Topological Morita contexts

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Abstract

In synthesis, this paper presents a generalization of the theory of Morita contexts from the case of abstract modules over abstract rings to that of complete l. t. (linearly topological) modules over complete l. t. rings.

To begin with, given three complete right l. t. rings $(R, \rho)$, $(S, \sigma)$ and $(T, \tau)$, and two complete l. t. bimodules $(RAS, \alpha)$ and $(SBT, \beta)$ satisfying suitable hypotheses, we introduce the “topological tensor product” $(A, \alpha) \otimes^u S(B, \beta)$. Next, we define a topological Morita context to be a family made up of two complete l. t. rings $(R, \rho)$ and $(S, \sigma)$, two bimodules $(SAR, \alpha)$ and $(RBS, \beta)$ of the above kind, and two continuous bilinear maps $\mu: (B, \beta) \otimes^u (A, \alpha) \to (R, \rho)$ and $\nu: (A, \alpha) \otimes^u (B, \beta) \to (S, \sigma)$; the context is called dense if both $\mu$ and $\nu$ have dense image. We then prove that such a dense Morita context yields an equivalence of categories between $\text{CLT}^{-}(R, \rho)$ and $\text{CLT}^{-}(S, \sigma)$, in such a way to induce an equivalence between $\text{Mod}^{-}(R, \rho)$ and $\text{Mod}^{-}(S, \sigma)$. Finally, we give a “topological” parallel of the notion of progenerator, and we show that such a “topological progenerator” gives rise to a dense context, and hence to an equivalence of the above-mentioned kind. Conversely, we show that every such equivalence arises in this way.

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Introduction

The fundamental, well known result of Morita’s theory on equivalences between categories of modules is that two rings \( R \) and \( S \) are similar if and only if there exists a progenerator \( P \) of \( \text{Mod}_R \) (or, equivalently, a progenerator \( Q \) of \( \text{Mod}_S \)) such that \( S = \text{End}(P_R) \) and \( R = \text{End}(S) \) and \( S = \text{End}(Q_S) \); in this case, the equivalence between \( \text{Mod}_R \) and \( \text{Mod}_S \) is represented by the pair of functors \( \text{Hom}_R(P, -) \) and \( - \otimes_R P \). Starting from the first half of the 1970s, some authors begun to develop weakened notions of similarity, studying equivalences between suitable subcategories of \( \text{Mod}_R \) and \( \text{Mod}_S \). Their research concentrated both on finding out what pairs of rings \( R \) and \( S \) admit such an equivalence, and on representing those equivalences by means of tensor products and Hom functors. Actually, the two problems are intimately correlated, and often they are both solved by finding a suitable bimodule \( S_P_R \), which is “almost” faithfully balanced, and such that, if \( (\mathcal{F}, \mathcal{G}) \) denotes the equivalence in question, there are natural equivalences of functors \( \mathcal{F} \cong \text{Hom}_R(P, -) \) and \( \mathcal{G} \cong - \otimes_S P \).

In this line of research, it is already classical Fuller’s work [4], where the author considered equivalences between \( \text{Mod}_S \) and a closed subcategory \( G_R \) of \( \text{Mod}_R \), i.e., a subcategory of \( \text{Mod}_R \) which is closed under taking submodules, epimorphic images, and forming arbitrary direct sums. Note that in this setting the two rings \( R \) and \( S \) no longer play symmetric roles. This work was then further generalized in many directions. An important generalization is the one that leads to the concept of \( \ast \)-module, introduced by G. Menini and A. Orsatti in [10], a setting which again treats the two rings asymmetrically. But another notable generalization had already been presented by E. Gregorio in [5]; his idea was that the symmetry broken by Fuller can be reinstated by looking at the problem from a topological point of view. Indeed, Gregorio considered two complete right linearly topological rings \((R, \rho)\) and \((S, \sigma)\) (according to the conventions stated at the end of this introduction, “linearly topological” always includes “Hausdorff”), and the categories \( \text{Mod}_R \) and \( \text{Mod}_S \) of torsion right modules over them. Observe that the categories \( G_R \) and \( \text{Mod}_S \) considered by Fuller fall within this framework, by taking as \( \sigma \) the discrete topology of \( S \), and as \( \rho \) the \( P \)-topology of \( R \), where \( P \) is the bimodule that represents the equivalence. Given an equivalence of categories between \( \text{Mod}_R \) and \( \text{Mod}_S \), Gregorio was able to represent it by means of a bimodule \( S_P_R \) endowed with a suitable \( R \)-linear Hausdorff topology \( \varepsilon \); more precisely, the equivalence is given by the pair of functors

\[
\text{CHom}_R((P, \varepsilon), -) : \text{Mod}_R \leftrightarrow \text{Mod}_S : \text{Hom}(S, \sigma) \cong - \otimes_S P, \tag{0.1}
\]

and \( (S, \sigma) \cong \text{CEnd}^\ast_R(P, \varepsilon) \) topologically (see Conventions and notation, below, for an explanation of the symbols). Moreover, \( (P, \varepsilon) \) has properties similar to those of a quasi-progenerator: Gregorio called such a topological module a \( \rho \)-progenerator. Conversely, if \( (P, \varepsilon) \) is a \( \rho \)-progenerator and \( (S, \sigma) = \text{CEnd}^\ast_R(P, \varepsilon) \), then Gregorio proved that (0.1) is an equivalence of categories.

The present work fits in this line of research. Encouraged by the results of [12], and especially by the machinery developed in order to obtain them, we wanted to extend...
Gregorio’s theorem to the setting of the complete linearly topological modules over \((R, \rho)\) and \((S, \sigma)\). We must at once inform the reader that this project was only partly successful, since we were not able to cover all equivalences

\[ \mathcal{F} : \mathrm{CLT}-(R, \rho) \leftrightarrow \mathrm{CLT}-(S, \sigma) : \mathcal{G}, \]

but only those which induce an equivalence between \(\mathrm{Mod}-(R, \rho)\) and \(\mathrm{Mod}-(S, \sigma)\). We have not been able to ascertain whether all equivalences between \(\mathrm{CLT}-(R, \rho)\) and \(\mathrm{CLT}-(S, \sigma)\) preserve discrete modules. In some cases this is obviously true: for instance, if one of the rings, say \(R\), is discrete, because then a discrete module can be characterized as an object \((A, \alpha) \in \mathrm{CLT}-R\) such that every mono-epic morphism with codomain \((A, \alpha)\) is an isomorphism (see also Proposition 5.13); but in general the problem is still open.

Let us describe in more detail the main results of this paper (for unknown symbols and terminology, see Conventions and notation). In Section 1, given two right l. t. rings \((R, \rho)\) and \((S, \sigma)\), we recall and extend some definitions and propositions of [12, Sections 1, 2 and 3], in particular the definition of the category \(\mathrm{CLT}-(R, \rho)\) of “topological \((R,S)\)-bimodules” and of the completed tensor product \((A, \alpha) \hat{\otimes}^u (B, \beta)\) (for three right l. t. rings \((R, \rho)\), \((S, \sigma)\) and \((T, \tau)\), and two objects \((A, \alpha) \in \mathrm{CLT}-(R, \rho)\) and \((B, \beta) \in \mathrm{CLT}-(S, \sigma)\)). This kind of tensor product is associative (up to topological isomorphisms), and satisfies

\[ (S, \sigma) \hat{\otimes}^u (B, \beta) \cong (B, \beta) \quad \text{and} \quad (A, \alpha) \hat{\otimes}^u (S, \sigma) \cong (A, \alpha) \]  

(0.2)

for every \((A, \alpha) \in \mathrm{CLT}-(R, \rho)\) and every \((B, \beta) \in \mathrm{CLT}-(S, \sigma)\). Then, in Section 2, we introduce the subject of our study, defining a topological Morita context as a natural generalization of [7, Definition 3.11]. This generalization makes sense because an object \((A, \alpha) \in \mathrm{CLT}-(R, \rho)\), where \((R, \rho)\) is a complete right l. t. ring, defines a topological Morita context (which we call the topological Morita context generated by \((A, \alpha)\)) as in [7, Section 3.12]. We say that a topological Morita context is dense if both the maps \(\mu\) and \(\nu\) (cf. [7, Definition 3.11]) have dense image. Our first result on topological Morita contexts is Theorem 4.3, where we prove that a dense Morita context \(((R, \rho), (S, \sigma), (A, \alpha), (B, \beta), \mu, \nu)\) yields an equivalence of categories between \(\mathrm{CLT}-(R, \rho)\) and \(\mathrm{CLT}-(S, \sigma)\); this equivalence has the property of inducing an equivalence between \(\mathrm{Mod}-(R, \rho)\) and \(\mathrm{Mod}-(S, \sigma)\).

A central notion, that parallels the classical notion of progenerator, is introduced in Section 5. Let \((R, \rho)\) be a complete right l. t. ring; an object \((P, \varepsilon) \in \mathrm{CLT}-(R, \rho)\) will be called a topological progenerator (of \(\mathrm{CLT}-(R, \rho)\)) if it is a generator of \(\mathrm{CLT}-(R, \rho)\) in the sense of category theory, it is topologically finitely generated, and the functor

\[ \mathcal{H} = \mathrm{CHom}^u_R((P, \varepsilon), -) : \mathrm{CLT}-(R, \rho) \to \mathrm{CLT}-\mathbb{Z} \]

preserves epimorphisms. Observe that \((R, \rho)\) is a topological progenerator of \(\mathrm{CLT}-(R, \rho)\).

A topological progenerator as defined here is very similar to a \(\rho\)-progenerator in the sense of [5], but it is not defined exactly in the same way. It is clear that a topological progenerator is a \(\rho\)-progenerator; the converse is not evident at a first sight, but it turns
out to be true too. Indeed, the simple fact that a functor of type $\text{CHom}^u_{(B, \beta)}(-)$ has a left adjoint can easily be used to extend Gregorio’s equivalence (0.1) to $\text{CLT}^u_{(R, \rho)}$ and $\text{CLT}^u_{(S, \sigma)}$: we show how this can be done in Section 8. This implies that the notions of topological progenerator and of $\rho$-progenerator coincide.

As already said, a big problem, that we have not been able to solve yet, is whether the notion of topological progenerator is categorical, that is, whether it is preserved by all equivalences of categories from $\text{CTL}^u_{(R, \rho)}$ to $\text{CTL}^u_{(S, \sigma)}$, where $(S, \sigma)$ is another complete right l. t. ring. We show that such an equivalence will preserve the notion of topological progenerator if it induces an equivalence between $\text{Mod}^u_{(R, \rho)}$ and $\text{Mod}^u_{(S, \sigma)}$, that is, if it is what we call a topological equivalence. The equivalence mentioned in Theorem 4.3 is topological in this sense.

The course of our theory comes to its logical end in Section 6, where we show that the topological Morita context defined by a topological progenerator is dense. Thus, in Section 7, we obtain a topological generalization of the results which are called “Morita I–III” in Sections 3.12 and 3.15 of [7]. Finally, in Section 9 we re-obtain the main result of [12] as a corollary of the theory developed in the present paper; in this way we provide examples of topological Morita contexts.

**Conventions and notation.** All linearly topological rings and modules will be supposed to be Hausdorff. When we need to speak of a ring or of a module which is endowed with a linear topology but is not Hausdorff, we shall say that its topology is linear, but we shall not call it “linearly topological”. We shall often abbreviate “linearly topological” as “l. t.”

All rings have unit $1 \neq 0$, and all modules are unitary; if $R$, $S$, . . . is the ring, its unit is ordinarily denoted by $1_R$, $1_S$, . . . , and its zero element by $0_R$, $0_S$, . . . . Almost always (the only exception will be in Section 9) a ring homomorphisms $f : R \to S$ carries $1_R$ into $1_S$; when this is not the case, an explicit advice is given. $\mathbb{Z}$ is the ring of integers, and $\mathbb{N}$ is the set of natural numbers.

If $(R, \rho)$ is a complete right l. t. (Hausdorff) ring, we denote by $\text{CLT}^u_{(R, \rho)}$ the category of all complete l. t. (Hausdorff) right $(R, \rho)$-modules and all continuous $R$-linear applications, and by $\text{Mod}^u_{(R, \rho)}$ the full subcategory of $\text{CLT}^u_{(R, \rho)}$ whose objects are the discrete modules.

Let $R$ and $S$ be two (abstract) rings. If $(R_A_S, \alpha)$ and $(R_B_S, \beta)$ are two topological $(R, S)$-bimodules, then $\text{CHom}_{R_S}^u((A, \alpha), (B, \beta))$ denotes the abelian group of all continuous, left $R$-linear and right $S$-linear applications $f : (A, \alpha) \to (B, \beta)$, and the symbol $\text{CHom}^u_{R_S}((A, \alpha), (B, \beta))$ denotes the same group endowed with the topology of uniform convergence. In particular, if $R = \mathbb{Z}$, so that $(A, \alpha)$ and $(B, \beta)$ are simply two right topological $S$-modules, then $\text{CHom}_S((A, \alpha), (B, \beta))$ denotes the abelian group of all continuous $S$-linear applications $f : (A, \alpha) \to (B, \beta)$, and $\text{CHom}^u_S((A, \alpha), (B, \beta))$ is the same group endowed with the topology of uniform convergence; also, we pose as usual $\text{CEnd}_S(A, \alpha) = \text{CHom}_S((A, \alpha), (A, \alpha))$, and $\text{CEnd}^u_S(A, \alpha) = \text{CHom}^u_S((A, \alpha), (A, \alpha))$.

As a rule, when we omit to show the topology of a module or of a ring it should be understood that module or that ring is endowed with the discrete topology. Nevertheless, sometimes we shall omit to show even non-discrete topologies simply for reasons of space; in this case the context, or our explicit notice, should suffice to solve the possible ambiguity.
The sign $\cong$ means “is isomorphic to”; when applied to topological modules, unless specified otherwise, the isomorphism is understood to be topological.

We shall only consider functors between additive categories, and we shall always suppose them to be additive.

1. A topological tensor product

This section is devoted to the introduction of some concepts that we are going to use throughout the rest this paper, and to state some basic propositions about their behaviour. Some of these definitions and propositions were given or proved in [12]; of course, in this case the proofs will not be repeated here.

1.1. Definition (cf. [12, Definition 1.1]). Let $(R,\rho)$ and $(S,\sigma)$ be two complete right l.t. rings; we denote by $(R,\rho)\mathcal{B}(S,\sigma)$ the following category. The objects of $(R,\rho)\mathcal{B}(S,\sigma)$ are l.t. (Hausdorff) and complete abelian groups $(A,\alpha)$ such that:

- $A$ has an $(R,S)$-bimodule structure $RAS$;
- $(AS,\alpha)\in\text{CLT}(S,\sigma)$;
- $R$ acts on $(A,\alpha)$ by means of continuous $S$-endomorphisms, that is, the left action of $R$ on $A$ defines a ring homomorphism $\omega:R\rightarrow \text{CEnd}_S(A,\alpha)$;
- $\omega:(R,\rho)\rightarrow \text{CEnd}_S^u(A,\alpha)$ is continuous (for the displayed topologies).

A morphism $f$ in $(R,\rho)\mathcal{B}(S,\sigma)$ from $(A,\alpha)$ to $(B,\beta)$ is a homomorphism of $(R,S)$-bimodules $f:(A,\alpha)\rightarrow (B,\beta)$ which is also continuous.

Note that $\mathcal{Z}(R,\rho)\mathcal{B}(S,\sigma) = \text{CLT}(R,\rho)$, and that for every complete right l.t. ring $(R,\rho)$ we have $(R,\rho)\mathcal{B}(S,\sigma)$.

For the rest of this section we fix the following setting: we have three complete right l.t. rings $(R,\rho)$, $(S,\sigma)$, and $(T,\tau)$, and two objects $(A,\alpha)\in(R,\rho)\mathcal{B}(S,\sigma)$ and $(B,\beta)\in(S,\sigma)\mathcal{B}(T,\tau)$.

1.2. (cf. [12, Section 1.2]) Let $\tau_1(\alpha)$ be the topology on $A\otimes S B$ having as a basis of neighbourhoods of zero the family of $T$-submodules

$$\{\text{Im}(A' \otimes S B): A' \text{ is an open } S\text{-submodule of } (A,\alpha)\},$$

where $\text{Im}(A' \otimes S B)$ denotes the image in $A \otimes S B$ of $A' \otimes S B \rightarrow A \otimes S B$; then, let $\tau_2(\beta)$ be the $(T$-linear$)$ inductive topology of the family $\{\text{``}a \otimes -\text{''}; a \in A\}$, where “$a \otimes -$”: $B \rightarrow A \otimes S B$ is the $T$-linear application sending $b$ to $a \otimes b$; finally, let $\tau(\alpha,\beta) = \tau_1(\alpha) \wedge \tau_2(\beta)$.

We endow $A \otimes S B$ with the topology $\tau(\alpha,\beta)$, and we denote by $(A,\alpha)\otimes S^u(B,\beta)$ the topological module (since $\tau$ is a topological module: see Proposition 1.6) obtained in this way. Note that $\tau(\alpha,\beta)$ is a generalization of the topology defined in [5, Section 3.3], to which it reduces when $\alpha$ is discrete.
1.3. Remark. A straightforward verification shows that a $T$-submodule $W$ of $A \otimes S B$ is open in $\tau(\alpha, \beta)$ if and only if it verifies both the following conditions:

(i) $\forall a \in A \quad "a \otimes -"^{-1}(W)$ is open in $(B, \beta)$;
(ii) $\bigcap_{b \in B} "- \otimes b"^{-1}(W)$ is open in $(A, \alpha)$.

In other words, $\tau(\alpha, \beta)$ is the finest $T$-linear topology on $A \otimes S B$ that makes, at the same time, continuous each application \("a \otimes -"\) (one by one) and equicontinuous the family \{""- \otimes b"": b \in B\} of applications.

1.4. Proposition. Let

\[ f : (A, \alpha) \to (C, \gamma) \quad \text{and} \quad g : (B, \beta) \to (D, \delta) \]

be two morphisms in $(R, \rho)$-u$\mathcal{B}$-(S, $\sigma$) and $(S, \sigma)$-u$\mathcal{B}$-(T, $\tau$), respectively; then

\[ f \otimes_S g : (A, \alpha) \otimes^u_S (B, \beta) \to (C, \gamma) \otimes^u_S (D, \delta) \]

is continuous.

Proof. See [12, Proposition 1.4]. \(\square\)

1.5. In general, the topology of $(A, \alpha) \otimes^u_S (B, \beta)$ is neither Hausdorff nor complete; so, we denote by $(A, \alpha) \widehat{\otimes}^u_S (B, \beta)$ the Hausdorff completion of $(A, \alpha) \otimes^u_S (B, \beta)$. Similarly, given two morphisms $f : (A, \alpha) \to (C, \gamma)$ and $g : (B, \beta) \to (D, \delta)$ in $(R, \rho)$-u$\mathcal{B}$-(S, $\sigma$) and $(S, \sigma)$-u$\mathcal{B}$-(T, $\tau$) respectively, we denote by

\[ f \widehat{\otimes}^u_S g : (A, \alpha) \widehat{\otimes}^u_S (B, \beta) \to (C, \gamma) \widehat{\otimes}^u_S (D, \delta) \]

the continuous morphism canonically associated with $f \otimes_S g$.

1.6. Proposition. $(A, \alpha) \widehat{\otimes}^u_S (B, \beta) \in (R, \rho)$-u$\mathcal{B}$-(T, $\tau$).

Proof. This is only a sequence of routine checks. Each step is either absolutely trivial or discussed in [11, Proof of Propositions 2.9 and 2.43]. \(\square\)

We have thus defined a functor

\[ \widehat{\otimes}^u_S : \quad (R, \rho)$-u$\mathcal{B}$-(S, $\sigma) \times (S, \sigma)$-u$\mathcal{B}$-(T, $\tau) \to (R, \rho)$-u$\mathcal{B}$-(T, $\tau) \] \quad (1.7)

In particular, for a fixed $(B, \beta) \in (S, \sigma)$-u$\mathcal{B}$-(T, $\tau$) we obtain a functor

\[ \widehat{\otimes}^u_S (B, \beta) : (R, \rho)$-u$\mathcal{B}$-(S, $\sigma) \to (R, \rho)$-u$\mathcal{B}$-(T, $\tau) \] \quad (1.8)

We now look at another familiar functor.
1.9. Proposition. Let \((B, \beta) \in (S, \sigma) - u B - (T, \tau)\) and \((C, \gamma) \in (R, \rho) - u B - (T, \tau)\); then \(\CHom^u_T((B, \beta), (C, \gamma)) \in (R, \rho) - u B - (S, \sigma)\).

Proof. This is again a sequence of routine checks, and again, each step is either absolutely trivial or discussed in [11, Proof of Proposition 2.11].

It is now natural to complete Proposition 1.9 and state the existence of a functor

\[ \CHom^u_T(-, -): (S, \sigma) - u B - (T, \tau) \times (R, \rho) - u B - (T, \tau) \to (R, \rho) - u B - (S, \sigma) \]

which acts on the morphisms in the obvious way. In particular, a fixed \((B, \beta) \in (S, \sigma) - u B - (T, \tau)\) yields a functor

\[ \CHom^u_T((B, \beta), -): (R, \rho) - u B - (T, \tau) \to (R, \rho) - u B - (S, \sigma) \]

(1.10)

We want now to see that some common canonical isomorphisms involving tensor products become topological isomorphisms when those tensor products are endowed with the topology defined in 1.2. The most important fact, which motivates the definitions of 1.2, is that (1.8) is a left adjoint of (1.10); the following proposition actually tells more than this.

1.11. Proposition. Let \((A, \alpha) \in (R, \rho) - u B - (S, \sigma)\), \((B, \beta) \in (S, \sigma) - u B - (T, \tau)\) and \((C, \gamma) \in (R, \rho) - u B - (T, \tau)\); there exist topological isomorphisms (of l. t. abelian groups)

\[ \Phi^A_C : \CHom^u_R(T(A \otimes_S B, C)) \to \CHom^u_R(A, \CHom^u_T(B, C)) \]

\[ f \mapsto \left[ a \mapsto \left[ b \mapsto f(a \otimes b) \right] \right] \]

and

\[ \Psi^A_C : \CHom^u_R(A, \CHom^u_T(B, C)) \to \CHom^u_R(T(A \otimes_S B, C)) \]

\[ g \mapsto \left[ \sum_i a_i \otimes b_i \mapsto \sum_i g(a_i)(b_i) \right] \]

(to ease the notations we have omitted to show some topologies even if they are not discrete), which are natural in \((A, \alpha)\) and \((C, \gamma)\) and are inverse one of each other.

Proof. Same as the proof of Proposition 1.8 of [12].

In the previous proposition \((C, \gamma)\) is Hausdorff and complete; therefore:

1.12. Corollary. Let \((A, \alpha) \in (R, \rho) - u B - (S, \sigma)\), \((B, \beta) \in (S, \sigma) - u B - (T, \tau)\) and \((C, \gamma) \in (R, \rho) - u B - (T, \tau)\); then

\[ \CHom^u_R(A \otimes_S B, C) \cong \CHom^u_R(A, \CHom^u_T(B, C)) \]

the isomorphism being topological and natural in \((A, \alpha)\) and in \((C, \gamma)\).
1.13. Remark. For future use, we point out the following obvious variation of Corollary 1.12: suppose you have a fourth right l. t. ring $(U,v)$, and assume the hypotheses of Corollary 1.12 with the exception that $(C,\gamma)$ is no longer an object of $(R,\rho)-^{uB}-(T,\tau)$, but of $(U,v)-^{uB}-(T,\tau)$; then you have topological $(U,R)$-bilinear isomorphisms
\[
\Phi^A_C : \text{CHom}^u_R(A \hat{\otimes}^u B, C) \to \text{CHom}^u_S(A, \text{CHom}^u_T(B, C))
\]
\[f \mapsto [a \mapsto b \mapsto f(a \otimes b)]\]
and
\[
\Psi^A_C : \text{CHom}^u_R(A, \text{CHom}^u_T(B, C)) \to \text{CHom}^u_S(A \otimes^u B, C)
\]
\[g \mapsto \left[\sum_i a_i \otimes b_i \mapsto \sum_i g(a_i)(b_i)\right]\]
natural in $(A, \alpha)$ and $(C, \gamma)$ and inverse one of each other; here the naturality is understood with respect to morphisms $f : A_1 \to A$ and $g : C \to C_1$ which are simply $S$-linear and $T$-linear, respectively (but of course still continuous).

1.14. Remark. Recall that if $(X, \xi)$ is a right topological module over $(R,\rho)$, then $X \cong \text{Hom}_R((R,\rho),(X,\xi))$ via the usual canonical isomorphism; if, moreover, $(X, \xi) \in (R,\rho)-^{uB}-(R,\rho)$, then such isomorphism is also a (topological) isomorphism $(X, \xi) \cong \text{CHom}_R^u((R,\rho),(X,\xi))$ in $(R,\rho)-^{uB}-(R,\rho)$, as one can readily verify.

Exploiting the adjunctions described by Corollary 1.12 and Remark 1.13, or by means of direct computation, it is easy to prove the following two propositions.

1.15. Proposition. The following hold:

(i) $(S,\sigma) \hat{\otimes}^u_S (B, \beta)$ and $(B, \beta)$ are naturally isomorphic in $(S,\sigma)-^{uB}-(T,\tau)$;
(ii) $(A, \alpha) \hat{\otimes}^u_S (S, \sigma)$ and $(A, \alpha)$ are naturally isomorphic in $(R,\rho)-^{uB}-(S,\sigma)$.

For our theory it is very important to have at hand a “topological” version of the associative property of the tensor product. We therefore take an interest in the situation in which four complete right l. t. rings $(R,\rho)$, $(S,\sigma)$, $(T,\tau)$ and $(U,v)$ are given, together with three bimodules $(A, \alpha) \in (R,\rho)-^{uB}-(S,\sigma)$, $(B, \beta) \in (S,\sigma)-^{uB}-(T,\tau)$, and $(C, \gamma) \in (T,\tau)-^{uB}-(U,v)$.

1.16. Proposition. In the situation just described, $(A, \alpha) \hat{\otimes}^u_S ((B, \beta) \hat{\otimes}^u_T ((C, \gamma)))$ and $((A, \alpha) \hat{\otimes}^u_S (B, \beta)) \hat{\otimes}^u_T (C, \gamma)$ are naturally isomorphic in $(R,\rho)-^{uB}-(U,v)$.

It is possible to establish other canonical and topological isomorphisms that express the exchange properties between tensor products and quotients, and between tensor products
and direct sums. These exchange properties, however, are not at all used in the sequel of this paper; for this reason, the interested reader is referred to [11, Chapter 2, Section 2].

2. Topological Morita contexts

For the following definition, cf. [7, Section 3.12, Definition 3.11].

2.1. Definition. A topological Morita context is defined giving a collection \(((R, \rho), (S, \sigma), (A, \alpha), (B, \beta), \mu, \nu)\) of six objects such that:

- \((R, \rho)\) and \((S, \sigma)\) are two complete right l.t. rings;
- \((A, \alpha) \in \text{CL T}_R (S, \sigma)\) and \((B, \beta) \in \text{CL T}_R (R, \rho)\);
- \(\mu : (B, \beta) \otimes^S (A, \alpha) \to (R, \rho)\) and \(\nu : (A, \alpha) \otimes^R (B, \beta) \to (S, \sigma)\) are two continuous bilinear maps, satisfying both the following conditions:
  
  (i) \(\forall a \in A, \forall b \in B, \forall a' \in A, b \cdot a' = a \cdot \mu(b \otimes a')\);
  
  (ii) \(\forall b \in B, \forall a \in A, \forall b' \in B, b \cdot b' = b \cdot v(a \otimes b')\).

In the sequel, we shall adopt the standard convention of writing \((b | a)\) in the place of \(\mu(b \otimes a)\), and \([a | b]\) in the place of \(\nu(a \otimes b)\); with this notation, conditions (i) and (ii) of Definition 2.1 assume the customary appearance:

\[\forall a \in A, \forall b \in B, \forall a' \in A, (b | a') = b(a \cdot a');\]  
\[\forall b \in B, \forall a \in A, \forall b' \in B, (b | a) b' = b(a | b').\]  

The typical example of a topological Morita context is the context generated by a complete l.t. module \((A, \alpha)\), which we are now going to describe.

2.2. Morita context generated by a complete l.t. module. Let \((R, \rho)\) be a complete right l.t. ring, and let \((A, \alpha) \in \text{CL T}_R (R, \rho)\); let us put \((S, \sigma) = \text{CEnd}_R^a(A, \alpha)\) and \((B, \beta) = \text{CHom}_R^a((A, \alpha), (R, \rho))\). It is immediate to check that \((S, \sigma)\) is a right l.t. (in particular, Hausdorff) ring, and it is well known that \((S, \sigma)\) is complete as soon as \((A, \alpha)\) is complete.

It is absolutely self-evident that \((A, \alpha) \in \text{CL T}_R (S, \sigma)\); but, as it is easily seen, we also have \((B, \beta) \in \text{CL T}_R (R, \rho)\), if we endow \((B, \beta)\) with its obvious structure of right \(S\)-module (for \(b \in B, s \in S\) put \(bs = b \circ s\)) and define the left action of \(R\) on \(B\) in the usual way (for \(r \in R, b \in B\), and \(a \in A\) put \(rb(a) = r(b(a))\)).

It make hence sense to form the topological tensor products \((B, \beta) \otimes^S (A, \alpha)\) and \((A, \alpha) \otimes^R (B, \beta)\); we then define the morphisms \(\mu\) and \(\nu\) as in the “classical” case:

\[\mu : B \otimes^S A \to R, \quad b \otimes a \mapsto (b | a) = b(a)\]
\[\nu : A \otimes^R B \to S, \quad a \otimes b \mapsto [a | b] = [a' \mapsto a \cdot b(a')]\]

It is easy to verify that \(\mu\) and \(\nu\) are well-defined and continuous (the details are available in [11, Section 2.3]), and it is immediate that \(\mu\) and \(\nu\) are linear and satisfy conditions (i)
and (ii) of Definition 2.1; therefore, \(((R, \rho), (S, \sigma), (A, \alpha), (B, \beta), \mu, \nu)\) is a topological Morita context. We shall call it the (topological) Morita context generated by \((A, \alpha)\).

3. A hypothesis of density

Throughout this section, \(((R, \rho), (S, \sigma), (A, \alpha), (B, \beta), \mu, \nu)\) is a fixed topological Morita context. In parallel with what is done in the theory of classical Morita contexts, we shall now draw some important consequences from the hypothesis that \(\mu\) and \(\nu\) have dense images.

In the following proposition, \(Z\) denotes the closure of zero in \((B, \beta) \otimes^u_S (A, \alpha)\):

\[
Z = \bigcap \{ W : W \text{ is an open submodule of } (B, \beta) \otimes^u_S (A, \alpha) \}.
\]

3.1. Proposition. If \(\text{Im } \mu\) is dense in \((R, \rho)\), then \(\text{Ker } \mu = Z\).

Proof. Since, by hypothesis, \(\mu\) is continuous and \((R, \rho)\) is Hausdorff, it is obvious that \(Z \subset \text{Ker } \mu\); the non-trivial part is the opposite inclusion.

Given \(t = \sum b_i \otimes a_i \in \text{Ker } \mu\) and an open \(R\)-submodule \(W\) of \((B, \beta) \otimes^u_S (A, \alpha)\), we have to show that \(t \in W\). There exists an open \(R\)-submodule \(A'\) of \((A, \alpha)\) such that \(a \in A' \Rightarrow \forall i, b_i \otimes a' \in W\); then, there exists an open right ideal \(U\) of \((R, \rho)\) such that \(r \in U \Rightarrow \forall i, a_i r \in A'\). Since \(\text{Im } \mu\) is dense in \((R, \rho)\) we can write

\[
1_R = \sum_j (y_j | x_j) + u \quad \text{with } y_j \in B, \ x_j \in A \text{ and } u \in U;
\]

then:

\[
t = \sum_i b_i \otimes a_i = \sum_i b_i \otimes (a_i \cdot 1_R) = \sum_i b_i \otimes \left( a_i \cdot \sum_j (y_j | x_j) + a_i u \right)
\]

\[
= \sum_{i,j} b_i \otimes a_j (y_j | x_j) + \sum_i b_i \otimes a_i u. \tag{*}
\]

But the first summand of the last member of \((*)\) is null; indeed:

\[
\sum_{i,j} b_i \otimes a_i (y_j | x_j) = \sum_{i,j} b_i \otimes [a_i | y_j] x_j = \sum_{i,j} b_i [a_i | y_j] \otimes x_j
\]

\[
= \sum_{i,j} (b_i | a_i) y_j \otimes x_j = \sum_j \left( \sum_i (b_i | a_i) \right) y_j \otimes x_j
\]

\[
= \sum_j 0_R \cdot y_j \otimes x_j = 0.
\]

Thus \(t = \sum_i b_i \otimes a_i u;\) but \(\forall i, a_i u \in A'\), and hence \(\forall i, b_i \otimes a_i u \in W\). □
3.2. Proposition. If \( \text{Im} \mu \) is dense in \((R, \rho)\) then \( \mu \) is open on its image.

**Proof.** Given an open \( R \)-submodule \( W \) of \((B, \beta) \otimes_S^u (A, \alpha)\), take an open \( S \)-submodule \( B' \) of \((B, \beta)\) such that \( B' \subseteq \bigcap_{a \in A} \left( \left( a \otimes \alpha \right)^{-1}(W) \right) \), and then an open right ideal \( U \) of \((R, \rho)\) such that \( UB \subseteq B' \); we claim that \( \mu(W) \supseteq U \cap \text{Im} \mu \).

So, let \( r \in U \cap \text{Im} \mu \); then \( r = \sum_i (b_i, a_i) \) for suitable elements \( b_i \in B, a_i \in A \). Moreover, since \( W \) is open in \( \tau_2(\alpha) \) we can find an open \( R \)-submodule \( A' \) of \((A, \alpha)\) such that \( a' \in A' \Rightarrow \forall i b_i \otimes a' \in W \). Then, we pick an open right ideal \( V \) of \((R, \rho)\) such that \( v \in V \Rightarrow \forall i a_i v \in A' \). Finally, we write

\[
1_R = \sum_j (y_j, x_j) + v \quad \text{with} \quad y_j \in B, \ x_j \in A \quad \text{and} \quad v \in V;
\]

then

\[
r = r \cdot 1_R = \sum_j (ry_j, x_j) + rv = \sum_j (ry_j, x_j) + \sum_i (b_i, a_i v).
\]

Now,

\[
r \in U \Rightarrow \forall j ry_j \in B' \Rightarrow \forall j ry_j \otimes x_j \in W,
\]

so that

\[
\sum_j (ry_j, x_j) = \mu \left( \sum_j ry_j \otimes x_j \right) \in \mu(W).
\]

On the other hand,

\[
v \in V \Rightarrow \forall i a_i v \in A' \Rightarrow \forall i b_i \otimes a_i v \in W,
\]

and hence

\[
\sum_i (b_i, a_i v) = \mu \left( \sum_i b_i \otimes a_i v \right) \in \mu(W).
\]

Therefore \( r \in \mu(W) \), as we wanted. \( \square \)

3.3. Definition. Given a topological Morita context as above, we shall denote by

\[
\tilde{\mu} : (B, \beta) \otimes_S^u (A, \alpha) \rightarrow (R, \rho) \quad \text{and} \quad \tilde{\nu} : (A, \alpha) \otimes_R^u (B, \beta) \rightarrow (S, \sigma)
\]

the continuous morphisms canonically associated with \( \mu \) and \( \nu \), respectively.
Since the conditions that define a topological Morita context are perfectly symmetric, results analogous to those stated in Propositions 3.1 and 3.2 hold for $v$. Thus we can summarize:

### 3.4. Theorem

Let $((R, \rho), (S, \sigma), (A, \alpha), (B, \beta), \mu, v)$ be, as above, a topological Morita context; if $\text{Im} \mu$ (resp., $\text{Im} v$) is dense in $(R, \rho)$ (resp., in $(S, \sigma)$), then

$$\hat{\mu} : (B, \beta) \hat{\otimes}_R^u (A, \alpha) \rightarrow (R, \rho)$$

(resp., $\hat{v} : (A, \alpha) \hat{\otimes}_S^u (B, \beta) \rightarrow (S, \sigma)$)

is a topological isomorphism.

It can be interesting to remark that the topology $\tau_1(\beta)$ is not at all mentioned in the proof of Proposition 3.1: it is only used in the proof of Proposition 3.2.

### 4. Dense contexts and equivalences

We are now in a position to prove our main theorem concerning equivalences determined by topological Morita contexts satisfying a suitable hypothesis.

#### 4.1. Definition

A topological Morita context $((R, \rho), (S, \sigma), (A, \alpha), (B, \beta), \mu, v)$ will be called dense if both the maps $\mu$ and $v$ have dense image.

Observe that $\mu$ and $v$ can have dense image one independently of the other.

**Notice.** Throughout this section, $((R, \rho), (S, \sigma), (A, \alpha), (B, \beta), \mu, v)$ will denote a dense Morita context.

In its fullest generality, our main theorem can be stated as follows:

#### 4.2. Theorem

Let $(T, \tau)$ be a complete right l. t. ring; the following hold:

(i) the pair of functors

$$(A, \alpha) \hat{\otimes}_R^u : (R, \rho)-^uB-(T, \tau) \leftrightarrow (S, \sigma)-^uB-(T, \tau) : (B, \beta) \hat{\otimes}_S^u$$

is an equivalence of categories;

(ii) the pair of functors

$$- \hat{\otimes}_R^u(B, \beta) : (T, \tau)-^uB-(R, \rho) \leftrightarrow (T, \tau)-^uB-(S, \sigma) : - \hat{\otimes}_S^u(A, \alpha)$$

is an equivalence of categories.
Proof. The claim is an immediate consequence of Proposition 1.16 and Theorem 3.4.

That was the “crude”, or “raw” version of the equivalence theorem; now it remains to “cook” it in order to obtain the results we are interested in.

4.3. Theorem. The following facts hold:

(i) the pair of functors

\[-\widehat{\otimes}_R^u (B, \beta) : \text{CLT}^u -(R, \rho) \leftrightarrow \text{CLT}^u -(S, \sigma) : -\widehat{\otimes}_S^u (A, \alpha)\]

is an equivalence of categories;

(ii) there is a lattice isomorphism between the lattice of closed right ideals of \((R, \rho)\) and the lattice of closed \(S\)-submodules of \((B, \beta)\), that induces an isomorphism between the lattice of closed two-sided ideals of \((R, \rho)\) and the lattice of closed sub-\((R, S)\)-bimodules of \((B, \beta)\);

(iii) there is a lattice isomorphism between the lattice of closed left ideals of \((R, \rho)\) and the lattice of closed \(S\)-submodules of \((A, \alpha)\), that induces an isomorphism between the lattice of closed two-sided ideals of \((R, \rho)\) and the lattice of closed sub-\((S, R)\)-bimodules of \((A, \alpha)\);

(iv) statements similar to (ii) and to (iii) hold for \((S, \sigma)\), but exchanging the rôles of \((A, \alpha)\) and \((B, \beta)\); as a consequence, the lattices of closed two-sided ideals of \((R, \rho)\) and of \((S, \sigma)\) are isomorphic.

Proof. (i) is obtained from Theorem 4.2(ii), putting \((T, \tau) = \mathbb{Z}\).

(ii) The lattice of closed right ideals of \((R, \rho)\) is precisely the lattice of sub-objects of \((R, \rho)\) in \(\text{CLT}^u -(R, \rho)\), and the lattice of closed \(S\)-submodules of \((B, \beta)\) is precisely the lattice of subobjects of \((B, \beta)\) in \(\text{CLT}^u -(S, \sigma)\); then one applies (i). The statement about two-sided ideals and sub-bimodules follows similarly from Theorem 4.2(ii), putting \((T, \tau) = (R, \rho)\).

(iii), (iv) Apply similar arguments, using Theorem 4.2 with \((T, \tau) = \mathbb{Z}, \ (T, \tau) = (R, \rho)\) or \((T, \tau) = (S, \sigma)\) as necessary.

4.4. Of course, the equivalence between \(\text{CLT}^u -(R, \rho)\) and \(\text{CLT}^u -(S, \sigma)\) can also be described by means of functors \(\text{CHom}^u\): in fact, for every \((X, \xi) \in \text{CLT}^u -(R, \rho)\) and every \((Y, \eta) \in \text{CLT}^u -(S, \sigma)\) there are natural and topological isomorphisms

\[\text{CHom}_R^u((B, \beta), \text{CHom}_R^u((A, \alpha), (X, \xi))) \cong \text{CHom}_R^u((B, \beta) \widehat{\otimes}_R^u (A, \alpha), (X, \xi))\]

\[\cong \text{CHom}_R^u((R, \rho), (X, \xi)) \cong (X, \xi)\]

and

\[\text{CHom}_R^u((A, \alpha), \text{CHom}_S^u((B, \beta), (Y, \eta))) \cong \text{CHom}_S^u((A, \alpha) \widehat{\otimes}_R^u (B, \beta), (Y, \eta))\]

\[\cong \text{CHom}_S^u((S, \sigma), (Y, \eta)) \cong (Y, \eta).\]
This means that the pair of functors
\[ \text{CHom}^R((A,\alpha),-):\text{CLT}-(R,\rho)\leftrightarrow \text{CLT}-(S,\sigma):\text{CHom}_S^S((B,\beta),-). \]
is an equivalence of categories.
In particular, the functors \( \text{CHom}^R((A,\alpha),-)) \) and \( \text{CHom}^S_S((B,\beta),-) \) are adjoint one of each other on both sides; by Proposition 1.12 we then have:

4.5. Proposition. There are natural and topological isomorphisms of functors
\[ -\hat{\otimes}^R_R(B,\beta)\cong \text{CHom}^R_R((A,\alpha),-) \quad \text{and} \quad -\hat{\otimes}^S_S(A,\alpha)\cong \text{CHom}^S_S((B,\beta),-). \]

4.6. Corollary. The equivalence described in Theorem 4.3 induces an equivalence of categories between \( \text{Mod}-(R,\rho) \) and \( \text{Mod}-(S,\sigma) \).

5. Topological progenerators

Throughout this section we fix a complete right l. t. ring \((R,\rho)\). Recall that, for two objects \((A,\alpha)\) and \((B,\beta)\) of \(\text{CLT}-(R,\rho)\), the continuous trace of \((A,\alpha)\) in \((B,\beta)\) is the closed submodule \(\text{CTr}_{(A,\alpha)}(B,\beta)\) of \((B,\beta)\) defined by
\[ \text{CTr}_{(A,\alpha)}(B,\beta) = \text{cl}_{(B,\beta)}\left(\sum \{ \text{Im} f : f \in \text{CHom}_R((A,\alpha),(B,\beta)) \} \right), \]
where \(\text{cl}_{(B,\beta)}(X)\) denotes the closure of \(X\) in \((B,\beta)\).

5.1. Definition. Let \((A,\alpha)\) and \((B,\beta)\) be two objects of \(\text{CLT}-(R,\rho)\). We shall say that \((A,\alpha)\) topologically generates \((B,\beta)\) if for every continuous \(R\)-linear application \(g:(B,\beta)\to(C,\gamma),\) \(g\neq0\), there exists a continuous \(R\)-linear application \(f:(A,\alpha)\to(B,\beta)\) such that \(g\circ f\neq0\) (i.e., if \((A,\alpha)\) generates \((B,\beta)\) in \(\text{CLT}-(R,\rho)\) in the sense of category theory).

Note the following facts:
(i) \((R,\rho)\) topologically generates every \((A,\alpha)\in\text{CLT}-(R,\rho)\).
(ii) \((A,\alpha)\) topologically generates \((B,\beta)\) if and only if \(\text{CTr}_{(A,\alpha)}(B,\beta) = B\).

Just for this paragraph, let us denote by \(\text{LT}-(R,\rho)\) the category of all l. t. right \((R,\rho)\)-modules (not necessarily complete) and of continuous \(R\)-linear applications. It is well known that, given an arbitrary set \(I\neq\emptyset\) and a family \(((A_i,\alpha_i))_{i\in I}\) of objects of \(\text{LT}-(R,\rho)\), the box topology of \(\bigoplus_{i\in I} A_i\), which we shall denote by \(\bigoplus_{i\in I} A_i\), is the topology having as a basis of neighbourhoods of zero those submodules \(A'\) of \(A = \bigoplus_{i\in I} A_i\) such that \(\pi_i(A')\) is open in \((A_i,\alpha_i)\) for all \(i\in I\), where \(\pi_i:A\to A_i\) is the canonical projection. It is likewise well known that, endowed with this topology, \(\bigoplus_{i\in I} A_i\) becomes the coproduct in \(\text{LT}-(R,\rho)\) of the family \(((A_i,\alpha_i))_{i\in I}\) (this strongly depends upon the fact that we are
dealing with linear topologies). The following lemma shows that the same construction yields coproducts in $\text{CLT}^*(R, \rho)$, too.

5.2. Lemma. Let $((A_\iota, \alpha_\iota))_{\iota \in I}$ be a family of objects of $\text{CLT}^*(R, \rho)$, and denote by $\bigoplus_{\iota \in I} (A_\iota, \alpha_\iota)$ the module $\bigoplus_{\iota \in I} A_\iota$ endowed with the box topology $\bigoplus_{\iota \in I} \alpha_\iota$; then

$$\bigoplus_{\iota \in I} (A_\iota, \alpha_\iota) \in \text{CLT}^*(R, \rho).$$

Proof. Set $(\tilde{A}, \tilde{\alpha}) = \bigoplus_{\iota \in I} (A_\iota, \alpha_\iota)$; the claim is that $(\tilde{A}, \tilde{\alpha})$ is complete. So, let $(\tilde{a}_\gamma)_{\gamma \in \Gamma}$ be a Cauchy net in $\tilde{A}$; for every $\gamma \in \Gamma$ $\tilde{a}_\gamma$ is a family $(a_{\gamma \iota})_{\iota \in I}$. It is obvious that for every $\iota \in I$ $(a_{\gamma \iota})_{\gamma \in \Gamma}$ is a Cauchy net in $(A_\iota, \alpha_\iota)$; let $a_\iota \in A_\iota$ be its limit. The family $\tilde{a} = (a_\iota)_{\iota \in I}$ is an element of $\prod_{\iota \in I} A_\iota$; we claim that $(\tilde{a}_\gamma)_{\gamma \in \Gamma}$ converges to $\tilde{a}$ in the box topology of $\prod_{\iota \in I} A_\iota$ (obvious definition).

Let $\prod_{\iota \in I} A_\iota'$, where for each $\iota \in I$ $A_\iota'$ is an open submodule of $(A_\iota, \alpha_\iota)$, be a typical neighbourhood of zero in the box topology of $\prod_{\iota \in I} A_\iota$; we have to find $\gamma \in \Gamma$ such that

$$\gamma, \delta \geq \tilde{\gamma} \Rightarrow \forall \iota \in I a_{\gamma \iota} - a_\iota \in A_\iota'.$$

(5.3)

By hypothesis, there exists $\tilde{\gamma} \in \Gamma$ such that

$$\gamma, \delta \geq \tilde{\gamma} \Rightarrow \forall \iota \in I a_{\gamma \iota} - a_\iota \in A_\iota';$$

this $\tilde{\gamma}$ satisfies (5.3). Indeed, let us fix $\iota \in I$ (after having fixed $\tilde{\gamma}$). Since $(a_{\gamma \iota})_{\gamma \in \Gamma}$ converges to $a_\iota$, there exists $\tilde{\delta}(\iota) \in \Gamma$ such that

$$\delta \geq \tilde{\delta}(\iota) \Rightarrow a_{\delta \iota} - a_\iota \in A_\iota'.$$

Now, write

$$a_{\gamma \iota} - a_\iota = a_{\gamma \iota} - a_{\delta \iota} + a_{\delta \iota} - a_\iota,$$

where $\delta \geq \tilde{\gamma}$ and $\delta \geq \tilde{\gamma}(\iota)$; (5.3) follows.

Now [3, Exercise 2.10.6] shows that $a_\iota = 0_{A_\iota}$ for almost all $\iota \in I$. \(\Box\)

Given $(A, \alpha) \in \text{CLT}^*(R, \rho)$ and a set $I$, we denote by $(A, \alpha)^{(I)}$ the module $A^{(I)}$ endowed with the box topology. By the above lemma, $(A, \alpha)^{(I)} \in \text{CLT}^*(R, \rho)$, and knowing this, the following proposition is trivial (cf. [11, Proposition 3.13]).

5.4. Proposition. Let $(A, \alpha), (B, \beta) \in \text{CLT}^*(R, \rho)$; the following are equivalent:

(a) $(A, \alpha)$ topologically generates $(B, \beta)$;
(b) there exists (of course in $\text{CLT}^*(R, \rho)$) an epimorphism $p : (A, \alpha)^{(I)} \to (B, \beta)$, where $I = \text{CHom}_{\rho}((A, \alpha), (B, \beta))$;
(c) there exists an epimorphism $p : (A, \alpha)^{(I)} \to (B, \beta)$, where $I$ is a set.
Having carried out this preliminary work, we can now introduce a central concept of our theory. The following definition is similar to Definition 2.4 of [5], where the notion of $\rho$-generator is given, but it is not exactly the same. In fact, our setting is somehow hybrid, in that it mixes objects of $\text{CLT}-(R, \rho)$ and objects of $\text{Mod}-(R, \rho)$ (and this, as we shall say in a moment, is its big deficiency).

5.5. Definition. Let $(P, \varepsilon) \in \text{CLT}-(R, \rho)$; we shall say that $(P, \varepsilon)$ is:

(i) **topologically finitely generated** (in $\text{CLT}-(R, \rho)$) (abbreviated in t. f. g.), if for every open submodule $P'$ of $(P, \varepsilon)$ the discrete module $P/P'$ is finitely generated as a right $R$-module;

(ii) **topologically projective** (in $\text{CLT}-(R, \rho)$), if every diagram in $\text{CLT}-(R, \rho)$ of the form

\[
\begin{array}{ccc}
(P, \varepsilon) & \xrightarrow{x} & (A, \alpha) \\
\downarrow \bar{x} & & \downarrow f \\
(B, \beta) & \xleftarrow{\bar{f}} & (P, \varepsilon)
\end{array}
\] (5.6)

with $(B, \beta)$ discrete and $f$ epimorphism, can be completed to a commutative diagram in $\text{CLT}-(R, \rho)$

\[
\begin{array}{ccc}
(P, \varepsilon) & \xrightarrow{x} & (P, \varepsilon) \\
\downarrow \bar{x} & & \downarrow x \\
(A, \alpha) & \xleftarrow{\bar{f}} & (B, \beta)
\end{array}
\]

(iii) **a topological generator** (of $\text{CLT}-(R, \rho)$), if it topologically generates every object of $\text{CLT}-(R, \rho)$.

Finally, $(P, \varepsilon)$ will be called a **topological progenerator** (of $\text{CLT}-(R, \rho)$) if it satisfies conditions (i)–(iii).

Of course, it suffices to verify condition (i) for a basis of neighbourhoods of zero in $(P, \varepsilon)$. Note that, clearly, a topological progenerator is also a $\rho$-progenerator in the sense of [5]; the converse is not evident at a first sight, but it turns out to be true too (see Theorem 8.4).

5.7. Remark. It easy to show (see [11, Remark 3.10]) that an object $(P, \varepsilon) \in \text{CLT}-(R, \rho)$ satisfies condition (ii) of Definition 5.5 if and only if the functor

$$
\mathcal{H} = \text{CHom}_{R}^{\eta}((P, \varepsilon), -) : \text{CLT}-(R, \rho) \to \text{CLT}-\mathbb{Z}
$$

preserves epimorphisms; but the stated condition is more useful for computations.
5.8. Proposition. \((R, \rho)\) is a topological progenerator of \(\text{CLT}-(R, \rho)\).

In the definition of topological progenerator given above, the notion of discrete module appears twice; we have not been able to ascertain whether this notion is categorical, that is, whether it is preserved by every equivalence of categories \(F: \text{CLT}-(R, \rho) \rightarrow \text{CLT}-(S, \sigma)\), where \((S, \sigma)\) is another complete right l. t. ring. Anyway, for our purposes it suffices to determine a particular class of equivalences between categories of complete l. t. modules that are guaranteed to send discrete modules into discrete modules; all the equivalences of this class do preserve the notion of topological progenerator.

For the rest of this section \((S, \sigma)\) denotes another complete right l. t. ring.

5.9. Definition. A functor \(F: \text{CLT}-(R, \rho) \rightarrow \text{CLT}-(S, \sigma)\) will be called topological if for every two objects \((A, \alpha), (B, \beta) \in \text{CLT}-(R, \rho)\) the map
\[
\text{CHom}_R^u \left( (A, \alpha), (B, \beta) \right) \rightarrow \text{CHom}_S^u \left( F(A, \alpha), F(B, \beta) \right)
\]
\[f \mapsto Ff\]
is a continuous homomorphism (of l. t. abelian groups).

Perhaps, a better adjective to define a functor that satisfies the condition of the above definition would be “continuous”. Unfortunately, both the terms “continuous functor” and “topological functor” are already widely used in category theory in a completely different meaning (cf. [9, Section V.4], [2, Section 7.3], and [6]). Since a conflict in terminology seems unavoidable (we simply cannot imagine another term), we prefer to stick to the term “topological”, because, as we shall see, a topological Morita context always yields what we have decided to call a topological equivalence (see below).

The following two propositions, whose proof is routine (cf. [11, Propositions 3.17, 3.18]), show that the notion of topological functor is well settled.

5.10. Proposition. Suppose that \(F, G: \text{CLT}-(R, \rho) \rightarrow \text{CLT}-(S, \sigma)\) are naturally equivalent functors; if \(F\) is topological, then \(G\) is topological too.

5.11. Proposition. Let \((P, \varepsilon)\) be any object of \((S, \sigma)-\text{uB}-(R, \rho)\); the functor
\[
\text{CHom}_R^u \left( (P, \varepsilon), - \right): \text{CLT}-(R, \rho) \rightarrow \text{CLT}-(S, \sigma)
\]
is topological.

Given a fully faithful functor \(F: \text{CLT}-(R, \rho) \rightarrow \text{CLT}-(S, \sigma)\), we shall call it bitopological iff for every two objects \((A, \alpha), (B, \beta) \in \text{CLT}-(R, \rho)\) the map
\[
\text{CHom}_R^u \left( (A, \alpha), (B, \beta) \right) \rightarrow \text{CHom}_S^u \left( F(A, \alpha), F(B, \beta) \right)
\]
\[f \mapsto Ff\]
is a homeomorphism (that is, an isomorphism of l. t. abelian groups).
5.12. Definition. An equivalence of categories
\[ F : \text{CLT}^{-}(R, \rho) \rightarrow \text{CLT}^{-}(S, \sigma), \quad G : \text{CLT}^{-}(S, \sigma) \rightarrow \text{CLT}^{-}(R, \rho) \]
will be called topological if \( F \) and \( G \) are both topological (hence bitopological).

5.13. Proposition. Suppose that an equivalence of categories
\[ F : \text{CLT}^{-}(R, \rho) \rightarrow \text{CLT}^{-}(S, \sigma), \quad G : \text{CLT}^{-}(S, \sigma) \rightarrow \text{CLT}^{-}(R, \rho) \]
is given; then the following are equivalent (here the symbol \( \cong \) denotes natural equivalence of functors):

(a) there exists \((P, \varepsilon) \in (S, \sigma)-uB-(R, \rho)\) such that \( F \cong \text{CHom}_{u}^{R}(\langle P, \varepsilon \rangle, -) \);
(b) \((A, \alpha) \in \text{Mod}^{-}(R, \rho) \Rightarrow F(A, \alpha) \in \text{Mod}^{-}(S, \sigma)\);
(c) \( F \) is bitopological;
(d) there exists \((Q, \xi) \in (R, \rho)-uB-(S, \sigma)\) such that \( G \cong \text{CHom}_{u}^{S}(\langle Q, \xi \rangle, -) \);
(e) \((B, \beta) \in \text{Mod}^{-}(S, \sigma) \Rightarrow G(B, \beta) \in \text{Mod}^{-}(R, \rho)\);
(f) \( G \) is bitopological.

5.14. Remark. The equivalence between (b) and (e) was first observed by E. Gregorio
(private communication to the author), with a more complicated proof.

Proof. The implications (a) \( \Rightarrow \) (b) and (d) \( \Rightarrow \) (e) are obvious. It is also clear that (c) \( \Leftrightarrow \) (f), anyway this will follow from what we are about to say.

(b) \( \Rightarrow \) (c) Let \((A, \alpha), (B, \beta) \in \text{CLT}^{-}(R, \rho)\), and denote by \( \mathfrak{B} \) the family of all open submodules of \((B, \beta)\). Since \( F \) is an equivalence of categories, since \( F \) sends discrete modules into discrete modules, and since any functor of the form \( \text{CHom}_{u}^{R}(X, \xi, -) \) has a left adjoint, we have the following chain of topological isomorphisms:

\[
\text{CHom}_{u}^{S}(F(A, \alpha), F(B, \beta)) \cong \text{CHom}_{u}^{S}(F(A, \alpha), F(\lim_{B' \in \mathfrak{B}} B/B'))
\cong \lim_{B' \in \mathfrak{B}} \text{CHom}_{u}^{S}(F(A, \alpha), F(B/B'))
\cong \lim_{B' \in \mathfrak{B}} \text{CHom}_{u}^{S}((A, \alpha), B/B')
\cong \text{CHom}_{u}^{S}((A, \alpha), (B, \beta)).
\]

One can verify that the composed isomorphism is just \( f \leftrightarrow F f \).

(c) \( \Rightarrow \) (d) It suffices to put \((Q, \xi) = F(R, \rho)\). Then, since \( F \) is an equivalence and it is bitopological, we have we following chain of natural isomorphisms in \( \text{CLT}^{-}(S, \sigma) \):
\[ G(B, \beta) \cong \text{CHom}_u^R(R, \rho, G(B, \beta)) \cong \text{CHom}_u^S(F(R, \rho), F(G(B, \beta)) \cong \text{CHom}_u^S((Q, \zeta), (B, \beta)) \]

(see also Proposition 1.14). In particular, \( \text{CEnd}_u^R(Q, \zeta) \cong G(Q, \zeta) \cong (R, \rho) \), so that it actually results \((Q, \zeta) \in (R, \rho) - u\beta - (S, \sigma)\).

The other implications are obtained by symmetry. □

In particular, by Proposition 4.5 the equivalence that arises from a dense Morita context is topological.

We can now state the result we were looking for:

5.15. Proposition. Let \( F : \text{CLT}-(R, \rho) \to \text{CLT}-(S, \sigma) \) be a topological equivalence, and let \((P, \varepsilon) \in \text{CLT}-(R, \rho)\); then:

(i) if \((P, \varepsilon)\) is t. f. g., then \( F(P, \varepsilon) \) is t. f. g.;
(ii) if \((P, \varepsilon)\) is topologically projective in \( \text{CLT}-(R, \rho) \), then \( F(P, \varepsilon) \) is topologically projective in \( \text{CLT}-(S, \sigma) \);
(iii) if \((P, \varepsilon)\) is a topological generator of \( \text{CLT}-(R, \rho) \), then \( F(P, \varepsilon) \) is a topological generator of \( \text{CLT}-(S, \sigma) \);
(iv) if \((P, \varepsilon)\) is a topological progenerator of \( \text{CLT}-(R, \rho) \), then \( F(P, \varepsilon) \) is a topological progenerator of \( \text{CLT}-(S, \sigma) \).

Proof. (i) It is well known that \((P, \varepsilon)\) is topologically isomorphic to \( \lim P/P' \) with \( P' \) varying among the open submodules of \((P, \varepsilon)\); therefore \((P, \varepsilon)\) is t. f. g. if and only if it results \((P, \varepsilon) \cong \lim D_i\) with \( D_i \in \text{Mod}-(R, \rho) \) and \( D_i\) f. g. for every \( i \). By the previous proposition, if \( D_i\) is discrete and f. g., then so is \( F(D_i)\); for, a discrete module is f. g. if and only if every directed family of submodules of it, the union of which is the whole module, has an element which is already the whole module, and expressed in these terms the property is evidently preserved by every equivalence of categories. Since an equivalence (even not topological) preserves limits, we are done.

(ii) It follows again by the previous proposition.
(iii) Obvious (it is not even necessary that the equivalence be topological).
(iv) It follows from (i)–(iii). □

6. Morita context generated by a topological progenerator

In this section we shall prove that the topological Morita context generated by a topological progenerator \((P, \varepsilon) \in \text{CLT}-(R, \rho)\) is dense; we shall then be able to apply the results of Section 4 to obtain a topological equivalence between \( \text{CLT}-(R, \rho) \) and \( \text{CLT}-(S, \sigma) \), where \((S, \sigma) = \text{CEnd}_u^R(P, \varepsilon)\).

In the following two propositions, \((R, \rho)\) is a complete right l. t. ring, \((P, \varepsilon)\) is an object of \( \text{CLT}-(R, \rho) \) and \( \mathcal{C} \cong ((R, \rho), (S, \sigma), (P, \varepsilon), (Q, \zeta), \mu, \nu) \) is the topological
Morita context generated by \((P, \varepsilon)\) (cf. 2.2); we recall that this means, in particular, that 
\((S, \sigma) = \text{CEnd}_R^u(P, \varepsilon)\) and that 
\((Q, \zeta) = \text{CHom}_R^u((P, \varepsilon), (R, \rho)).\)

6.1. Proposition. \(\mu: (Q, \zeta) \otimes_R^u (P, \varepsilon) \to (R, \rho)\) has dense image if and only if \((P, \varepsilon)\) is a 
topological generator of \(\text{CL}_T-(R, \rho)\).

Proof. This follows immediately from the definitions. \(\square\)

6.2. Proposition. \(\nu: (P, \varepsilon) \otimes_R^u (Q, \zeta) \to (S, \sigma)\) has dense image if and only if \((P, \varepsilon)\) is 
t.f.g. and topologically projective in \(\text{CL}_T-(R, \rho)\).

Proof. Recall that \(\nu\) is defined as follows:

\[
v: (P, \varepsilon) \otimes_R^u (Q, \zeta) \to (S, \sigma) = \text{CEnd}_R^u(P, \varepsilon)
\]

\[
x \otimes y \mapsto \left[p \mapsto x \cdot y(p)\right].
\]

First, assume that \((P, \varepsilon)\) is t. f. g. and topologically projective in \(\text{CL}_T-(R, \rho)\).

To show that \(\text{Im} \nu\) is dense, it suffices to verify that, having fixed at will an open 
submodule \(P'\) of \((P, \varepsilon)\), and having put 
\(V = W(P; P') = \{s \in S: sP \subseteq P'\}\), 
there exists \(t = \sum_i x_i \otimes y_i \in P \otimes_R Q\) such that \(1_S - \nu(t) \in V\), that is 
\[
\forall p \in P \ p = \sum_i x_i(y_i | p) \in P'. \quad (6.3)
\]

We therefore proceed to construct such a \(t\). We denote by \(\pi_P: P \to P / P'\) the canonical 
projection. Since \(P / P'\) is finitely generated, there exist \(n \in \mathbb{N}\) and \(x_1, \ldots, x_n \in P\) such that 
the continuous map
\[
f: (R, \rho)^n \to P / P' 
\]
\[
(r_1, \ldots, r_n) \mapsto \sum_{i=1}^n \pi_P(x_i) \cdot r_i
\]

\((R, \rho)^n\) is the module \(R^n\) endowed with the power topology of \(\rho\) is surjective. Let then 
\(\pi_i: R^n \to R\) be the canonical projection on the \(i\)th component; since \((P, \varepsilon)\) is topologically 
projective, there exists \(\tilde{f}\) that makes the following diagram commute.
For every \( i = 1, \ldots, n \) put \( y_i = \pi_i \circ \tilde{f} \): evidently, \( y_i \in Q \). Now, \( \forall p \in P \) one has:

\[
\pi_{P'} \left( \sum_{i=1}^{n} x_i(y_i | p) \right) = \sum_{i=1}^{n} \pi_{P'}(x_i) \cdot (y_i | p) = f(\tilde{f}(p)) = \pi_{P'}(p);
\]

clearly this proves (6.3).

Conversely, suppose that \( \text{Im} \nu \) is dense in \( (S, \sigma) \). If \( P' \) is an open \( R \)-submodule of \( (P, \varepsilon) \), taken an open right ideal \( V \) of \( (S, \sigma) \) such that \( VP \subseteq P' \), we write

\[
1_S = \sum_{i=1}^{n} [x_i | y_i] + v,
\]

where \( n \in \mathbb{N}, v \in V \), and, for all \( i = 1, \ldots, n, x_i \in P \) and \( y_i \in Q \). Therefore, \( \forall p \in P \) we have

\[
p = 1_S \cdot p = \sum_{i=1}^{n} [x_i | y_i]p + vp = \sum_{i=1}^{n} x_i(y_i | p) + vp;
\]

(6.4)

but \( vp \in P' \) because \( v \in V \), and this, in view of the arbitrary choice of \( P' \), proves that \( (P, \varepsilon) \) is t. f. g. in \( \text{CLT}(R, \rho) \). Moreover, suppose we are given a diagram in \( \text{CLT}(R, \rho) \)

\[
\begin{array}{ccc}
(P, \varepsilon) & \xrightarrow{f} & (A, \alpha) \\
\downarrow & & \downarrow h \\
(A, \alpha) & \xrightarrow{h} & (B, \beta)
\end{array}
\]

with \( \beta \) discrete and \( h \) epimorphism; put \( P' = \text{Ker} f \); since \( f \) is continuous with discrete codomain, \( P' \) is an open \( R \)-submodule of \( (P, \varepsilon) \). The foregoing discussion hence applies to \( P' \), so that, by 6.4 and with the same symbols as above, we have

\[
f(p) = \sum_{i=1}^{n} f(x_i) \cdot (y_i | p)
\]

(indeed, \( f(vp) = 0 \) since \( vp \in P' \)). Clearly, \( h \) is surjective, so we can find elements \( a_1, \ldots, a_n \in A \) such that \( \forall i = 1, \ldots, n \ h(a_i) = f(x_i) \). Then we define

\[
\tilde{f} : (P, \varepsilon) \to (A, \alpha) \\
p \mapsto \sum_{i=1}^{n} a_i \cdot (y_i | p);
\]
clearly \( \tilde{f} \) is continuous, and moreover, \( \forall p \in P \),

\[
h(\tilde{f}(p)) = \sum_{i=1}^{n} h(a_i) \cdot (y_i | p) = \sum_{i=1}^{n} f(x_i) \cdot (y_i | p) = f(p). \quad \square
\]

7. Conclusive results

We are now ready to state our conclusive results, that constitute a “topological”
generalization of the classical theory of Morita contexts. The next theorem follows the
pattern of the result called “Morita I” in Section 3.12 of [7].

7.1. Theorem. Suppose that \( \mathcal{C} = ((R, \rho), (S, \sigma), (P, \varepsilon), (Q, \zeta), \mu, \nu) \) is a dense Morita
context; let \( \hat{\mu} : (Q, \zeta) \hat{\otimes}_R (P, \varepsilon) \to (R, \rho) \) and \( \hat{\nu} : (P, \varepsilon) \hat{\otimes}_S (Q, \zeta) \to (S, \sigma) \) be the
continuous morphisms canonically associated with \( \mu \) and \( \nu \), respectively; then the
following hold:

(i) \( \hat{\mu} \) and \( \hat{\nu} \) are topological isomorphisms;
(ii) \( (P, \varepsilon) \) and \( (Q, \zeta) \) are topological progenerators of \( \text{CLT}-(R, \rho) \) and \( \text{CLT}-(S, \sigma) \),
respectively;
(iii) \( (S, \sigma) \) is topologically isomorphic to \( \text{CEnd}^u_R(P, \varepsilon) \), and \( (R, \rho) \) is topologically
isomorphic to \( \text{CEnd}^u_S(Q, \zeta) \);
(iv) \( (Q, \zeta) \) is topologically isomorphic to \( \text{CHom}^u_R((P, \varepsilon), (R, \rho)) \), and \( (P, \varepsilon) \) is topolog-
ically isomorphic to \( \text{CHom}^u_S((Q, \zeta), (S, \sigma)) \);
(v) the pair of functors

\[
- \hat{\otimes}_R(Q, \zeta) : \text{CLT}-(R, \rho) \leftrightarrow \text{CLT}-(S, \sigma) : - \hat{\otimes}_S(P, \varepsilon)
\]

is a topological equivalence;
(vi) statements (ii)–(iv) of Theorem 4.3 hold for \( \mathcal{C} \): in particular, \( (R, \rho) \) and \( (S, \sigma) \) have
isomorphic lattices of closed two-sided ideals.

Proof. (i) is Theorem 3.4, and (v) follows from Theorem 4.3, from Proposition 4.5 and
from Proposition 5.13.
(ii) Since \( (P, \varepsilon) \cong (S, \sigma) \hat{\otimes}_S^u(P, \varepsilon) \) and \( (S, \sigma) \) is a topological progenerator of \( \text{CLT}-(S, \sigma) \) (cf. Proposition 5.8), the claim ensues from (v) and Proposition 5.14; similarly for
(\( Q, \zeta \)).
(iii) By (v), Proposition 4.5 and Proposition 1.15(i) we have

\[
(S, \sigma) \cong \text{CHom}^u_R((P, \varepsilon), (S, \sigma) \hat{\otimes}_S^u(P, \varepsilon)) \cong \text{CHom}^u_R((P, \varepsilon), (P, \varepsilon)).
\]

Similarly for \( (R, \rho) \).
(iv) It follows from Propositions 1.15(i) and 4.5; for instance, for \( (Q, \zeta) \):

\[
(Q, \zeta) \cong (R, \rho) \hat{\otimes}_R(Q, \zeta) \cong \text{CHom}^u_R((P, \varepsilon), (R, \rho)).
\]
Conversely (the following theorem parallels “Morita II” in [7, Section 3.15]):

**7.2. Theorem.** Let \((R, \rho)\) and \((S, \sigma)\) be two complete right l. t. rings; suppose that a topological equivalence

\[
\mathcal{F}: \text{CLT}(R, \rho) \leftrightarrow \text{CLT}(S, \sigma): \mathcal{G}
\]

is given, and put \((P, \varepsilon) = \mathcal{G}(S, \sigma), (Q, \zeta) = \mathcal{F}(R, \rho)\); then the following hold:

(i) \((P, \varepsilon)\) and \((Q, \zeta)\) are topological progenerators of \(\text{CLT}(R, \rho)\) and \(\text{CLT}(S, \sigma)\), respectively;

(ii) there exist maps \(\mu: (Q, \zeta) \otimes^u_R (P, \varepsilon) \to (R, \rho)\) and \(\nu: (P, \varepsilon) \otimes^u_R (Q, \zeta) \to (S, \sigma)\) such that the topological Morita context \(((R, \rho), (S, \sigma), (P, \varepsilon), (Q, \zeta), \mu, \nu)\) is dense, so that Theorem 7.1 holds for it; in particular,

\[
(R, \rho) \cong \text{CEnd}^u_S(Q, \zeta) \quad \text{and} \quad (S, \sigma) \cong \text{CEnd}^u_R(P, \varepsilon),
\]

the isomorphisms being topological;

(iii) there are natural and topological equivalences of functors

\[
\mathcal{F} \cong -\widehat{\otimes}^u_R(Q, \zeta), \quad \mathcal{G} \cong -\widehat{\otimes}^u_S(P, \varepsilon).
\]

**Proof.** (i) holds by Proposition 5.14. The topological Morita context generated by \((P, \varepsilon)\), call it \(((R, \rho), (S_1, \sigma_1), (P, \varepsilon), (Q_1, \zeta_1), \mu_1, \nu_1)\), is dense, so the pair of functors

\[
-\widehat{\otimes}^u_R(Q_1, \zeta_1) \cong \text{CHom}^u_R((P, \varepsilon), -): \text{CLT}(R, \rho) \to \text{CLT}(S_1, \sigma_1)
\]

and

\[
-\widehat{\otimes}^u_S(P, \varepsilon) \cong \text{CHom}^u_S((Q_1, \zeta_1), -): \text{CLT}(S_1, \sigma_1) \to \text{CLT}(R, \rho)
\]

is a topological equivalence. Now, since \(\mathcal{G}\) is a topological equivalence too, we have topological isomorphisms

\[
(S_1, \sigma_1) = \text{CEnd}^u_S(P, \varepsilon) = \text{CEnd}^u_R(\mathcal{G}(S, \sigma)) \cong \text{CEnd}^u_S(S, \sigma) \cong (S, \sigma).
\]

Moreover, arguing as in the proof of implication (c) \(\Rightarrow\) (d) of Proposition 5.13 one finds that

\[
(Q_1, \zeta_1) = \text{CHom}^u_R((P, \varepsilon), (R, \rho)) \cong \mathcal{F}(R, \rho) = (Q, \zeta);
\]

(ii) should now be evident, and (iii) is again an immediate consequence of the proof of implication (c) \(\Rightarrow\) (d) of Proposition 5.13. □

Before stating the analogue of “Morita III”, we need to fix a definition.
7.3. Definition. Let \((R, \rho)\) and \((S, \sigma)\) be two complete right l. t. rings; we say that an object of \((P, \epsilon) \in (S, \sigma)\text{-}uB\text{-}(R, \rho)\) is (topologically) invertible if there exists an object \((Q, \zeta) \in (R, \rho)\text{-}uB\text{-}(S, \sigma)\) such that \((P, \epsilon) \otimes^u_R (Q, \zeta) \cong (S, \sigma)\) and \((Q, \zeta) \otimes^u_S (P, \epsilon) \cong (R, \rho)\).

If \((T, \tau)\) is a third complete right l. t. rings, and if \((A, \alpha) \in (R, \rho)\text{-}uB\text{-}(S, \sigma)\) and \((B, \beta) \in (S, \sigma)\text{-}uB\text{-}(T, \tau)\) are both invertible, then it immediately ensues from the associative property of \(\otimes^u\) that \((A, \alpha) \otimes^u_S (B, \beta)\) is invertible too.

The following theorem deliberately imitates even the wording of [7, “Morita III”].

7.4. Theorem. The map \((P, \epsilon) \mapsto -\otimes^u_S(P, \epsilon)\) defines a bijection of the class of isomorphism classes of invertible objects of \((S, \sigma)\text{-}uB\text{-}(R, \rho)\) and the class of natural and topological equivalences of functors giving topological equivalences of \(CLT\text{-}(S, \sigma)\) and \(CLT\text{-}(R, \rho)\). In this correspondence, composition of equivalences corresponds to topological tensor products of invertible objects, as denoted by 1.7.

Proof. If one bears in mind Proposition 1.15, the proof runs exactly as that of “Morita III” in Section 3.15 of [7]. 

Before concluding this section, we want to refer the reader to [5, Section 7] for some interesting results about the commutative case.

8. Comparison with \(\rho\)-progenerators

The theory we have been developing is intimately correlated with the results of Gregorio’s paper [5]. Given two right l. t. rings \((R, \rho)\) and \((S, \sigma)\), Gregorio considered equivalences between \(\text{Mod}\text{-}(R, \rho)\) and \(\text{Mod}\text{-}(S, \sigma)\), characterizing the topological bimodules \((S_P, \epsilon)\) that determine them. The key notion in this respect is the definition of \(\rho\)-progenerator (for a complete right l. t. ring \((R, \rho)\)):

8.1. Definition (cf. [5, Definition 2.4]). Let \((R, \rho)\) be a complete right l. t. ring and \((P, \epsilon) \in \text{CLT}\text{-}(R, \rho)\); we say that \((P, \epsilon)\) is:

(i) topologically quasi-projective, if for every open submodule \(P'\) of \((P, \epsilon)\) and every continuous morphism \(f: (P, \epsilon) \to P/P'\) (where \(P/P'\) has the discrete topology), there exists a continuous endomorphism \(g: (P, \epsilon) \to (P, \epsilon)\) that makes the following diagram commute (the bottom line is the canonical projection):

\[
\begin{array}{c}
(P, \epsilon) \\
\downarrow g \\
(P, \epsilon)
\end{array}
\begin{array}{c}
\downarrow f \\
(P, \epsilon) \to P/P' \to 0
\end{array}
\]

(ii) a self generator, if \((P, \epsilon)\) topologically generates all its open submodules (cf. Definition 5.1);
Finally, \((P, \varepsilon)\) is called a \(\rho\)-progenerator if it is topologically finitely generated (cf. Definition 5.5.(i)) and satisfies the three conditions above.

The main theorem of [5] is the following:

8.2. **Theorem** (cf. [5, Theorem 4.9]).

1) Let \((R, \rho)\) and \((S, \sigma)\) be two complete right t. r. rings, and suppose that an equivalence
\[
\mathcal{F}: \text{Mod-}(R, \rho) \leftrightarrow \text{Mod-}(S, \sigma): \mathcal{G}
\]
is given; then there exists a \(\rho\)-progenerator \((P, \varepsilon) \in \text{CLT-}(R, \rho)\) such that:

(i) \((S, \sigma) \cong \text{CEnd}_R^u(P, \varepsilon)\) topologically;
(ii) \(\mathcal{F} \cong \text{CHom}_R^u((P, \varepsilon), -);
(iii) \(\mathcal{G} \cong -\hat{\otimes}_S^u(P, \varepsilon)\).

2) Conversely, let \((R, \rho)\) be a complete right l. t. ring, \((P, \varepsilon) \cong \text{CLT-}(R, \rho)\) a \(\rho\)-progenerator, and \((S, \sigma) = \text{CEnd}_R^u(P, \varepsilon)\); then we have an equivalence
\[
\text{CHom}_R^u((P, \varepsilon), -): \text{Mod-}(R, \rho) \leftrightarrow \text{Mod-}(S, \sigma): -\hat{\otimes}_S^u(P, \varepsilon).
\]

Remark. In part (1), \((P, \varepsilon)\) is given by
\[
(P, \varepsilon) = \lim_{V \in \mathcal{G}} \mathcal{G}(S/V), \tag{8.3}
\]
where \(\mathcal{G}\) denotes the filter of all open right ideals of \((S, \sigma)\), and \(\varepsilon\) is the limit topology of the discrete topologies on \(\mathcal{G}(S/V)\).

Combining Theorem 8.2 with Theorem 7.2 we obtain the following result, which, among other things, establishes the non-trivial fact that \((P, \varepsilon) \in \text{CLT-}(R, \rho)\) is a topological progenerator if (and only if) it is a \(\rho\)-progenerator.

8.4. **Theorem.** Let \((P, \varepsilon) \in \text{CLT-}(R, \rho)\) and \((S, \sigma) = \text{CEnd}_R^u(P, \varepsilon)\); the following are equivalent:

(a) \((P, \varepsilon)\) is a \(\rho\)-progenerator;
(b) there is an equivalence of categories
\[
\text{CHom}_R^u((P, \varepsilon), -): \text{Mod-}(R, \rho) \leftrightarrow \text{Mod-}(S, \sigma): -\hat{\otimes}_S^u(P, \varepsilon);
\]
(c) there is an equivalence of categories
\[
\text{CHom}_R^u((P, \varepsilon), -): \text{CLT-}(R, \rho) \leftrightarrow \text{CLT-}(S, \sigma): -\hat{\otimes}_S^u(P, \varepsilon);
\]
(d) \((P, \varepsilon)\) is a topological progenerator.

**Proof.** (d) \(\Rightarrow\) (a) is obvious.

(a) \(\Rightarrow\) (b) This is Theorem 8.2.

(b) \(\Rightarrow\) (c) Repeating the construction of (8.3), put

\[
(Q, \xi) = \lim_{U \in \mathcal{F}} \mathcal{F}(R/U),
\]

where \(\mathcal{F}\) denotes the filter of all open right ideals of \((R, \rho)\)\(,\) and \(\xi\) is the limit topology of the discrete topologies on \(\mathcal{F}(R/U)\). It can then be shown (see [5, Theorem 1.3]) that\(\sim\)\(\mathcal{H}\)\(\mathcal{M}\)

\[
\hat{\mathcal{H}}_{\mathcal{M}}((P, \varepsilon), -) \cong \mathcal{H}\mathcal{M}((Q, \xi), -).
\]

Now, we can regard \(\mathcal{H}\mathcal{M}((P, \varepsilon), -)\) and \(\mathcal{H}\mathcal{M}((Q, \xi), -)\) as functors

\[
\mathcal{H}\mathcal{M}((P, \varepsilon), -) : \text{CLT}(R, \rho) \leftrightarrow \text{CLT}(S, \sigma) : \mathcal{H}\mathcal{M}((Q, \xi), -).
\]

Let then \((A, \alpha) \in \text{CLT}(R, \rho)\), and denote by \(\mathcal{F}\) the filter of all open submodules of \((A, \alpha)\); since the functor \(\mathcal{H}\mathcal{M}((P, \varepsilon), -)\) is a right adjoint, we have the following chain of natural and topological isomorphisms (to ease the notation, we put \(\mathcal{F} = \mathcal{H}\mathcal{M}((P, \varepsilon), -)\) and \(\mathcal{G} = \mathcal{H}\mathcal{M}((Q, \xi), -)\):

\[
\mathcal{G}(\mathcal{F}(A, \alpha)) \cong \mathcal{G}(\lim_{A' \in \mathcal{F}} A/A') \cong \lim_{A' \in \mathcal{F}} \mathcal{G}(\mathcal{F}(A/A')) \cong \lim_{A' \in \mathcal{F}} A/A' \cong (A, \alpha).
\]

Similarly one proves that \(\mathcal{F}(\mathcal{G}(B, \beta)) \cong (B, \beta)\).

(c) \(\Rightarrow\) (d) This follows from Theorem 7.2. \(\square\)

9. **Ideals generated by dense idempotents**

We still have to show that dense Morita contexts do exist. Both as an example and an application, we shall re-obtain the main result of our paper [12] by means of the theory of topological Morita contexts that we have developed so far (actually, the results of Section 4 suffice for this).

Let \((R, \rho)\) be a right complete l. t. ring, \(e\) be an idempotent of \(R, S = eRe;\) note that \(S\) is a subring of \(R,\) but its unit is \(1_S = e \neq 1_R\) (in general); we shall endow \(S\) with its induced topology, which we denote by \(\sigma.\) These symbols will be used throughout this section; moreover, if we have a module \((A, \alpha) \in \text{LT}(R, \rho)\), we shall denote by \(\alpha e\) the topology induced by \(\alpha\) on the \(S\)-submodule \(Ae,\) and a similar notation will be used in analogous situations (for instance, \(\sigma = e\rho e).\)

Observe that if \((A, \alpha) \in \text{CLT}(R, \rho),\) then \((Ae, \alpha e)\) is complete as well. The same is true of \(Re, eR,\) and also of \(S = eRe.\)

9.1 **Definition** (cf. [12, Definition 2.7]). An idempotent element \(e \in R\) will be called a dense idempotent of \((R, \rho)\) if \(ReR\) is dense in \((R, \rho).\)
Throughout this section, we shall assume that \( e \) is a dense idempotent of \( (R, \rho) \). An example of a complete l. t. ring with dense idempotents is the topological ring of the matrices indexed by an infinite set \( \Lambda \) and with coefficients in a right complete l. t. ring \((R, \rho)\), introduced by Leptin [8] and described also in Chapter 1 of [11], or, more briefly, in Section 4 of [12].

9.2. Consider the two obvious bilinear maps
\[
\mu : Re \otimes_S eR \to R, \quad y \otimes x \mapsto xy \\
\nu : eR \otimes_R Re \to S, \quad x \otimes y \mapsto xy.
\]

It is trivial that \((S, \sigma) \cong \text{CEnd}^u_R(eR, e\rho)\) topologically (cf. [1, Proposition 4.15] for a similar statement), and that the obvious ring homomorphism \( R \to \text{CEnd}^u_S(Re, \rho e) \) is continuous; from [12, Proposition 2.10 and Corollary 2.2] and from the fact that \( e \) is a dense idempotent, it follows that \(((R, \rho), (S, \sigma), (eR, e\rho), (Re, \rho e), \mu, \nu)\) is a dense Morita context.

We can therefore re-obtain the main result of our paper [12] as a mere corollary of Theorem 4.3.

9.3. Theorem (cf. [12, Theorem 3.6]). Let \((R, \rho)\) be a complete right l. t. ring, \( e \) be a dense idempotent of \((R, \rho)\), and \((S, \sigma)\) be the ring \( S = eRe \) endowed with the relative topology of \( \rho \). The pair of functors
\[
\hat{-} \otimes^u_R eRe : \text{CLT}-(R, \rho) \leftrightarrow \text{CLT}-(S, \sigma) : \hat{-} \otimes^u_S eR
\]
is an equivalence of categories.

Similarly, Corollary 4.6 and Proposition 4.5 imply the two following propositions.

9.4. Proposition (cf. [12, Proposition 3.9]). The following facts hold:

(i) \( A \in \text{Mod}-(R, \rho) \Rightarrow A \hat{\otimes}^u_R eRe \in \text{Mod}-(S, \sigma) \);
(ii) \( B \in \text{Mod}-(S, \sigma) \Rightarrow B \hat{\otimes}^u_S eR \in \text{Mod}-(R, \rho) \).

9.5. Proposition (cf. [12, Proposition 3.11]). The functors
\[
\hat{-} \otimes^u_R (eR, \rho e) \quad \text{and} \quad \text{CHom}^u_R((eR, \rho e), -)
\]
\[
(\text{resp.,} \quad \hat{-} \otimes^u_S (eR, \rho e) \quad \text{and} \quad \text{CHom}^u_S((Re, \rho e), -))
\]
are naturally and topologically equivalent, that is, isomorphic in the category of functors from \text{CLT}-(R, \rho) to \text{CLT}-(S, \sigma) (resp., from \text{CLT}-(S, \sigma) to \text{CLT}-(R, \rho)).
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