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The Cohomology of Restricted Lie Algebras and of Hopf Algebras[1,2]

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1. INTRODUCTION

In theory, the bar construction suffices to calculate the homology groups of an augmented algebra. In practice, the bar construction is generally too large (has too many generators) to allow computation of higher dimensional homology groups. In this paper, we develop a procedure which simplifies the calculation of the homology and cohomology of Hopf algebras.

Let A be a (graded) Hopf algebra over a field K of characteristic p. Filter A by $F_qA = A$ if $q \geqslant 0$ and $F_qA = I(A)^{-q}$ if $q < 0$, where $I(A)$ is the augmentation ideal. The associated bigraded algebra E^0A, $E^0_{q,r}A = (F_qA/F_{q-1}A)_{q+r}$, is clearly a primitively generated Hopf algebra over K. By a theorem due to Milnor and Moore [*10*], this implies that E^0A is isomorphic to the universal enveloping algebra of its restricted Lie algebra of primitive elements if $p > 0$, and to the universal enveloping algebra of its Lie algebra of primitive elements if $p = 0$.

We will develop machinery which takes advantage of this structural theorem to facilitate the computation of the cohomology of A. In order to use the structural theorem, we require first a procedure for calculating the cohomology of A, knowing that of E^0A. This is obtained by the construction of a spectral sequence passing from the cohomology of E^0A to that of A. Second, we require a method of calculating the cohomology of E^0A. This is obtained by the construction of a reasonably small canonical E^0A-free resolution of K.

The plan of the paper is as follows. After developing the properties of the bar and cobar constructions in Section III, we construct the desired spectral

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sequence in Section IV. This is done by filtering the bar construction of A in such a manner that the resulting E^1 is the bar construction of E^0A. We thereby obtain a spectral sequence passing from the homology of E^0A to that of A, and the dual spectral sequence passes from the cohomology of E^0A to that of A. In sections V and VI, which are completely independent of III and IV, we construct the E^0A-free resolution of K. We first construct a $U(L)$-free resolution of K, where $U(L)$ is the universal enveloping algebra of a Lie algebra L over K. If L is restricted, we then show how to extend the resulting $V(L)$-complex to obtain a $V(L)$-free resolution of K, where $V(L)$ is the universal enveloping algebra of the restricted Lie algebra L. Finally, in section VII we describe embeddings of these resolutions in the bar constructions of $U(L)$ and $V(L)$, respectively.

Now if A is a Hopf algebra over a field of characteristic p, then $E^0A = U(PE^0A)$ if $p = 0$ and $E^0A = V(PE^0A)$ if $p > 0$. We can calculate the homology of E^0A using the resolutions developed in Sections V and VI. The embeddings of these resolutions in the bar construction of E^0A give representative cycles there for elements of the homology of E^0A. Therefore these embeddings allow explicit computation of the differentials in the homology spectral sequence. Dualization then gives the structure of the cohomology spectral sequence.

The main results of this paper have been announced in [8]. The methods developed here have been used by the author to compute part of the cohomology of the Steenrod algebra. Due to the existence of the Adams' spectral sequence, this knowledge gives immediate corollaries on the stable homotopy groups of spheres. A partial summary of these results may be found in [9]. Details and complete results will appear in a Memoir of the Am. Math. Soc.

The author would like to express his deep gratitude to J. C. Moore, who suggested this approach to the problem of calculating the cohomology of Hopf algebras.

2. Preliminaries

Throughout this paper, K will denote a fixed commutative ring. For simplicity, we assume that all K-modules which arise, in any context whatever, are free of finite type. The symbols $\otimes$ and Hom without subscripts will mean the corresponding functors taken over K. All K-modules will be graded on the nonnegative integers, and if M and N are K-modules, then $M \otimes N$ is graded by $(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j$ and M^* is graded by $M^{*n} = \operatorname{Hom}(M_n, K)$.

Whenever two objects a and b are permuted, the sign $(-1)^{\deg a \deg b}$ will be introduced, the degree involved being the total degree if a or b belong to bigraded or trigraded K-modules. For example, if M and N are K-modules, we define isomorphisms:

$$f : M \to M^{**}, \qquad f(m)(\mu) = (-1)^{\deg m \deg \mu} \mu(m); \tag{2.1}$$

and

$$g : M^* \otimes N^* \to (M \otimes N)^*,$$
$$g(\mu \otimes \nu)(m \otimes n) = (-1)^{\deg \nu \deg m} \mu(m)\, \nu(n). \tag{2.2}$$

If X is a complex over K with boundary d, the boundary δ in X^* is defined by

$$\delta(\xi)(x) = (-1)^{\deg \xi + 1} \xi d(x), \qquad x \in X, \qquad \xi \in X^*. \tag{2.3}$$

$H(X^*) = H(X)^*$ and $H(X^*)^* = H(X)$. If X and Y are complexes, then $H(X) \otimes H(Y) = H(X \otimes Y)$.

By an algebra will be meant an augmented associative K-algebra, the term quasi-algebra being used for a not necessarily associative algebra. Similarly, coalgebra will mean augmented coassociative K-coalgebra, and quasi-coalgebra will mean not necessarily coassociative coalgebra. If A is an algebra, a K-module M may be given a structure of left A-module either by K-morphisms $\phi : A_q \otimes M_n \to M_{q+n}$ or by K-morphisms $\phi : A_q \otimes M_n \to M_{n-q}$. We must allow the second type of A-module structure for the dual of a left A-module to have a natural structure of right A-module. We note that the following are functorial equivalences on right A-modules N and left A-modules M:

$$\eta : (N \otimes_A M)^* \xrightarrow{\simeq} \mathrm{Hom}_A(N, M^*), \qquad \eta(f)(n)(m) = f(n \otimes m), \tag{2.4}$$

and

$$\eta' : (N \otimes_A M)^* \xrightarrow{\simeq} \mathrm{Hom}_A(M, N^*), \qquad \eta(f)(m)(n) = (-1)^{\deg m \deg n} f(n \otimes m). \tag{2.5}$$

If A is both a complex and an algebra, then A is a differential algebra if $d(xy) = d(x)\, y + (-1)^{\deg x} x d(y)$. If M is both a complex and a left A-module, then M is a differential A-module if $d(xm) = d(x)\, m + (-1)^{\deg x} x d(m)$. Differential coalgebras and comodules are defined dually.

The reader is referred to Milnor and Moore [*12*] for the definitions and properties of (graded) Hopf algebras, Lie algebras, and restricted Lie algebras. We state here a version of the Poincaré-Birkhoff-Witt theorem which will be needed in section VI.

THEOREM 1. *Let L be a Lie algebra or a restricted Lie algebra over K. Let $Z(L)$ denote the universal enveloping algebra $U(L)$ or $V(L)$. Filter $Z(L)$ by $F_iZ(L) = 0$ if $i < 0$, $F_0Z(L) = K$, $F_1Z(L) = K + L$, and $F_iZ(L) = (F_1Z(L))^i$*

if $i > 1$. Let $L^{\#}$ denote the underlying K-module of L regarded as an Abelian Lie algebra, restricted with restriction zero if L is restricted. Since $U(L^{\#})$ is the free commutative algebra generated by L and

$$\sum_s E^0_{1,s}Z(L^{\#}) = L = \sum_s E^0_{1,s}Z(L),$$

there is a natural map $E^0Z(L^{\#}) \to E^0Z(L)$. This map is an isomorphism of Hopf algebras, and it follows that $Z(L)$ and $Z(L^{\#})$ are isomorphic as K-modules.

A proof of the theorem may be found in [*10*].

3. The Bar and Cobar Constructions

In this section, A will denote a fixed algebra over K. $\epsilon : A \to K$ will denote the augmentation, and $I(A) = \operatorname{Ker} \epsilon$.

We recall the definition of the two-sided bar construction. Let $B(A, A) = A \otimes T(sI(A)) \otimes A$, where $sI(A)$ denotes a copy of $I(A)$ with all elements being given a second degree, the (homological) dimension, of one. We write elements of $B(A, A)$ in the form $a[a_1 \mid \cdots \mid a_n]\, b$. Define $\epsilon : B(A, A) \to A$ by $\epsilon(a[\ ]\, b) = ab$ and $\epsilon(a[a_1 \mid \cdots \mid a_n]\, b) = 0$. Define $s : B(A, A) \to B(A, A)$ *and* $\sigma : A \to B(A, A)$ by

$$s(a[a_1 \mid \cdots \mid a_n]\, b) = [a \mid a_1 \mid \cdots \mid a_n]\, b; \qquad \sigma(a) = [\ ]\, a. \tag{3.1}$$

Define $t : B(A, A) \to B(A, A)$ and $\tau : A \to B(A, A)$ by

$$t(a_0[a_1 \mid \cdots \mid a_n]\, b) = (-1)^p\, a_0[a_1 \mid \cdots \mid a_n \mid b],$$

$$p = \sum_{i=0}^{n} (\deg a_i + 1); \qquad \tau(a) = a[\ ]. \tag{3.2}$$

Now define $d : B(A, A) \to B(A, A)$ by either $ds + sd = 1 - \sigma\epsilon$ or $dt + td = 1 - \tau\epsilon$, and by requiring d to be an A-A-bimodule morphism, $d(axb) = (-1)^{\deg a}\, ad(x)\, b$, $x \in T(sIA)$. Explicit calculation shows that the same differential is obtained by either definition, namely:

$$d[a_1 \mid \cdots \mid a_n] = a_1[a_2 \mid \cdots \mid a_n] + \sum_{i=1}^{n-1} (-1)^{\lambda(i)}\, [a_1 \mid \cdots \mid a_i a_{i+1} \mid \cdots a_n]$$
$$- (-1)^{\lambda(n-1)}\, [a_1 \mid \cdots \mid a_{n-1}]\, a_n\,, \tag{3.3}$$

where

$$\lambda(i) = i + \sum_{j=1}^{i} \deg a_i = \deg\, [a \mid \cdots \mid a_i].$$

Let M be a left, N a right A-module. Since s is a morphism of right A-modules, $B(A, M) = B(A, A) \otimes_A M$ is an A-free resolution of M. Similarly, since t is a morphism of left A-modules, $B(N, A) = N \otimes_A B(A, A)$ is an A-free resolution of N. Let $B(N, M) = N \otimes_A B(A, A) \otimes_A M$. We define the homology and cohomology of A with coefficients in M or N by:

$$\begin{aligned} H_*(A; M) &= H(B(K, M)) = \operatorname{Tor}^A(K, M) \\ H_*(A; N) &= H(B(N, K)) = \operatorname{Tor}^A(N, K). \end{aligned} \tag{3.4}$$

$$\begin{aligned} H^*(A; M) &= H(\operatorname{Hom}_A(B(A, K), M)) = \operatorname{Ext}_A(K, M) \\ H^*(A; N) &= H(\operatorname{Hom}_A(B(K, A), N)) = \operatorname{Ext}_A(K, N). \end{aligned} \tag{3.5}$$

We abbreviate $H_*(A) = H_*(A, K)$ and $H^*(A) = H^*(A, K)$. Here K may be regarded as either a trivial left or a trivial right A-module, since 2.4 and 2.5 imply functorial equivalences

$$H^*(A; M^*) \cong H(B(K, M)^*) \cong \operatorname{Ext}_A(M, K).$$

Choosing bases for $I(A)^*$ and M^* dual to given bases for $I(A)$ and M and noting that $C(K, M) = T(sI(A)^*) \otimes M^*$, we may write elements of $C(K, M)$ in the form $[\alpha_1 \mid \cdots \mid \alpha_n]\mu$, $\alpha_i \in I(A)^*$, $\mu \in M^*$. Using 2.2 and 2.3, we find that the coboundary on $C(K, M)$ is given by

$$\begin{aligned} \delta[\alpha_1 \mid \cdots \mid \alpha_n]\mu = &-\sum_{i=1}^{n}(-1)^{\lambda(i)}[\alpha_1 \mid \cdots \mid \alpha_i' \mid \alpha_i'' \mid \cdots \mid \alpha_n]\mu \\ &-(-1)^{\lambda(n+1)}[\alpha_1 \mid \cdots \mid \alpha_n \mid \alpha']\mu', \end{aligned} \tag{3.6}$$

where

$$\lambda(i) = \deg[\alpha_1 \mid \cdots \mid \alpha_i'], \qquad \lambda(n+1) = \deg[\alpha_1 \mid \cdots \mid \alpha_n \mid \alpha'].$$

Here the coproduct ϕ^* on $I(A)^*$ is given by $\phi^*(\alpha_i) = \Sigma\, \alpha_i' \otimes \alpha_i''$ and the A^*-comodule structure $\phi_m{}^*$ on M^* is given by $\phi_m{}^*(\mu) = \Sigma\, \alpha' \otimes \mu'$, where in both cases the index of summation is understood and in the formula for δ, summation over each such sum is understood. $C(K, M)$ is of course the cobar construction.

For the remainder of this section, we suppose that A is a Hopf algebra with coproduct ψ. In this case we can give $B(A, A)$ a structure of differential coalgebra. We use ψ to give $B(A, A)$ a structure of A-A-bimodule. If we let $\partial = d \otimes 1 + 1 \otimes d$ and $\epsilon' = \epsilon \otimes \epsilon$, then $B(A, A) \otimes B(A, A)$ becomes an A-A-bimodule complex over $A \otimes A$. If $S = s \otimes 1 + \sigma\epsilon \otimes s$, $\sigma' = \sigma \otimes \sigma$, $T = t \otimes \tau\epsilon + 1 \otimes t$, and $\tau' = \tau \otimes \tau$, then both $S\partial + \partial S = 1 - \sigma'\epsilon'$ and $\partial T + T\partial = 1 - \tau'\epsilon'$. Define $D : B(A, A) \to B(A, A) \otimes B(A, A)$ by either $Ds = SD$ or $Dt = TD$, by $D[\,] = [\,] \otimes [\,]$, and by requiring D to be an A-A-bimodule morphism. Explicit calculation shows that the same D is obtained by both definitions and that $\partial D = Dd$. D is given by

$$D[a_1 | \cdots | a_n] = \sum_{r=0}^{n} (-1)^{\mu(r)} [a_1' | \cdots | a_r'] a_{r+1}' \cdots a_n' \otimes a_1'' \cdots a_r''[a_{r+1}'' | \cdots | a_n''], \quad (3.7)$$

where

$$\mu(r) = \sum_{i<j} \deg a_i'' \deg a_j' + \sum_{i=1}^{r} (r-i) \deg a_i'' + \sum_{j=r+1}^{n} (j-r) \deg a_j'.$$

Here $\psi(a_i) = \Sigma\, a_i' \otimes a_i''$, the index of summation being understood, and summation over each such sum is understood in the expression for D. We remark that D may be considered as the composition:

$$B(A, A) \xrightarrow{f} B(A \otimes A, A \otimes A) \xrightarrow{g} B(A, A) \otimes B(A, A),$$

where f is induced in the obvious way by ψ, and where g is given by formula (3.7) under the interpretation $a_i = a_i' \otimes a_i''$ (with no summation); of course g is defined for any augmented algebra.

D induces a coproduct on each of $B(A, K)$, $B(K, A)$, and $B(K, K)$. The coproduct on $B(A, K)$ agrees with that defined by Adams [*1*, p. 35]. If M is a left A-module, D induces a structure of left differential $B(K, K)$-comodule on $B(K, M)$. Then $H_*(A; M)$ becomes a (left) $H_*(A)$-comodule.

Dualization gives $C(K, M)$ a structure of differential left $C(K, K)$-module. The product is denoted by $\cup$ and is given by:

$$[\alpha_1 | \cdots | \alpha_p] \cup [\beta_1 | \cdots | \beta_q]\, \mu = [\alpha_1 | \cdots | \alpha_p | \beta_1 | \cdots | \beta_q]\, \mu. \quad (3.8)$$

This cup product of cochains induces a structure of (left) $H^*(A)$-module on $H^*(A; M^*)$. The dual of the $H^*(A)$-module structure on $H^*(A; M^*)$ is the $H_*(A)$-comodule structure on $H_*(A; M)$.

Note that the product (3.8) actually commutes with the differential even without the assumption that A is a Hopf algebra. When A is a Hopf algebra, the resulting product on $H^*(A)$ is commutative.

4. The Spectral Sequences

In this section, A denotes a filtered K-algebra and M a filtered left A-module. If M is any A-module, M may be filtered by $F_pM = (F_pA) \cdot M$. We let E^0M denote the associated bigraded E^0A-module of M,

$$E^0_{p,q}M = \left(\frac{F_pM}{F_{p-1}M}\right)_{p+q}.$$

We assume that for $N = A$ and $N = M$:

$$F_pN = N \quad \text{if} \quad p \geqslant 0, \quad F_{-1}A = I(A), \quad \text{and} \quad \bigcap_p F_pN = 0 \quad (4.1)$$

or

$$F_pN = 0 \quad \text{if} \quad p < 0, \quad F_0A = K, \quad \text{and} \quad \bigcup_P F_pN = N. \quad (4.1')$$

Under these hypotheses, we can obtain a spectral sequence passing from $H_*(E^0A; E^0M)$ to $H_*(A; M)$. Let $B(A; M)$ denote $B(K, M)$. Filter $B(A; M) = T(sI(A)) \otimes M$ by

$$F_pB(A; M) = F_{p_1}I(A) \otimes \cdots \otimes F_{p_n}I(A) \otimes F_{p_0}M \text{ summed}$$

$$\text{over all sequences } \{p_0, \cdots, p_n\} \text{ such that } n + \sum_{i=0}^{n} p_i \leqslant p$$

$$\text{and, in case (4.1), } p_0 \leqslant 0 \text{ and } p_i \leqslant -1 \text{ for } i > 0. \quad (4.2)$$

Since $F_pA \cdot F_qA \subset F_{p+q}A$, it is clear that $d(F_pB(A; M)) \subset F_{p-1}B(A; M)$, so that $E^0 = E^1$ in the resulting spectral sequence. We consider the spectral sequence to commence with E^1, and we denote the rth term by $E^r(M)$, $r \geqslant 1$. (The symbol E^0M will continue to denote the associated bigraded module of M). Observe that $E^1(M)$ is trigraded, with

$$E^1_{p,q,t}(M) = \left[\frac{F_pB_{p+q}(A; M)}{F_{p-1}B_{p+q}(A; M)}\right]_t,$$

where $p + q$ is the homological dimension. As a K-module, we may identify $E^1_{p,q,t}(M)$ with $B_{p+q}(E^0A; E^0M)_{-q,q+t}$, where $B_{p+q}(E^0A; E^0M)$ has its natural bigrading. It is convenient to regrade $B(E^0A; E^0M)$ by

$$B(E^0A; E^0M)_{p,q,t} = B_{p+q}(E^0A; E^0M)_{-q,q+t},$$

and then $E^1(M) = B(E^0A; E^0M)$ as a trigraded K-module. By using the definition of the differential in the bar construction (formula 3.3), it is easily verified that the differential in $E^1(M)$ is the same as that in $B(E^0A; E^0M)$. It follows that $E^2(M) = H_*(E^0A; E^0M)$ (with the appropriate grading). Thus we have

THEOREM 3. *Let A be a filtered K-algebra and M a filtered left A-module. Suppose A and M satisfy (4.1) or (4.1′). Then the filtration (4.2) of $B(A; M)$ gives rise to a spectral sequence $\{E^r(M)\}$ which converges to $H_*(A; M)$ and satisfies*:

$$E^2_{p,q,t}(M) = H_{p+q}(E^0A; E^0M)_{-q,q+t}.$$

Proof. We need only observe that (4.1) or (4.1′) implies that the filtration (4.2) is finite in each degree and that the spectral sequence therefore converges.

We now obtain the dual spectral sequence passing from $H^*(E^0A; (E^0M)^*)$ to $H^*(A; M^*)$. $F^pN^* = (N/F_{p-1}N)^*$ defines a filtration on N^*, $N = A$ and $N = M$, and the resulting associated bigraded objects are $(E^0A)^*$ and $(E^0M)^*$. Let $C(A; M)$ denote $B(A; M)^*$ and filter $C(A; M) = T(sI(A)^*) \otimes M^*$ by

$$F^pC(A; M) = F^{p_1}I(A)^* \otimes \cdots \otimes F^{p_n}I(A)^* \otimes F^{p_0}M^* \text{ summed}$$

$$\text{over all sequences } \{p_0, \cdots, p_n\} \text{ such that } n + \sum_{i=0}^{n} p_i \geqslant p$$

$$\text{and, in case (4.1′), } p_0 \geqslant 0 \text{ and } p_i \geqslant 1 \text{ if } i > 0. \tag{4.3}$$

This is equivalent to $F^pC(A; M) = [B(A; M)/F_{p-1}B(A; M)]^*$. $E_0 = E_1$ in the resulting spectral sequence. We consider the spectral sequence to commence with E_1, and we denote the rth term by $E_r(M)$, $r \geqslant 1$. Trigrading $C(E^0A; E^0M)$ by $C[E^0A; E^0M)_{p,q,t} = [B(E^0A; E^0M)_{p,q,t}]^*$, we find that $E_1(M) = C(E^0A; E^0M)$ as a trigraded K-module. It is easily verified that the differential in $E_1(M)$ is the same as that in $C(E^0A; E^0M)$. Therefore

$$E_2(M) = H^*(E^0A; (E^0M)^*)$$

(with the appropriate grading).

Arguing as in Massey [*6*], we find that each $E_r(M)$ is a differential $E_r(K)$-module. Clearly this structure is dual to an $E^r(K)$-comodule structure on $E^r(M)$. The spectral sequence $\{E_r(M)\}$ converges to $H^*(A; M^*)$, regarded as an $H^*(A)$-module. This means that if $E^0H^*(A; M^*)$ denotes the associated graded object of $H^*(A; M^*)$ with respect to the filtration

$$F^pH^*(A; M^*) = \text{Image } [H(F^pC(A; M))],$$

then

$$E_\infty(M) = E^0H^*(A; M^*)$$

and the $E_\infty(K)$-module structure on $E_\infty(M)$ agrees with the $E^0H^*(A)$-module structure on $E^0H^*(A; M^*)$ obtained by passing to quotients from the $H^*(A)$-module structure on $H^*(A; M^*)$. Summarizing, we have the

THEOREM 4. *Let A be a filtered K-algebra and M a filtered left A-module. Suppose A and M satisfy (4.1) or (4.1′). Then the filtration (4.3) of $C(A; M)$ gives rise to a spectral sequence $\{E_r(M)\}$ which satisfies:*

(i) *Each $E_r(M)$ is a differential $E_r(K)$-module.*

(ii) $E_2(M) = H^*(E^0A; (E^0M)^*)$ *as a module over* $E_2(K) = H^*(E^0A)$.

(iii) $\{E_r(M)\}$ *converges to* $H^*(A; M^*)$, *regarded as an* $H^*(A)$*-module.*

(iv) *The spectral sequences* $\{E^r(M)\}$ *and* $\{E_r(M)\}$ *are dual to each other.*

5. The Homology of Lie Algebras

In this section, L will denote a (graded or bigraded) Lie algebra over K and $U(L)$ will denote its universal enveloping algebra.

We will construct a $U(L)$-free resolution of K regarded as a trivial right $U(L)$-module. As a K-module, our resolution will be $Y(L) = \bar{Y}(L) \otimes U(L)$, where $\bar{Y}(L) = \Gamma(sL^-) \otimes E(sL^+)$. Here Γ denotes a divided polynomial algebra and E an exterior algebra. L^- denotes the K-submodule of L consisting of the elements of odd (total) degree and L^+ denotes the K-submodule of even degree elements. sL denotes a copy of L with all elements being given a new degree, the (homological) dimension, of one. We adopt the convention that $L^+ = L$ and L^- is void if $2 = 0$ in K.

In the next section, we will rely heavily on the fact that $Y(L)$ can be given a structure of differential K-algebra. Our procedure will be to first define an algebra structure on $Y(L)$ and then to impose the appropriate differential. The author is indebted to W. S. Massey for suggesting the approach to be followed in this section.

It should be more or less clear that the standard algebra structure on $Y(L)$ is not appropriate for our purposes. We will need the concept of semi-tensor product, introduced by Massey and Peterson [7], and defined as follows. Suppose B is an algebra over a Hopf algebra A. This means that B is a left A-module and that the product on B is a morphism of A-modules, where the coproduct ψ on A is used to define the A-module structure on $B \otimes B$. Then $B \otimes A$ may be given a product by the formula:

(a) $$(b_1 \otimes a_1)(b_2 \otimes a_2) = \sum (-1)^{\deg a_1'' \deg b_2} b_1(a_1' b_2) \otimes a_1'' a_2 ,$$

where

$$\psi(a_1) = \sum a_1' \otimes a_1'' .$$

The resulting object is an algebra called the semi-tensor product of B and A and denoted $B \odot A$. A proof that the product is associative may be found in the cited paper of Massey and Peterson. B and A are imbedded as subalgebras of $B \odot A$ via $b \to b \otimes 1$ and $a \to 1 \otimes a$. The product is completely determined by the formula for $(1 \otimes a)(b \otimes 1)$, where a and b are indecomposable elements of A and B, and by the products on A and B.

To apply this concept to $Y(L)$, we must give $\bar{Y}(L)$ a structure of algebra over the Hopf algebra $U(L)$. If suffices to define $u\langle y\rangle$ and $u\gamma_r(x)$ for $u \in L$, $\langle y\rangle \in E(sL^+)$ and $\gamma_r(x) \in \Gamma(sL^-)$. Then the definition is extended to $\bar{Y}(L)$ by

(b) $uz_1z_2 = (uz_1)\, z_2 + (-1)^{\deg u \deg z_1}\, z_1(uz_2), \qquad z_1, z_2 \in \bar{Y}(L)\; u \in L.$

To simplify statements, we will sometimes write sz for an element of $\bar{Y}(L)$ of dimension one. Now define

(c) $u\langle y\rangle = (-1)^{\deg u}\, s[u, y]; \qquad u\gamma_r(x) = (-1)^{\deg u}\, \gamma_{r-1}(x)\, s[u, x].$

It is easily seen that (b) and (c) are consistent on $\gamma_i(x)\,\gamma_j(x) = (i, j)\,\gamma_{i+j}(x)$. We must prove that (b) and (c) define a structure of L-module on $\bar{Y}(L)$, that is,

$$u_1(u_2z) - (-1)^{\deg u_1 \deg u_2}\, u_2(u_1z) = [u_1, u_2]\, z, \qquad z \in \bar{Y}(L).$$

Using (b), we find that it suffices to prove this for $z = \langle y\rangle$ and $z = \gamma_r(x)$. In these cases, the result is obtained by explicit calculations using the Jacobi identity:

$$(-1)^{pr}\,[x, [y, z]] + (-1)^{qp}\,[y[z, x]] + (-1)^{rq}\,[z, [x, y]] = 0,$$
$$x, y, z \in L, \qquad \deg x = p, \qquad \deg y = q, \qquad \deg z = r.$$

Since $U(L)$ is primitively generated, the elements of L being primitive, it follows that (b) and (c) do give $\bar{Y}(L)$ a structure of algebra over the Hopf algebra $U(L)$.

Now $Y(L)$ may be considered as the K-algebra $\bar{Y}(L) \odot U(L)$. We identify $\bar{Y}(L)$ and $U(L)$ with their images in $Y(L)$ and write the product by juxtaposition. Thus the symbol uz, $u \in U(L)$, $z \in \bar{Y}(L)$ will henceforward mean the product $(1 \otimes u)(z \otimes 1)$ in $\bar{Y}(L) \odot U(L)$ and not the $U(L)$-module product in $\bar{Y}(L)$, the latter structure being of no further direct concern to us. We can now state the following lemma and theorem:

Lemma 4. *$Y(L)$ may be given an algebra structure by requiring the product to agree with the natural one on $\bar{Y}(L)$ and on $U(L)$ and to satisfy the relations:*

$$u\langle y\rangle = (-1)^{\deg u}\,\langle y\rangle\, u + (-1)^{\deg u}\, s[u, y], \qquad u \in L,$$
$$\langle y\rangle = sy \in sL^+. \tag{5.1}$$

$$u\gamma_r(x) = \gamma_r(x)\, u + (-1)^{\deg u}\,\gamma_{r-1}(x)\, s[u, x], \qquad u \in L,$$
$$\gamma_1(x) = sx \in sL^-. \tag{5.2}$$

$Y(L)$ may be given a Hopf algebra structure with coproduct D by requiring D to be a morphism of algebras and to agree with the natural coproduct on $\bar{Y}(L)$ and on $U(L)$.

Proof. The lemma correctly describes the algebra $Y(L) = \bar{Y}(L) \odot U(L)$, relations (5.1) and (5.2) following from (a) and (c).

THEOREM 5. *Define a differential d on $Y(L)$ by*

$$d(ab) = d(a)\, b + (-1)^{\deg a}\, a\, d(b), \qquad a, b \in Y(L), \tag{5.3}$$

and

$$d(u) = 0, \qquad d\langle y\rangle = y, \qquad d\gamma_r(x) = \gamma_{r-1}(x)\, x - \tfrac{1}{2}\gamma_{r-2}(x)\, \langle [x, x]\rangle, \tag{5.4}$$

where $u \in U(L)$, $\langle y\rangle \in sL^+$, $\gamma_1(x) \in sL^-$, and $\gamma_{-1}(x) = 0$.

Then $Y(L)$ is a $U(L)$-free resolution of K, and is also a differential coalgebra over $U(L)$.

Proof. d is easily seen to be well-defined, that is, d is consistent with relations (5.1) and (5.2). Clearly d and D are morphisms of right $U(L)$-modules. Explicit calculation gives $d^2 = 0$ and $Dd = (d \otimes 1 + 1 \otimes d)\, D$ on generators of $\bar{Y}(L)$, noting for the latter that

$$D\langle y\rangle = \langle y\rangle \otimes 1 + 1 \otimes \langle y\rangle$$

and

$$D\gamma_r(x) = \sum_{i+j=r} \gamma_i(x) \otimes \gamma_j(x).$$

It follows that $d^2 = 0$ and $Dd = (d \otimes 1 + 1 \otimes d)\, D$ on $Y(L)$. We omit the proof of exactness, as it is quite similar to that given in Cartan and Eilenberg [3, p. 281] for the classical case of Lie algebras concentrated in degree zero and to the proof to be given in the next section for restricted Lie algebras.

We remark that $Y(L)$ is a differential Hopf algebra over K, but is not an algebra over $U(L)$. Had we started with $Y(L) = U(L) \otimes \bar{Y}(L)$, the lemma and theorem would still be true precisely as stated. An explicit formula for d is easily obtained. Let $f = \gamma_{r_1}(x_1) \cdots \gamma_{r_m}(X_m)$ and $g = \langle y_1, \cdots, y_n\rangle$ denote typical elements of $\Gamma(sL^-)$ and $E(sL^+)$. Let f_i result from f by replacing r_i by $r_i - 1$, and let $f_{i,j} = (f_j)_i$, $i \leqslant j$, unless $i = j$ and $r_i = 1$, when $f_{i,i} = 0$. Let g_i result from g by omission of y_i, and let $g_{i,j} = (g_j)_i$, $i < j$. With this notation, we have:

$$\begin{aligned} d(fg) = {} & \sum_{i=1}^{n} (-1)^{i+1} f g_i y_i + (-1)^n \sum_{j=1}^{m} f_j g x_j \\ & + \tfrac{1}{2}(-1)^{n+1} \sum_{j=1}^{m} f_{j,j} g \langle [x_j, x_j]\rangle \\ & + (-1)^{n+1} \sum_{1 \leqslant i < j \leqslant m} f_{i,j} g \langle [x_i, x_j]\rangle \\ & + (-1)^{n+1} \sum_{1 \leqslant i < j \leqslant n} (-1)^{i+j} f g_{i,j} \langle [y_i, y_j]\rangle \\ & + \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^j f_i \gamma_1([x_i, y_j])\, g_j . \end{aligned} \tag{5.5}$$

6. The Homology of Restricted Lie Algebras

In this section, K will be an algebra over Z_p for some prime number p. L will denote a (graded or bigraded) restricted Lie algebra over K with restriction (pth power) $\xi : L^+ \to L^+$. $V(L)$ will denote the universal enveloping algebra of L. Recall that $V(L) = U(L)/J$, where J is the ideal in $U(L)$ generated by $\{y^p - \xi(y) \mid y \in L^+\}$.

Let $W(L) = \bar{Y}(L) \otimes V(L)$. Recalling the definition of $\bar{Y}(L)$ as an algebra over the Hopf algebra $U(L)$, it is easily verified that $J \cdot \bar{Y}(L) = 0$, and therefore $\bar{Y}(L)$ may be considered as an algebra over the Hopf algebra $V(L)$. Then $W(L)$ has an algebra structure as the semi-tensor product $\bar{Y}(L) \odot V(L)$. Lemma 4 is true with $W(L)$ replacing $Y(L)$ and $V(L)$ replacing $U(L)$. If we define a differential on $W(L)$ by 5.3 and 5.4, $W(L)$ becomes a $V(L)$-free complex over K, and 5.5 remains valid. $W(L)$ is a coalgebra over $V(L)$ and a Hopf algebra over K. $W(L)$ is not a resolution of K since $\langle y \rangle y^{p-1} - \langle \xi(y) \rangle$ is a nonbounding cycle, $y \in L^+$. We wish to enlarge $W(L)$ to obtain a resolution. Let $s^2\pi L^+$ denote a copy of L^+ with degrees multiplied by p and with all elements having (homological) dimension two. As a K-module, our resolutions will be

$$X(L) = \Gamma(s^2\pi L^+) \otimes W(L)$$

and, writing $\tilde{y} = s^2\pi y$, we will have

$$d\gamma_1(\tilde{y}) = \langle y \rangle y^{p-1} - \langle \xi(y) \rangle.$$

If L is Abelian with restriction zero, then

$$H_*(V(L)) = \Gamma(s^2\pi L^+) \otimes \bar{Y}(L),$$

and therefore no smaller resolution could be obtained canonically.

The construction[3] of $X(L)$ is a simple application of the theory of twisted tensor products developed by Brown in [2]. We recall the definition. Let B be a differential coalgebra with coproduct D, let G be a differential algebra with product π, and let F be a left differential G-module with module product σ. Let $R = \mathrm{Hom}\,(B, G)$ and give R a structure of differential algebra with differential δ and product $\cup$ defined by

$$\delta(r)\,(b) = d(r(b)) + (-1)^{\deg r + 1}\, r(d(b)) \tag{6.1}$$

and

$$(r \cup r')\,(b) = \pi(r \otimes r')\, D(b). \tag{6.2}$$

[3] The construction of $X(L)$ given here replaces that outlined in [8]. N. Shimada pointed out an error in the previous construction.

Then give $B \otimes F$ a structure of left differential R-module with module product $\cap$ defined by

$$r \cap (b \otimes f) = (1 \otimes \sigma)(1 \otimes r \otimes 1)(D \otimes 1)(b \otimes f). \tag{6.3}$$

The statement as to differential algebra and module structure on R and $B \otimes F$ are easily proven by verifying the identities:

$$(r \cup r') \cup r'' = r \cup (r' \cup r''), \tag{6.4}$$

$$\delta(r \cup r') = \delta(r) \cup r' + (-1)^{\deg r}\, r \cup \delta(r'), \tag{6.5}$$

$$(r \cup r') \cap (b \otimes f) = r \cap (r' \cap (b \otimes f)), \tag{6.6}$$

and

$$d(r \cap (b \otimes f)) = \delta(r) \cap (b \otimes f) + (-1)^{\deg r}\, r \cap d(b \otimes f). \tag{6.7}$$

Now let $t \in R^1$, so that $t_n : B_n \to G_{n-1}$, $n \geqslant 1$.

Define $d_t : B \otimes F \to B \otimes F$ by

$$d_t(b \otimes f) = d(b \otimes f) + t \cap (b \otimes f). \tag{6.8}$$

Using (6.6) and (6.7), we find $d_t^2(b \otimes f) = (\delta(t) + t \cup t) \cap (b \otimes f)$. t is said to be a twisting cochain if $\epsilon t_1 = 0$, $\epsilon : G \to K$, and if $\delta(t) + t \cup t = 0$, that is, if

$$dt_n + t_{n-1}d + \sum_{i=1}^{n-1} t_i \cup t_{n-i} = 0, \qquad n > 1. \tag{6.9}$$

Then $B \otimes F$ furnished with the differential d_t is called a twisted tensor product and is denoted by $B \otimes_t F$.

Under additional hypotheses on B, F, and G, we can formulate a procedure for defining a structure of differential quasi-coalgebra on $B \otimes_t F$. First, give $F \otimes F$ a structure of left differential $G \otimes G$-module with module product $(\sigma \otimes \sigma)(1 \otimes T \otimes 1)$ where $(\pi \otimes \pi)(1 \otimes T \otimes 1)$ defines the product on $G \otimes G$. Then if $S = \operatorname{Hom}(B, G \otimes G)$, we can give $B \otimes (F \otimes F)$ a structure of left S-module precisely as above. Now suppose that F is a differential coalgebra and define

$$\Lambda : S \otimes (B \otimes F) \to B \otimes F \otimes B \otimes F$$

by

$$s\Lambda(b \otimes f) = (1 \otimes T \otimes 1)(D \otimes 1 \otimes 1)(s \cap (b \otimes D(f))). \tag{6.10}$$

Then Λ is a morphism of complexes, that is,

$$d(s\Lambda(b \otimes f)) = \delta(s)\, \Lambda(b \otimes f) + (-1)^{\deg s}\, s\Lambda d(b \otimes f). \tag{6.11}$$

Of course, the tensor product of coalgebras is a coalgebra with coproduct

$(1 \otimes T \otimes 1)(D \otimes D)$. Suppose that G is a differential Hopf algebra and that F is a left G-module coalgebra, so that

$$(\pi \otimes \pi)(1 \otimes T \otimes 1)(D \otimes D) = D\pi : G \otimes G \to G \otimes G$$

and

$$(\sigma \otimes \sigma)(1 \otimes T \otimes 1)(D \otimes D) = D\sigma : G \otimes F \to F \otimes F.$$

Let ϕ, ϕ_r, and ϕ_ℓ denote the morphisms of differential algebras $R \to S$ defined by $\phi(r)(b) = D(r(b))$, $\phi_r(r)(b) = 1 \otimes r(b)$, and $\phi_\ell(r)(b) = r(b) \otimes 1$. Tedious calculations then yield

$$D(r \cap (b \otimes f)) = \phi(r) A(b \otimes f), \tag{6.12}$$

$$sA(r \cap (b \otimes f)) = (s \cup \phi(r)) A(b \otimes f), \tag{6.13}$$

$$(1 \otimes 1 \otimes r \cap) D(b \otimes f) = \phi_r(r) A(b \otimes f), \tag{6.14}$$

$$(1 \otimes 1 \otimes r \cap)(sA(b \otimes f)) = (\phi_r(r) \cup s) A(b \otimes f), \tag{6.15}$$

and, provided that B is cocommutative,

$$(r \cap \otimes 1 \otimes 1) D(b \otimes f) = \phi_\ell(r) A(b \otimes f), \tag{6.16}$$

and

$$(r \cap \otimes 1 \otimes 1)(sA(b \otimes f)) = (\phi_\ell(r) \cup s) A(b \otimes f). \tag{6.17}$$

Now let $s \in S^0$, $s_n : B_n \to (G \otimes G)_n$, and suppose $s_0 = 0$. Define

$$D_s = D + sA : B \otimes_t F \to (B \otimes_t F) \otimes (B \otimes_t F).$$

If G is a Hopf algebra, F is a left G-module coalgebra, and B is a cocommutative coalgebra, then, letting $\bar{\phi} = \phi - \phi_\ell - \phi_r$, formulas (6.10) through (6.17) imply:

$$((d_t \otimes 1 + 1 \otimes d_t) D_s - D_s d_t)(b \otimes f)$$
$$= [\delta(s) - \bar{\phi}(t) - s \cup \phi(t) + \phi_\ell(t) \cup s + \phi_r(t) \cup s] A(b \otimes f).$$

We say that s is a t-twisting coproduct if the bracketed element of S vanishes, that is, if

$$\delta(s_n) = \bar{\phi}(t_n) + \sum_{i=1}^{n-1} (s_i \cup \phi(t_{n-i}) - \phi_\ell(t_{n-i}) \cup s_i - \phi_r(t_{n-i}) \cup s_i), \tag{6.18}$$

and then D_s gives $B \otimes_t F$ a structure of differential quasi-coalgebra.

To apply this theory to the construction of $X(L)$, we let $B = \Gamma(s^2\pi L^+)$, $G = Y(L^+)$, and $F = W(L)$. $\Gamma(s^2\pi L^+)$ is given its natural coproduct and zero differential. The left differential $Y(L^+)$-module structure on $W(L)$ is determined by the epimorphism of differential algebras $Y(L^+) \to W(L^+)$ and by the algebra structure of $W(L)$. We must define a twisting cochain

$t : \Gamma(s^2\pi L^+) \to Y(L^+)$, that is, we require $t_n : \Gamma_n(s^2\pi L^+) \to Y_{n-1}(L^+)$ satisfying (6.9). (Of course, t_n is to be of degree zero with respect to the grading derived from that of L.) Note that $t_n = 0$ if n is odd, since $\Gamma_n(s^2\pi L^+)$ is then zero.

LEMMA 6. *There exists a twisting cochain t: $\Gamma(s^2\pi L^+) \to Y(L^+)$ such that*

$$t_2(\gamma_1(\bar{y})) = \langle y \rangle y^{p-1} - \langle \xi(y) \rangle, \qquad y \in L^+.$$

Proof. We must define t_{2n}, $n \geqslant 2$, so as to have $dt_{2n} = -r$, where

$$r = \sum_{i=1}^{n-1} t_{2i} \cup t_{2(n-i)} .$$

Suppose inductively that t_{2i} has been defined for $i < n$. Since $Y(L^+)$ is acyclic, it clearly suffices to prove that $d(r(b)) = 0$ for all $b \in \Gamma_{2n}(s^2\pi L^+)$. We have

$$\begin{aligned} dr &= \sum_{i=1}^{n-1} dt_{2i} \cup t_{2(n-i)} - \sum_{i=1}^{n-1} t_{2i} \cup dt_{2(n-i)} \\ &= dt_2 \cup t_{2(n-1)} - \sum_{i=2}^{n-1} \sum_{k=1}^{i-1} t_{2k} \cup t_{2(i-k)} \cup t_{2(n-i)} \\ &\qquad - t_{2(n-1)} \cup dt_2 + \sum_{i=1}^{n-2} \sum_{k=1}^{n-i-1} t_{2i} \cup t_{2k} \cup t_{2(n-i-k)} \\ &= dt_2 \cup t_{2(n-1)} - t_{2(n-1)} \cup dt_2 . \end{aligned}$$

Now since $\Gamma(s^2\pi L^+)$ is cocommutative and since $dt_2(\gamma_1(\bar{y})) = y^p - \xi(y)$, we easily see that for any $b \in \Gamma_{2n}(s^2\pi L^+)$, $dr(b)$ is a sum of commutators of the form $[y^p - \xi(y), z]$, $z \in Y(L^+)$. But for $z \in L^+$, we have:

$$[y^p, z] = (ady)^p (z) = [\xi(y), z]$$

and, using (1) of the previous section,

$$[y^p, \langle z \rangle] = \langle (ady)^p (z) \rangle = \langle [\xi(y), z] \rangle = [\xi(y), \langle z \rangle],$$

where $(ady)^p(z) = [y[y[\cdots[y, z]\cdots]]]$, p factors of y. Thus $y^p - \xi(y)$ is central in $Y(L^+)$, and it follows that $d(r(b)) = 0$, as was to be shown.

Since $Y(L^+)$ is a differential Hopf algebra, $W(L)$ is a left $Y(L^+)$-module coalgebra, and $\Gamma(s^2\pi L^+)$ is a cocommutative coalgebra, it makes sense to seek a t-twisting coproduct $s : \Gamma(s^2\pi L^+) \to Y(L^+) \otimes Y(L^+)$. Again, we must have $s_n = 0$ if n is odd.

LEMMA 7. *Let $t : \Gamma(s^2\pi L^+) \to Y(L^+)$ be a twisting cochain such that $t_2(\gamma_1(\tilde{y})) = \langle y\rangle y^{p-1} - \langle \xi(y)\rangle$, $y \in L^+$. Then there exists a t-twisting coproduct $s : \Gamma(s^2\pi L^+) \to Y(L^+) \otimes Y(L^+)$ such that*

$$s_2(\gamma_1(\tilde{y})) = \sum_{i=1}^{p-1} (-1)^i \langle y\rangle y^{i-1} \otimes \langle y\rangle y^{p-1-i}, \qquad y \in L^+.$$

Proof. An easy calculation gives $ds_2(\gamma_1(\tilde{y})) = \bar{\phi}(t_2)(\gamma_1(\tilde{y}))$. We must define s_{2n}, $n \geq 2$, satisfying (18). Suppose inductively that s_{2i} has been defined for $i < n$, $n \geq 2$. It suffices to prove that $d(u(b)) = 0$ for all $b \in \Gamma_{2n}(s^2\pi L^+)$, where

$$u = \bar{\phi}(t_{2n}) + \sum_{i=1}^{n-1} (s_{2i} \cup \phi(t_{2(n-i)}) - \phi_\ell(t_{2(n-i)}) \cup s_{2i} - \phi_r(t_{2(n-i)}) \cup s_{2i}).$$

A long, but straightforward, calculation proves that

$$du = s_{2(n-1)} \cup \phi(dt_2) - \phi_\ell(dt_2) \cup s_{2(n-1)} - \phi_r(dt_2) \cup s_{2(n-1)}.$$

Now $dt_2(\gamma_1(\tilde{y})) = y^p - \xi(y)$, which is primitive in $U(L)$, hence

$$\phi(dt_2) = \phi_\ell(dt_2) + \phi_r(dt_2).$$

Therefore, for any $b \in \Gamma_{2n}(s^2\pi L^+)$, $du(b)$ is a sum of commutators of the form

$$[(y^p - \xi(y)) \otimes 1 + 1 \otimes (y^p - \xi(y)), z \otimes z'], \; z \otimes z' \in Y(L^+) \otimes Y(L^+).$$

The fact that $y^p - \xi(y)$ is central in $Y(L^+)$ implies that $(y^p - \xi(y)) \otimes 1$ and $1 \otimes (y^p - \xi(y))$ are central in $Y(L^+) \otimes Y(L^+)$, and it follows that $du(b)$ is zero for all b.

From now on, we suppose given a fixed twisting cochain t and t-twisting coproduct s on $\Gamma(s^2\pi L^+)$, and we let d and D denote the differential d_t and the coproduct D_s on $\Gamma(s^2\pi L^+) \otimes_t W(L)$. We suppose that t and s are so chosen that $t(b) = 0$ if $r(b) = 0$ and $s(b) = 0$ if $u(b) = 0$, where r and u are the maps defined in the proofs of the lemmas above.

THEOREM 8. *Let $X(L) = \bar{X}(L) \otimes V(L)$, $\bar{X}(L) = \Gamma(s^2\pi L^+) \otimes \bar{Y}(L)$, and regard $X(L)$ as the twisted tensor product $\Gamma(s^2\pi L^+) \otimes_t W(L)$. Then $X(L)$ is a $V(L)$-free resolution of K, and is also a differential quasi-coalgebra over $V(L)$.*

Proof. It is easily verified that d and D are morphisms of right $V(L)$-modules, and it remains to prove that $X(L)$ is acyclic. We will first filter $X(L)$ in such a manner that $E^0X(L)$ in the resulting spectral sequence is $X(L^\#)$, where $L^\#$ is the underlying K-module of L regarded as an Abelian restricted

Lie algebra with restriction zero. For convenience, if L is bigraded, we regrade it by total degree. Give $V(L)$ the filtration defined in Theorem 1 and filter $\bar{X}(L)$ by

$$F_i\bar{X}(L) = \bigoplus_{pm+n\leqslant i} \Gamma_m(s^2\pi L^+) \otimes \bar{Y}_n(L).$$

Then define

$$F_uX(L) = \bigoplus_{i+j=u} F_i\bar{X}(L) \otimes F_jV(L),$$

and filter $Y(L^+)$ similarly. It is easy to see that t, and therefore d, is filtration-preserving. In fact, an examination of the definition of t shows that

$$t_2(\gamma_1(\tilde{y})) \equiv \langle y\rangle\, y^{p-1} \bmod F_{p-1}Y(L^+)$$

and

$$t_{2m}(\Gamma_m(s^2\pi L^+)) \equiv 0 \bmod F_{pm-1}Y(L^+) \qquad \text{if} \qquad m > 1.$$

Now define

$$E^0_{u,v,t}X(L) = \left(\frac{F_uX_{u+v}(L)}{F_{u-1}X_{u+v}(L)}\right)_t$$

and then regrade by

$$E^0_{n,t}X(L) = \bigoplus_{u+v=n} E^0_{u,v,t}X(L).$$

By Theorem 1, $V(L^\#) = E^0_{0,*}X(L)$, and we therefore find that $E^0X(L) = X(L^\#)$ as a K-module. Inspection of the differentials in $E^0X(L)$ and $X(L^\#)$ shows that we actually have $E^0X(L) = X(L^\#)$ as a complex. It remains to prove that $H(X(L)) = K$ under the assumption that L is Abelian with restriction zero. In this case, the differential in $X(L)$ is given explicitly by the requirement that $X(L)$ with its natural algebra structure be a differential K-algebra and by $d(u) = 0$ if $u \in V(L)$,

$$d\gamma_r(x) = \gamma_{r-1}(x)x \qquad \text{if} \qquad x \in L^-,$$

and

$$d\langle y\rangle = y \qquad \text{and} \qquad d\gamma_r(\tilde{y}) = \gamma_{r-1}(\tilde{y})\,\langle y\rangle\, y^{p-1} \qquad \text{if} \qquad y \in L^+.$$

We will prove the result by obtaining a contracting homotopy $s: X(L) \to X(L)$. Suppose first that L has one generator $y \in L^+ = L$. Define:

$$\begin{array}{lll}
s(1) = 0 & & \\
s(\gamma_i(\tilde{y})\, y^j) = \gamma_i(\tilde{y})\,\langle y\rangle\, y^{j-1}, & 0 \leqslant i, & 1 \leqslant j \leqslant p-1 \\
s(\gamma_i(\tilde{y})\,\langle y\rangle\, y^{j-1}) = 0, & 0 \leqslant i, & 1 \leqslant j \leqslant p-1 \\
s(\gamma_i(\tilde{y})\,\langle y\rangle\, y^{p-1}) = \gamma_{i+1}(\tilde{y}), & 0 \leqslant i & \\
s(\gamma_{i+1}(\tilde{y})) = 0, & 0 \leqslant i. &
\end{array}$$

Clearly s satisfies $ds + sd = i - \epsilon$ (where i is the identity and ϵ is the augmentation $X(L) \to K$ followed by the inclusion $K \to X_0(L)$). Next, suppose that L has one generator $x \in L^- = L$. Then $s(1) = 0$, $s(\gamma_i(x)x) = \gamma_{i+1}(x)$ and $s(\gamma_{i+1}(x)) = 0$, $i \geqslant 0$, defines the desired contracting homotopy. Now suppose $L = L_1 \oplus L_2$, where L_1 has dimension one. As a complex, $X(L) = X(L_1) \otimes X(L_2)$ and, given contracting homotopies on $X(L_1)$ and $X(L_2)$, $s \otimes 1 + \epsilon \otimes s$ defines a contracting homotopy on $X(L)$. By finite and transfinite induction, this completes the proof.

COROLLARY 9. *There exists a spectral sequence $\{E^rL\}$ of differential coalgebras which converges to $H_*(V(L))$ and satisfies*

$$E^2L = \Gamma(s^2\pi L^+) \otimes H_*(U(L)).$$

The dual spectral sequence $\{E_rL\}$ of differential algebras converges to $H^*(V(L))$ and satisfies

$$E_2L = P((s^2\pi L^+)^*) \otimes H^*(U(L)).$$

Proof. Filter the complex $\bar{X}(L) = X(L) \otimes_{V(L)} K$ by

$$F_i\bar{X}_n(L) = \bigoplus_{m \leqslant i} \Gamma_m(s^2\pi L^+) \otimes \bar{Y}_{n-m}(L).$$

Observe that the operations $t \cap$ and $s \wedge$ lower filtration by at least two. Thus if $\{E^rL\}$ denotes the resulting spectral sequence, we have

$$E^0L = \Gamma(s^2\pi L^+) \otimes \bar{Y}(L)$$

as a differential coalgebra. Since the homology of $\bar{Y}(L)$ is $H_*(U(L))$ and $d_1 = 0$, the result follows. Note that the quasi-coalgebra structure on $\bar{X}(L)$ gives rise to a coassociative coproduct on each E^rL.

Remarks 10. If char $K = 2$, $X(L)$ takes on a quite simple form. An easy calculation proves that

$$d\langle y, z, [y, z]\rangle = (t_2 \cup t_2)\,(\gamma_1(\tilde{y})\,\gamma_1(\tilde{z})), \qquad y, z \in L = L^+.$$

Thus

$$t_4(\gamma_1(\tilde{y})\,\gamma_1(\tilde{z})) = \langle y, z, [yz]\rangle, \qquad \text{and} \qquad t_4(\gamma_2(\tilde{y})) = 0.$$

Further, $t_4 \cup t_2 + t_2 \cup t_4 = 0$, hence we must take $t_n = 0$ if $n > 4$. Another computation shows that we may define a t-twisting coproduct s by:

$$s(\gamma_{r_1}(\tilde{y}_1) \cdots \gamma_{r_n}(\tilde{y}_n)) = 0$$

unless each $r_i = 1$ when

$$s(\gamma_1(\tilde{y}_1) \cdots \gamma_1(\tilde{y}_n)) = \langle y_1, \cdots, y_n\rangle \otimes \langle y_1, \cdots, y_n\rangle .$$

Now $\bar{X}(L) = \Gamma(s^2\pi L) \otimes E(sL)$ is naturally isomorphic (as an algebra) to $\Gamma(sL)$ via the map $\langle y \rangle \to \gamma_1(y)$ and $\gamma_r(\tilde{y}) \to \gamma_{2r}(y)$, $y \in L$. Identify $\bar{X}(L)$ with $\Gamma(sL)$ and let $f = \gamma_{r_1}(y_1) \cdots \gamma_{r_n}(y_n)$ denote a typical element. Let f_i result from f by replacing r_i with $r_i - 1$, and let $f_{i,j} = (f_j)_i$, $i \leqslant j$, unless $i = j$ and $r_i = 1$ when $f_{i,i} = 0$. Then we find that d is given explicitly by:

$$d(f) = \sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n} f_{i,i}\gamma_1(\xi(y_i)) + \sum_{1 \leqslant i < j \leqslant n} f_{i,j}\gamma_1([y_i, y_j]). \tag{6.19}$$

The coproduct turns out to be the natural one:

$$D(f) = \prod_{i=1}^{n} D\gamma_{r_i}(y_i), \qquad \text{where} \qquad D\gamma_r(y) = \sum_{i+j=r} \gamma_i(y) \otimes \gamma_j(y). \tag{6.20}$$

In particular, $X(L)$ is coassociative in this case.

Remarks 11. If char $K > 2$, explicit determination of t and s in the general case is prohibitively difficult. The complex $\bar{X}(L)$ is usually most efficiently studied by means of the spectral sequences of Corollary 9. This is particularly true since D is not coassociative. In fact, we find that $t_{2n}(\gamma_n(\tilde{y})) = 0$ and $s_{2n}(\gamma_n(\tilde{y})) = 0$ if $n > 1$, and we then see that

$$(D \otimes 1)\, D\gamma_n(\tilde{y}) \neq (1 \otimes D)\, D\gamma_n(\tilde{y}) \qquad \text{if} \qquad n > 1.$$

7. Embedding of Resolutions in the Bar Construction

In this section, A will denote a K-algebra. $B(A) = \bar{B}(A) \otimes A$ will denote the right bar construction of A, that is $B(A) = B(K, A)$ in the notation of section *III*. The differential d and contracting homotopy t in $B(A)$ are derived from formulas 3.2 and 3.3, and, if A is a Hopf algebra, $B(A)$ has the coproduct D given by formula 3.7. We will find sufficient conditions for an A-free complex over K to be canonically embeddable in $B(A)$. The result will be used to embed $Y(L)$ in $B(U(L))$ and $X(L)$ in $B(V(L))$, where $Y(L)$ and $X(L)$ are the resolutions obtained in the previous sections. We will need the following property of $B(A)$.

Lemma 12. *Let $x \in Z_qB(A) \cap \operatorname{Ker} \epsilon$. Then there exists one and only one $y \in \bar{B}(A)$ such that $d(y) = x$, namely $y = t(x)$.*

Proof. Since $d(x) = 0 = \epsilon(x)$, $x = (dt + td - \tau\epsilon)(x) = dt(x)$. If $y' \in \bar{B}(A)$ also satisfies $d(y') = x$, then $d(y'') = 0$, $y'' = y' - t(x)$. Since $t(y'') = 0 = \epsilon(y'')$, $y'' = (dt + td - \tau\epsilon)\, y'' = 0$.

PROPOSITION 13. *Let $X = \bar{X} \otimes A$ be an A-free resolution of K. Suppose $X_0 = A$ and $\epsilon : X_0 \to K$ is the augmentation of A. Then there exists a unique morphism of A-complexes $\mu : X \to B(A)$ lying over the identity map of K and satisfying $\mu(\bar{X}) \subset \bar{B}(A)$. μ is determined inductively by the formula $\mu_n = t\mu_{n-1}d$ on $\bar{X}_n$, $n \geqslant 1$. If $d(\bar{X}_n) \cap \bar{X}_{n-1} = 0$ for all $n \geqslant 1$, then μ is a monomorphism.*

Proof. That μ is a morphism of complexes and is unique follows immediately from the lemma. If $d(\bar{X}_n) \cap \bar{X}_{n-1} = 0$ for all $n \geqslant 1$, then since $\bar{B}(A) = \ker t$, $\mu_{n-1}d(\bar{X}_n) \cap \operatorname{Ker} t = 0$ and therefore μ is a monomorphism. We remark that μ is the "canonical comparison" of MacLane [5, p. 267].

Remarks 14. Suppose X satisfies all the hypotheses of Proposition 13. Observe that each of the following is a split exact sequence of K-modules:

$$0 \to Z_0X \underset{i}{\overset{k}{\leftrightarrows}} X_0 \underset{\epsilon}{\overset{\sigma}{\leftrightarrows}} K \to 0,$$

and

$$0 \to Z_nX \underset{i}{\overset{k}{\leftrightarrows}} X_n \underset{\delta}{\overset{j}{\leftrightarrows}} B_{n-1}X \to 0, \qquad n \geqslant 1.$$

Here $d = i\delta$ and σ, j, and k are K-morphisms satisfying $ki = 1$, $\delta j = 1$ and $\epsilon\sigma = 1$, $kj = 0$ and $k\sigma = 0$, $ik + j\delta = 1$ and $ik + \sigma\epsilon = 1$. Let $s = jk$. Then we have $ds + sd = 1 - \sigma\epsilon$ and $s^2 = 0$. We may define an epimorphism of A-complexes $\nu : B(A) \to X$ by letting $\nu = s\nu d$ on $\bar{B}_n(A)$, $n \geqslant 1$. Then $\nu\mu = 1$ and therefore X is a direct summand of $B(A)$ as an A-complex. Now suppose $A = U(L)$ or $A = V(L)$ and $X = Y(L)$ of Section V or $X = X(L)$ of Section VI. The argument above applies. If M is a left A-module, then $(\nu \otimes_A 1)^*$ gives an embedding of $(X \otimes_A M)^* = \bar{X}^* \otimes M^*$ in

$$C(A; M) = \bar{B}(A)^* \otimes M^*.$$

If A is the associated graded algebra of a given Hopf algebra, it would seem that this embedding allows direct computation of the differentials in the cohomology spectral sequence defined in Section IV. However, the author has not been able to obtain a canonical definition of s and therefore of ν^*, and this procedure appears to be unworkable in the applications.

In order to obtain a more explicit description of the embeddings of $Y(L)$ in $B(U(L))$ and $X(L)$ in $B(V(L))$, we need some further properties of the bar construction. We define an (m, n)-shuffle to be a permutation π of the $m + n$ integers $1, 2, \cdots, m + n$ that satisfies $\pi(i) < \pi(j)$ if either $1 \leqslant i < j < m$ or $m + 1 \leqslant i < j \leqslant m + n$. Using this concept, we define a commutative product $*$ in $\bar{B}(A)$ by $[\,] * x = x$ and by

$$[a_1 \mid \cdots \mid a_m] * [a_{m+1} \mid \cdots \mid a_{m+n}] = \sum_{\pi} (-1)^{\sigma(\pi)} [a_{\pi(1)} \mid \cdots \mid a_{\pi(m+n)}], \qquad (7.1)$$

where the sum is taken over all (m, n)-shuffles and

$$\sigma(\pi) = \sum \deg [a_i] \deg [a_{m+j}]$$

summed over all pairs $(i, m+j)$ such that $\pi(i) > \pi(m+j)$, that is, such that π moves a_i past a_{m+j}. If A is commutative and $\bar{B}(A)$ is regarded as $B(A) \otimes_A K$, then $\bar{B}(A)$ is a differential K-algebra.

Now suppose L is a sub Lie algebra of $I(A)$ and define $Y(L) = \bar{Y}(L) \otimes A$, where $\bar{Y}(L) = \Gamma(sL^-) \otimes E(sL^+)$. We adopt the notation for elements of $Y(L)$ given above formula (5) of section V, and then that formula defines a structure of A-complex on $Y(L)$. Define a monomorphism of K-algebras $\mu : \bar{Y}(L) \to \bar{B}(A)$ by

$$\mu(fg) = (-1)^{\dim fg} [x_1]^{r_1} * \cdots * [x_m]^{r_m} * [y_1] * \cdots * [y_n], \tag{7.2}$$

where

$$[x_i]^{r_i} = [x_i \mid \cdots \mid x_i], \qquad r_i \quad \text{factors} \quad x_i\,,$$

and

$$\dim fg = n + \sum_{i=1}^{m} r_i\,.$$

(Here $f = \gamma_{r_1}(x_1) \cdots \gamma_{r_m}(x_m)$, $x_i \in L^-$, and $g = \langle y_1, \cdots, y_n \rangle$, $y_j \in L^+$). Extend μ to a map $Y(L) \to B(A)$ by requiring μ to be a morphism of A-modules. We will prove that the image of μ is a subcomplex of $B(A)$ and that μ is a morphism of A-complexes. It will follow by uniqueness that μ is the map defined in Proposition 13. If $A = U(L)$, μ is of course the desired embedding of $Y(L)$ in $B(U(L))$. We will need two lemmas.

Lemma 15. *Let $y \in I(A)$ and suppose that either the degree of y is even or* $2 = 0$ in K. Define A-morphisms $\alpha(y)$ and $\beta(y)$ of $B(A)$ into itself by:

(i) $\alpha(y)([a_1 \mid \cdots \mid a_n]) = [y] * [a_1 \mid \cdots \mid a_n]$, and

(ii) $\beta(y)([a_1 \mid \cdots \mid a_n]) = [a_1 \mid \cdots \mid a_n]\, y + \sum_{i=1}^{n} [a_1 \mid \cdots \mid [y, a_i] \mid \cdots \mid a_n]$.

Then $d\alpha(y) + \alpha(y)\, d + \beta(y) = 0$ on $B(A)$.

Proof. Let $w(y) = d\alpha(y) + \alpha(y)\, d + \beta(y)$. It suffices to prove that $w(y) = 0$ on $\bar{B}(A)$. $w(y)([\ ]) = 0$ is clear. Suppose that $w(y) = 0$ on $B_{n-1}(a)$ and consider $w(y)(z)$, $z \in \bar{B}_n(A)$, $n \geqslant 1$. $w(y)(z) = (td + dt)\, w(y)(z)$. A simple calculation proves that $w(y)(z) \in \operatorname{Ker} t$, hence it suffices to prove $tdw(y)(z) = 0$. By the induction hypothesis,

$$dw(y)(z) = (d\alpha(y)\, d + d\beta(y))(z) = (d\beta(y) - \beta(y)\, d)(z).$$

Now explicit calculation shows that $dw(y)(z) \in \operatorname{Ker} t$.

LEMMA 16. *Let $x \in I(A)$ and suppose that either the degree of x is odd or $2 = 0$ in K. For each integer $k \geq 1$, define A-morphisms $\alpha_k(x)$ and $\beta_k(x)$ of $B(A)$ into itself by:*

(i) $\alpha_k(x)([a_1 \mid \cdots \mid a_n]) = [x]^k * [a_1 \mid \cdots \mid a_n]$

(ii) $\beta_k(x)([a_1 \mid \cdots \mid a_n]) = (-1)^{\lambda(n)} [x]^{k-1} * [a_1 \mid \cdots \mid a_n]\, x$

$$- [x^2] * [x]^{k-2} * [a_1 \mid \cdots \mid a_n]$$

$$- \sum_{i=1}^{n} (-1)^{\lambda(i-1)} [x]^{k-1} * [a_1 \mid \cdots \mid a_{i-1} \mid [x, a_i] \mid a_{i+1} \mid \cdots \mid a_n],$$

where

$$\lambda(i) = \deg [a_1 \mid \cdots \mid a_i].$$

Then $d\alpha_k(x) - \alpha_k(x)\, d + \beta_k(x)$ is zero on $B(A)$.

The proof is analogous to that of Lemma 15. Note that $x^2 = \frac{1}{2}[x, x]$ if $\deg x$ is odd and $2 \neq 0$ in K.

Now we can prove the following theorem:

THEOREM 17. *Let L be a sub Lie algebra of $I(A)$ and let $Y(L)$ denote the A-complex $\Gamma(sL^-) \otimes E(sL^+) \otimes A$. Define $\mu : Y(L) \to B(A)$ by formula (7.2). Then μ is a monomorphism of A-complexes and is therefore the map defined in Proposition 13. If $\bar{Y}(L) = Y(L) \otimes_A K$ and $\bar{B}(A) = B(A) \otimes_A K$, then $\bar{\mu} : \bar{Y}(L) \to \bar{B}(A)$ is a morphism of differential coalgebras.*

Proof. Let f, g, f_i and g_i be as defined above 5.5. If $y \in L^+$, $\alpha(y)$ and $\beta(y)$ of Lemma 15 take the following forms on $\mu(Y(L))$:

$$\alpha(y)\, \mu(fg) = -\mu(f\langle y\rangle g). \tag{7.3}$$

$$\beta(y)\, \mu(fg) = \mu \left[fgy + \sum_{i=1}^{m} f_i \gamma_1([y, x_i])\, g + \sum_{j=1}^{n} (-1)^{n-j} fg_j\langle [y, y_j]\rangle \right]. \tag{7.4}$$

If $x \in L^-$, $\alpha_k(x)$ and $\beta_k(x)$ of Lemma 16 take the forms:

$$\alpha_k(x)\, \mu(fg) = (-1)^k \mu(\gamma_k(x) fg). \tag{7.5}$$

$$\beta_k(x)\, \mu(fg) = (-1)^{k-1} \mu\Big[(-1)^n \gamma_{k-1}(x) fgx + \tfrac{1}{2}(-1)^{n+1} \gamma_{k-2}(x) fg\langle [x, x]\rangle$$

$$+ (-1)^{n+1} \sum_{j=1}^{m} \gamma_{k-1}(x) f_j g\langle [x, x_j]\rangle$$

$$+ \sum_{j=1}^{n} (-1)^j \gamma_{k-1}(x) f\gamma_1([x, y_j])\, g_j\Big]. \tag{7.6}$$

These maps satisfy the relations:

$$d\alpha(y) = -\alpha(y)\,d - \beta(y) \qquad \text{and} \qquad d\alpha_k(x) = \alpha_k(x)\,d - \beta_k(x). \tag{7.7}$$

$d\mu = \mu d$ is easily verified on elements of the forms $\langle y \rangle$ and $\gamma_k(x)$. By using induction on $m + n$, a comparison of formula 5.5 with (7.3) through (7.7) proves that $d\mu = \mu d$ on $Y(L)$. The fact that $D\bar{\mu} = (\bar{\mu} \otimes \bar{\mu})\,D$ on $\bar{Y}(L)$ follows from the definitions.

We remark that the proof above is a generalization of that in Cartan and Eilenberg [3, p. 277-281] for the case of Lie algebras concentrated in degree zero. Were the sign omitted in the definition of μ, we would have $d\mu = -\mu d$.

It remains to discuss the embedding of $X(L)$ in $B(V(L))$, L a restricted Lie algebra. Suppose first that char $K = 2$. Identity $X(L)$ with $\Gamma(sL) \otimes V(L)$ as in Remarks 10. Define a morphism of algebras $\mu : \bar{X}(L) \to \bar{B}(V(L))$ by

$$\mu(\gamma_{r_1}(y_1) \cdots \gamma_{r_n}(y_n)) = [y_1]^{r_1} * \cdots * [y_n]^{r_n}\,. \tag{7.8}$$

Extend μ to $X(L)$ by requiring μ to be a morphism of $V(L)$-modules. Then Lemma 16 may be used to prove:

THEOREM 18. *Let L be a restricted Lie algebra over K, where char $K = 2$. Define $\mu : X(L) \to B(V(L))$ by formula (7.8). Then μ is a monomorphism of A-complexes. Further, $\bar{\mu} : \bar{X}(L) \to \bar{B}(V(L))$ is a morphism of differential coalgebras, where*

$$\bar{X}(L) = X(L) \otimes_{V(L)} K \qquad \textit{and} \qquad \bar{B}(V(L)) = B(V(L)) \otimes_{V(L)} K.$$

If char $K > 2$, Theorem 17 gives the embedding of $\bar{Y}(L) \otimes V(L)$ in $B(V(L))$. Proposition 13 must be used to obtain the extension to $X(L)$. Theorem 7 tells how to compute d on $X(L)$, and the formula $\mu = t\mu d$ on $\bar{X}(L)$ allows the determination of μ. For example,

$$t\mu d\, \gamma_1(\tilde{y}) = -[y \mid y^{p-1}]; \qquad \mu(\gamma_r(\tilde{y})) = -[y \mid y^{p-1}]^r$$

implies

$$\mu(\gamma_r(\tilde{y})\, \langle y \rangle) = [y \mid y^{p-1}]^r * [y],$$

which in turn implies

$$\mu(\gamma_{r+1}(\tilde{y})) = -[y \mid y^{p-1}]^{r+1}.$$

Therefore

$$\mu(\gamma_r(\tilde{y})) = -[y \mid y^{p-1}]^r \qquad \text{and} \qquad \mu(\gamma_r(\tilde{y})\, \langle y \rangle) = [y \mid y^{p-1}]^r * [y], \tag{7.9}$$

where $[y \mid y^{p-1}]^r = [y \mid y^{p-1} \mid \cdots \mid y \mid y^{p-1}]$, r factors of $[y \mid y^{p-1}]$. Of course, $\bar{\mu} : \bar{X}(L) \to \bar{B}(V(L))$ is not a morphism of differential coalgebras in this case, and $\bar{\mu}(\bar{X}(L))$ is not closed under the coproduct of $\bar{B}(V(L))$.

REFERENCES

1. ADAMS, J. F. On the non-existence of elements of Hopf invariant one. *Ann. Math.* **72** (1960), 20-104.
2. BROWN, E. H., JR. Twisted tensor products I. *Ann. Math.* **69** (1959), 223-246.
3. CARTAN, H. AND EILENBERG, S. "Homological Algebra." Princeton University Press, Princeton, 1956.
4. JACOBSON, N. "Lie Algebras." Wiley (Interscience), New York, 1962.
5. MACLANE, S. "Homology." Academic Press, New York, 1963.
6. MASSEY, W. S. Products in exact couples. *Ann. Math.* **59** (1954), 558-569.
7. MASSEY, W. S. AND PETERSON, F. P. The cohomology structure of certain fiber spaces I. *Topology* **4** (1965), 47-65.
8. MAY, J. P. The cohomology of restricted Lie algebras and of Hopf algebras. *Bull. Am. Math. Soc.* **71** (1965), 130-137.
9. MAY, J. P. The cohomology of the Steenrod algebra; stable homotopy groups of spheres. *Bull. Am. Math. Soc.* **71** (1965), 138-142.
10. MILNOR, J. AND MOORE, J. C. On the structure of Hopf algebras. *Ann. Math.* **81** (1965), 211-264.