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The $w_0(p)$–$w_0(q)$ mapping problem for factorable matrices II

P. Spanos, B. Thorpe *

The University of Birmingham, Department of Mathematics, Edgbaston, Birmingham B15 2TT, UK

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Abstract

We find necessary and sufficient conditions for the class of factorable matrices $M(\boldsymbol{a},\boldsymbol{b})$ to map $w_0(p)$ into $w_0(q)$ for $0<q<p\leqslant 1$.

Keywords: Mapping theorem; Sequence spaces

1. Introduction

In this paper we complete our investigation of the $w_0(p)$–$w_0(q)$ mapping problem for factorable matrices started in [5], by examining the remaining case when $0<q<p\leqslant 1$, which seems to be the most difficult one. We are indebted to K. Grosse-Erdmann for his suggestion of the following approach.

2. Preliminaries

We shall use the conventions found in [5]. In particular, we let ϕ denote the space of finitely non-zero sequences and we define r by the equation $1/r = 1/q - 1/p$. If $\boldsymbol{a} = (a_n)_{n\geqslant 1}$ and $\boldsymbol{b} = (b_n)_{n\geqslant 1}$ are taken to be non-negative sequences, the factorable matrix $M := M(\boldsymbol{a},\boldsymbol{b}) = (m_{nk})$ is defined as follows:

$$m_{nk} = \begin{cases} a_n b_k & \text{if } k = 1,\dots,n, \\ 0 & \text{if } k > n; \end{cases}$$

and without loss of generality, we can assume that $\boldsymbol{b}$ has at least one non-zero coordinate.

For $0<p<\infty$, $w(p)$ will denote the space of strongly Cesàro summable sequences of order 1 and index p; i.e. the space of sequences $\boldsymbol{x}$ such that there is a number L depending on $\boldsymbol{x}$, satisfying

$$\sum_{k=1}^{n} |x_k - L|^p = \mathrm{o}(n) \quad \text{as } n\to\infty.$$

* Corresponding author.
E-mail addresses: pispanos@for.mat.bham.ac.uk (P. Spanos), b.thorpe@bham.ac.uk (B. Thorpe).

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A partial summation shows that we have the equivalent definition

$$\sum_{k=2^n}^{2^{n+1}-1} |x_k - L|^p = \mathrm{o}\big(2^n\big) \quad \text{as } n \to \infty,$$

and this leads to the reasonable interpretation of $w(\infty)$ as c, the space of convergent sequences. For $L = 0$, we get the space $w_0(p)$. Similarly, we take $w_0(\infty) = c_0$, the space of convergent to zero sequences, equipped with the sup norm. If $p \geqslant 1$, $w(p)$ is a BK-space (i.e. a Banach sequence space with continuous coordinate mappings $\boldsymbol{x} \mapsto x_k$) when equipped with the norm

$$\sup_{n\geqslant 0}\left(2^{-n}\sum_{\nu\in D_n}|x_\nu|^p\right)^{1/p},$$

where henceforth we shall write $D_n := [2^n, 2^{n+1}) \cap \mathbb{N}_0$. If $0 < p < 1$, it is a complete p-normed K-space with p-norm

$$\|\boldsymbol{x}\|_{w_0(p)} = \sup_{n\geqslant 0}\left(2^{-n}\sum_{\nu\in D_n}|x_\nu|^p\right).$$

Note that when $q \leqslant p$, we have $w_0(q) \supset w_0(p)$, so that $w_0(p)$ is also a q-normed space under the q-norm $\|.\|_{w_0(q)}$.

We shall denote the positive orthant of $w(p)$ by $w(p)_+$ and now describe how we simplify our problem as per the discussion of [5, p. 7].

Let $0 < q < p \leqslant 1$. We can assume without loss of generality that $\boldsymbol{x} \in w_0(p)_+$, since $(x_n)_{n\geqslant 1} \in w_0(p)$ if and only if $(|x_n|)_{n\geqslant 1} \in w_0(p)$. We need to find necessary and sufficient conditions in order that

$$\boldsymbol{x} \in w_0(p)_+ \quad \Rightarrow \quad \sum_{\mu\in D_n} a_\mu^q\left(\sum_{k=1}^{2^n-1} b_k x_k + \sum_{k=2^n}^{\mu} b_k x_k\right)^q = \mathrm{o}\big(2^n\big).$$

This holds if and only if

$$\boldsymbol{x} \in w_0(p)_+ \quad \Rightarrow \quad \sum_{\mu\in D_n} a_\mu^q\left(\sum_{k=1}^{2^n-1} b_k x_k\right)^q = \mathrm{o}\big(2^n\big) \tag{1}$$

and

$$\boldsymbol{x} \in w_0(p)_+ \quad \Rightarrow \quad \sum_{\mu\in D_n} a_\mu^q\left(\sum_{k=2^n}^{\mu} b_k x_k\right)^q = \mathrm{o}\big(2^n\big)$$

both hold. Our problem thus splits into examining each of these cases separately.

Set for each $n \in \mathbb{N}$,

$$a_{nk} = \begin{cases} (2^{-n}\sum_{\mu\in D_n} a_\mu^q)^{1/q} b_k & \text{if } k = 1, 2, \ldots, 2^n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $0 < p < 1$, we can apply Theorem 7(a) of Maddox [3, p. 172]. In this case, (1) becomes equivalent to $\boldsymbol{a} \in w_0(q)$ and

$$\left(2^{-n}\sum_{k\in D_n} a_k^q\right)^{1/q}\sum_{k=0}^{n-1} 2^{k/p}\max_{j\in D_k} b_j = \mathrm{O}(1). \tag{2}$$

To complete the solution, it therefore suffices to find necessary and sufficient conditions for

$$\boldsymbol{x} \in w_0(p)_+ \quad \Rightarrow \quad \sum_{\mu\in D_n} a_\mu^q\left(\sum_{k=2^n}^{\mu} b_k x_k\right)^q = \mathrm{o}\big(2^n\big). \tag{3}$$

We show in Theorem 1 that

$$2^{-n}\sum_{j\in D_n}\left[\left(\sum_{k=j}^{2^{n+1}-1} a_k^q\right)^{1/p}\max_{2^n\leqslant k\leqslant j} b_k\right]^r a_j^q = \mathrm{O}(1) \tag{4}$$

is the "truncated" part's desired condition.

3. Main result

Theorem 1. *Let $0<q<p\leqslant 1$. Then $M(\boldsymbol{a},\boldsymbol{b})$ maps $w_0(p)$ into $w_0(q)$ if and only $\boldsymbol{a}\in w_0(q)$ and conditions* (2) *and* (4) *hold.*

Proof. We first note that, since all the terms are non-negative, (3) can be re-written as

$$\sum_{k\in D_n}\left(k^{-1/p}x_k\right)^p = \mathrm{o}(1) \quad\Rightarrow\quad \sum_{\mu\in D_n} a_\mu^q\left(\sum_{k=2^n}^{\mu}\left(k^{-1/q}b_k\right)x_k\right)^q = \mathrm{o}(1),$$

and this is equivalent to

$$\sum_{k\in D_n} x_k^p = \mathrm{o}(1) \quad\Rightarrow\quad \sum_{\mu\in D_n} a_\mu^q\left(\sum_{k=2^n}^{\mu} k^{-1/r}b_k x_k\right)^q = \mathrm{o}(1). \tag{5}$$

If we consider the complete p-normed K-space

$$c_0\left(2^n,p\right) = \left\{\boldsymbol{x}=(x_k)_{k\geqslant 1}\colon \sum_{k\in D_n}|x_k|^p\to 0 \text{ as } n\to\infty\right\},$$

which has p-norm $\|\boldsymbol{x}\| = \sup_{n\geqslant 0}\sum_{k\in D_n}|x_k|^p$ (cf. [2]), then we seek necessary and sufficient conditions on the matrix $B=(b_{\mu k}) : c_0(2^n,p)\to c_0(2^n,q)$, where, for $n\in\mathbb{N}_0$ and $\mu\in D_n$,

$$b_{\mu k} = \begin{cases} a_\mu k^{-1/r} b_k & \text{for } k=2^n,\dots,\mu,\\ 0 & \text{otherwise.}\end{cases}$$

Consider the vector-valued sequence space $c_0(\ell^p)$, where

$$c_0\left(\ell^p\right) = \left\{\boldsymbol{X}=(\boldsymbol{x}_n)_{n\geqslant 0}\colon \boldsymbol{x}_n=(x_{nk})_{k\geqslant 1}\in\ell^p \text{ and } \|\boldsymbol{x}_n\|_{\ell^p}=\sum_{k=1}^{\infty}|x_{nk}|^p\to 0 \text{ as } n\to\infty\right\}$$

with p-norm $\|\boldsymbol{X}\| = \sup_{n\geqslant 0}\sum_{k=1}^{\infty}|x_{nk}|^p$, under which it is complete. We can embed $c_0(2^n,p)$ as a linear subspace of $c_0(\ell^p)$ by mapping $\boldsymbol{x}=(x_k)_{k\geqslant 1}\mapsto \boldsymbol{X}=(\boldsymbol{x}_n)_{n\geqslant 0}$, where $\boldsymbol{x}_n=(x_{nk})_{k\geqslant 1}$ and

$$x_{nk} = \begin{cases} x_k & \text{if } k\in D_n,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\|\boldsymbol{X}\| = \|\boldsymbol{x}\|$, this embedding is an isometry and if E_p denotes the image of $c_0(2^n,p)$ in $c_0(\ell^p)$, then E_p is a closed linear subspace of $c_0(\ell^p)$.

Now we consider an infinite diagonal matrix of linear operators $A=\operatorname{diag}\{A_n\}$, where $A_n=(a_{\mu k}^{(n)}) : \ell^p\to\ell^q$ (note that if the general infinite matrix of operators is given by $A=(A_{nk})$, then in this case its diagonal is the matrix $(A_{n,n+1})$). It follows from [4, p. 372] that $A : c_0(\ell^p)\to c_0(\ell^q)$ if and only if $\sup_{n\geqslant 0}\|A_n\|_{p,q}<\infty$, where $\|A_n\|_{p,q} = \sup_{\|\boldsymbol{x}\|_p=1}\sum_{\mu=1}^{\infty}|\sum_{k=1}^{\infty} a_{\mu k}^{(n)}x_k|^q$.

In the special case that A_n is defined so that if $\mu\notin D_n$, then $a_{\mu k}^{(n)}=0$, for all $k\geqslant 1$, and if $\mu\in D_n$, then

$$a_{\mu k}^{(n)} = \begin{cases} a_\mu k^{-1/r} b_k & \text{for } k=2^n,\dots,\mu,\\ 0 & \text{otherwise,}\end{cases}$$

we see that

$$\|A_n\|_{p,q} = \sup_{\|\boldsymbol{x}\|_p=1} \sum_{\mu\in D_n} \left| a_\mu \sum_{k=2^n}^{\mu} k^{-1/r} b_k x_k \right|^q = \sup_{\|\boldsymbol{x}\|_{\ell^p(D_n)}=1} \sum_{\mu\in D_n} \left| a_\mu \sum_{k=2^n}^{\mu} k^{-1/r} b_k x_k \right|^q = \|B'\|_{\ell^p(D_n),\ell^q(D_n)},$$

where B' is the restriction of the matrix B above to the finite dimensional subspace $\{\boldsymbol{x} = (0,\dots,x_{2^n},\dots,x_{2^{n+1}-1}, 0,\dots)\colon x_i \in \mathbb{R}\}$. Moreover, for this choice, $A : c_0(\ell^p) \to c_0(\ell^q)$ if and only if $A : E_p \to E_q$ and this is equivalent to the map $B : c_0(2^n, p) \to c_0(2^n, q)$. Thus the necessary and sufficient condition for (5) to hold is

$$\sup_{n\geqslant 0} \|B'\|_{\ell^p(D_n),\ell^q(D_n)} < \infty. \tag{6}$$

We note that

$$\sup_{n\geqslant 0} \sum_{\mu\in D_n} a_\mu^q \left(\sum_{k=\mu}^{2^{n+1}-1} a_k^q \right)^{r/p} \max_{2^n\leqslant\nu\leqslant\mu} \left(\frac{b_\nu^r}{\nu} \right) < \infty \tag{7}$$

is equivalent to (4).

In order to prove the necessity and sufficiency of (4), it is enough to show that, for any factorable matrix $M(\boldsymbol{a},\boldsymbol{b})$, we have

$$\left\| M(\boldsymbol{a},\boldsymbol{b}) \right\|_{\ell^p,\ell^q}^{1/q} \asymp \left[\sum_{n=1}^{\infty} a_n^q \left(\sum_{k=n}^{\infty} a_k^q \right)^{r/p} \max_{1\leqslant k\leqslant n} b_k^r \right]^{1/r}, \tag{8}$$

where $M \asymp N$ means that there are positive constants k, K independent of $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $kM \leqslant N \leqslant KM$; and then use (6) and (7).

We first prove that (8) holds with the extra assumption that $\boldsymbol{a} \notin \phi$. To see this, by Observation 9.1 of [1, p. 50],

$$\left\| M(\boldsymbol{a},\boldsymbol{b}) \right\|_{\ell^p,\ell^q} = \|\boldsymbol{b}\|_{\mathcal{M}(\ell^p, c(\boldsymbol{a},1,q))} = \sup_{\|\boldsymbol{x}\|_p=1} \|\boldsymbol{b}\boldsymbol{x}\|_{c(\boldsymbol{a},1,q)},$$

where $\mathcal{M}(\ell^p, c(\boldsymbol{a},1,q)) = \{z\colon\ \boldsymbol{z}\boldsymbol{x} \in c(\boldsymbol{a},1,q),\ \forall \boldsymbol{x} \in \ell^p\}$ denotes the set of multipliers from ℓ^p to $c(\boldsymbol{a},1,q)$; and since $\boldsymbol{a} \notin \phi$,

$$c(\boldsymbol{a},1,q) = \left\{ \boldsymbol{y}\colon\ \|\boldsymbol{y}\|_{c(\boldsymbol{a},1,q)} = \sum_{n=1}^{\infty} a_n^q \left(\sum_{k=1}^{n} |y_k| \right)^q < \infty \right\}$$

is a complete q-normed K-space (see [1, p. 26]). By Theorem 7.7 of [1]

$$\mathcal{M}\big(\ell^p, c(\boldsymbol{a},1,q)\big) = c(\boldsymbol{h},\infty,r),$$

where

$$c(\boldsymbol{h},\infty,r) = \left\{ \boldsymbol{y}\colon\ \|\boldsymbol{y}\|_{c(\boldsymbol{h},\infty,r)} = \left[\sum_{n=1}^{\infty} h_n \max_{1\leqslant k\leqslant n} |y_k|^r \right]^{\min(1,1/r)} < \infty \right\}$$

and $h_n = a_n^q (\sum_{k=n}^{\infty} a_k^q)^{r/p}$, so that $c(\boldsymbol{h},\infty,r)$ is an r-normed space if $0 < r < 1$ and a normed space if $r \geqslant 1$.

We now use a slight extension of Theorem 15 of [6, p. 64].

Proposition 2. *Let X be a complete p-normed K-space with $\phi \subset X$ and Y be a complete q-normed K-space. Let $Z = \mathcal{M}(X,Y) = \{z\colon\ \boldsymbol{z}\boldsymbol{x} \in Y,\ \forall \boldsymbol{x} \in X\}$. Then Z is a complete q-normed K-space under the q-norm $\|z\| = \sup_{\|\boldsymbol{x}\|_X=1} \|\boldsymbol{z}\boldsymbol{x}\|_Y$.*

Proof. This follows exactly as in [6] making the necessary changes to take into account that we have complete p-normed and q-normed spaces instead of Banach spaces. For example, the set of continuous linear maps from X to Y, $B(X,Y)$ is a complete q-normed space under the q-norm $\|T\| = \sup_{\|\boldsymbol{x}\|_X=1} \|T\boldsymbol{x}\|_Y$ and since X is a p-normed K-space, for each $n \in \mathbb{N}$ the mapping $\boldsymbol{x} \mapsto x_n$ is continuous, so there exists $K_n \in \mathbb{R}$ such that $|x_n| \leqslant K_n \|\boldsymbol{x}\|_X^{1/p}$. □

By the remarks before Proposition 2 we see that $c(\boldsymbol{h},\infty,r)$ has two topologies defined on it: one topology generated by the q-norm on the multiplier space and the other generated by the r-norm or norm on $c(\boldsymbol{h},\infty,r)$. Under each of these topologies, $c(\boldsymbol{h}, \infty, r)$ is a complete metrisable K-space. Since the topology on such spaces is unique (see Corollary 4 of [6, p. 56] noting that the Closed Graph Theorem also holds between complete metrisable spaces) these topologies must be the same. Hence, the identity map $I : \mathcal{M}(\ell^p, c(\boldsymbol{a}, 1, q)) \to c(\boldsymbol{h}, \infty, r)$ and its inverse are both continuous, so that

$$\|I\boldsymbol{b}\|_{c(\boldsymbol{h},\infty,r)} = \|\boldsymbol{b}\|_{c(\boldsymbol{h},\infty,r)} \leqslant \|I\| \|\boldsymbol{b}\|^{\min(1/q,r/q)}_{\mathcal{M}(\ell^p,c(\boldsymbol{a},1,q))} \quad \text{and}$$
$$\left\|I^{-1}\boldsymbol{b}\right\|_{\mathcal{M}(\ell^p,c(\boldsymbol{a},1,q))} = \|\boldsymbol{b}\|_{\mathcal{M}(\ell^p,c(\boldsymbol{a},1,q))} \leqslant \left\|I^{-1}\right\| \|\boldsymbol{b}\|^{q/\min(1,r)}_{c(\boldsymbol{h},\infty,r)}.$$

Hence (8) holds in the case that $\boldsymbol{a} \notin \phi$, since $\frac{\min(1,1/r)}{\min(1,r)} = 1/r$.

In the case that $\boldsymbol{a} \in \phi$, although we can prove directly that (8) holds, since we have already shown on [5, p. 4] that $M(\boldsymbol{a}, \boldsymbol{b})$ maps $w_0(p)$ into $w_0(q)$ for arbitrary sequences $\boldsymbol{b}$, and it is clear that the conditions (2) and (4) hold for arbitrary $\boldsymbol{b}$, the result is proved. □

In conclusion we add that the methods used in this paper can also be used for the cases covered in [5] and so give an alternative approach to the main results proved there.

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