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ABOUT RECOGNIZING (α, β) CLASSES OF POLAR GRAPHS

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Let G be a graph with a vertex set VG. If there exists such a partition $VG = A \cup B$ that all connected components of the induced subgraph $G(B)$ and of the complementary induced subgraph $\overline{G(A)}$ are complete graphs, orders of which do not exceed α and β, respectively, then G are defined to belong to (α, β) class of polar graphs.

In this paper it is proved that the decision problem of membership in (∞, β), where β is fixed and $\beta > 1$, is NP-complete. It is also proved that the decision problem of membership in (∞, ∞) is NP-complete and the corresponding search problem of constructing the polar partition is NP-equivalent. Thereby all formerly unsolved variations of the problem of recognizing (α, β) classes are exhausted.

All graphs considered are finite, undirected, without loops and multiple edges.

The class of polar graphs has been introduced in [8]. Let G be a graph with a vertex set VG. If there exists such a partition $VG = A \cup B$ that all connected components of the induced subgraph $G(B)$ and of the complementary induced subgraph $\overline{G(A)}$ are complete graphs, then G and this partition are defined to be polar. A is the top part, B is the bottom part of G. One of the parts may be empty.

A polar partition is not unique, and so it is naturally to consider polar graphs together with the fixed top and bottom parts, i.e., to call the triple (G, A, B) a polar graph, where G, A, B are such as above.

Let α and β be positive integers. We say that $(G, A, B) \in (\alpha, \beta)$ if orders of connected components of $\overline{G(A)}$ and $G(B)$ do not exceed α and β, respectively. We shall also say that G belongs to (α, β) class of polar graphs, if there exists a polar partition $VG = A \cup B$ such that $(G, A, B) \in (\alpha, \beta)$ [8]. Note that

$$(\beta, \alpha) = \overline{(\alpha, \beta)} = \{(\bar{G}, B, A) : (G, A, B) \in (\alpha, \beta)\}$$

and the following implication is true:

$$(\alpha \leqslant \alpha', \beta \leqslant \beta') \Rightarrow (\alpha, \beta) \subseteq (\alpha', \beta'),$$

the inclusion is proper if at least one of inequalities is strict [8].

The class of polar graphs is wide enough and contains certain well-known classes of graphs (for example, split [3], bipartite, threshold [2], domishold [1] graphs).

Naturally, there arises the problem of recognizing the (α, β) classes of polar graphs and constructing the corresponding polar partitions. In [5] it has been obtained a linear-time algorithm recognizing the $(1, 1)$ class. For arbitrary fixed $\alpha < \infty$, $\beta < \infty$ the problem mentioned above has been solved in [6] for $O(n^{2\alpha+2\beta+3})$ steps, where n is an order of a graph. It has been shown in [7] that the $(1, \beta)$ class is recognized for $O(n^3)$ steps, where $1 \leqslant \beta \leqslant \infty$, $(\alpha, \infty) = \bigcup_{\beta}^{\infty} (\alpha, \beta)$.

In this paper it is proved that the decision problem of membership in (∞, β) referred to as the POLAR(β) problem (in (β, ∞), respectively), where β is fixed and $\beta > 1$, is NP-complete. It is also proved that the decision problem of membership in $(\infty, \infty) = \bigcup_{\alpha,\beta}^{\infty} (\alpha, \beta)$, referred to as the POLAR problem, is NP-complete and the corresponding search problem of constructing the polar partition is NP-equivalent. Thereby all variations of the problem of recognizing (α, β) classes are exhausted.

Notation: $N_H(v)$ is the set of vertices of a graph H adjacent to v; $N_H[v] = N_H(v) \cup \{v\}$; VG and EG are the set of vertices and edges of a graph G; for $U \subseteq VG$, $G(U)$ is the induced subgraph; $\mathcal{K}_\beta$ is the set of graphs, all components of which are K_n, $n \leqslant \beta$; $K_{n,m}$ is the complete bipartite graph with parts of orders n, m respectively; K_n is the complete n-vertex graph. We say that $G \in C$, if there is a partition $VG = A \cup B$ that $G(A)$ is an edgeless graph and vertex degrees of $G(B)$ do not exceed 1. Obviously, $C \subset (\infty, 2)$. We will also use the terminology of [4].

Theorem 1. *The decision problem of membership of G in (∞, β) is* NP-*complete* $(\beta > 1)$.

Proof. POLAR(β) problem is easily seen to be in NP. It is known as the following NP-complete decision problem [4]:

Instance: A set U of variables, a collection C of sets of literals over U such that each $c_i \in C$ has $|c_i| = 3$ and does not contain a negated literal.

Question: Is there a function $t: U \to \{0, 1\}$ such that each c_i has exactly one true literal?

We transform this problem to POLAR($\beta - 1$) problem, $\beta > 2$. Let

$$C = \{c_i: i = 1, \ldots, m\}, \qquad c_i = \{u_{i1}, u_{i2}, u_{i3}\}, \quad m > 1.$$

Let H_i, $i = 1, \ldots, m$, be pairwise disjoint complete β-vertex graphs, $VH_i = v_{i1}, \ldots, v_{i\beta}$. Let vertices v_{ij}, v_{kl} be joined by the chain v_{ij}, v_{ijkl}, v_{kl} if and only if $i \neq k$ and one of the following holds:

(1) $j < 4$, $i < 4$, $u_{ij} = u_{kl}$; (2) $j > 3$, $l > 3$.

Additionally we join vertices v_{14}, v_{24} by the chain

$$v_{14}, v_1, v_2, v_3, v_4, v_{24}$$

and add a vertex v_5 adjacent to v_2 and v_3. We consider all v_{ijkl}, v_t, v_{rs} to be distinct. The graph obtained in the result is denoted by G.

Now we show that desired function t exists if and only if $G \in (\infty, \beta - 1)$.

First, suppose that t exists. Let $v_{ij} \in A$, if and only if $j<4$, $t(u_{ij})=1$; let $v_{ijkl} \in A$, if and only if either $j>3$ or $j<4$, $1<4$, $t(u_{ij})=0$; let $v_1, v_4, v_5 \in A$, $B = VG \setminus A$. Now we have: $G(A)$ is an edgeless graph, $G(B) \in \mathcal{K}_{\beta-1}$, i.e., $(G, A, B) \in (\infty, \beta - 1)$.

Conversely let $(G, A, B) \in (\infty, \beta - 1)$. $G(B)$ does not contain a clique of size β, hence A must contain at least one vertex v_{ij} for each $i = 1, \ldots, m$. If $G(A)$ contained an edge from some graph H_i, $G(A)$ would have an induced subgraph $K_1 \cup K_2$ (disjoint union), a contradiction. So A contains exactly one vertex from each H_i.

Let $v_{ij} \in A$, $j>3$. Then $v_{kl} \in A$ for all $k \neq i$, $l>3$, since otherwise $G(A)$ will contain an induced subgraph $K_1 \cup K_2$ whenever $v_{ijkl} \in A$, and $G(B)$ will contain an induced subgraph $K_{1,2}$ whenever $v_{ijkl} \in B$, a contradiction. Hence $\beta = 4$, v_{14}, $v_{24} \in A$, $v_1, v_4 \in B$. Further, if $v_2 \in A$ then $v_3, v_5 \in B$; if $v_2 \in B$, then $v_3, v_5 \in A$. In any case we have a contradiction. So $v_{ij} \in A$ implies $j<4$. Assuming now

$$t(u_{ij}) = \begin{cases} 1, & \text{if } v_{ij} \in A, \\ 0, & \text{if } v_{ij} \in B, \end{cases}$$

we obtain desired function t. □

Corollary 1. *The decision problem of membership of G in C is* NP-*complete.*

Lemma 1. *If $G \in C$, then G does not contain subgraphs H_i shown in Fig.* 1, $i = 1, 2, 3, 4$.

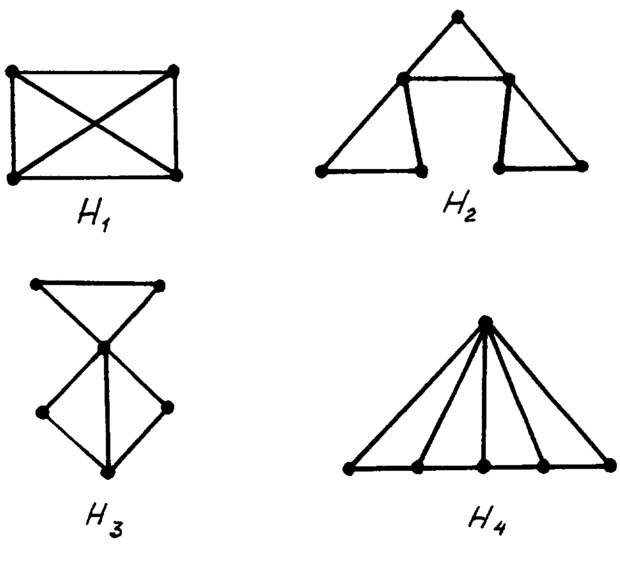

Fig. 1.

Proof. In any graph H_i there exists a triangle T_i, each edge of which is incident to exactly one vertex of some triangle of H_i. Suppose that $(G, A, B) \in C$ and G contains H_i, $i \in \{1, 2, 3, 4\}$. Then $VT_i \cap B = \{u, v\}$. If now $VT_i' \cap \{u, v\} = \{v\}$, where T_i' is a triangle of H_i and $T_i' \neq T_i$, then $VT_i' \cap B = \{v\}$ and therefore $|VT_i' \cap A| = 2$, a contradiction. □

From Corollary 1 and Lemma 1 we obtain NP-completeness of the following decision problem, referred to as ADDIT problem:

Instance: A graph G without subgraphs (not necessarily induced) isomorphic to H_i, $i = 1, 2, 3, 4$.
Question: Does G belong to C?

Lemma 2. *If G does not contain subgraphs H_i, $i = 1, 2, 3, 4$,* then a minimal set X of pairwise disjoint edges can be found in polynomial time such that the graph *$G - X$ has no triangle.*

Proof. We prove this lemma by induction on $|EG|$. The case $|EG| = 1$ is trivial. Let $|EG| > 1$ and suppose the lemma holds for all graphs on fewer than $|EG|$ edges. Let T be a triangle of G, $VT = \{u, v, w\}$. We now consider two cases.

Case 1. At least one edge of T, say uv, belongs to a triangle $T' \neq T$, that is G contains a subgraph $K_4 - zw$. If the edge uv has a common vertex with some triangle not containing uv then G contains one of the subgraphs H_1, H_3, H_4. Otherwise the validity of the lemma follows from inductive hypothesis for $G - uv$.

Case 2. No edge of T belongs to any triangle other than T. If at least two vertices u, v of T do not belong to other tiangles then the validity of the lemma follows from inductive hypothesis for $G - uv$. Otherwise G contains H_2, a contradiction. □

Theorem 2. *POLAR problem is* NP-*complete.*

Proof. It is easy to see that POLAR $\in$ NP. We transform ADDIT problem to POLAR problem. From Lemma 2 we know that there exists a minimal set X of pairwise disjoint edges such that $G - X$ has not triangles. Every edge $e = uv$ from X is associated to a graph H_e with vertex set $\{a_1, \ldots, a_{11}\}$ (see Fig. 2). (All graphs are vertex disjoint.)

Now we identify the vertex v with a_1 and the vertex u with a_2, respectively. Then we make a_{10} to be adjacent to every vertex of the set M_e and a_{11} to be adjacent to every vertex of the set N_e, where

$$M_e = N_G(v) \setminus N_G[u], \qquad N_e = N_G(u) \setminus N_G[v].$$

We repeat these steps for each edge $e \in X$. Next removing all edges of X and

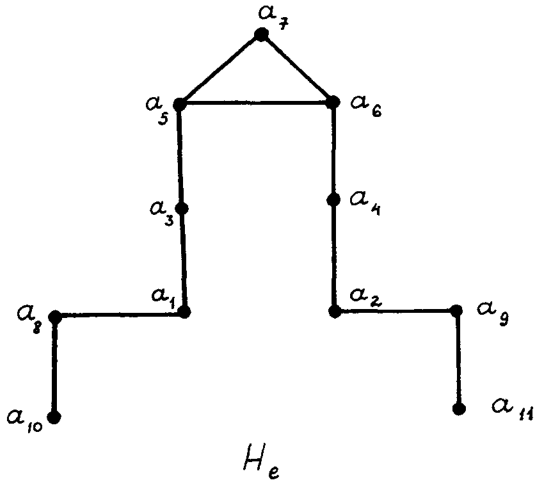

Fig. 2.

adding the only component $F = K_{1,2}$ we obtain a graph H. Obviously, H can be constructed in polynomial time. Now we shall prove that $G \in C$ if and only if $H \in (\infty, \infty)$.

Suppose $(G, A, B) \in C$, $e = uv$ is an arbitrary edge of X. We construct a polar partition of H. At first the sets A and B are included in the top and bottom parts, respectively. Moreover, one vertex of VF is included in the top part and two others are included in the bottom part. Next if $u, v \in B$, then we include the vertices a_3, a_4, a_8, a_9 in the top part, but the rest vertices of H_e include in the bottom part. If $v \in B$, $u \in A$, then we include the vertices a_3, a_6, a_8, a_{11} in the top part, but the rest vertices of $VH_e \setminus \{u, v\}$ include in the bottom part. (Note there exists a triangle T of G with an edge e, i.e. $|VT \cap B| = 2$ and hence $M_e \cap B = \emptyset$, $M_e \subseteq A$.) Repeating these steps for each edge of X we obtain the polar partition of the graph H.

Let now $(H, W_1, W_2) \in (\infty, \infty)$. $VF \cap W_1 \neq \emptyset$ for otherwise $H(W_2)$ will contain an induced subgraph $K_{1,2}$. It follows that $H(W_1)$ is edgeless, for otherwise $H(W_1)$ will contain an induced subgraph $K_1 \cup K_2$. Assuming $A = VG \cap W_1$, $B = VG \cap W_2$ we prove that $(G, A, B) \in C$. Since $G - X$ has no triangles, $H(B)$ contains only components K_1, K_2 and it follows that $(H(A \cup B), A, B) \in C$. Consider an arbitrary edge $e = uv$ of X. If $u, v \in A$, then $a_3, a_4 \in W_2$, and a_5, a_6 cannot belong to the same part. It follows that $a_5 \in W_1$, $a_6 \in W_2$ (or vice versa), and hence $a_7 \in W_2$, a contradiction.

Suppose now that $u, v \in B$ and B contains at least one vertex $w = u, v$ such that $w \in N_G(v) \cup N_G(u)$. Obviously, either $w \in M_e$ or $w \in N_e$. Assume, for example, $w \in M_e$. Then a_10 and w are adjacent and $a_{10} \in W_1$. But $a_8 \in W_1$ and a_8 is adjacent to a_{10}, a contradiction. Thus, either $|\{u, v\} \cap A| = 1$ or u, v are isolated vertices in $H(B)$. Since this is true for each $e \in X$ and all edges of X are disjoint, it now follows that $(G, A, B) = H(A \cup B) \cup X \in C$. □

Corollary 2. *The search problem of constructing polar partition of a graph G is* NP-*equivalent.*

From Theorem 2 follows that this search problem is NP-hard. The approach similar to one presented in [4, p. 116–117] can be used to prove that this problem is NP-easy.

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