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Spanning trees with small degrees and few leaves☆

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ABSTRACT

We give an Ore-type condition sufficient for a graph G to have a spanning tree with small degrees and with few leaves.

1. Introduction

From a classical result by Ore [1] it is well-known that if a simple graph G with $n \geq 2$ vertices is such that $d(u) + d(v) \geq n - 1$ for each pair u, v of non-adjacent vertices of G, then G contains a Hamiltonian path.

A *leaf* of a tree T is a vertex of T with degree one. A natural generalisation of Hamiltonian paths are spanning trees with a small number of leaves. In this direction, Ore's result was generalised by Broersma and Tuinstra [2] to the following theorem.

Theorem 1 ([2]). *Let $s \geq 2$ and $n \geq 2$ be integers. If G is a connected simple graph with n vertices such that $d(u) + d(v) \geq n - s + 1$, for each pair u, v of non-adjacent vertices, then G contains a spanning tree with at most s leaves.*

Further related results have been obtained by Egawa et al. [3] and by Tsugaki and Yamashita [4]. See also [5] for a survey on spanning trees with specific properties.

In this note we consider spanning trees with small degrees as well as with a small number of leaves. Our result is the following.

Theorem 2. *Let n, k and $d_1, d_2, \ldots, d_n$ be integers with $1 \leq k \leq n - 1$ and $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n - 1$. If G is a k-connected simple graph with vertex set $V(G) = \{w_1, w_2, \ldots, w_n\}$ such that $d(u) + d(v) \geq n - 1 - \sum_{j=1}^{k} (d_i - 2)$ for any non-adjacent vertices u and v of G, then G has a spanning tree T with at most $2 + \sum_{j=1}^{k} \left(d_j - 2\right)$ leaves and such that $d_T\left(w_j\right) \leq d_j$ for $j = 1, 2, \ldots, n$.*

2. Proof of Theorem 2

Let T be a largest subtree of G with at most $2 + \sum_{j=1}^{k} \left(d_j - 2\right)$ leaves and such that if $w_j \in V(T)$, then $d_T\left(w_j\right) \leq d_j$. Since G is k-connected and $n \geq 2$, it contains a path with at least $k + 1$ vertices. Therefore, we may assume that tree T has at least $k + 1$ vertices.

If T is not a spanning tree, there is a vertex w of G not in T. By Menger's theorem, there are k internally disjoint paths $\pi_1, \pi_2, \ldots, \pi_k$ in G joining w to k different vertices $r_1, r_2, \ldots, r_k$ of T.

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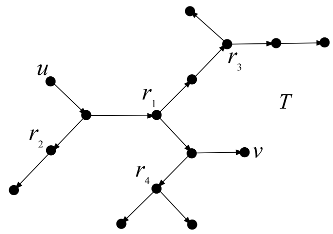

Fig. 1. $n = 15, k = 4, d_1 = d_2 = \cdots = d_{15} = 3$.

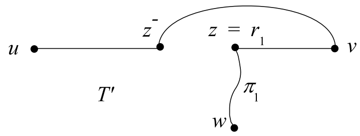

Fig. 2. $T' = \left((T + z^- v) - z^- z\right) \cup \pi_1$.

Let n_1 denote the number of leaves of T. We claim $n_1 = 2 + \sum_{j=1}^{k} \left(d_j - 2\right)$, otherwise there is a vertex r_i such that $d_T(r_i) < d_{j_i}$ where $w_{j_i} = r_i$. Then $T' = T \cup \pi_i$ is a subtree of G with more vertices than T such that $d_{T'}\left(w_j\right) \le d_j$ for each $w_j \in V\left(T'\right)$ and with at most $n_1 + 1 \le 2 + \sum_{j=1}^{k}\left(d_j - 2\right)$ leaves, which contradicts our assumption on the maximality of T.

Because of Ore's theorem, we can assume $d_i \ge 3$ for some $i = 1, 2, \ldots, k$. Since T has $n_1 = 2 + \sum_{j=1}^{k}\left(d_j - 2\right) \ge 3$ leaves, as shown above, there is a vertex w_j of T such that $d_T\left(w_j\right) \ge 3$. Suppose there are vertices x and y of degree one in T such that $xy \in E(G)$. Since T is not a path, there is an edge zz' in the unique xy path contained in T with $d_T(z) \ge 3$. Let $T' = \left(T - zz'\right) + xy$ and notice that T' is a subtree of G with $V\left(T'\right) = V(T)$, with less than $2 + \sum_{j=1}^{k}\left(d_j - 2\right)$ leaves and such that $d_{T'}\left(w_j\right) \le d_j$ for each $w_j \in V\left(T'\right)$. As above, this is a contradiction and therefore no leaves of T are adjacent in G.

Notice that $d_T(r_1) \ge 2$, otherwise $T' = T \cup \pi_1$ would be a tree larger than T, with the same number of leaves and with $d_{T'}\left(w_j\right) \le d_j$ for each vertex w_j of T'. Let u and v be any two leaves of T with the property that the vertex r_1 lies in the unique uv path T_{uv}, contained in T. Orient the edges of T in such a way that the corresponding directed tree $\overrightarrow{T}$ is outdirected with root u (see Fig. 1).

For each vertex $z \ne u$ in T let z^- be the unique vertex of T such that $z^- z$ is an arc of $\overrightarrow{T}$. Let

$$A = \{y \in V(T) : yv \in E(G)\} \quad \text{and} \quad B = \left\{x^- \in V(T) : ux \in E(G)\right\}.$$

Because of the way the tree T was chosen, all vertices of G adjacent to u or to v lie in T and therefore $|A| = d(v)$. Let x_1 and x_2 be vertices of T adjacent to u in G, if $x_1^- = x_2^- = z$ for some vertex z of T, let $T' = (T + ux_1) - zx_1$. Since zx_1 and zx_2 are edges of T, $d_{T'}(z) \ge 2$ and T' is a subtree of G with $V\left(T'\right) = V(T)$, with less than $2 + \sum_{j=1}^{k}\left(d_{i_j} - 2\right)$ leaves and such that $d_{T'}\left(w_j\right) \le d_j$ for each $w_j \in V\left(T'\right)$. Again, this is a contradiction, therefore $|B| = d(u)$.

Since no vertex in $A \cup (B \setminus \{u\})$ is a leaf of T,

$$|A \cup B| \le |V(T)| - n_1 + 1 \le (n - 1) - n_1 + 1 = n - 2 - \sum_{j=1}^{k}\left(d_j - 2\right).$$

Also

$$|A \cup B| = |A| + |B| - |A \cap B| = d(u) + d(v) - |A \cap B| \ge n - 1 - \sum_{j=1}^{k}\left(d_j - 2\right) - |A \cap B|.$$

Therefore $|A \cap B| \ge 1$; let $z^- \in A \cap B$. We consider two cases:

Case 1. Edge $z^- z$ lies on the path T_{uv}.

If $z = r_1$(see Fig. 2), let

$$T' = \left((T + z^- v) - z^- z\right) \cup \pi_1 \quad \text{and}$$

and if $r_1 \ne z$ (see Fig. 3), let

$$T' = \left(\left(\left((T + uz) + z^- v\right) - r_1^- r_1\right) - z^- z\right) \cup \pi_1.$$

Both situations lead to a contradiction since T' is a subtree of G larger than T, with at most $2 + \sum_{j=1}^{k}\left(d_j - 2\right)$ leaves and such that $d_T\left(w_j\right) \le d_j$ for each $w_j \in V\left(T'\right)$.

Case 2. Edge $z^- z$ does not lie on the path T_{uv}.

If z^- lies in T_{uv}, let $T'' = (T + uz) - z^- z$ (see Fig. 4).

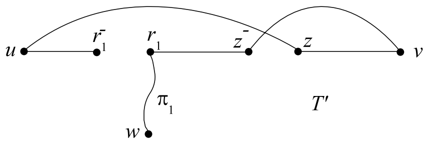

Fig. 3. $T' = \left(\left(\left((T + uz) + z^- v\right) - r_1^- r_1\right) - z^- z\right) \cup \pi_1$.

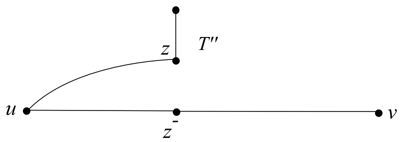

Fig. 4. $T'' = (T + uz) - z^- z$.

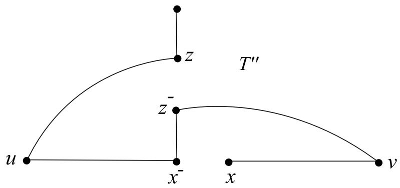

Fig. 5. $T'' = \left(\left(\left((T + uz) + z^- v\right) - x^- x\right) - z^- z\right)$.

And if z^- does not lie in T_{uv}, let x be a vertex in T_{uv} not in T_{uz^-} such that x^- is a vertex in T_{uz^-} (see Fig. 5). Let

$$T'' = \left(\left(\left((T + uz) + z^- v\right) - x^- x\right) - z^- z\right).$$

In this case T'' is a subtree of G with $V\left(T''\right) = V(T)$, with at most $n_1 - 1 = 1 + \sum_{j=1}^{k} \left(d_j - 2\right)$ leaves and such that $d_{T''}\left(w_j\right) \le d_j$ for each $w_j \in V\left(T''\right)$. As seen above, this is not possible.

Cases 1 and 2 cover all possibilities, therefore T is a spanning tree of G. □

Let $k \geqslant 1$ and $d_1, d_2, \ldots, d_n$ be integers with $3 \le d_1 \le d_2 \le \cdots \le d_n$ and $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_{2-k+d_1+\cdots+d_k}\}$ be sets of vertices. The complete bipartite graph G with bipartition (X, Y) is k-connected, has $n = 2 + \sum_{j=1}^{k} d_i$ vertices and is such that $d(u) + d(v) \ge 2k = n - 2 - \sum_{j=1}^{k} (d_i - 2)$ for any vertices u and v of G. Nevertheless, if T is a spanning tree of G, then $d_T\left(x_j\right) > d_j$ for some $j = 1, 2, \ldots, k$. This shows that the condition in Theorem 2 is tight.

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