi}, \quad \mathscr{A}_{\Phi}^{f} = \sqrt[*]{\Phi}F, \tag{57}$$

in which $f = [F, F^*]\colon \underline{\mathscr{A}}_{\Phi} \xrightarrow{\sim} \underline{\mathscr{A}}_{\Phi}$ is a selfadjoint automorphism.

(d) *Let $\mathbb{F}$ be a field and let $\underline{\mathscr{A}}_{\Phi} := (\sqrt[*]{\Phi}, (\sqrt[*]{\Phi})^*) \in \operatorname{ind}_0(\underline{P})$, in which Φ is a nonsingular matrix over $\mathbb{F}$ that is indecomposable for similarity* (*thus, its characteristic polynomial is a power of some irreducible polynomial p_{Φ}*).

(i) *The ring* $\operatorname{End}(\underline{\mathscr{A}}_{\Phi})$ *of endomorphisms of $\underline{\mathscr{A}}_{\Phi}$ consists of the matrix pairs*

$$[f(\Phi), f(\Phi^{-*})], \quad f(x) \in \mathbb{F}[x], \tag{58}$$

and the involution on $\operatorname{End}(\underline{\mathscr{A}}_{\Phi})$ *is*

$$[f(\Phi), f(\Phi^{-*})]^{\circ} = [\bar{f}(\Phi^{-1}), \bar{f}(\Phi^{*})].$$

(ii) $\mathbb{T}(\mathscr{A}_\Phi)$ *can be identified with the field*

$$\mathbb{F}(\kappa) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x], \quad \kappa := x + p_\Phi(x)\mathbb{F}[x], \tag{59}$$

with involution

$$f(\kappa)^\circ = \bar{f}(\kappa^{-1}). \tag{60}$$

Each element of $\mathbb{T}(\mathscr{A}_\Phi)$ *on which this involution acts identically is uniquely represented in the form* $q(\kappa)$ *for some nonzero function* (17). *The representations*

$$\mathscr{A}_\Phi^{q(\kappa)} : \bullet \circlearrowright \sqrt[*]{\Phi}q(\Phi) \tag{61}$$

(*see* (46)) *constitute the orbit of* $\mathscr{A}_\Phi$.

Proof. (a) Each representation of $\underline{P}$ can be decomposed into a direct sum of representations of the form (52) using the reduction from [8, Section 1.8]. The uniqueness of this decomposition follows from the Krull-Schmidt theorem [1, Chapter I, Theorcm 3.6]. A detailed proof is given in [29], in which an analogous canonical form was constructed for a pair consisting of a linear mapping and a pseudolinear mapping over a field or skew field with an automorphism φ and a φ-differentiation δ ($\varphi = 1$ and $\delta = 0$ in our case). A canonical form of matrix pencils over the skew field of real quaternions is given in [22].

(b) and (c) Let $\Phi, \Psi \in \mathscr{O}_\mathbb{F}$ be n-by-n. In view of (50), $(\Phi, I_n)^\circ = (I_n, \Phi^*) \simeq (\Phi^{-*}, I_n)$ and so

$$(\Psi, I_n) \simeq (\Phi, I_n)^\circ \quad \Longleftrightarrow \quad \Psi \text{ is similar to } \Phi^{-*}. \tag{62}$$

Suppose (Φ, I_n) is isomorphic to a selfadjoint representation:

$$[F_1, F_2]\colon (\Phi, I_n) \xrightarrow{\sim} (B, B^*). \tag{63}$$

Define a selfadjoint representation (A, A^*) by the congruence

$$[F_1^{-1}, F_1^*]\colon (B, B^*) \xrightarrow{\sim} (A, A^*). \tag{64}$$

The composition of (63) and (64) is the isomorphism

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\Phi"] \arrow[r, bend right, "I_n"'] \arrow[d, "I_n"'] & \bullet \arrow[d, "F:=F_1^*F_2"] \\ \bullet \arrow[r, bend left, "A"] \arrow[r, bend right, "A^*"'] & \bullet \end{tikzcd}$$

By (51), $A = F\Phi$ and $A^* = F$. Thus $A = A^*\Phi$. Taking $A = \sqrt[*]{\Phi}$, we obtain

$$[I_n, (\sqrt[*]{\Phi})^*]\colon (\Phi, I_n) \xrightarrow{\sim} (\sqrt[*]{\Phi}, (\sqrt[*]{\Phi})^*).$$

This means that if $(\Phi, I_n) \in \operatorname{ind}(\underline{P})$ is isomorphic to a selfadjoint representation, then (Φ, I_n) is isomorphic to (56). Hence, the representations (56) comprise $\operatorname{ind}_0(\underline{P})$. Due to (62), we can identify isomorphic representations in the set of remaining representations $(\Phi, I_n) \in \operatorname{ind}(\underline{P})$ by imposing the condition (53); we then obtain $\operatorname{ind}_1(\underline{P})$ from Lemma 4.1(b).

To verify (55), we prove that $J_m(0)$ is permutationally similar to

$$\begin{cases} (M_n, N_n)^+ = [M_n \backslash N_n^{\mathrm{T}}] & \text{if } m = 2n - 1, \\ (J_n(0), I_n)^+ = [J_n(0) \backslash I_n] & \text{if } m = 2n \end{cases}$$

(see (44)). The units of $J_m(0)$ are at the positions $(1,2), (2,3), \dots, (m-1,m)$; so it suffices to prove that there is a permutation f on $\{1,2,\dots,m\}$ such that

$$(f(1), f(2)), (f(2), f(3)), \dots, (f(m-1), f(m))$$

are the positions of the unit entries in $[M_n \backslash N_n^{\mathrm{T}}]$ or $[J_n(0)\backslash I_n]$. This becomes clear if we arrange the positions of the unit entries in the $(2n-1)\times(2n-1)$ matrix

$$[M_n\backslash N_n^{\mathrm{T}}] = \left[\begin{array}{ccc|ccc} & & & 0 & & 0 \\ & 0 & & 1 & \ddots & \\ & & & & \ddots & 0 \\ & & & 0 & & 1 \\ \hline 1 & 0 & 0 & & & \\ \ddots & \ddots & & & 0 & \\ 0 & 1 & 0 & & & \end{array}\right]$$

as follows:

$$(n, 2n-1), (2n-1, n-1), (n-1, 2n-2), (2n-2, n-2), \dots, (2, n+1), (n+1, 1),$$

and the positions of the unit entries in the $2n \times 2n$ matrix $[J_n(0)\backslash I_n]$ as follows:

$$(1, n+1), (n+1, 2), (2, n+2), (n+2, 3), \dots, (2n-1, n), (n, 2n).$$

(d) Let $\mathbb{F}$ be a field. If Φ is a square matrix over $\mathbb{F}$ that is indecomposable for similarity, then each matrix over $\mathbb{F}$ that commutes with Φ is a polynomial in Φ. To verify this, we may assume that Φ is an $n \times n$ Frobenius block (18). Then the vectors

$$e := (1, 0, \dots, 0)^{\mathrm{T}},\ \Phi e, \dots, \Phi^{n-1}e \tag{65}$$

form a basis of $\mathbb{F}^n$. Let $S \in \mathbb{F}^{n\times n}$ commute with Φ, let

$$Se = a_0 e + a_1 \Phi e + \cdots + a_{n-1}\Phi^{n-1}e, \quad a_0, \dots, a_{n-1} \in \mathbb{F},$$

and let $f(x) := a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \in \mathbb{F}[x]$. Then $Se = f(\Phi)e$ and

$$S\Phi e = \Phi S e = \Phi f(\Phi)e = f(\Phi)\Phi e, \dots, S\Phi^{n-1}e = f(\Phi)\Phi^{n-1}e.$$

Since (65) is a basis, $S = f(\Phi)$.

(i) Let $\underline{\mathscr{A}}_\Phi := (A, A^*) \in \mathrm{ind}_0(\underline{P})$, in which Φ is a nonsingular matrix over $\mathbb{F}$ that is indecomposable for similarity and $A := \sqrt[*]{\Phi}$. Let $g = [G_1, G_2] \in \mathrm{End}(\underline{\mathscr{A}}_\Phi)$. Then (51) ensures that

$$G_2 A = A G_1, \qquad G_2 A^* = A^* G_1, \tag{66}$$

and so

$$\Phi G_1 = A^{-*}AG_1 = A^{-*}G_2A = G_1A^{-*}A = G_1\Phi. \tag{67}$$

Since G_1 commutes with Φ, we have $G_1 = f(\Phi)$ for some $f(x) \in \mathbb{F}[x]$, and

$$G_2 = AG_1A^{-1} = f(A\Phi A^{-1}) = f(AA^{-*}AA^{-1}) = f(\Phi^{-*}). \tag{68}$$

Consequently, the ring $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$ of endomorphisms of $\underline{\mathscr{A}}_\Phi$ consists of the matrix pairs (58), and the involution (42) has the form

$$[f(\Phi), f(\Phi^{-*})]^\circ = [f(\Phi^{-*})^*, f(\Phi)^*] = [\bar{f}(\Phi^{-1}), \bar{f}(\Phi^*)].$$

(ii) The first equality in (68) ensures that each endomorphism $[f(\Phi), f(\Phi^{-*})]$ is completely determined by $f(\Phi)$. Thus, the ring $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$ can be identified with

$$\mathbb{F}[\Phi] = \{f(\Phi) | f \in \mathbb{F}[x]\} \quad \text{with involution } f(\Phi) \mapsto \bar{f}(\Phi^{-1}),$$

which is isomorphic to $\mathbb{F}[x]/p_\Phi(x)^s\mathbb{F}[x]$, in which $p_\Phi(x)^s$ is the characteristic polynomial (19) of Φ. Thus, the radical of the ring $\mathbb{F}[\Phi]$ is generated by $p_\Phi(\Phi)$ and $\mathbb{T}(\mathscr{A}_\Phi)$ can be identified with the field (59) with involution $f(\kappa)^\circ = \bar{f}(\kappa^{-1})$.

According to Lemma 2.2, each element of the field (59) on which the involution acts identically is uniquely representable in the form $q(\kappa)$ for some nonzero function $q(x)$ of the form (17). The pair $[q(\Phi), Aq(\Phi)A^{-1}]$ is an endomorphism of $\underline{\mathscr{A}}_\Phi$ due to (66). This endomorphism is selfadjoint since the function (17) satisfies $q(x^{-1}) = \bar{q}(x)$, and so

$$Aq(\Phi)A^{-1} = q(\Phi^{-*}) = \bar{q}(\Phi^*) = q(\Phi)^*.$$

Since distinct functions $q(x)$ give distinct $q(\kappa)$ and

$$q(\Phi) \in q(\kappa) = q(\Phi) + p_\Phi(\Phi)\mathbb{F}[\Phi],$$

in (46) we may take $f_{q(\kappa)} := [q(\Phi), q(\Phi)^*] \in \mathrm{End}(\underline{\mathscr{A}}_\Phi)$. By (57), the corresponding representations $\mathscr{A}_\Phi^{q(\kappa)} = \mathscr{A}_\Phi^{f_{q(\kappa)}}$ have the form (61) and constitute the orbit of $\mathscr{A}_\Phi$. □

Proof of Theorem 2.2. (a) Each square matrix A gives the representation (49) of the pograph (48). Theorem 3.1 ensures that each representation of (48) over a field $\mathbb{F}$ of characteristic different from 2 is isomorphic to a direct sum of representations of the form $\mathscr{M}^+$ and $\mathscr{A}^a$, where $\mathscr{M} \in \mathrm{ind}_1(\underline{P})$, $\underline{\mathscr{A}} \in \mathrm{ind}_0(\underline{P})$, and $0 \neq a = a^\circ \in \mathbb{T}(\mathscr{A})$. This direct sum is determined uniquely up to permutation of summands and replacement of the whole group of summands $\mathscr{A}^{a_1} \oplus \cdots \oplus \mathscr{A}^{a_s}$ with the same $\mathscr{A}$ by $\mathscr{A}^{b_1} \oplus \cdots \oplus \mathscr{A}^{b_s}$, provided that the Hermitian forms $a_1x_1^\circ x_1 + \cdots + a_sx_s^\circ x_s$ and $b_1x_1^\circ x_1 + \cdots + b_sx_s^\circ x_s$ are equivalent over $\mathbb{T}(\mathscr{A})$, which is a field by (59).

This proves (a) since we can use the sets $\mathrm{ind}_1(\underline{P})$ and $\mathrm{ind}_0(\underline{P})$ from Lemma 4.1; the field $\mathbb{T}(\mathscr{A})$ is isomorphic to (59), and the representations $\mathscr{M}^+$ and $\mathscr{A}^a$ have the form (54), (55), and (61).

(b) Let $\mathbb{F}$ be a real closed field and let $\Phi \in \mathcal{O}_\mathbb{F}$ be such that $\sqrt[*]{\Phi}$ exists. Let us identify $\mathbb{T}(\mathscr{A}_\Phi)$ with the field (59). Then $\mathbb{T}(\mathscr{A}_\Phi)$ is either $\mathbb{F}$ or its algebraic closure. In the latter case, the involution (60) on $\mathbb{T}(\mathscr{A}_\Phi)$ is not the identity; otherwise $\kappa = \kappa^{-1}$, $\kappa^2 - 1 = 0$, i.e., $p_\Phi(x) = x^2 - 1$, which contradicts the irreducibility of $p_\Phi(x)$.

Applying Theorem 3.2, we complete the proof of (b). □

5. Proof of Theorem 2.1

5.1. *Proof of Theorem 2.1(a)*

Let $\mathbb{F}$ be an algebraically closed field of characteristic different from 2 with the identity involution. Take $\mathcal{O}_\mathbb{F}$ to be all nonsingular Jordan blocks.

The summands (i)–(iii) of Theorem 2.1(a) can be obtained from the summands (i)–(iii) of Theorem 2.2 because for nonzero $\lambda, \mu \in \mathbb{F}$

$$J_n(\lambda) \text{ is similar to } J_n(\mu)^{-\mathrm{T}} \iff \lambda = \mu^{-1},$$
$$\sqrt[T]{J_n(\lambda)} \text{ exists} \iff \lambda = (-1)^{n+1}.$$

The first of these two equivalences is obvious.

Let us prove the second. By (21) and (22), if $\sqrt[T]{J_n(\lambda)}$ exists then $\lambda = (-1)^{n+1}$. Conversely, let $\lambda = (-1)^{n+1}$. It suffices to prove the following useful statement:

$$\text{the cosquares of } \Gamma_n \text{ and } \Gamma_n' \text{ are similar to } J_n((-1)^{n+1}), \tag{69}$$

which implies that $\sqrt[T]{J_n((-1)^{n+1})}$ exists by (20) with $\sqrt[T]{\Phi} = \Gamma_n$ and $\Psi = J_n((-1)^{n+1})$.

To verify the first similarity in (69), compute

$$\Gamma_n^{-1} = (-1)^{n+1}\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \iddots \\ -1 & -1 & -1 & -1 & \\ 1 & 1 & 1 & & \\ -1 & -1 & & & \\ 1 & & & & 0 \end{bmatrix}$$

and

$$\Gamma_n^{-\mathrm{T}}\Gamma_n = (-1)^{n+1}\begin{bmatrix} 1 & 2 & & * \\ & 1 & \ddots & \\ & & \ddots & 2 \\ 0 & & & 1 \end{bmatrix}. \tag{70}$$

To verify the second similarity in (69), there are two cases to consider: If n is even then

$$(\Gamma_n')^{-1} = \left[\begin{array}{cccc|cccc} \vdots & \vdots & & \vdots & \vdots & \iddots & -1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & \iddots & \iddots & \\ -1 & -1 & \cdots & -1 & -1 & 1 & & \\ 1 & 1 & \cdots & 1 & 1 & & & \\ \hline -1 & -1 & \cdots & -1 & & & & \\ \vdots & \vdots & \iddots & & & & 0 & \\ -1 & -1 & & & & & & \\ -1 & & & & & & & \end{array}\right]$$

and

$$(\Gamma_n')^{-\mathrm{T}}\Gamma_n' = \begin{bmatrix} -1 & \pm 2 & & * \\ & -1 & \ddots & \\ & & \ddots & \pm 2 \\ 0 & & & -1 \end{bmatrix}.$$

If n is odd then

$$(\Gamma_n')^{-1} = \begin{bmatrix} & & & \pm 1 & \ldots & -1 & 1 \\ & 0 & & \vdots & \iddots & \iddots & \\ & & & -1 & 1 & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & \iddots & & & & & \\ 1 & & & & & & 0 \end{bmatrix} \tag{71}$$

and

$$(\Gamma_n')^{-\mathrm{T}}\Gamma_n' = \begin{bmatrix} 1 & \pm 1 & & * \\ & 1 & \ddots & \\ & & \ddots & \pm 1 \\ 0 & & & 1 \end{bmatrix}. \tag{72}$$

We have proved that all direct sums of matrices of the form (i)–(iii) are canonical matrices for congruence. Let us prove the last statement of Theorem 2.1(a). If two nonsingular matrices over $\mathbb{F}$ are congruent then their cosquares are similar. The converse statement is correct too because the cosquares of distinct canonical matrices for congruence have distinct Jordan canonical forms. Due to (69), Γ_n and Γ_n' are congruent to $\sqrt[T]{J_n((-1)^{n+1})}$.

5.2. *Proof of Theorem 2.1(b)*

Let $\mathbb{F}$ be an algebraically closed field of characteristic 2.

According to [30], each square matrix over $\mathbb{F}$ is congruent to a matrix of the form

$$\bigoplus_i [J_{m_i}(\lambda_i)\backslash I_{m_i}] \oplus \bigoplus_j \sqrt[T]{J_{n_j}(1)} \oplus \bigoplus_k J_{r_k}(0), \tag{73}$$

in which $\lambda_i \neq 0$, n_j is odd, and $J_{m_i}(\lambda_i) \neq J_{n_j}(1)$ for all i and j. This direct sum is determined uniquely up to permutation of summands and replacement of any $J_{m_i}(\lambda_i)$ by $J_{m_i}(\lambda_i^{-1})$.

The matrix $\sqrt[T]{J_n(1)}$ was constructed in [30, Lemma 1] for any odd n, but it is cumbersome. Let us prove that Γ_n' is congruent to $\sqrt[T]{J_n(1)}$. Due to (71) and (72) (with $-1 = 1$), the cosquare of Γ_n' is similar to $J_n(1)$. Let Σ be the canonical matrix of the form (73) for Γ_n'. Then the cosquares of Σ and Γ_n' are similar, and so $\Sigma = \sqrt[T]{J_n(1)}$.

5.3. *Proof of Theorem 2.1(c)*

Let $\mathbb{F} = \mathbb{P} + \mathbb{P}i$ be an algebraically closed field with nonidentity involution represented in the form (5). Take $\mathcal{O}_{\mathbb{F}}$ to be all nonsingular Jordan blocks.

The summands (i)–(iii) of Theorem 2.1(c) can be obtained from the summands (i)–(iii) of Theorem 2.2 because for nonzero $\lambda, \mu \in \mathbb{F}$

$$\begin{gathered} J_n(\lambda) \text{ is similar to } J_n(\mu)^{-*} \iff \lambda = \bar{\mu}^{-1}, \\ \sqrt[*]{J_n(\lambda)} \text{ exists } \iff |\lambda| = 1 \text{ (see (7))}. \end{gathered} \tag{74}$$

Let us prove (74). By (21), if $\sqrt[*]{J_n(\lambda)}$ exists for $\lambda = a + bi (a, b \in \mathbb{P})$ then $x - \lambda = x - \bar{\lambda}^{-1}$. Thus, $\lambda = \bar{\lambda}^{-1}$ and $1 = \lambda\bar{\lambda} = a^2 + b^2 = |\lambda|^2$. Conversely, let $|\lambda| = 1$. It suffices to show that the *cosquare of $i^{n+1}\sqrt{\lambda}\Gamma_n$ is similar to $J_n(\lambda)$ since then $\sqrt[*]{J_n(\lambda)}$ exists by (20) with $\Psi = J_n(\lambda)$. To verify this similarity, observe that for each unimodular $\lambda \in \mathbb{F}$,

$$(i^{n+1}\sqrt{\lambda}\Gamma_n)^{-*}(i^{n+1}\sqrt{\lambda}\Gamma_n) = \lambda(-1)^{n+1}\Gamma_n^{-\mathrm{T}}\Gamma_n; \tag{75}$$

by (70), $\lambda(-1)^{n+1}\Gamma_n^{-\mathrm{T}}\Gamma_n$ is similar to $\lambda J_n(1)$, which is similar to $J_n(\lambda)$.

It remains to prove that each of the matrices (9) can be used instead of (iii) in Theorem 2.1(c). Let us show that if $\lambda \in \mathbb{F}$ is unimodular, then $J_n(\lambda)$ is similar to the *cosquare of each of the matrices

$$\sqrt{\lambda}\sqrt[*]{J_n(1)}, \quad i^{n+1}\sqrt{\lambda}\Gamma_n, \quad i^{n+1}\sqrt{\lambda}\Gamma_n', \quad \sqrt{\lambda}\Delta_n(1). \tag{76}$$

The first similarity is obvious. The second was proved in (75). The third can be proved analogously since $(\Gamma_n')^{-\mathrm{T}}\Gamma_n'$ is similar to $\Gamma_n^{-\mathrm{T}}\Gamma_n$ by (69). The fourth similarity holds since $J_n(1)$ is similar to the *cosquare of $\Delta_n(1)$ as a consequence of the following useful property: for each $\mu \in \mathbb{F}$ with $\bar{\mu}^{-1}\mu \neq -1$,

$$J_n(\bar{\mu}^{-1}\mu) \text{ is similar to the } *\text{cosquare of } \Delta_n(\mu). \tag{77}$$

To verify this assertion, compute

$$\Delta_n(\mu)^{-*}\Delta_n(\mu) = \begin{bmatrix} * & & i\bar{\mu}^{-2} & \bar{\mu}^{-1} \\ & \iddots & \iddots & \\ i\bar{\mu}^{-2} & \bar{\mu}^{-1} & & \\ \bar{\mu}^{-1} & & & 0 \end{bmatrix} \Delta_n(\mu)$$

$$= \begin{bmatrix} \mu\bar{\mu}^{-1} & i\bar{\mu}^{-1}u & & * \\ & \mu\bar{\mu}^{-1} & \ddots & \\ & & \ddots & i\bar{\mu}^{-1}u \\ 0 & & & \mu\bar{\mu}^{-1} \end{bmatrix}$$

with $u := \bar{\mu}^{-1}\mu + 1 \neq 0$.

Therefore, the *cosquare of each of the matrices (76) can replace $J_n(\lambda)$ in $\mathcal{O}_{\mathbb{F}}$, and so each of the matrices (76) may be used as $\sqrt[*]{\Phi}$ in (iii) of Theorem 2.2(a). Thus, instead of $\pm\sqrt[*]{J_n(\lambda)}$ in (iii) of Theorem 2.1(c) we may use any of the matrices (76) multiplied by ± 1; and hence any of the matrices (9) except for μA since each $\sqrt{\lambda}$ can be represented in the form $a + bi$ with $a, b \in \mathbb{P}$, $b \geqslant 0$, and $a + bi \neq -1$. Let A be any nonsingular $n \times n$ matrix whose *cosquare is similar to a Jordan block. Then A is *congruent to some matrix of type (iii), and hence A is *congruent to $\mu_0\Gamma_n$ for some unimodular μ_0. Thus, μA is *congruent to $\mu\mu_0\Gamma_n$, and so we may use μA instead of $\pm\sqrt[*]{J_n(\lambda)}$ in (iii).

5.4. *Proof of Theorem 2.1(d)*

Let $\mathbb{P}$ be a real closed field. Let $\mathbb{K} := \mathbb{P} + \mathbb{P}i$ be the algebraic closure of $\mathbb{P}$ represented in the form (3) with involution $a + bi \mapsto a - bi$. By [9, Theorem 3.4.5], we may take $\mathcal{O}_{\mathbb{P}}$ to be all $J_n(a)$ with $a \in \mathbb{P}$, and all $J_n(\lambda)^{\mathbb{P}}$ with $\lambda \in \mathbb{K} \setminus \mathbb{P}$ determined up to replacement by $\bar{\lambda}$.

Let $a \in \mathbb{P}$. Reasoning as in the proof of Theorem 2.1(a), we conclude that

- $J_n(a)$ is similar to $J_n(b)^{-\mathrm{T}}$ with $b \in \mathbb{P}$ if and only if $a = b^{-1}$;
- $\sqrt[T]{J_n(a)}$ exists if and only if $a = (-1)^{n+1}$.

Thus, the summands (i)–(iii) of Theorem 2.2 give the summands (i)–(iii) in Theorem 2.1(d). Due to (69), we may take $(\Gamma_n)^{-\mathrm{T}}\Gamma_n$ or $(\Gamma_n')^{-\mathrm{T}}\Gamma_n'$ instead of $J_n((-1)^{n+1})$ in $\mathcal{O}_{\mathbb{P}}$. Thus, we may use $\pm\Gamma_n$ or $\pm\Gamma_n'$ instead of $\pm\sqrt[T]{J_n((-1)^{n+1})}$ in Theorem 2.1(d).
Let $\lambda, \mu \in (\mathbb{P} + \mathbb{P}i) \setminus \mathbb{P}$. Then

$$\begin{aligned} &J_n(\lambda)^{\mathbb{P}} \text{ is similar to } (J_n(\mu)^{\mathbb{P}})^{-\mathrm{T}} \iff \lambda \in \{\mu^{-1}, \bar{\mu}^{-1}\}, \\ &\sqrt[T]{J_n(\lambda)^{\mathbb{P}}} \text{ exists} \iff |\lambda| = 1. \end{aligned} \tag{78}$$

Let us prove (78). For $\Phi := J_n(\lambda)^{\mathbb{P}}$, we have

$$p_\Phi(x) = (x-\lambda)(x-\bar{\lambda}) = x^2 - (\lambda+\bar{\lambda}) + |\lambda|^2.$$

If $\sqrt[T]{\Phi}$ exists then $|\lambda| = 1$ by (21) and (30).

Conversely, let $|\lambda| = 1$. We can take

$$\sqrt[T]{J_n(\lambda)^{\mathbb{P}}} = \left(\sqrt[*]{J_n(\lambda)}\right)^{\mathbb{P}}. \tag{79}$$

Indeed, $M := \sqrt[*]{J_n(\lambda)}$ exists by (74); it suffices to prove

$$(M^{\mathbb{P}})^{-T} M^{\mathbb{P}} = J_n(\lambda)^{\mathbb{P}}. \tag{80}$$

If M is represented in the form $M = A + Bi$ with A and B over $\mathbb{P}$, then its realification $M^{\mathbb{P}}$ (see (8)) is permutationally similar to

$$M_{\mathbb{P}} := \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

Applying the same transformation of permutation similarity to the matrices of (80) gives

$$(M_{\mathbb{P}})^{-T} M_{\mathbb{P}} = J_n(\lambda)_{\mathbb{P}}. \tag{81}$$

Since

$$\begin{bmatrix} A+Bi & 0 \\ 0 & A-Bi \end{bmatrix} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} = \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix},$$

we have

$$M_{\mathbb{P}} = S^{-1}(M \oplus \bar{M})S = S^*(M \oplus \bar{M})S$$

with

$$S := \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} = S^{-*}.$$

Thus, (81) is represented in the form

$$(S^*(M \oplus \bar{M})S)^{-*} S^*(M \oplus \bar{M})S = S^{-1}(J_n(\lambda) \oplus J_n(\bar{\lambda}))S.$$

This equality is equivalent to the pair of equalities

$$M^{-*}M = J_n(\lambda), \qquad \bar{M}^{-*}\bar{M} = J_n(\bar{\lambda}),$$

which are valid since $M = \sqrt[*]{J_n(\lambda)}$. This proves (79), which completes the proof of (78).

Thus, the summands (ii) and (iii) of Theorem 2.2 give the summands (ii′) and (iii′) in Theorem 2.1(d).

It remains to prove that each of the matrices (10) can be used instead of (iii′). Every unimodular $\lambda = a + bi \in \mathbb{P} + \mathbb{P}i$ with $b > 0$ can be expressed in the form

$$\lambda = \frac{e+i}{e-i}, \quad e \in \mathbb{P},\ e > 0. \tag{82}$$

Due to (69), the *cosquares

$$((e+i)\Gamma_n)^{-*}(e+i)\Gamma_n = \lambda \Gamma_n^{-*}\Gamma_n, \quad ((e+i)\Gamma_n')^{-*}(e+i)\Gamma_n' = \lambda(\Gamma_n')^{-*}\Gamma_n'$$

are similar to $\lambda J_n((-1)^{n+1})$, which is similar to $(-1)^{n+1}J_n(\lambda)$. Theorem 2.2 ensures that the matrix $\pm\sqrt[T]{J_n(\lambda)^{\mathbb{P}}}$ in (iii′) can be replaced

$$\text{by } \pm((e+i)\Gamma_n)^{\mathbb{P}} \text{ and also by } \pm((e+i)\Gamma_n')^{\mathbb{P}} \text{ with } e>0. \tag{83}$$

For each square matrix A over $\mathbb{P}+\mathbb{P}i$ we have

$$S^{\mathrm{T}}A^{\mathbb{P}}S=\overline{A}^{\mathbb{P}}, \qquad S:=\operatorname{diag}(1,-1,1,-1,\ldots), \tag{84}$$

and so $-((e+i)\Gamma_n)^{\mathbb{P}}$ is congruent to

$$-\overline{(e+i)\Gamma_n}^{\mathbb{P}}=-((e-i)\Gamma_n)^{\mathbb{P}}=((-e+i)\Gamma_n)^{\mathbb{P}}.$$

Therefore, the matrices (83) are congruent to $((c+i)\Gamma_n)^{\mathbb{P}}$ and $((c+i)\Gamma_n')^{\mathbb{P}}$ with $0\neq c\in\mathbb{P}$ and $|c|=e$.

Let us show that the summands (iii′) can be also replaced by $\Delta_n(c+i)$ with $0\neq c\in\mathbb{P}$. By (77), the *cosquare of $\Delta_n(e+i)$ with $e>0$ is similar to $J_n(\lambda)$, in which λ is defined by (82). Reasoning as in the proof of (80), we find that the cosquare of $\Delta_n(e+i)^{\mathbb{P}}$ is similar to $J_n(\lambda)^{\mathbb{P}}$. Hence, $\pm\Delta_n(e+i)^{\mathbb{P}}$ with $e>0$ can be used instead of (iii′). Due to (84), the matrix $-\Delta_n(e+i)^{\mathbb{P}}$ is congruent to

$$-\overline{\Delta_n(e+i)}^{\mathbb{P}}=\Delta_n(-e+i)^{\mathbb{P}}.$$

5.5. Proof of Theorem 2.1(e)

Lemma 5.1. *Let $\mathbb{H}$ be the skew field of quaternions over a real closed field $\mathbb{P}$. Let $\mathcal{O}_{\mathbb{H}}$ be a maximal set of nonsingular indecomposable canonical matrices over $\mathbb{H}$ for similarity.*

(a) *Each square matrix over $\mathbb{H}$ is *congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the form:*
   (i) $J_n(0)$.
   (ii) $(\Phi, I_n)^+=[\Phi\backslash I_n]$, *in which $\Phi\in\mathcal{O}_{\mathbb{H}}$ is an $n\times n$ matrix such that $\sqrt[*]{\Phi}$ does not exist; Φ is determined up to replacement by the unique $\Psi\in\mathcal{O}_{\mathbb{F}}$ that is similar to Φ^{-*}.*
   (iii) $\varepsilon_\Phi\sqrt[*]{\Phi}$, *in which $\Phi\in\mathcal{O}_{\mathbb{H}}$ is such that $\sqrt[*]{\Phi}$ exists; $\varepsilon_\Phi=1$ if $\sqrt[*]{\Phi}$ is *congruent to $-\sqrt[*]{\Phi}$ and $\varepsilon_\Phi=\pm1$ otherwise. This means that $\varepsilon_\Phi=1$ if and only if $\mathbb{T}(\mathscr{A}_\Phi)$ is an algebraically closed field with the identity involution or $\mathbb{T}(\mathscr{A}_\Phi)$ is a skew field of quaternions with involution different from quaternionic conjugation (1).*

(b) *If $\varepsilon_\Phi=1$ and Φ is similar to Ψ, then $\varepsilon_\Psi=1$.*

Proof. (a) Theorem 3.2 ensures that any given representation of any pograph P over $\mathbb{H}$ decomposes uniquely, up to isomorphism of summands, into a direct sum of indecomposable representations. Hence the problem of classifying representations of P reduces to the problem of classifying indecomposable representations. By Theorem 3.2 and Lemma 4.1, the matrices (i)–(iii) form a maximal set of nonisomorphic indecomposable representations of the pograph (48).

(b) On the contrary, assume that $\varepsilon_\Psi=\pm1$. Then $\sqrt[*]{\Psi}$ and $-\sqrt[*]{\Psi}$ have the same canonical form $\sqrt[*]{\Phi}$, a contradiction. □

Let $\mathbb{P}$ be a real closed field and let $\mathbb{H}$ be the skew field of $\mathbb{P}$-quaternions with quaternionic conjugation (1) or quaternionic semiconjugation (2). These involutions act as complex conjugation on the algebraically closed subfield $\mathbb{K} := \mathbb{P} + \mathbb{P}i$. By [14, Section 3, §12], we can take $\mathcal{O}_{\mathbb{F}}$ to be all $J_n(\lambda)$, in which $\lambda \in \mathbb{K}$ and λ is determined up to replacement by $\bar{\lambda}$. For any nonzero $\mu \in \mathbb{K}$, the matrix $J_n(\mu)^{-*}$ is similar to $J_n(\bar{\mu}^{-1})$. Since $\bar{\mu}^{-1}$ is determined up to replacement by μ^{-1},

$$J_n(\lambda) \text{ is similar to } J_n(\mu)^{-*} \Longleftrightarrow \lambda \in \{\mu^{-1}, \bar{\mu}^{-1}\}.$$

Let us prove that for a nonzero $\lambda \in \mathbb{K}$

$$\sqrt[*]{J_n(\lambda)} \text{ exists } \Longleftrightarrow |\lambda| = 1.$$

If $\sqrt[*]{J_n(\lambda)}$ exists then by (21) $x - \lambda = x - \bar{\lambda}^{-1}$ and so $|\lambda| = 1$. Conversely, let $|\lambda| = 1$. In view of (70), the *cosquare of $A := \sqrt{\lambda(-1)^{n+1}}\Gamma_n$ is

$$\Phi := A^{-*}A = \lambda F, \qquad F := (-1)^{n+1}\Gamma_n^{-\mathrm{T}}\Gamma_n = \begin{bmatrix} 1 & 2 & & * \\ & 1 & \ddots & \\ & & \ddots & 2 \\ 0 & & & 1 \end{bmatrix}, \tag{85}$$

and so Φ is similar to $J_n(\lambda)$. Thus, $\sqrt[*]{J_n(\lambda)}$ exists by (20) with $\sqrt[*]{\Phi} = A$.

Lemma 5.1(a) ensures the summands (i)–(iii) in Theorem 2.1(e); the coefficient ε in (iii) is defined in Lemma 5.1(a). Let us prove that ε can be calculated by (11). By Lemma 5.1(b) and since Φ in (85) is similar to $J_n(\lambda)$, we have $\varepsilon = \varepsilon_\Phi$, so it suffices to prove (11) for ε_Φ.

Two matrices $G_1, G_2 \in \mathbb{H}^{n\times n}$ give an endomorphism $[G_1, G_2]$ of $\underline{\mathscr{A}}_\Phi = (A, A^*)$ if and only if they satisfy (66). By (67), the equalities (66) imply

$$G_1\Phi = \Phi G_1. \tag{86}$$

Case $\lambda \neq \pm 1$. Represent G_1 in the form $U + Vj$ with $U, V \in \mathbb{K}^{n\times n}$. Then (86) implies two equalities

$$U\Phi = \Phi U, \qquad V\overline{\Phi}j = \Phi Vj. \tag{87}$$

By the second equality and (85), $\bar{\lambda}VF = \lambda FV$,

$$(\bar{\lambda} - \lambda)V = \lambda(F - I)V - \bar{\lambda}V(F - I).$$

Thus $V = 0$ since $\lambda \neq \bar{\lambda}$ and $F - I$ is nilpotent upper triangular. By the first equality in (87) (which is over the field $\mathbb{K}$), $G_1 = U = f(\lambda F) = f(\Phi)$ for some $f \in \mathbb{K}[x]$; see the beginning of the proof of Lemma 4.1(d). Since A is over $\mathbb{K}$, the identities (66) imply (68).

Because $G_2 = AG_1A^{-1}$, the homomorphism $[G_1, G_2] \in \mathrm{End}(\underline{\mathscr{A}}_\Psi)$ is completely determined by $G_1 = f(\Phi)$. The matrix $\Phi = \lambda F$ is upper triangular, so the mapping $f(\Phi) \mapsto f(\lambda)$ on $\mathbb{K}[\Phi]$ defines an endomorphism of rings $\mathrm{End}(\underline{\mathscr{A}}_\Phi) \to \mathbb{K}$; its kernel is the radical of $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$. Hence $\mathbb{T}(\mathscr{A}_\Phi)$ can be identified with $\mathbb{K}$. Using (50), we see that the involution on $\mathbb{T}(\mathscr{A}_\Phi)$ is induced by the mapping $G_1 \mapsto G_2^*$ of the form

$$f(\lambda F) \mapsto f((\lambda F)^{-*})^* = \bar{f}((\lambda F)^{-1}).$$

Therefore, the involution is

$$f(\lambda) \longmapsto \bar{f}(\lambda^{-1}) = \bar{f}(\bar{\lambda}) = \overline{f(\lambda)}$$

and coincides with the involution $a + bi \mapsto a - bi$ on $\mathbb{K}$. The statement (iii) in Lemma 5.1(a) now implies $\varepsilon_\Phi = \pm 1$; this proves (11) in the case $\lambda \neq \pm 1$.

Case $\lambda = \pm 1$. Then

$$A = \sqrt{\lambda(-1)^{n+1}}\,\Gamma_n = \begin{cases} \Gamma_n & \text{if } \lambda = (-1)^{n+1}, \\ i\Gamma_n & \text{if } \lambda = (-1)^n. \end{cases} \tag{88}$$

Define

$$\begin{aligned} \check{h} &:= a + bi - cj - dk \quad \text{for each } h = a + bi + cj + dk \in \mathbb{H}, \\ \check{f}(x) &:= \sum_l \check{h}_l x^l \quad \text{for each } f(x) = \sum_l h_l x^l \in \mathbb{H}[x]. \end{aligned}$$

Because $\lambda = \pm 1$ and by (86), G_1 has the form

$$G_1 = \begin{bmatrix} a_1 & a_2 & \ddots & a_n \\ & a_1 & \ddots & \ddots \\ & & \ddots & a_2 \\ 0 & & & a_1 \end{bmatrix}, \quad a_1, \dots, a_n \in \mathbb{H}.$$

Thus, $G_1 = f(\Phi)$ for some polynomial $f(x) \in \mathbb{H}[x]$.

Using the first equality in (66), the identity $if(x) = \check{f}(ix)$, and (88), we obtain

$$G_2 = AG_1A^{-1} = Af(\Phi)A^{-1} = \begin{cases} f(A\Phi A^{-1}) = f(\Phi^{-*}) & \text{if } \lambda = (-1)^{n+1}, \\ \check{f}(A\Phi A^{-1}) = \check{f}(\Phi^{-*}) & \text{if } \lambda = (-1)^n. \end{cases}$$

Since the homomorphism $[G_1, G_2]$ is completely determined by $G_1 = f(\Phi)$ and Φ has the upper triangular form (85) with $\lambda = \pm 1$, we conclude that the mapping $f(\Phi) \mapsto f(\lambda)$ defines an endomorphism of rings $\mathrm{End}(\underline{\mathscr{A}}_\Phi) \to \mathbb{H}$; its kernel is the radical of $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$. Hence $\mathbb{T}(\mathscr{A}_\Phi)$ can be identified with $\mathbb{H}$. The involution on $\mathbb{T}(\mathscr{A}_\Phi)$ is induced by the mapping $G_1 \mapsto G_2^*$; i.e., by

$$f(\Phi) \mapsto \begin{cases} \bar{f}(\Phi^{-1}) & \text{if } \lambda = (-1)^{n+1}, \\ \hat{f}(\Phi^{-1}) & \text{if } \lambda = (-1)^n, \end{cases}$$

in which the involution $h \mapsto \bar{h}$ on $\mathbb{F}$ is either quaternionic conjugation (1) or quaternionic semiconjugation (2), and $h \mapsto \hat{h}$ denotes the other involution (2) or (1). Thus the involution on $\mathbb{T}(\mathscr{A}_\Phi)$ is $h \mapsto \bar{h}$ if $\lambda = (-1)^{n+1}$ and is $h \mapsto \hat{h}$ if $\lambda = (-1)^n$. Due to (iii) in Lemma 5.1(a), this proves (11) in the case $\lambda = \pm 1$.

It remains to prove that the matrices (12) and (13) can be used instead of (iii) in Theorem 2.1(e).

Let us prove this statement for the first matrix in (12). For each unimodular $\lambda \in \mathbb{K}$, the *cosquare (85) of $A = \sqrt{\lambda(-1)^{n+1}}\,\Gamma_n$ is similar to $J_n(\lambda)$, so we can replace $J_n(\lambda)$ by Φ in $\mathcal{O}_{\mathbb{H}}$ and conclude by Lemma 5.1(a) that εA can be used instead of (iii) in Theorem 2.1(e).

First, let the involution on $\mathbb{H}$ be quaternionic conjugation. By (11) the matrix εA is

$$\text{either } i\Gamma_n, \text{ or } \pm\mu\Gamma_n \quad \text{with } \mu := \sqrt{\lambda(-1)^{n+1}} \neq i. \tag{89}$$

Since λ is determined up to replacement by $\bar{\lambda}$ and $\sqrt{\lambda(-1)^{n+1}} \neq i$, we can take $\lambda(-1)^{n+1} = u + vi \neq -1$ with $v \geqslant 0$, and obtain $\mu = \sqrt{\lambda(-1)^{n+1}} = a + bi$ with $a > 0$ and $b \geqslant 0$. Replacing the matrices $-\mu\Gamma_n = (-a - bi)\Gamma_n$ in (89) by the *congruent matrices $\bar{j} \cdot (-a - bi)\Gamma_n \cdot j = (-a + bi)\Gamma_n$, we get the first matrix in (12).

Now let the involution be quaternionic semiconjugation. By (11) the matrix εA is

$$\text{either } \Gamma_n, \text{ or } \pm\mu\Gamma_n \quad \text{with } \mu := \sqrt{\lambda(-1)^{n+1}} \neq 1. \tag{90}$$

In (90) we can take $\lambda(-1)^{n+1} = u + vi \neq 1$ with $v \geqslant 0$. Then $\mu = \sqrt{\lambda(-1)^{n+1}} = a + bi$ with $a \geqslant 0$ and $b > 0$. Replacing the matrices $-\mu\Gamma_n = (-a - bi)\Gamma_n$ in (90) by the *congruent matrices $\bar{j} \cdot (-a - bi)\Gamma_n \cdot j = (a - bi)\Gamma_n$ ($\bar{j} = j$ since the involution is quaternionic semiconjugation), we get the first matrix in (12).

The same reasoning applies to the second matrix in (12).

Let us prove that the matrix (13) can be used instead of (iii) in Theorem 2.1(e). By (77), $J_n(\lambda)$ with a unimodular $\lambda \in \mathbb{K}$ is similar to the *cosquare of $\sqrt{\lambda}\Delta_n$ with $\Delta_n := \Delta_n(1)$. Therefore, $\varepsilon\sqrt[*]{J_n(\lambda)}$ in (iii) can be replaced by $\varepsilon\sqrt{\lambda}\Delta_n$.

Suppose that either the involution is quaternionic conjugation and n is odd, or that the involution is quaternionic semiconjugation and n is even. Then $\bar{j} = (-1)^n j$. By (11), $\varepsilon = 1$ if $\lambda = -1$ and $\varepsilon = \pm 1$ if $\lambda \neq -1$. So each $\varepsilon\sqrt{\lambda}\Delta_n$ is either $i\Delta_n$ or $\pm\mu\Delta_n$, in which $\mu := \sqrt{\lambda}$ and $\lambda = u + vi \neq -1$. We can suppose that $v \geqslant 0$ since λ is determined up to replacement by $\bar{\lambda}$. Because μ is represented in the form $a + bi$ with $a > 0$ and $b \geqslant 0$, the equality

$$S_n\Delta_n S_n = (-1)^n\Delta_n, \qquad S_n := \operatorname{diag}(j, -j, j, -j, \ldots),$$

shows that we can replace $-\mu\Delta_n = (-a - bi)\Delta_n$ by the *congruent matrix

$$S_n^*(-a - bi)\Delta_n S_n = (-1)^n S_n(-a - bi)\Delta_n S_n = (-a + bi)\Delta_n$$

and obtain the matrix (13).

Now suppose that the involution is quaternionic conjugation and n be even, or that the involution is quaternionic semiconjugation and n is odd. Then $\bar{j} = (-1)^{n+1} j$. By (11), each $\varepsilon\sqrt{\lambda}\Delta_n$ is either Δ_n or $\pm\mu\Delta_n$, in which $\mu := \sqrt{\lambda}$ and $\lambda = u + vi \neq 1$ with $v \geqslant 0$. Since μ is represented in the form $a + bi$ with $a \geqslant 0$ and $b > 0$, we can replace $-\mu\Delta_n = (-a - bi)\Delta_n$ by the *congruent matrix

$$S_n^*(-a - bi)\Delta_n S_n = (-1)^{n+1} S_n(-a - bi)\Delta_n S_n = (a - bi)\Delta_n$$

and obtain the matrix (13).

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