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# Vector-valued Littlewood–Paley–Stein theory for semigroups

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## Abstract

We develop a generalized Littlewood–Paley theory for semigroups acting on  $L^p$ -spaces of functions with values in uniformly convex or smooth Banach spaces. We characterize, in the vector-valued setting, the validity of the one-sided inequalities concerning the generalized Littlewood–Paley–Stein  $g$ -function associated with a subordinated Poisson symmetric diffusion semigroup by the martingale cotype and type properties of the underlying Banach space. We show that in the case of the usual Poisson semigroup and the Poisson semigroup subordinated to the Ornstein–Uhlenbeck semigroup on  $\mathbb{R}^n$ , this general theory becomes more satisfactory (and easier to be handled) in virtue of the theory of vector-valued Calderón–Zygmund singular integral operators.

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### 1. Introduction and preliminaries

Given a martingale  $\{f_n\}$  with values in a Banach space  $\mathbb{B}$ , its generalized “square” function is defined as

$$S_q f = \left( \sum_{n=1}^{\infty} \|f_n - f_{n-1}\|_{\mathbb{B}}^q \right)^{1/q}.$$

Then  $\mathbb{B}$  is said to have martingale cotype  $q$ ,  $2 \leq q < \infty$  if there exist  $p \in (1, \infty)$  and a constant  $C > 0$  such that  $\|S_q f\|_{L^p} \leq C \sup_n \|f_n\|_{L^p_{\mathbb{B}}}$  for every bounded  $\mathbb{B}$ -valued  $L^p$ -martingale  $\{f_n\}$ . The validity of the reverse inequality defines martingale type  $q$ ,  $1 < q \leq 2$ . Recall that if the inequality above (or its inverse) holds for one  $p \in (1, \infty)$ , so does it for all  $p \in (1, \infty)$ . These notions were introduced and studied in depth by Pisier (see [Pi1, Pi2]). His renorming theorem states that there are geometric properties of the underlying Banach space, characterized by the existence of an equivalent norm in the space which is uniformly convex of power type  $q$  or uniformly smooth of power type  $q$ . We also recall that  $\mathbb{B}$  is of martingale cotype  $q$  iff  $\mathbb{B}^*$  is of martingale type  $q'$ , where  $q'$  is the index conjugate to  $q$ . For  $p \in (1, \infty)$ ,  $L^p$  is of martingale cotype  $\max\{2, p\}$  and of martingale type  $\min\{2, p\}$ .

On the other hand, it is well known that martingale inequalities involving square function are closely related to the corresponding inequalities concerning the Littlewood–Paley or Lusin square function in harmonic analysis. It is in this spirit that a generalized Littlewood–Paley theory is developed in [Xu] for functions with values in uniformly convex Banach spaces. Let us recall the main results of [Xu]. Let  $f$  be a function in  $L^1(\mathbb{T})$ , where  $\mathbb{T}$  denotes the torus equipped with normalized Haar measure  $d\theta$ . The classical Littlewood–Paley  $g$ -function is defined for  $z \in \mathbb{T}$  as

$$Gf(z) = \left( \int_0^1 (1-r)^2 \|\nabla P_r * f(z)\|^2 \frac{dr}{1-r} \right)^{1/2}.$$

In this notation,

$$\|\nabla P_r * f(z)\| = \left( \left| \frac{\partial P_r}{\partial r} * f(z) \right|^2 + \left| \frac{1}{r} \frac{\partial P_r}{\partial \theta} * f(z) \right|^2 \right)^{1/2}, \tag{1.1}$$

with

$$P_r(\theta) = \frac{1-r^2}{1+r^2-2r \cos \theta},$$

being the Poisson kernel for the disk. It is a classical result that for any  $p \in (1, \infty)$  there exist positive constants  $c_p$  and  $C_p$  such that

$$c_p \|f\|_{L^p(\mathbb{T})} \leq |\hat{f}(0)| + \|Gf\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})}. \tag{1.2}$$

One can extend the definition of  $G$  to functions that take values in a Banach space  $\mathbb{B}$ , just by replacing absolute value by norm in (1.1). In this case, (1.2) holds if and only if  $\mathbb{B}$  is isomorphic to a Hilbert space. However, one of the two inequalities in (1.2) can be true in non-Hilbertian spaces. The study of these one-sided inequalities is the main objective of [Xu]. More generally, we can introduce the following generalized “Littlewood–Paley  $g$ -function”:

$$G_q f(z) = \left( \int_0^1 (1-r)^q \|\nabla P_r * f(z)\|_{\mathbb{B}}^q \frac{dr}{1-r} \right)^{1/q}.$$

Then  $\mathbb{B}$  is said to be of Lusin cotype  $q$  (resp. Lusin type  $q$ ) if there exist  $p \in (1, \infty)$  and a positive constant  $C$  such that

$$\|G_q f\|_{L^p(\mathbb{T})} \leq C \|f\|_{L^p_{\mathbb{B}}(\mathbb{T})} \quad \left( \text{resp. } \|f\|_{L^p_{\mathbb{B}}(\mathbb{T})} \leq C \left( \|\hat{f}(0)\|_{\mathbb{B}} + \|G_q f\|_{L^p(\mathbb{T})} \right) \right).$$

It is not difficult to see that if  $\mathbb{B}$  is of Lusin cotype  $q$  (resp. Lusin type  $q$ ), then  $2 \leq q \leq \infty$  (resp.  $1 \leq q \leq 2$ ). It is proved in [Xu] that the definition above is independent of  $p$ , that is, if one of the inequalities above holds for one  $p \in (1, \infty)$ , then so does it for every  $p \in (1, \infty)$  (with a different constant depending on  $p$ ). The main result of [Xu] states that a Banach space  $\mathbb{B}$  is of Lusin type  $q$  (resp. Lusin cotype  $q$ ) iff  $\mathbb{B}$  is of martingale type  $q$  (resp. martingale cotype  $q$ ).

The main goal of the present paper is to extend the results in [Xu] to general symmetric diffusion semigroups, and thus to develop a generalized Littlewood–Paley theory for these semigroups on  $L^p$ -spaces of functions with values in uniformly convex or smooth Banach spaces. Recall that a symmetric diffusion semigroup is a collection of linear operators  $\{\mathcal{T}_t\}_{t \geq 0}$  defined on  $L^p(\Omega, d\mu)$  over a measure space  $(\Omega, d\mu)$  satisfying the following properties:

$$\mathcal{T}_0 = \text{Id}, \quad \mathcal{T}_t \mathcal{T}_s = \mathcal{T}_{t+s}, \quad \|\mathcal{T}_t\|_{L^p \rightarrow L^p} \leq 1 \quad \forall p \in [1, \infty], \tag{1.3}$$

$$\lim_{t \rightarrow 0} \mathcal{T}_t f = f \text{ in } L^2 \quad \forall f \in L^2, \tag{1.4}$$

$$\mathcal{T}_t^* = \mathcal{T}_t \text{ on } L^2, \quad \mathcal{T}_t f \geq 0 \text{ if } f \geq 0, \quad \mathcal{T}_t 1 = 1. \tag{1.5}$$

The subordinated Poisson semigroup  $\{\mathcal{P}_t\}_{t \geq 0}$  is defined as

$$\mathcal{P}_t f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{T}_{t^2/4u} f \, du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/4u}}{u^{3/2}} \mathcal{T}_u f \, du. \tag{1.6}$$

$\{\mathcal{P}_t\}_{t \geq 0}$  is again a symmetric diffusion semigroup, see [St1]. Recall that if  $A$  denotes the infinitesimal generator of  $\{\mathcal{T}_t\}_{t \geq 0}$ , then that of  $\{\mathcal{P}_t\}_{t \geq 0}$  is  $-(-A)^{1/2}$ .

It is well know (and easy to check) that any bounded operator  $T$  on  $L^p(\Omega)$  for all  $p \in [1, \infty]$  naturally and boundedly extends to  $L^p_{\mathbb{B}}(\Omega)$  for every Banach space  $\mathbb{B}$ , where  $L^p_{\mathbb{B}}(\Omega)$  denotes the usual Bochner–Lebesgue  $L^p$ -space of  $\mathbb{B}$ -valued functions defined on  $\Omega$ . More precisely, the extension is  $T \otimes \text{Id}_{\mathbb{B}}$ . Indeed, this is clear for  $p = 1$  (via projective tensor product); the case  $p = \infty$  is done by duality, and then  $1 < p < \infty$  by interpolation. With a slight abuse of notation (which will not cause any ambiguity), we shall denote these extensions still by the same symbol  $T$ .

Thus  $\mathcal{T}_t$  and  $\mathcal{P}_t$  have straightforward extensions to  $L^p_{\mathbb{B}}(\Omega)$  for every Banach space  $\mathbb{B}$ ; moreover, these extensions are also contractive. (Note that we can also justify these extensions by the positivity of  $\mathcal{T}_t$  and  $\mathcal{P}_t$ .) According to the convention above, we shall consider  $\{\mathcal{T}_t\}_{t \geq 0}$  and  $\{\mathcal{P}_t\}_{t \geq 0}$  as semigroups on  $L^p_{\mathbb{B}}(\Omega)$  too.

In these circumstances we can define the generalized “Littlewood–Paley  $g$ -function” associated to the semigroup as

$$\mathfrak{G}_q(f)(x) = \left( \int_0^\infty \left\| t \frac{\partial \mathcal{P}_t f(x)}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q}.$$

The first result of this paper, see Theorem 2.1, states that a Banach space  $\mathbb{B}$  is of martingale cotype  $q$  iff for every symmetric diffusion semigroup  $\{\mathcal{T}_t\}_{t \geq 0}$  with subordinated semigroup  $\{\mathcal{P}_t\}_{t \geq 0}$ , the generalized  $g$ -function operator  $\mathfrak{G}_q$  is bounded from  $L^p_{\mathbb{B}}(\Omega)$  to  $L^p(\Omega)$ , namely

$$\|\mathfrak{G}_q(f)\|_{L^p(\Omega)} \leq C \|f\|_{L^p_{\mathbb{B}}(\Omega)} \quad \forall f \in L^p_{\mathbb{B}}(\Omega). \tag{1.7}$$

The validity of the reverse inequality (with a necessary additional term) characterizes the martingale type  $q$  (see Theorem 2.2). These results are proved in Section 2. The main ingredient of our arguments is the classical Rota theorem on the dilation of a positive contraction on  $L^p$  by conditional expectations. This theorem allows to reduce (1.7) (after a discretization) to a corresponding inequality for martingales.

This approach via Rota’s theorem is also efficacious in studying (1.7) and its dual form for an individual semigroup. We shall show in Section 3 that for a given subordinated Poisson semigroup  $\{\mathcal{P}_t\}$ , (1.7) is equivalent to its dual form, which is an inequality reverse to (1.7) with  $\mathbb{B}, p$  and  $q$  replaced by  $\mathbb{B}^*, p'$  and  $q'$ , respectively (and with an additional term). The key to this is the existence of a certain projection, whose proof, using Rota’s theorem once more, is unfortunately rather technical and complicated.

Our proof for the implication “(1.7)  $\Rightarrow$  martingale cotype  $q$ ” uses the Poisson semigroup on the torus modulo the results in [Xu] quoted previously. (Note however that this Poisson semigroup on the torus is a multiplicative semigroup on  $(0, 1)$ .) Thus it would be interesting to know the family of semigroups  $\{\mathcal{P}_t\}_{t \geq 0}$  for which the validity of (1.7) implies martingale cotype  $q$ . One of the aims of the remainder of the paper (after Section 3) is to show that this is indeed the case for the Poisson semigroup on

$\mathbb{R}^n$ . Such a result is, of course, conceivable after [Xu]. For any  $q \geq 1$ , the  $n$ -dimensional generalized “Littlewood–Paley  $g$ -function” is defined as

$$\mathcal{G}_q(f)(x) = \left( \int_0^\infty t^q \|\nabla P_t * f(x)\|_{\ell_{\mathbb{B}}^2}^q \frac{dt}{t} \right)^{1/q},$$

where

$$\|\nabla P_t * f(x)\|_{\ell_{\mathbb{B}}^2}^2 = \left( \left\| \frac{\partial P_t}{\partial t} * f(x) \right\|_{\mathbb{B}}^2 + \sum_{k=1}^n \left\| \frac{\partial P_t}{\partial x_k} * f(x) \right\|_{\mathbb{B}}^2 \right)^{1/2},$$

with

$$P_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}},$$

the kernel of the Poisson semigroup for the upper half-space. Note that we use the same symbol  $P_t$  to denote the Poisson kernels both on  $\mathbb{T}$  and on  $\mathbb{R}^n$ . This should not have any confusion in the concrete context. Then  $\mathbb{B}$  is of martingale cotype  $q$  (resp. martingale type  $q$ ) iff for some (equivalently every)  $p \in (1, \infty)$  there is a constant  $C$  such that

$$\|\mathcal{G}_q(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \quad \left( \text{resp. } \|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq C \|\mathcal{G}_q(f)\|_{L^p(\mathbb{R}^n)} \right).$$

This result, among some others on  $\mathcal{G}_q$ -function on  $\mathbb{R}^n$ , is proved in Sections 4 and 5 and achieved by viewing the operators  $\mathcal{G}$  as vector-valued Calderón–Zygmund operators. For these operators, under suitable conditions, one can get the equivalence of the strong type  $(p, p)$  and the boundedness  $\text{BMO} - \text{BMO}$  (see Theorem 4.1). As a consequence, we obtain the characterization of the Lusin cotype in terms of  $\text{BMO}$  boundedness of the  $g$ -functions (Corollary 4.2 and Theorems 5.2 and 5.3). These two sections extend most of the results in [Xu] for  $\mathbb{T}$  to  $\mathbb{R}^n$ .

The previous results for the usual Poisson semigroup on  $\mathbb{R}^n$  can be extended to the Poisson semigroup subordinated to the Ornstein–Uhlenbeck semigroup on  $\mathbb{R}^n$ . This is done in Section 6 (see Theorems 6.1 and 6.2).

The last section contains a further characterization of Lusin cotype property in terms of almost sure finiteness of the generalized Littlewood–Paley  $g$ -functions (Theorems 7.1 and 7.4).

## 2. One-sided vector-valued Littlewood–Paley–Stein inequalities for semigroups

We shall consider general symmetric diffusion semigroups, that is, the collections of linear operators  $\{\mathcal{T}_t\}_{t \geq 0}$  defined on  $L^p(\Omega)$ , satisfying (1.3)–(1.5). Given such a semigroup  $\{\mathcal{T}_t\}_{t \geq 0}$  we consider its subordinated semigroup  $\{\mathcal{P}_t\}_{t \geq 0}$ , defined as in

(1.6). Let  $\mathbb{F} \subset L^2(\Omega)$  be the subspace of the fix points of  $\{\mathcal{P}_t\}_{t \geq 0}$ , i.e., the subspace of all  $f$  such that  $\mathcal{P}_t(f) = f$  for all  $t > 0$ . Let  $F : L^2(\Omega) \rightarrow \mathbb{F}$  be the orthogonal projection. It is clear that  $F$  extends to a contractive projection (still denoted by  $F$ ) on  $L^p(\Omega)$  for every  $1 \leq p \leq \infty$  and that  $F(L^p(\Omega))$  is exactly the fix point space of  $\{\mathcal{P}_t\}_{t \geq 0}$  on  $L^p(\Omega)$ . Moreover, for any Banach space  $\mathbb{B}$ ,  $F$  extends to a contractive projection on  $L^p_{\mathbb{B}}(\Omega)$  for every  $1 \leq p \leq \infty$  and that  $F(L^p_{\mathbb{B}}(\Omega))$  is again the fix point space of  $\{\mathcal{P}_t\}_{t \geq 0}$  considered as a semigroup on  $L^p_{\mathbb{B}}(\Omega)$ . According to our convention, in the sequel, we shall use the same symbol  $F$  to denote any of these contractive projections.

Recall that the generalized Littlewood–Paley  $g$ -function associated with  $\{\mathcal{P}_t\}_{t \geq 0}$  is defined by

$$\mathfrak{G}_q(f)(x) = \left( \int_0^\infty \left\| t \frac{\partial \mathcal{P}_t f(x)}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q}.$$

The main results of this section are the following two theorems.

**Theorem 2.1.** *Given a Banach space  $\mathbb{B}$  and  $2 \leq q < \infty$ , the following statements are equivalent:*

- (i)  $\mathbb{B}$  is of martingale cotype  $q$ .
- (ii) For every symmetric diffusion semigroup  $\{\mathcal{T}_t\}_{t \geq 0}$  with subordinated semigroup  $\{\mathcal{P}_t\}_{t \geq 0}$  and for every (or, equivalently, for some)  $p \in (1, \infty)$  there is a constant  $C$  such that

$$\|\mathfrak{G}_q f\|_{L^p(\Omega)} \leq C \|f\|_{L^p_{\mathbb{B}}(\Omega)} \quad \forall f \in L^p_{\mathbb{B}}(\Omega). \tag{2.1}$$

**Theorem 2.2.** *Given a Banach space  $\mathbb{B}$  and  $1 < q \leq 2$ , the following statements are equivalent:*

- (i)  $\mathbb{B}$  is of martingale type  $q$ .
- (ii) For every symmetric diffusion semigroup  $\{\mathcal{T}_t\}_{t \geq 0}$  with subordinated semigroup  $\{\mathcal{P}_t\}_{t \geq 0}$  and for every (or, equivalently, for some)  $p \in (1, \infty)$  there is a constant  $C$  such that

$$\|f\|_{L^p_{\mathbb{B}}(\Omega)} \leq C \left( \|F(f)\|_{L^p_{\mathbb{B}}(\Omega)} + \|\mathfrak{G}_q f\|_{L^p(\Omega)} \right) \quad \forall f \in L^p_{\mathbb{B}}(\Omega). \tag{2.2}$$

The rest of this section is essentially devoted to the proof of these theorems. The difficult part is the implication “(i)  $\Rightarrow$  (ii)” in Theorem 2.1. Then the same implication in Theorem 2.2 will follow by duality. Both converse implications will be done by using the Poisson semigroup on the torus with the help of [Xu]. For the main part of the proof we shall need the following result, which has independent interest.

**Theorem 2.3.** *Let  $\mathbb{B}$  be a Banach space of martingale cotype  $q \in [2, \infty)$  and  $\{\mathcal{T}_t\}_{t \geq 0}$  a symmetric diffusion semigroup. Then for any  $p \in (1, \infty)$ ,*

$$\left\| \left( \int_0^\infty \left\| t \frac{\partial M_t f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\Omega)} \leq C_{p,q,\mathbb{B}} \|f\|_{L^p_{\mathbb{B}}(\Omega)} \quad \forall f \in L^p_{\mathbb{B}}(\Omega),$$

where  $M_t = \frac{1}{t} \int_0^t \mathcal{T}_s ds$ .

The pattern of our proof for the theorem above is borrowed from [St1, Chapter IV]. As in [St1], the key ingredient is Rota’s dilation theorem (see Theorem 2.5 below), which allows to reduce the inequality in Theorem 2.3 to a similar inequality for martingales.

Given a  $\sigma$ -finite measure space  $(M, \mathcal{F}, dm)$  and a sub- $\sigma$ -algebra  $\mathcal{G} \subset \mathcal{F}$ , we denote as usual by  $E(\cdot | \mathcal{G})$  the conditional expectation with respect to  $\mathcal{G}$ . (Note that our measure space  $(M, \mathcal{F}, dm)$  is no longer a probability one; however all usual properties on conditional expectations in the probabilistic case are still valid in the present setting.) Recall that  $E(\cdot | \mathcal{G})$  is a positive contraction on  $L^p(M, \mathcal{F}, dm)$  for every  $p \in [1, \infty]$  and naturally extends to  $L^p_{\mathbb{B}}(M, \mathcal{F}, dm)$  for every Banach space  $\mathbb{B}$ . The classical Doob maximal inequality is also valid in the vector-valued setting. Let  $(\mathcal{F}_n)$  be an increasing filtration of sub- $\sigma$ -algebras of  $\mathcal{F}$ . For  $f \in L^1_{\mathbb{B}}(M, \mathcal{F}, dm)$  we define its maximal function as

$$f^* = \sup_{n \geq 1} \|E(f | \mathcal{F}_n)\|_{\mathbb{B}}.$$

Then we have the following Doob maximal weak type  $(1, 1)$  inequality

$$\lambda m\{f^* > \lambda\} \leq \int_{\{f^* > \lambda\}} \|f(x)\|_{\mathbb{B}} dm(x)$$

for every Banach space  $\mathbb{B}$ . Similarly, we can also extend the results of [MT]; in particular, we get that for every  $1 < p, q < \infty$  and every sequence  $(f_k) \subset L^p_{\mathbb{B}}(M, \mathcal{F}, dm)$

$$\left\| \left( \sum_{k=1}^\infty ((f_k)^*)^q \right)^{1/q} \right\|_{L^p} \leq C_{p,q} \left\| \left( \sum_{k=1}^\infty \|f_k\|_{\mathbb{B}}^q \right)^{1/q} \right\|_{L^p}. \tag{2.3}$$

We shall use the following lemma, motivated by [St1], inequality (\*\*) on p. 115.

**Lemma 2.4.** Let  $\mathbb{B}$  be a Banach space of martingale cotype  $q \in [2, \infty)$ ,  $(M, dm)$  be any  $\sigma$ -finite measure space and  $\{E_n\}$  be an arbitrary monotone sequence of conditional expectations on  $(M, dm)$ . Then, for every  $p$ ,  $1 < p < \infty$ ,

$$\left\| \left( \sum_{n=1}^{\infty} n^{q-1} \|(\sigma_n - \sigma_{n-1})f\|_{\mathbb{B}}^q \right)^{1/q} \right\|_{L^p} \leq C_{p,q,\mathbb{B}} \|f\|_{L^p_{\mathbb{B}}},$$

where

$$\sigma_n = \frac{E_0 + \dots + E_n}{n + 1}.$$

**Proof.** Observe that it is enough to prove the inequality taking the summation in  $n \geq n_0$  for any fixed  $n_0$ . If we define  $d_n = E_n - E_{n-1}$  for  $n \geq 0$  (with the convention that  $E_{-1} = 0$ ), for  $j \geq 1$  we have

$$\Delta_j = E_{2^j} - E_{2^{j-1}} = \sum_{k=0}^{2^j} d_k - \sum_{k=0}^{2^{j-1}} d_k = \sum_{k=2^{j-1}+1}^{2^j} d_k.$$

Consider, for each  $n \geq 5$ ,  $J_n$  the unique integer such that  $2^{J_n} < n \leq 2^{J_n+1}$ . Then

$$\begin{aligned} \sigma_n - \sigma_{n-1} &= \frac{1}{n(n+1)} \sum_{j=0}^n j d_j \\ &= \frac{1}{n(n+1)} \left( d_1 + 2d_2 + \sum_{k=2}^{J_n} \sum_{j=2^{k-1}+1}^{2^k} j d_j + \sum_{k=2^{J_n}+1}^n j d_j \right). \end{aligned}$$

Now, for each  $k$ ,  $2 \leq k \leq J_n$ ,

$$\begin{aligned} \sum_{j=2^{k-1}+1}^{2^k} j d_j &= 2^k \Delta_k - \sum_{j=2^{k-1}+1}^{2^k} (2^k - j) d_j \\ &= 2^k \Delta_k - \sum_{j=2^{k-1}+1}^{2^k-1} \sum_{i=j}^{2^k-1} d_j = 2^k \Delta_k - \sum_{i=2^{k-1}+1}^{2^k-1} E_i(\Delta_k). \end{aligned}$$



We can treat the rest of the terms in a similar way, and then we get

$$\begin{aligned} \sigma_n - \sigma_{n-1} = \frac{1}{n(n+1)} & \left[ d_1 + 2d_2 + \sum_{k=2}^{J_n} \left( 2^k \Delta_k - \sum_{i=2^{k-1}+1}^{2^k-1} E_i(\Delta_k) \right) \right. \\ & \left. + nE_n(\Delta_{J_n+1}) - \sum_{k=2^{J_n}+1}^{n-1} E_k(\Delta_{J_n+1}) \right]. \end{aligned}$$

Thus,

$$\begin{aligned} \left( \sum_{n=5}^{\infty} n^{q-1} \|(\sigma_n - \sigma_{n-1})f\|_{\mathbb{B}}^q \right)^{1/q} & \leq \|d_1 f + 2d_2 f\|_{\mathbb{B}} \left( \sum_{n=5}^{\infty} \frac{n^{q-1}}{n^q(n+1)^q} \right)^{1/q} \\ & + \left( \sum_{n=5}^{\infty} \frac{n^{q-1}}{n^q(n+1)^q} \left\| \sum_{k=2}^{J_n} 2^k \Delta_k f \right\|_{\mathbb{B}}^q \right)^{1/q} \\ & + \left( \sum_{n=5}^{\infty} \frac{n^{q-1}}{n^q(n+1)^q} \left\| \sum_{k=2}^{J_n} \sum_{i=2^{k-1}+1}^{2^k-1} E_i(\Delta_k f) \right\|_{\mathbb{B}}^q \right)^{1/q} \\ & + \left( \sum_{n=5}^{\infty} \frac{n^{q-1}}{n^q(n+1)^q} \|nE_n(\Delta_{J_n+1} f)\|_{\mathbb{B}}^q \right)^{1/q} \\ & + \left( \sum_{n=5}^{\infty} \frac{n^{q-1}}{n^q(n+1)^q} \left\| \sum_{k=2^{J_n}+1}^{n-1} E_k(\Delta_{J_n+1} f) \right\|_{\mathbb{B}}^q \right)^{1/q} \\ & = C \|d_1 f + 2d_2 f\|_{\mathbb{B}} + I + II + III + IV. \end{aligned}$$

Using  $|\sum_{i=1}^n 2^i a_i|^q \leq 2^{n(q-1)} \sum_{i=1}^n 2^i |a_i|^q$ , we have that

$$\begin{aligned} I^q & \leq \sum_{n=5}^{\infty} \frac{1}{n^{q+1}} 2^{J_n(q-1)} \sum_{k=2}^{J_n} 2^k \|\Delta_k f\|_{\mathbb{B}}^q \leq \sum_{n=5}^{\infty} \frac{1}{n^2} \sum_{k=2}^{J_n} 2^k \|\Delta_k f\|_{\mathbb{B}}^q \\ & \leq \sum_{k=2}^{\infty} \|\Delta_k f\|_{\mathbb{B}}^q. \end{aligned}$$

Since  $\mathbb{B}$  is of martingale cotype  $q$ ,  $\|I\|_{L^p} \leq C_{p,q,\mathbb{B}} \|f\|_{L^p_{\mathbb{B}}}$ .

In order to handle the second term, let us call  $k(j) = k$  when  $2^{k-1} \leq j < 2^k$ . Then,

$$\begin{aligned} II^q &\leq \sum_{n=5}^{\infty} \frac{1}{n^{q+1}} \left\| \sum_{j=3}^{2^{J_n}-1} E_j(\Delta_{k(j)} f) \right\|_{\mathbb{B}}^q \leq \sum_{n=5}^{\infty} \frac{1}{n^{q+1}} 2^{J_n(q-1)} \sum_{j=3}^{2^{J_n}-1} \|E_j(\Delta_{k(j)} f)\|_{\mathbb{B}}^q \\ &\leq \sum_{n=5}^{\infty} \frac{1}{n^2} \sum_{j=3}^{n-1} \|E_j(\Delta_{k(j)} f)\|_{\mathbb{B}}^q \leq \sum_{j=3}^{\infty} \frac{1}{j+1} \|E_j(\Delta_{k(j)} f)\|_{\mathbb{B}}^q. \end{aligned}$$

Using (2.3) and the martingale cotype  $q$  of  $\mathbb{B}$ , we obtain that

$$\begin{aligned} \|II\|_{L^p} &\leq \left\| \left( \sum_{j=3}^{\infty} \frac{1}{j+1} \|E_j(\Delta_{k(j)} f)\|_{\mathbb{B}}^q \right)^{1/q} \right\|_{L^p} \\ &\leq C_{p,q} \left\| \left( \sum_{k=2}^{\infty} \|\Delta_k f\|_{\mathbb{B}}^q \sum_{j=2^{k-1}+1}^{2^k} \frac{1}{j+1} \right)^{1/q} \right\|_{L^p} \leq C_{p,q,\mathbb{B}} \|f\|_{L^p_{\mathbb{B}}}. \end{aligned}$$

Analogously, one can show that  $\|III\|_{L^p} + \|IV\|_{L^p} \leq C_{p,q} \|f\|_{L^p_{\mathbb{B}}}$ . We leave the details to the reader. Thus the lemma is proved.  $\square$

We shall also need the following result due to Rota, see [St1, Chapter IV]. Let  $Q$  be a linear operator on  $L^p(\Omega, \mathcal{A}, d\mu)$  satisfying the conditions

- (i)  $\|Q\|_{L^p \rightarrow L^p} \leq 1$  for every  $p \in [1, \infty]$ ,
- (ii)  $Q = Q^*$  in  $L^2$ ,
- (iii)  $Qf \geq 0$  for every  $f \geq 0$ ,
- (iv)  $Q1 = 1$ .

**Theorem 2.5.** *For any  $Q$  as above, there exist a measure space  $(M, \mathcal{F}, dm)$ , a decreasing collection of  $\sigma$ -algebras  $\dots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \dots \subset \mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F}$ , and another  $\sigma$ -algebra  $\hat{\mathcal{F}} \subset \mathcal{F}$  such that*

- (a) *there exists an isomorphism  $i : (\Omega, \mathcal{A}, d\mu) \rightarrow (M, \hat{\mathcal{F}}, dm)$  (which induces an isomorphism between  $L^p$  spaces, also denoted by  $i$ ,  $i(f)(m) = f(i^{-1}m)$ ),*
- (b) *for every  $f \in L^p(M, \hat{\mathcal{F}}, dm)$ , we have*

$$Q^{2^n}(i^{-1} f)(x) = \hat{E}(E_n(f))(i(x)), \quad x \in \Omega,$$

where  $\hat{E}(f) = E(f|\hat{\mathcal{F}})$  and  $E_n(f) = E(f|\mathcal{F}_n)$ .

This theorem holds in the scalar-valued case. For the vector-valued case, the validity of the second statement is a consequence of the fact that all operators in consideration extend to contractions on  $\mathbb{B}$ -valued  $L^p$ -spaces. Indeed, the linearity implies that the

formula in the statement (b) above holds for all  $\mathbb{B}$ -valued simple functions, and so for all  $\mathbb{B}$ -valued  $p$ -integrable functions.

**Proof of Theorem 2.3.** Observe that it is enough to prove

$$\left\| \left( \int_{a^b} \left\| t \frac{\partial M_t f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\Omega)} \leq C_{p,q,\mathbb{B}} \|f\|_{L^p_{\mathbb{B}}(\Omega)}$$

for any  $0 < a < b < \infty$ , and also that it is enough if we restrict ourselves to functions  $f$  in the algebraic tensor product  $\mathbb{B} \otimes L^p(\Omega)$ . Take then  $f = \sum_{k=1}^K v_k \phi_k$ . By the results in [St1], see the lemma in p. 72 and its proof, it is not difficult to observe that for every  $t_0 \in (0, \infty)$ , there exists  $\varepsilon_0 > 0$  such that

$$\mathcal{T}_t f(x) = \sum_{j=0}^{\infty} f_j(x)(t - t_0)^j \tag{2.4}$$

for  $t \in (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$  and almost every  $x$ , where  $\sum \|f_j\|_{L^p_{\mathbb{B}}} \varepsilon_0^j < \infty$  and where  $f_j$  depend on  $t_0$ . Since we can cover  $(a, b)$  with a finite collection of such intervals, we can split  $(a, b)$  into a finite collection of subintervals  $(a_i, b_i)$  of  $(a, b)$  in which a expression like (2.4) holds for a fixed  $t_0$  (and therefore, with the same  $f_j$ ) for every  $t \in (a_i, b_i)$ . Then, splitting the integral between  $a$  and  $b$  into the integrals corresponding to such subintervals, we can handle all the functions appearing in the integral as power series with vector-valued coefficients. In these circumstances, we can replace the integral by Riemann sums, and all derivatives by difference quotients. The first step is choosing  $\varepsilon$  small. Then, we approximate the integral as follows:

$$\left\| \left( \int_{a^b} \left\| t \frac{\partial M_t f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p} \sim \left\| \left( \sum_{n=0}^{n_1} (n\varepsilon)^{q-1} \left\| \frac{\partial M_t f}{\partial t} \right|_{t=n\varepsilon} \right)^{1/q} \right\|_{L^p},$$

where the sign  $\sim$  means that the difference term goes to zero as  $\varepsilon \rightarrow 0$ . The next step is substituting the partial derivative inside the sum by the difference quotient

$$\frac{M_{(n+1)\varepsilon} f - M_{n\varepsilon} f}{\varepsilon} = \frac{1}{\varepsilon} \frac{1}{(n+1)\varepsilon} \int_0^{(n+1)\varepsilon} \mathcal{T}_s f \, ds - \frac{1}{\varepsilon} \frac{1}{n\varepsilon} \int_0^{n\varepsilon} \mathcal{T}_s f \, ds,$$

and then each of the integrals by its Riemann sums, getting then that

$$\left\| \left( \int_{a^b} \left\| t \frac{\partial M_t f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p}$$

$$\begin{aligned} &\sim \left\| \left( \sum_{n=n_0}^{n_1} n^{q-1} \left\| \frac{1}{n+1} \sum_{j=0}^n \mathcal{T}_{j\varepsilon} f - \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{T}_{j\varepsilon} f \right\|_{\mathbb{B}}^q \right)^{1/q} \right\|_{L^p} \\ &= \left\| \left( \sum_{n=n_0}^{n_1} n^{q-1} \|\tilde{\sigma}_n f - \tilde{\sigma}_{n-1} f\|_{\mathbb{B}}^q \right)^{1/q} \right\|_{L^p}, \end{aligned}$$

where  $\tilde{\sigma}_n f = \frac{1}{n+1} \sum_{j=0}^n \mathcal{T}_{j\varepsilon} f$ . Now, observe that by our hypothesis,  $\mathcal{T}_{\varepsilon/2}$  satisfies assumptions (i)–(iv) of Rota’s theorem and  $\mathcal{T}_{n\varepsilon} f = \hat{E}(E_n(f))$ . Hence,  $\tilde{\sigma}_n f = \hat{E}(\sigma_n f)$  where  $\sigma_n$  is as in Lemma 2.4. Therefore, by the properties of conditional expectation and Lemma 2.4, we get

$$\begin{aligned} \left\| \left( \int_{a^b} \left\| t \frac{\partial M_t f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p} &\sim \left\| \left( \sum_{n=n_0}^{n_1} n^{q-1} \|\hat{E}(\sigma_n f) - \hat{E}(\sigma_{n-1} f)\|_{\mathbb{B}}^q \right)^{1/q} \right\|_{L^p} \\ &\leq \left\| \left\{ n^{1-1/q} (\sigma_n f - \sigma_{n-1} f) \right\} \right\|_{L^p_{\mathbb{B}}} \\ &= \left\| \left( \sum_{n=1}^{\infty} n^{q-1} \|\sigma_n f - \sigma_{n-1} f\|_{\mathbb{B}}^q \right)^{1/q} \right\|_{L^p} \\ &\leq C_{p,q,\mathbb{B}} \|f\|_{L^p_{\mathbb{B}}}. \end{aligned}$$

Therefore, we have achieved the proof of Theorem 2.3.  $\square$

The following lemma says that the boundedness of  $G_q f = \|Tf\|_{L^q_{\mathbb{B}}((0,1), dr/(1-r))}$  from  $L^p_{\mathbb{B}}(\mathbb{T})$  in  $L^p(\mathbb{T})$  is equivalent to the boundedness of the operator  $T$  when the kernel is restricted to values of  $r$  close to one and  $\theta$  close to zero.

**Lemma 2.6.** *Let  $\mathbb{B}$  be a Banach space and  $p, q \in (1, \infty)$ . Let  $\delta > 0$  (close to 0). Then there is a constant  $C_\delta$  (depending only on  $\delta$ ) such that for any  $f \in L^p_{\mathbb{B}}(\mathbb{T})$*

$$\left\| \left[ (1-r) \frac{\partial P_r}{\partial r} \chi_{(0,1-\delta)}(r) \right] * f \right\|_{L^p_{L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})}(\mathbb{T})} \leq C_\delta \|f\|_{L^p_{\mathbb{B}}(\mathbb{T})},$$

and

$$\left\| \left[ (1-r) \frac{\partial P_r}{\partial r} \chi_{(1-\delta,1)}(r) \chi_{(-\delta,\delta)}(\theta) \right] * f \right\|_{L^p_{L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})}(\mathbb{T})} \leq C_\delta \|f\|_{L^p_{\mathbb{B}}(\mathbb{T})}.$$

**Proof.** The proof is very easy. We show only the first inequality. Its left-hand side is a convolution of the  $\mathbb{B}$ -valued function  $f$  with an  $L^q((0, 1), \frac{dr}{1-r})$ -valued function.

Therefore it is enough to prove that the latter is in  $L^1_{L^q((0,1), \frac{dr}{1-r})}$ , namely, we have to prove

$$\int_0^{2\pi} \left\| (1-r) \frac{\partial P_r(\gamma)}{\partial r} \chi_{(0,1-\delta)}(r) \right\|_{L^q((0,1), \frac{dr}{1-r})} d\gamma < \infty.$$

But this follows immediately if we observe that

$$\begin{aligned} & \left\| (1-r) \frac{\partial P_r(\gamma)}{\partial r} \chi_{(0,1-\delta)}(r) \right\|_{L^q((0,1), \frac{dr}{1-r})}^q \\ &= \int_0^{1-\delta} \left| (1-r) \frac{2(1-r)^2 - 2(r^2 + 1) \sin^2(\gamma/2)}{((1-r)^2 + 2r \sin^2(\gamma/2))^2} \right|^q \frac{dr}{1-r} \leq C_\delta^q. \end{aligned}$$

Hence the lemma is proved.  $\square$

The following easy lemma is proved in a similar way as in [St1, p. 49].

**Lemma 2.7.** *Let  $\mathbb{B}$  be a Banach space and  $p, q \in (1, \infty)$ . Then for any  $f \in L^p(\Omega) \otimes \mathbb{B}$  we have*

$$\left( \int_0^\infty \left\| t \frac{\partial \mathcal{P}_t f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q} \leq C_0 \left( \int_0^\infty \left\| t \frac{\partial M_t f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q},$$

where  $C_0$  is an absolute constant.

**Proof.** Call  $\varphi(s) = \frac{1}{2\sqrt{\pi}} \frac{e^{-1/4s}}{s^{3/2}}$ . Using integration by parts we have

$$\mathcal{P}_t = \frac{1}{t^2} \int_0^\infty \varphi\left(\frac{s}{t^2}\right) \left( \frac{\partial}{\partial s} s M_s \right) ds = - \int_0^\infty \frac{s}{t^4} \varphi'\left(\frac{s}{t^2}\right) M_s ds = - \int_0^\infty s \varphi'(s) M_{t^2 s} ds.$$

Therefore (with  $M'_s = \frac{\partial M_s}{\partial s}$ ),

$$t \frac{\partial}{\partial t} \mathcal{P}_t = -2 \int_0^\infty t^2 s^2 \varphi'(s) M'_{t^2 s} ds = -2 \int_0^\infty s \varphi'(s) \left[ t^2 s M'_{t^2 s} \right] ds. \tag{2.5}$$

Thus

$$\begin{aligned} \left[ \int_0^\infty \left\| t \frac{\partial \mathcal{P}_t f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right]^{1/q} &\leq 2 \int_0^\infty s |\varphi'(s)| \left[ \int_0^\infty \left\| t^2 s M'_{t^2 s} f \right\|_{\mathbb{B}}^q \frac{dt}{t} \right]^{1/q} ds \\ &= 2^{1-1/q} K \left[ \int_0^\infty \left\| t M'_t f \right\|_{\mathbb{B}}^q \frac{dt}{t} \right]^{1/q}, \end{aligned}$$

where

$$K = \int_0^\infty s |\varphi'(s)| ds.$$

Hence the lemma is proved.  $\square$

Now we are well prepared for the proofs of Theorems 2.1 and 2.2.

**Proof of Theorem 2.1.** (i)  $\Rightarrow$  (ii). This is an immediate consequence of Theorem 2.3 and Lemma 2.7.

(ii)  $\Rightarrow$  (i). We shall prove that the operator  $f \mapsto G^1_q(f)$  is bounded from  $L^p_{\mathbb{B}}(\mathbb{T})$  to  $L^p(\mathbb{T})$  for  $p \in (1, \infty)$ . Recall that

$$G^1_q(f)(z) = \left( \int_0^1 (1-r)^q \left\| \frac{\partial P_r}{\partial r} * f(z) \right\|_{\mathbb{B}}^q \frac{dr}{1-r} \right)^{1/q}, \quad z \in \mathbb{T}.$$

By [Xu], this is equivalent to the martingale cotype  $q$  of  $\mathbb{B}$ . Observe that if in the Poisson kernel  $P_r, 0 < r < 1$ , we change the parameter according to  $r = e^{-t}$ , we obtain the kernel  $\tilde{P}_t$  of the Poisson semigroup subordinated to the heat semigroup in the torus. Fix a  $\delta \in (0, 1)$  (very close to 1). By the same change of parameter and the fact that for any  $t \in (0, -\log \delta)$ ,  $\frac{1-e^{-t}}{e^{-t}} \sim t$ , then we have

$$\begin{aligned} \int_\delta^1 \left\| (1-r) \frac{\partial P_r}{\partial r} * f(\theta) \right\|^q \frac{dr}{1-r} &= \int_0^{-\log \delta} \left\| \frac{1-e^{-t}}{e^{-t}} \frac{\partial \tilde{P}_t}{\partial t} * f(\theta) \right\|^q \frac{e^{-t} dt}{1-e^{-t}} \\ &\leq C_{\delta,q}^q \int_0^{-\log \delta} \left\| t \frac{\partial \tilde{P}_t}{\partial t} * f(\theta) \right\|^q \frac{dt}{t} \leq C_{\delta,q}^q \int_0^\infty \left\| t \frac{\partial \tilde{P}_t}{\partial t} * f(\theta) \right\|^q \frac{dt}{t}. \end{aligned}$$

Therefore, by hypothesis (ii), we have that

$$\left\| \left[ (1-r) \frac{\partial P_r}{\partial r} \chi_{(\delta,1)}(r) \right] * f \right\|_{L^p_{L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})}(\mathbb{T})} \leq C'_{\delta,q} \|f\|_{L^p_{\mathbb{B}}(\mathbb{T})}.$$

Then by Lemma 2.6 we get

$$\|G_q^1(f)\|_{L^p(\mathbb{T})} \leq C'' \|f\|_{L_{\mathbb{B}}^p(\mathbb{T})}.$$

By [Xu], this implies that  $\mathbb{B}$  is of Lusin cotype  $q$ , and so of martingale cotype  $q$  too. Thus the proof of Theorem 2.1 is finished.  $\square$

**Proof of Theorem 2.2.** (i)  $\Rightarrow$  (ii). Write the spectral decomposition of the semigroup  $\{\mathcal{P}_t\}_{t \geq 0}$ : for any  $f \in L^2(\Omega)$

$$\mathcal{P}_t f = \int_0^\infty e^{-\lambda t} d e_\lambda f,$$

where  $\{e_\lambda\}$  is a resolution of the identity. Thus

$$\frac{\partial \mathcal{P}_t f}{\partial t} = - \int_{0+}^\infty \lambda e^{-\lambda t} d e_\lambda f.$$

It is easy to deduce from this formula that for any  $f, g \in L^2(\Omega)$  (recalling that  $F$  is the projection on the fix point subspace of  $\{\mathcal{P}_t\}_{t \geq 0}$ )

$$\int_\Omega (f - F(f))(g - F(g)) d\mu = 4 \int_\Omega \int_0^\infty \left[ t \frac{\partial \mathcal{P}_t f}{\partial t} \right] \left[ t \frac{\partial \mathcal{P}_t g}{\partial t} \right] \frac{dt}{t} d\mu. \tag{2.6}$$

Now we use duality. Fix two functions  $f \in L_{\mathbb{B}}^p(\Omega)$  and  $g \in L_{\mathbb{B}^*}^{p'}(\Omega)$ , where  $p'$  denotes the conjugate index of  $p$ . Without loss of generality we may assume that  $f$  and  $g$  are in the algebraic tensor products  $(L^p(\Omega) \cap L^2(\Omega)) \otimes \mathbb{B}$  and  $(L^{p'}(\Omega) \cap L^2(\Omega)) \otimes \mathbb{B}^*$ , respectively. With  $\langle \cdot, \cdot \rangle$  denoting the duality between  $\mathbb{B}$  and  $\mathbb{B}^*$ , we have

$$\int_\Omega \langle f, g \rangle d\mu = \int_\Omega \langle F(f), F(g) \rangle d\mu + \int_\Omega \langle f - F(f), g - F(g) \rangle d\mu.$$

The first term on the right is easy to be estimated:

$$\left| \int_\Omega \langle F(f), F(g) \rangle d\mu \right| \leq \|F(f)\|_{L_{\mathbb{B}}^p(\Omega)} \|F(g)\|_{L_{\mathbb{B}^*}^{p'}(\Omega)} \leq \|F(f)\|_{L_{\mathbb{B}}^p(\Omega)} \|g\|_{L_{\mathbb{B}^*}^{p'}(\Omega)}.$$

For the second one, by (2.6) and Hölder’s inequality

$$\begin{aligned} \left| \int_{\Omega} \langle f - F(f), g - F(g) \rangle d\mu \right| &= 4 \left| \int_{\Omega} \int_0^{\infty} \langle t \frac{\partial \mathcal{P}_t f}{\partial t}, t \frac{\partial \mathcal{P}_t g}{\partial t} \rangle \frac{dt}{t} d\mu \right| \\ &\leq 4 \int_{\Omega} \int_0^{\infty} \left\| t \frac{\partial \mathcal{P}_t f}{\partial t} \right\| \left\| t \frac{\partial \mathcal{P}_t g}{\partial t} \right\| \frac{dt}{t} d\mu \\ &\leq 4 \| \mathfrak{G}_q(f) \|_{L^p(\Omega)} \| \mathfrak{G}_{q'}(g) \|_{L^{p'}(\Omega)}. \end{aligned}$$

Now since  $\mathbb{B}$  is of martingale type  $q$ ,  $\mathbb{B}^*$  is of martingale cotype  $q'$ . Thus by Theorem 2.1,

$$\| \mathfrak{G}_{q'}(g) \|_{L^{p'}(\Omega)} \leq C \| g \|_{L_{\mathbb{B}^*}^{p'}(\Omega)}.$$

Combining the preceding inequalities, we get

$$\left| \int_{\Omega} \langle f, g \rangle d\mu \right| \leq \left( \| F(f) \|_{L_{\mathbb{B}}^p(\Omega)} + C \| \mathfrak{G}_q(f) \|_{L^p(\Omega)} \right) \| g \|_{L_{\mathbb{B}^*}^{p'}(\Omega)},$$

which gives (ii), taking the supremum over all  $g$  as above such that  $\| g \|_{L_{\mathbb{B}^*}^{p'}(\Omega)} \leq 1$ .

(ii)  $\Rightarrow$  (i). As in the corresponding proof of Theorem 2.1, we use again the Poisson semigroup on the torus. We keep the notations introduced there. Recall that  $\tilde{P}_t = P_{e^{-t}}$ . By the calculations done there,

$$\int_{\delta}^1 \left\| (1-r) \frac{\partial P_r}{\partial r} * f(\theta) \right\|^q \frac{dr}{1-r} \approx \int_0^{-\log \delta} \left\| t \frac{\partial \tilde{P}_t}{\partial t} * f(\theta) \right\|^q \frac{dt}{t},$$

where the equivalence constants depend only on  $\delta$  and  $q$ . On the other hand, on the interval  $(0, \delta)$ , we have

$$\begin{aligned} \int_0^{\delta} \left\| (1-r) \frac{\partial P_r}{\partial r} * f(\theta) \right\|^q \frac{dr}{1-r} &= \int_{-\log \delta}^{\infty} \left\| \frac{1-e^{-t}}{e^{-t}} \frac{\partial \tilde{P}_t}{\partial t} * f(\theta) \right\|^q \frac{e^{-t} dt}{1-e^{-t}} \\ &\geq C_{\delta,q}^q \int_{-\log \delta}^{\infty} e^{(q-1)t} \left\| \frac{\partial \tilde{P}_t}{\partial t} * f(\theta) \right\|^q dt \\ &\geq (C'_{\delta,q})^q \int_{-\log \delta}^{\infty} t^{q-1} \left\| \frac{\partial \tilde{P}_t}{\partial t} * f(\theta) \right\|^q dt. \end{aligned}$$



Therefore,

$$\int_0^\infty t^{q-1} \left\| \frac{\partial \tilde{P}_t}{\partial t} * f(\theta) \right\|^q dt \leq C_{\delta,q}^q \int_0^1 \left\| (1-r) \frac{\partial P_r}{\partial r} * f(\theta) \right\|^q \frac{dr}{1-r}.$$

Thus by hypothesis (ii),

$$\|f\|_{L^p_{\mathbb{B}}(\mathbb{T})} \leq C_{\delta,q} \left( \|\hat{f}(0)\|_{\mathbb{B}} + \|G_q^1(f)\|_{L^p(\mathbb{T})} \right).$$

Hence by [Xu],  $\mathbb{B}$  is of Lusin type  $q$ , and so of martingale type  $q$  too.  $\square$

We end this section with some remarks and questions.

**Remark 2.8.** Checking back the proofs above of Theorems 2.1 and 2.2, we see that under the condition that  $\mathbb{B}$  is of martingale cotype  $q$  (resp. martingale type  $q$ ), (2.1) (resp. (2.2)) is true for more general semigroups  $\{\mathcal{P}_t\}$  associated to a symmetric diffusion semigroup other than the subordinated Poisson semigroup given by (1.6). ( $\{\mathcal{P}_t\}$  needs not be even a semigroup for the validity of (2.1).) What we need is that  $\{\mathcal{P}_t\}$  is defined by

$$\mathcal{P}_t = \int_0^\infty \varphi(s) \mathcal{T}_{t\beta_s} ds,$$

where  $\beta$  is a non-zero real number, and where  $\varphi$  is a derivable function on  $\mathbb{R}_+$  such that both  $\varphi$  and  $t\varphi'$  are integrable on  $\mathbb{R}_+$  and such that the two limits  $\lim_{t \rightarrow 0} t\varphi(t)$  and  $\lim_{t \rightarrow \infty} t\varphi(t)$  exist. For instance, this is the case when the infinitesimal generator of  $\{\mathcal{P}_t\}$  is  $-(A)^\alpha$  with  $0 < \alpha < 1$ , where  $A$  is the infinitesimal generator of a symmetric diffusion semigroup  $\{\mathcal{T}_t\}$ . Indeed, by [Y, IX],  $\{\mathcal{P}_t\}$  can be represented as above with  $\beta = 1/\alpha$  and  $\varphi$  given by

$$\varphi(s) = \int_0^\infty \exp[st \cos \theta - t^\alpha \cos(\alpha\theta)] \times \sin[st \sin \theta - t^\alpha \cos(\alpha\theta) + \theta] dt,$$

where  $\theta$  can be any number in  $[\pi/2, \pi]$ .

**Remark 2.9.** The proof of (i)  $\Rightarrow$  (ii) in Theorem 2.2 implicitly shows the following: Given a Banach space  $\mathbb{B}$  and  $p, q \in (1, \infty)$ , if (2.1) holds for a given subordinated semigroup  $\{\mathcal{P}_t\}$ , then (2.2) holds for the same semigroup with  $\mathbb{B}$  and  $p, q \in (1, \infty)$  replaced by  $\mathbb{B}^*$  and  $p', q' \in (1, \infty)$ , respectively. The converse to this latter statement is also true. This will be the objective of the next section. With this converse, we can prove (ii)  $\Rightarrow$  (i) in Theorem 2.2 directly by duality and Theorem 2.1 without using the Poisson semigroup on  $\mathbb{T}$ . Note that such an approach is inevitable when one wishes to study the duality between (2.1) and (2.2) for an individual semigroup.

All semigroups considered in this paper are Markovian, that is,  $\mathcal{T}_t 1 = 1$ . We do not know whether Theorems 2.1 and 2.2 still hold for symmetric sub-Markovian semigroups (which are those satisfying (1.3)–(1.5) except the Markovianity).

**Problem 1.** Let  $\mathbb{B}$  be a Banach space of martingale cotype  $q$ . For which semigroups  $\{\mathcal{P}_t\}$  is the corresponding  $g$ -function mapping  $f \mapsto \mathfrak{G}_q(f)$  of weak type (1,1)?

We shall see later that the answer is positive for the usual Poisson semigroup and the subordinated Poisson Ornstein–Uhlenbeck semigroup on  $\mathbb{R}^n$ . We shall also show that (2.1) (resp. (2.2)) for one of these Poisson semigroups implies the martingale cotype  $q$  (resp. martingale type  $q$ ) of  $\mathbb{B}$ , like for the Poisson semigroup on the torus.

In general, it would be interesting to find conditions on a given semigroup  $\{\mathcal{P}_t\}$  which guarantee that the validity of (2.1) (resp. (2.2)) for  $\{\mathcal{P}_t\}$  implies martingale cotype  $q$  (resp. martingale type  $q$ ).

We state another problem about (2.1) for any symmetric diffusion semigroup (not necessarily subordinated to another one).

**Problem 2.** Let  $\mathbb{B}$  be a Banach space of martingale cotype  $q$  (resp. martingale type  $q$ ) and  $p \in (1, \infty)$ . Does (2.1) (resp. (2.2)) hold for any symmetric diffusion semigroup  $\{\mathcal{P}_t\}_{t \geq 0}$ ?

Problem 2 has an affirmative solution when  $\mathbb{B}$  is further a Banach lattice. Let us consider only the cotype case. It is well known that a Banach lattice  $\mathbb{B}$  is of martingale cotype  $q$  (with  $2 < q < \infty$ ) iff  $\mathbb{B}$  is  $q$ -concave and  $p$ -convex for some  $p > 1$ . For  $q = 2$ , the “if” part is still true; the “only if” part admits only a weaker form:  $\mathbb{B}$  is  $r$ -concave and  $p$ -convex for some  $p > 1$  and for any  $r > q = 2$ . See [LT]. Let  $\mathbb{B}$  be a  $q$ -concave and  $p$ -convex Banach lattice with  $1 < p \leq 2 \leq q < \infty$ . Then by [Pi3],  $\mathbb{B}$  can be written as a complex interpolation space between a Hilbert space and another lattice, i.e., there are a Hilbert space  $H$  and a Banach lattice  $\mathbb{B}_0$  such that

$$\mathbb{B} = (H, \mathbb{B}_0)_\theta.$$

Moreover,  $\mathbb{B}_0$  is  $q_0$ -concave and  $p_0$ -convex with  $p_0 > 1$  and  $q_0$  satisfying  $1/q = \theta/2 + (1 - \theta)/q_0$ . Now given a symmetric diffusion semigroup  $\{\mathcal{T}_t\}$ , following [St1, pp. 116–119], we consider the fractional averages of  $\{\mathcal{T}_t\}$ :

$$M_t^\alpha(f) = \frac{t^{-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T_s f ds.$$

$M_t^\alpha$  is well defined for  $\alpha \in \mathbb{C}$  with  $\operatorname{Re}(\alpha) > 0$ , and is continued analytically into the whole complex plane. Note that  $M_t^1$  is the usual average  $M_t$  and  $M_t^0 = T_t$ . By [St1],

for all  $\alpha \in \mathbb{C}$  and  $f \in L^p_H(\Omega)$  ( $1 < p < \infty$ ), we have

$$\left\| \left[ \int_0^\infty \left\| t \frac{\partial M_t^\alpha f}{\partial t} \right\|_H^2 \frac{dt}{t} \right]^{1/2} \right\|_{L^p(\Omega)} \leq C_{p,\alpha} \|f\|_{L^p_H(\Omega)}.$$

(This is proved in [St1] for the scalar-valued case; but the same arguments work as well for Hilbert-space-valued functions.) On the other hand, using Theorem 2.3, one can easily check that for any  $\alpha \in \mathbb{C}$  with  $\text{Re}(\alpha) > 1$  and  $f \in L^p_{\mathbb{B}_0}(\Omega)$  ( $1 < p < \infty$ )

$$\left\| \left[ \int_0^\infty \left\| t \frac{\partial M_t^\alpha f}{\partial t} \right\|_{\mathbb{B}_0}^{q_0} \frac{dt}{t} \right]^{1/q_0} \right\|_{L^p(\Omega)} \leq C_{p,\mathbb{B}_0,\alpha} \|f\|_{L^p_{\mathbb{B}_0}(\Omega)}.$$

Then interpolating these inequalities, we deduce that for any  $f \in L^p_{\mathbb{B}}(\Omega)$  ( $1 < p < \infty$ )

$$\left\| \left[ \int_0^\infty \left\| t \frac{\partial M_t^0 f}{\partial t} \right\|_{\mathbb{B}}^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\Omega)} \leq C_{p,\mathbb{B}} \|f\|_{L^p_{\mathbb{B}}(\Omega)}.$$

This is the desired inequality on  $T_t$  (recalling that  $T_t = M_t^0$ ).

### 3. Duality

Throughout this section  $\{\mathcal{T}_t\}_{t \geq 0}$  will be a fixed symmetric diffusion semigroup defined on  $L^p(\Omega, d\mu)$ , and  $\{\mathcal{P}_t\}_{t \geq 0}$  its subordinated Poisson semigroup. We shall keep all notations introduced in the previous section for these semigroups. In particular,  $F$  is the contractive projection from  $L^p(\Omega)$  (also from  $L^p_{\mathbb{B}}(\Omega)$ ) onto the fix point subspace of  $\{\mathcal{P}_t\}_{t \geq 0}$ . The following is the main result of this section.

**Theorem 3.1.** *Let  $\mathbb{B}$  be a Banach space and  $1 < p, q < \infty$ . Then the following statements are equivalent:*

(i) *There is a constant  $C > 0$  such that*

$$\|\mathfrak{G}_q f\|_{L^p(\Omega)} \leq C \|f\|_{L^p_{\mathbb{B}}(\Omega)} \quad \forall f \in L^p_{\mathbb{B}}(\Omega).$$

(ii) *There is a constant  $C > 0$  such that*

$$\|g\|_{L^{p'}_{\mathbb{B}^*}(\Omega)} \leq C \left( \|F(g)\|_{L^{p'}_{\mathbb{B}^*}(\Omega)} + \|\mathfrak{G}_{q'} g\|_{L^{p'}(\Omega)} \right) \quad \forall g \in L^{p'}_{\mathbb{B}^*}(\Omega).$$

The proof of the implication (i)  $\Rightarrow$  (ii) of Theorem 2.2 shows in fact (i)  $\Rightarrow$  (ii) in the theorem above. The inverse implication needs much more effort (as usually in such a situation). Let  $\mathbb{A} = L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})$ . An element  $h$  in  $L^p_{\mathbb{A}}(\Omega)$  is a function of two variables  $x \in \Omega$  and  $t \in \mathbb{R}_+$ , i.e.,  $h : (x, t) \mapsto h_t(x)$ . The key to the implication (ii)  $\Rightarrow$  (i) above is the existence of a bounded projection from  $L^p_{\mathbb{A}}(\Omega)$  onto the subspace of all functions  $h$  which can be written as  $h_t(x) = t \frac{\partial \mathcal{P}_t f}{\partial t}(x)$  for some function  $f$  on  $\Omega$ . Formally, the desired projection is given by  $h \mapsto t \frac{\partial \mathcal{P}_t(Qh)}{\partial t}(x)$ , where  $Qh$  is defined by

$$Qh(x) = \int_0^\infty t \frac{\partial \mathcal{P}_t h_t}{\partial t}(x) \frac{dt}{t}, \quad x \in \Omega. \tag{3.1}$$

Note that  $Q(h)$  is well-defined for nice functions  $h \in L^p_{\mathbb{A}}(\Omega)$ , for instance, for all compactly supported continuous functions from  $\mathbb{R}_+$  to  $L^p_{\mathbb{B}}(\Omega)$ . By the density of all such functions in  $L^p_{\mathbb{A}}(\Omega)$ , to prove the boundedness of  $Q$  we need only to estimate the relative norm of  $Qh$  for all such  $h$ .

**Theorem 3.2.** *Let  $\mathbb{B}$ ,  $p, q$  be as in Theorem 3.1. Then for any (nice)  $h$*

$$\|\mathfrak{G}_q(Qh)\|_{L^p(\Omega)} \leq C_{p,q} \|h\|_{L^p_{\mathbb{A}}(\Omega)}.$$

Consequently,  $\mathfrak{G}_q Q$  extends to a bounded operator from  $L^p_{\mathbb{A}}(\Omega)$  to  $L^p(\Omega)$  with norm controlled by a constant depending only on  $p$  and  $q$ .

Admitting this theorem, we can easily prove Theorem 3.1.

**Proof of Theorem 3.1.** (i)  $\Rightarrow$  (ii). The proof for this is similar to that for (i)  $\Rightarrow$  (ii) in Theorem 2.2.

(ii)  $\Rightarrow$  (i). Fix an  $f \in L^p_{\mathbb{B}}(\Omega)$ . Choose  $h \in L^{p'}_{L^{q'}_{\mathbb{B}^*}(\mathbb{R}_+, \frac{dt}{t})}(\Omega)$  of unit norm such

$$\|\mathfrak{G}_q f\|_{L^p(\Omega)} = \int_{\Omega} \int_{\mathbb{R}_+} \langle t \frac{\partial \mathcal{P}_t f}{\partial t}(x), h_t(x) \rangle \frac{dt}{t} dx.$$

Now we apply Theorem 3.2 to  $\mathbb{B}^*$ ,  $p'$  and  $q'$ . (We may assume  $f$  and  $h$  are nice enough to legitimate the calculations below.) We have, by hypothesis (ii) and Theorem 3.2

$$\begin{aligned} \|\mathfrak{G}_q f\|_{L^p(\Omega)} &= \int_{\mathbb{R}_+} \int_{\Omega} \langle f(x), t \frac{\partial \mathcal{P}_t h_t}{\partial t}(x) \rangle \frac{dt}{t} dx, \\ &= \int_{\Omega} \langle f(x), Q(h)(x) \rangle dx, \end{aligned}$$

$$\begin{aligned} &\leq \|f\|_{L^p_{\mathbb{B}}(\Omega)} \|Q(h)\|_{L^{p'}_{\mathbb{B}^*}(\Omega)} \\ &\leq C \|f\|_{L^p_{\mathbb{B}}(\Omega)} \|\mathfrak{G}_{q'}(Qh)\|_{L^{p'}(\Omega)} \leq C' \|f\|_{L^p_{\mathbb{B}}(\Omega)}. \end{aligned}$$

This yields (i).  $\square$

As for Theorem 2.3, we shall reduce Theorem 3.2 to an analogous inequality for martingales via Rota’s theorem. Let  $\{E_n\}$  be a monotone sequence of conditional expectations as in Lemma 2.4. Let us maintain the notations in that lemma and its proof. In the remainder of this section  $l^q$  denotes the usual  $\ell^q$  space over  $\mathbb{N}$  but with weight  $\{\frac{1}{n}\}$ , i.e., the norm of a sequence  $a$  is given by

$$\|a\|_{l^q} = \left( \sum_{n \geq 1} |a_n|^q \frac{1}{n} \right)^{1/q}.$$

The corresponding  $\mathbb{B}$ -valued version is  $l^q_{\mathbb{B}}$ , denoted by  $\mathbb{D} = l^q_{\mathbb{B}}$  in the sequel. Now we consider the discrete version of  $Q$  defined by (3.1). As before, the elements in  $L^p_{\mathbb{D}}(M)$  are regarded as sequences with values in  $L^p_{\mathbb{B}}(M)$ . Given  $h \in L^p_{\mathbb{D}}(M)$  we define

$$Rh = \sum_{n \geq 1} n \Delta \sigma_n(h_n) \frac{1}{n}, \tag{3.2}$$

where  $\Delta \sigma_n = \sigma_n - \sigma_{n-1}$ . Recall that  $\sigma_n = \frac{E_0 + \dots + E_n}{n + 1}$ .  $Rh$  is clearly well-defined for finite sequences  $(h_n)_n$ .

**Lemma 3.3.** *Let  $\mathbb{B}, p, q$  be as in Theorem 3.1. Let  $\{E_n\}$  be an arbitrary monotone sequence of conditional expectations on a measure space  $(M, dm)$ . Then for any finite sequence  $h = (h_n) \in L^p_{\mathbb{D}}(M)$*

$$\|(n \Delta \sigma_n(Rh))_{n \geq 1}\|_{L^p_{\mathbb{D}}(M)} \leq C_{p,q} \|h\|_{L^p_{\mathbb{D}}(M)}.$$

Consequently,  $h \mapsto (n \Delta \sigma_n(Rh))_{n \geq 1}$  extends to a bounded operator on  $L^p_{\mathbb{D}}(M)$  with norm majorized by a constant depending only on  $p$  and  $q$ .

**Proof.** Without loss of generality, we may assume  $\{E_n\}$  increasing. What we have to prove is the following inequality:

$$\left\| \left[ \sum_{m \geq 1} \left\| \sum_{n \geq 1} m \Delta \sigma_m \Delta \sigma_n h_n \right\|^q \frac{1}{m} \right]^{\frac{1}{q}} \right\|_{L^p(M)} \leq C_{p,q} \left\| \left[ \sum_{n \geq 1} \|h_n\|^q \frac{1}{n} \right]^{\frac{1}{q}} \right\|_{L^p(M)}. \tag{3.3}$$

Given  $m, n \geq 0$  we have

$$\Delta \sigma_m \Delta \sigma_n = \sigma_m \sigma_n - \sigma_{m-1} \sigma_n - \sigma_m \sigma_{n-1} + \sigma_{m-1} \sigma_{n-1}. \tag{3.4}$$

A simple calculation yields (with  $m \wedge n = \min(m, n)$ )

$$\sigma_m \sigma_n = \frac{1}{(m+1)(n+1)} \sum_{i=0}^m \sum_{j=0}^n E_i E_j = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m \wedge n} (m+n-2i+1) E_i.$$

Setting  $d_i = E_i - E_{i-1}$  (with the convention that  $E_{-1} = 0$ ) and using Abel summation, we get

$$\sigma_m \sigma_n = \sum_{j=0}^{m \wedge n} \left( 1 - \frac{j(m+n+2-j)}{(m+1)(n+1)} \right) d_j.$$

Thus by (3.4)

$$\Delta \sigma_m \Delta \sigma_n = \frac{1}{mn(m+1)(n+1)} \sum_{j=1}^{m \wedge n} j^2 d_j. \tag{3.5}$$

To prove (3.3) we use martingale transforms with vector-valued kernel as in [MT]. See the end of this paper (after Theorem 7.4) for a brief discussion on this subject. The martingales we consider here are those defined on  $(M, dm)$  relative to  $\{E_n\}$  with values in  $\mathbb{D} = l^q_{\mathbb{B}}$ . Because of (3.5), we want to express the mapping

$$T : h = \{h_n\}_{n \geq 1} \mapsto \left\{ \sum_{n \geq 1} \frac{1}{n(m+1)(n+1)} \sum_{j=1}^{m \wedge n} j^2 d_j h_n \right\}_{m \geq 1}$$

as a martingale transform, namely, we have to find a multiplying sequence  $\{v_j\}$  such that

$$Th = \sum_j v_j d_j h.$$

This is clear from the above definition of  $Th$ . In fact, each  $v_j$  is a constant (an element in  $\mathcal{L}(\mathbb{D})$ ) and its matrix is given by

$$A_j = \left[ \frac{j^2}{n(m+1)(n+1)} \chi_{[j, \infty)}(m) \chi_{[j, \infty)}(n) \right]_{m \geq 1, n \geq 1}.$$

More precisely,  $v_j$  is the operator  $A_j \otimes \text{Id}_{\mathbb{B}}$ , where  $A_j$  is considered as an operator on  $l^q$ . Therefore,

$$\|v_j\|_{\mathcal{L}(\mathbb{D})} = j^2 \left( \sum_{m \geq j} \frac{1}{m(m+1)^q} \right)^{1/q} \left( \sum_{n \geq j} \frac{1}{n(n+1)^{q'}} \right)^{1/q'} \leq C_0.$$

Therefore by [MT], it suffices to prove (3.3) for  $p = q$ . This is the main part of the proof. In the following  $C > 0$  denotes a constant which may depend on  $q$  and vary from line to line.

Let us first rewrite (3.3) in the case  $p = q$  (by using (3.5)):

$$L \equiv \int_M \sum_{m \geq 1} \left\| \sum_{n \geq 1} \frac{1}{n(m+1)(n+1)} \sum_{j=1}^{m \wedge n} j^2 d_j h_n \right\|^q \frac{1}{m} \leq C \int_M \sum_{n \geq 1} \|h_n\|^q \frac{1}{n}. \tag{3.6}$$

It is easy to see that it is enough to prove the above inequality (3.6) for  $m \wedge n \geq 5$ . We have

$$\begin{aligned} L &\leq \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{5 \leq n \leq m} \frac{1}{n(n+1)} \sum_{j=1}^n j^2 d_j h_n \right\|^q \\ &\quad + \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{n > m} \frac{1}{n(n+1)} \sum_{j=1}^m j^2 d_j h_n \right\|^q \equiv A + B. \end{aligned} \tag{3.7}$$

Let us first estimate  $A$ . To this end, we use the notations  $\Delta_k$  and  $J_n$  introduced during the proof of Lemma 2.4. Note that for  $2^{k-1} < j \leq 2^k$

$$d_j = E_j(\Delta_k) - E_{j-1}(\Delta_k).$$

Thus by Abel’s summation we have

$$\sum_{j=2^{k-1}+1}^{2^k} j^2 d_j = 2^{2k} \Delta_k - \sum_{j=2^{k-1}+1}^{2^k-1} (2j+1) E_j(\Delta_k), \quad k \geq 2$$

and so for  $n \geq 5$

$$\begin{aligned} \sum_{j=1}^n j^2 d_j &= d_1 + 4d_2 + \sum_{k=2}^{J_n} \sum_{j=2^{k-1}+1}^{2^k} j^2 d_j + \sum_{j=2^{J_n}+1}^n j^2 d_j \\ &= d_1 + 4d_2 + \sum_{k=2}^{J_n} 2^{2k} \Delta_k - \sum_{k=2}^{J_n} \sum_{j=2^{k-1}+1}^{2^k-1} (2j+1) E_j(\Delta_k) \\ &\quad + n^2 E_n(\Delta_{J_n+1}) - \sum_{j=2^{J_n}+1}^{n-1} (2j+1) E_j(\Delta_{J_n+1}). \end{aligned} \tag{3.8}$$

Inserting this decomposition of  $\sum_{j=1}^n j^2 d_j$  in the expression of  $A$  in (3.7) and by triangle inequality, we see that  $A$  is majorized by a sum of five terms,  $A_0 + A_1 + A_2 + A_3 + A_4$ , corresponding respectively to the five terms of the last member in (3.8). The term  $A_0$  is handled easily. Fix  $0 < \alpha < 1/2$  and put  $\beta = 1 - \alpha$ . For  $A_1$  we have

$$\begin{aligned} A_1 &= \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{5 \leq n \leq m} \frac{1}{n(n+1)} \sum_{k=2}^{J_n} 2^{2k} \Delta_k h_n \right\|^q \\ &= \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{k=2}^{J_m} 2^{2k} \Delta_k \left[ \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right] \right\|^q \\ &\leq C \sum_{m \geq 5} \frac{m^{2\alpha q}}{m^{q+1}} \sum_{k=2}^{J_m} 2^{2\beta q k} \int_M \left\| \Delta_k \left[ \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right] \right\|^q \\ &\leq C \int_M \sum_{m \geq 5} \frac{m^{2\alpha q}}{m^{q+1}} \sum_{k=2}^{J_m} 2^{2\beta q k} \left\| \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right\|^q \\ &\leq C \int_M \sum_{m \geq 5} \frac{1}{m^{(1-2\alpha)q+1}} \sum_{k=2}^{J_m} 2^{2\beta q k} 2^{-k} \sum_{n: J_n \geq k} \frac{1}{n^q} \|h_n\|^q \\ &\leq C \int_M \sum_{m \geq 5} \frac{1}{m^{(1-2\alpha)q+1}} \sum_{5 \leq n \leq m} \frac{1}{n^q} \|h_n\|^q \sum_{k=2}^{J_n} 2^{(2\beta q - 1)k} \\ &\leq C \int_M \sum_{n \geq 5} n^{2\beta q - q - 1} \|h_n\|^q \sum_{m \geq n} \frac{1}{m^{(1-2\alpha)q+1}} \quad (\text{since } 2\beta q - 1 > 0) \\ &\leq C \int_M \sum_{n \geq 1} \frac{\|h_n\|^q}{n} \quad (\text{since } 1 - 2\alpha > 0). \end{aligned}$$



We pass to  $A_2$ :

$$\begin{aligned}
 A_2 &= \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{5 \leq n \leq m} \frac{1}{n(n+1)} \sum_{k=2}^{J_n} \sum_{j=2^{k-1}+1}^{2^k-1} (2j+1) E_j \Delta_k h_n \right\|^q \\
 &= \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{k=2}^{J_m} \left[ \sum_{j=2^{k-1}+1}^{2^k-1} (2j+1) E_j \Delta_k \right] \left( \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right) \right\|^q.
 \end{aligned}$$

Let us handle the expression concerning the internal norm. As for  $A_1$  previously, we have

$$\begin{aligned}
 &\left\| \sum_{k=2}^{J_m} \left[ \sum_{j=2^{k-1}+1}^{2^k-1} (2j+1) E_j \Delta_k \right] \left( \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right) \right\|^q \\
 &\leq C m^{2\alpha q} \sum_{k=2}^{J_m} 2^{2\beta q k} \left\| \sum_{j=2^{k-1}+1}^{2^k-1} 2^{-2k} (2j+1) E_j \Delta_k \left( \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right) \right\|^q \\
 &\leq C m^{2\alpha q} \sum_{k=2}^{J_m} 2^{2\beta q k} \sum_{j=2^{k-1}+1}^{2^k-1} 2^{-2k} (2j+1) \left\| E_j \Delta_k \left( \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right) \right\|^q.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &\int_M \left\| \sum_{k=2}^{J_m} \left[ \sum_{j=2^{k-1}+1}^{2^k-1} (2j+1) E_j \Delta_k \right] \left( \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right) \right\|^q \\
 &\leq C m^{2\alpha q} \sum_{k=2}^{J_m} 2^{2\beta q k} \sum_{j=2^{k-1}+1}^{2^k-1} 2^{-2k} (2j+1) \int_M \left\| E_j \Delta_k \left( \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right) \right\|^q \\
 &\leq C m^{2\alpha q} \sum_{k=2}^{J_m} 2^{2\beta q k} \sum_{j=2^{k-1}+1}^{2^k-1} 2^{-2k} (2j+1) \int_M \left\| \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right\|^q \\
 &\leq C \int_M m^{2\alpha q} \sum_{k=2}^{J_m} 2^{2\beta q k} \left\| \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right\|^q.
 \end{aligned}$$

Combining the preceding inequalities, we get

$$\begin{aligned}
 A_2 &\leq C \int_M \sum_{m \geq 5} \frac{1}{m^{(1-2\alpha)q+1}} \sum_{k=2}^{J_m} 2^{2\beta qk} \left\| \sum_{n: J_n \geq k} \frac{1}{n(n+1)} h_n \right\|^q \\
 &\leq C \int_M \sum_{n \geq 1} \frac{\|h_n\|^q}{n}.
 \end{aligned}$$

It is easier to estimate  $A_3$  and  $A_4$ . Indeed, for  $A_3$  we have

$$\begin{aligned}
 A_3 &= \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{5 \leq n \leq m} \frac{1}{n(n+1)} n^2 E_n \Delta_{J_{n+1}} h_n \right\|^q \\
 &\leq C \sum_{m \geq 5} \frac{1}{m^2} \sum_{5 \leq n \leq m} \int_M \|E_n \Delta_{J_{n+1}} h_n\|^q \leq C \int_M \sum_{n \geq 1} \frac{\|h_n\|^q}{n}.
 \end{aligned}$$

$A_4$  is similarly estimated. Therefore, we get

$$A \leq C \|h\|_{L^q_{\mathbb{D}}(M)}^q. \tag{3.9}$$

Now we turn to  $B$ . This time we fix  $\alpha$  such that  $\frac{q-1}{2q} < \alpha < \frac{q-1}{q}$  and put again  $\beta = 1 - \alpha$ . As for  $A$  previously, using (3.8) with  $n$  replaced by  $m$ , we see that  $B$  is less than or equal to  $B_0 + B_1 + B_2 + B_3 + B_4$ , corresponding to the decomposition in (3.8). Thus

$$\begin{aligned}
 B_1 &= \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{n>m} \frac{1}{n(n+1)} \sum_{k=2}^{J_m} 2^{2k} \Delta_k h_n \right\|^q \\
 &= \int_M \sum_{m \geq 5} \frac{1}{m^{q+1}} \left\| \sum_{k=2}^{J_m} 2^{2k} \Delta_k \left[ \sum_{n>m} \frac{1}{n(n+1)} h_n \right] \right\|^q \\
 &\leq C \sum_{m \geq 5} m^{q-3} \sum_{k=2}^{J_m} 2^{2k} \int_M \left\| \Delta_k \left[ \sum_{n>m} \frac{1}{n(n+1)} h_n \right] \right\|^q \\
 &\leq C \int_M \sum_{m \geq 5} m^{q-1} \left( \sum_{n>m} \frac{1}{n^{2\alpha q'}} \right)^{q-1} \sum_{n>m} \frac{1}{n^{2\beta q}} \|h_n\|^q \\
 &\leq C \int_M \sum_{m \geq 5} m^{2(q-1)-2\alpha q} \sum_{n>m} \frac{1}{n^{2\beta q}} \|h_n\|^q \quad (\text{since } 2\alpha q' > 1) \\
 &\leq C \int_M \sum_{n \geq 1} \frac{\|h_n\|^q}{n} \quad (\text{since } q - 1 > \alpha q).
 \end{aligned}$$

In a similar way (also as for  $A_2, A_3$  and  $A_4$ ), we obtain the same bound for  $B_2, B_3$  and  $B_4$ . Therefore, we have

$$B \leq C \|h\|_{L^q_{\mathbb{D}}(M)}^q. \tag{3.10}$$

Finally combining (3.7), (3.9) and (3.10), we get (3.6). Therefore, the lemma is proved.  $\square$

**Proof of Theorem 3.2.** For notational simplicity we put

$$\Phi_t = t \frac{\partial P_t}{\partial t} \quad \text{and} \quad \Psi_t = t \frac{\partial M_t}{\partial t}.$$

(Recall that  $M_t = \frac{1}{t} \int_0^t T_s ds$ .) By Lemma 2.7,

$$\int_{\Omega} \left[ \int_0^{\infty} \left\| \int_0^{\infty} \Phi_s \Phi_t h_t \frac{dt}{t} \right\|^q \right]^{\frac{p}{q}} \frac{ds}{s} \leq C^p \int_{\Omega} \left[ \int_0^{\infty} \left\| \int_0^{\infty} \Psi_s \Phi_t h_t \frac{dt}{t} \right\|^q \right]^{\frac{p}{q}} \frac{ds}{s}.$$

By (2.5),

$$\begin{aligned} \int_0^{\infty} \Psi_s \Phi_t h_t \frac{dt}{t} &= -2 \int_0^{\infty} \int_0^{\infty} \left(\frac{u}{t^2}\right)^2 \varphi'\left(\frac{u}{t^2}\right) \Psi_s \Psi_u h_t \frac{du}{u} \frac{dt}{t} \\ &= \int_0^{\infty} \int_0^{\infty} \left[t^2 \varphi'(t)\right] \Psi_s \Psi_u h_{u^{1/2}t^{-1/2}} \frac{du}{u} \frac{dt}{t}. \end{aligned}$$

By the triangle inequality (recalling that  $\mathbb{A} = L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{ds}{s})$ ),

$$\begin{aligned} &\left\| \int_0^{\infty} \Psi_s \Phi_t h_t \frac{dt}{t} \right\|_{L^p_{\mathbb{A}}(\Omega)} \\ &\leq \int_0^{\infty} \left\| \int_0^{\infty} \Psi_s \Psi_u h_{u^{1/2}t^{-1/2}} \frac{du}{u} \right\|_{L^p_{\mathbb{A}}(\Omega)} t^2 |\varphi'(t)| \frac{dt}{t}. \end{aligned}$$

Now using the same discretization arguments as in the proof of Theorem 2.3, we deduce from Lemma 3.3 that

$$\begin{aligned} \left\| \int_0^{\infty} \Psi_s \Psi_u h_{u^{1/2}t^{-1/2}} \frac{du}{u} \right\|_{L^p_{\mathbb{A}}(\Omega)}^p &\leq C^p \int_{\Omega} \left[ \int_0^{\infty} \|h_{u^{1/2}t^{-1/2}}\|^q \frac{du}{u} \right]^{\frac{p}{q}} \\ &= 2^{p/q} C^p \|h\|_{L^p_{\mathbb{A}}(\Omega)}^p. \end{aligned}$$

Combining the preceding inequalities, we obtain

$$\left\| \int_0^\infty \Phi \cdot \Phi_t h_t \frac{dt}{t} \right\|_{L^p_{\mathbb{A}}(\Omega)} \leq C' \|h\|_{L^p_{\mathbb{A}}(\Omega)}.$$

This is the desired inequality. Thus we have achieved the proof of Theorem 3.2.  $\square$

#### 4. Poisson semigroup on $\mathbb{R}^n$

This section and the next are devoted to the study of the Littlewood–Paley  $g$ -function on  $\mathbb{R}^n$  in the vector-valued case. Our main goal is to prove the implications (ii)  $\Rightarrow$  (i) in Theorems 2.1 and 2.2 in the particular case of the Poisson semigroup on  $\mathbb{R}^n$ . This section collects some results on the  $g$ -function operator on  $\mathbb{R}^n$ , represented as a Calderón–Zygmund operator. It can be considered as preparatory, although some of these results are of general interest. The proof of the mentioned implications will be done in the next section.

Let  $\mathbb{B}$  be a Banach space and  $1 < q < \infty$ . Recall the generalized Littlewood–Paley  $g$ -function on  $\mathbb{R}^n$ :

$$\mathcal{G}_q(f)(x) = \left( \int_0^\infty t^q \|\nabla P_t * f(x)\|_{\ell^2_{\mathbb{B}}}^q \frac{dt}{t} \right)^{1/q}, \quad x \in \mathbb{R}^n.$$

It is often easier to consider the corresponding  $g$ -function defined only by the derivative in time, which is the following

$$\mathcal{G}_q^1(f)(x) = \left( \int_0^\infty t^q \left\| \frac{\partial P_t}{\partial t} * f(x) \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q}.$$

Similarly, we define  $\mathcal{G}_q^2(f)$  as the part of  $\mathcal{G}_q(f)$  corresponding to the gradient in the space variable:

$$\mathcal{G}_q^2(f)(x) = \left( \int_0^\infty t^q \|\nabla_x P_t * f(x)\|_{\ell^2_{\mathbb{B}}}^q \frac{dt}{t} \right)^{1/q},$$

where

$$\nabla_x = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right).$$

These  $g$ -functions can be treated as Calderón–Zygmund operators. To this end we first recall briefly the definition of these operators. Given a pair of Banach spaces  $\mathbb{B}_1$

and  $\mathbb{B}_2$ , a linear operator  $T$  is a Calderón–Zygmund operator on  $\mathbb{R}^n$ , with associated Calderón–Zygmund kernel  $k$  if  $T$  maps  $L_{c, \mathbb{B}_1}^\infty(\mathbb{R}^n)$ , the space of the essentially bounded functions on  $\mathbb{R}^n$  with compact support, into  $\mathbb{B}_2$ -valued strongly measurable functions on  $\mathbb{R}^n$ , and for any function  $f \in L_{c, \mathbb{B}_1}^\infty(\mathbb{R}^n)$  we have

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y)f(y)dy, \quad \text{for a.e. } x \text{ outside the support of } f,$$

where the kernel,  $k(x, y) \in \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$  satisfies

- (a)  $\|k(x, y)\| \leq C|x - y|^{-n}$ ,
- (b)  $\|\nabla_x k(x, y)\| + \|\nabla_y k(x, y)\| \leq C|x - y|^{-(n+1)}$ .

We shall always assume that there is  $\Lambda \in \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$  such that  $T(c) \equiv \Lambda(c)$  for all  $c \in \mathbb{B}_1$ .

Let us recall the BMO and  $H^1$  spaces on  $\mathbb{R}^n$ . Let  $\mathbb{B}$  be a Banach space.  $\text{BMO}_{\mathbb{B}}(\mathbb{R}^n)$  is the space of  $\mathbb{B}$ -valued functions  $f$  defined on  $\mathbb{R}^n$  such that

$$\|f\|_{\text{BMO}_{\mathbb{B}}(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q \|f(x) - f_Q\|_{\mathbb{B}} dx < \infty,$$

where  $f_Q = \frac{1}{|Q|} \int_Q f(x) dx$  and the supremum is taken over the cubes  $Q \subset \mathbb{R}^n$ . The space  $H^1$  is defined in the atomic sense. Namely, we say that a function  $a \in L_{\mathbb{B}}^\infty(\mathbb{R}^n)$  is a  $\mathbb{B}$ -atom if there exists a cube  $Q \subset \mathbb{R}^n$  containing the support of  $a$ , and such that  $\|a\|_{L_{\mathbb{B}}^\infty(\mathbb{R}^n)} \leq |Q|^{-1}$ , and  $\int_Q a(x) dx = 0$ . Then, we say that a function  $f$  is in  $H_{\mathbb{B}}^1(\mathbb{R}^n)$  if it admits a decomposition  $f = \sum_i \lambda_i a_i$ , where  $a_i$  are  $\mathbb{B}$ -valued atoms and  $\sum_i |\lambda_i| < \infty$ . We define  $\|f\|_{H_{\mathbb{B}}^1} = \inf \{ \sum_i |\lambda_i| \}$ , where the infimum runs over all those such decompositions (see [BG]).

The following theorem is a kind of folklore. We give a sketch of its proof for the convenience of the reader.  $\text{BMO}_{c, \mathbb{B}}(\mathbb{R}^n)$  denotes the subspace of  $\text{BMO}_{\mathbb{B}}(\mathbb{R}^n)$  consisting of functions with compact support.

**Theorem 4.1.** *Let  $\mathbb{B}_1, \mathbb{B}_2$  be two Banach spaces and  $T$  a Calderón–Zygmund operator with an associated kernel  $k$  as above. Let  $S$  be defined as  $S(f) = \|T(f)\|_{\mathbb{B}_2}$ . Then, the following statements are equivalent*

- (i)  $T$  maps  $L_{c, \mathbb{B}_1}^\infty(\mathbb{R}^n)$  into  $\text{BMO}_{\mathbb{B}_2}(\mathbb{R}^n)$ .
- (ii)  $S$  maps  $L_{c, \mathbb{B}_1}^\infty(\mathbb{R}^n)$  into  $\text{BMO}(\mathbb{R}^n)$
- (iii)  $T$  maps  $H_{\mathbb{B}_1}^1(\mathbb{R}^n)$  into  $L_{\mathbb{B}_2}^1(\mathbb{R}^n)$ .
- (iv)  $T$  maps  $L_{\mathbb{B}_1}^p(\mathbb{R}^n)$  into  $L_{\mathbb{B}_2}^p(\mathbb{R}^n)$  for any (or equivalently, for some)  $p \in (1, \infty)$ .
- (v)  $T$  maps  $L_{\mathbb{B}_1}^1(\mathbb{R}^n)$  into weak- $L_{\mathbb{B}_2}^1(\mathbb{R}^n)$ .
- (vi)  $T$  maps  $\text{BMO}_{c, \mathbb{B}_1}(\mathbb{R}^n)$  into  $\text{BMO}_{\mathbb{B}_2}(\mathbb{R}^n)$ .
- (vi)  $S$  maps  $\text{BMO}_{c, \mathbb{B}_1}(\mathbb{R}^n)$  into  $\text{BMO}(\mathbb{R}^n)$ .

**Proof.** The structure of the proof is the following: first, (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i). Then, we prove (i)  $\Rightarrow$  (iv)  $\Rightarrow$  (vi)  $\Rightarrow$  (vii)  $\Rightarrow$  (ii) and (iv)  $\Rightarrow$  (v)  $\Rightarrow$  (i).

The fact that  $L^\infty_{\mathbb{B}_1}$  is contained in  $\text{BMO}_{\mathbb{B}_1}$  gives that (vi) implies (i). Since the norm of a function in  $\text{BMO}_{\mathbb{B}}$  is a function in  $\text{BMO}$ , then we have (i)  $\Rightarrow$  (ii) and (vi)  $\Rightarrow$  (vii).

To get (ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (i), we can proceed as in [Jou, p. 49] with minor modifications due to considering the operator  $Sf = \|Tf\|_{\mathbb{B}_2}$ .

As we already know that (i)  $\Rightarrow$  (iii), we can apply interpolation (see [BX]) and we have that  $T$  maps  $L^p_{\mathbb{B}_1}$  into  $L^p_{\mathbb{B}_2}$  for  $1 < p < \infty$ , so we have (i)  $\Rightarrow$  (iv).

The proof of (iv)  $\Rightarrow$  (vi) is where the condition that  $T(c)(x) = \Lambda(c)$  plays a role. Let  $f$  be a function in  $\text{BMO}_{c, \mathbb{B}_1}$ . Given a cube  $Q$  with center  $x_0$ , let  $\tilde{Q}$  be its doubled cube. We decompose

$$\begin{aligned} & \frac{1}{|Q|} \int_Q \|Tf(x) - Tg_2(x_0)\|_{\mathbb{B}_2} dx \\ & \leq \frac{1}{|Q|} \int_Q \|Tg_1(x)\|_{\mathbb{B}_2} dx + \frac{1}{|Q|} \int_Q \|Tg_2(x) - Tg_2(x_0)\|_{\mathbb{B}_2} dx, \end{aligned}$$

where  $f = g_1 + g_2$ ,  $g_1 = (f - f_Q)\chi_{\tilde{Q}}$  and  $g_2 = (f - f_Q)\chi_{\mathbb{R}^n \setminus \tilde{Q}} + f_Q$ . By using Jensen and the  $L^p$  boundedness of  $T$ , we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q \|Tg_1(x)\| dx & \leq \left( \frac{1}{|Q|} \int_Q \|Tg_1(x)\|^p dx \right)^{1/p} \\ & \leq C \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} \|g_1(x)\|_{\mathbb{B}_1}^p dx \right)^{1/p} \\ & = C \left( \frac{1}{|Q|} \int_{\tilde{Q}} \|f(x) - f_Q\|_{\mathbb{B}_1}^p dx \right)^{1/p} \leq C \|f\|_{\text{BMO}_{\mathbb{B}_1}}, \end{aligned}$$

where in the last inequality we have used the John–Nirenberg theorem. On the other hand, using  $T(c)(x) = \Lambda(c)$ , we have

$$\begin{aligned} Tg_2(x) - Tg_2(x_0) & = T((f - f_Q)\chi_{\mathbb{R}^n \setminus \tilde{Q}})(x) - T((f - f_Q)\chi_{\mathbb{R}^n \setminus \tilde{Q}})(x_0) \\ & = \int_{\mathbb{R}^n \setminus \tilde{Q}} (k(x, y) - k(x_0, y))(f(y) - f_Q) dy. \end{aligned}$$

Now, using the hypothesis on the kernel  $k$  we have that for  $x \in Q$ ,  $y \in \mathbb{R}^n \setminus \tilde{Q}$ ,  $\|k(x, y) - k(x_0, y)\| \leq \frac{|x-x_0|}{|y-x_0|^{n+1}}$ , and therefore

$$\begin{aligned} \|Tg_2(x) - Tg_2(x_0)\|_{\mathbb{B}_2} & \leq \sum_{j=1}^{\infty} 2^{-j} \int_{2^j Q \setminus 2^{j-1} Q} \frac{1}{|y-x_0|^n} \|f(y) - f_Q\|_{\mathbb{B}_1} dy \\ & \leq \sum_{j=1}^{\infty} 2^{-j} \frac{1}{|2^j Q|} \int_{2^j Q} \|f(y) - f_Q\|_{\mathbb{B}_1} dy \leq C \|f\|_{\text{BMO}_{\mathbb{B}_1}}. \end{aligned}$$

By [GR, Theorem V.3.4], the strong type  $(p, p)$  implies that  $T$  is of weak type  $(1, 1)$ . This gives (iv)  $\Rightarrow$  (v). From (v), statement (i) can be achieved by using a slight modification of the argument given in the proof of Lemma 5.11 of [GR, p. 199]. The key is using Kolmogorov’s inequality [GR, Lemma V.2.8] relating  $L^{1,\infty}$  norm with  $L^q$  norm for  $0 < q < 1$  and the fact that  $BMO_q = BMO$ .  $\square$

It is well known that the various Littlewood–Paley  $g$ -functions can be expressed as Calderón–Zygmund operators with regular vector-valued kernels (see [Xu] for the case of the torus; also see [St2] for the scalar case). Therefore, we immediately get the following

**Corollary 4.2.** *Given a Banach space  $\mathbb{B}$ ,  $q \in (1, \infty)$ , the following statements are equivalent.*

- (i)  $\mathcal{G}_q$  maps  $L^\infty_{c,\mathbb{B}}(\mathbb{R}^n)$  into  $BMO(\mathbb{R}^n)$ .
- (ii)  $\mathcal{G}_q$  maps  $H^1_{\mathbb{B}}(\mathbb{R}^n)$  into  $L^1(\mathbb{R}^n)$ .
- (iii)  $\mathcal{G}_q$  maps  $L^p_{\mathbb{B}}(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$  for any (equivalently for some)  $p \in (1, \infty)$ .
- (iv)  $\mathcal{G}_q$  maps  $BMO_{c,\mathbb{B}}(\mathbb{R}^n)$  into  $BMO(\mathbb{R}^n)$ .
- (v)  $\mathcal{G}_q$  maps  $L^1_{\mathbb{B}}(\mathbb{R}^n)$  into  $L^{1,\infty}(\mathbb{R}^n)$ .

These statements are also equivalent if we replace  $\mathcal{G}_q$  by  $\mathcal{G}_q^1$  or  $\mathcal{G}_q^2$ .

The following result reduces the boundedness on  $\mathcal{G}_q$ ,  $\mathcal{G}_q^1$  and  $\mathcal{G}_q^2$  to that on one of them.

**Proposition 4.3.** *Let  $\mathbb{B}$  be a Banach space and  $p, q \in (1, \infty)$ . Then for any  $f \in L^p_{\mathbb{B}}(\mathbb{R}^n)$*

$$\|\mathcal{G}_q^1(f)\|_{L^p(\mathbb{R}^n)} \approx \|\mathcal{G}_q^2(f)\|_{L^p(\mathbb{R}^n)},$$

where the equivalence constants depend only on  $p, q$  and  $n$ .

**Proof.** Set, for simplicity,

$$\partial = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i} \quad \text{and} \quad \Phi_t = t \partial P_t, \quad \Phi_t^i = t \partial_i P_t, \quad i = 1, \dots, n.$$

Given  $t = t_1 + t_2$  we have

$$P_t * f = P_{t_1} * P_{t_2} * f.$$

Differentiating the two sides first in  $t_2$  and then in  $x_i$ , we get

$$t^2 \partial_i \partial P_{2t} * f = \Phi_t^i * \Phi_t * f. \tag{4.1}$$

(Here  $\partial_i \partial P_{2t} * f = [\partial_{x_i} \partial_s (P_s * f(x))] |_{s=2t}$ .) We use again the singular integral theory. Let  $\mathbb{A} = L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})$ . Given  $x \in \mathbb{R}^n$  let  $k(x) : \mathbb{A} \rightarrow \mathbb{A}$  be the operator defined by  $k(x)(\varphi)(t) = \Phi_t^i(x) \cdot \varphi(t)$  for  $\varphi \in \mathbb{A}$ . It is easy to check that  $(x, y) \mapsto k(x - y)$  is a Calderón–Zygmund kernel (satisfying additionally the condition in Theorem 4.1). Let  $T$  be the associated operator. We claim that  $T$  is bounded on  $L^q_{\mathbb{A}}(\mathbb{R}^n)$ . Indeed, fix  $h \in L^q_{\mathbb{A}}(\mathbb{R}^n)$ . Note that  $h$  can be regarded as a function of two variables  $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ . Then

$$\|T(h)\|_{L^q_{\mathbb{A}}(\mathbb{R}^n)}^q = \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} \left\| \int_{\mathbb{R}^n} \Phi_t^i(y) h(x - y, t) dy \right\|^q \frac{dt}{t} dx.$$

However,

$$\begin{aligned} & \left( \int_{\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} \Phi_t^i(y) h(x - y, t) dy \right\|^q dx \right)^{\frac{1}{q}} \\ & \leq \int_{\mathbb{R}^n} |\Phi_t^i(y)| \left( \int_{\mathbb{R}^n} \|h(x - y, t)\|^q dx \right)^{\frac{1}{q}} dy \\ & \leq C \left( \int_{\mathbb{R}^n} \|h(x, t)\|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

Therefore,

$$\|T(h)\|_{L^q_{\mathbb{A}}(\mathbb{R}^n)} \leq C \|h\|_{L^q_{\mathbb{A}}(\mathbb{R}^n)}.$$

This gives our claim. Then by Theorem 4.1, we deduce that  $T$  is bounded on  $L^p_{\mathbb{A}}(\mathbb{R}^n)$  for all  $p \in (1, \infty)$ .

Applying this boundedness of  $T$  to  $h(x, t) = \Phi_t * f(x)$  and using (4.1), we get

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+} \|t^2 \partial_i \partial P_t * f(x)\|^q \frac{dt}{t} \right)^{p/q} dx \leq C^p \|\mathcal{G}_q^1(f)\|_{L^p(\mathbb{R}^n)}^p.$$

Assume, without loss of generality, that  $f$  is good enough such that  $\partial_i P_t * f(x) \rightarrow 0$  as  $t \rightarrow \infty$  (for instance,  $f$  is compactly supported). Then

$$\partial_i P_t * f(x) = - \int_t^\infty \partial \partial_i P_s * f(x) ds.$$



Therefore

$$\begin{aligned} \int_0^\infty \|t \partial_i P_t * f(x)\|^q \frac{dt}{t} &\leq \int_0^\infty t^q \left( \int_t^\infty s \|\partial \partial_i P_s * f(x)\| \frac{ds}{s} \right)^q \frac{dt}{t} \\ &\leq C \int_0^\infty t^{q/2} \int_t^\infty s^{3q/2} \|\partial \partial_i P_s * f(x)\|^q \frac{ds}{s} \frac{dt}{t} \\ &= C' \int_0^\infty s^{2q} \|\partial \partial_i P_s * f(x)\|^q \frac{ds}{s}. \end{aligned}$$

It then follows that

$$\begin{aligned} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+} \|t \partial_i P_t * f(x)\|^q \frac{dt}{t} \right)^{p/q} dx \\ \leq C^p \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+} \|t^2 \partial_i \partial P_t * f(x)\|^q \frac{dt}{t} \right)^{p/q} dx. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+} \|t \partial_i P_t * f(x)\|^q \frac{dt}{t} \right)^{p/q} dx \leq C^p \|\mathcal{G}_q^1(f)\|_{L^p(\mathbb{R}^n)}^p.$$

Adding the  $n$  inequalities so obtained over  $i = 1, \dots, n$ , we get

$$\|\mathcal{G}_q^2(f)\|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n} \|\mathcal{G}_q^1(f)\|_{L^p(\mathbb{R}^n)}.$$

To prove the reverse inequality, we first observe that the same arguments as above show that

$$\begin{aligned} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+} \|t^2 \partial_i^2 P_t * f(x)\|^q \frac{dt}{t} \right)^{p/q} dx \\ \leq C^p \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+} \|t \partial_i P_t * f(x)\|^q \frac{dt}{t} \right)^{p/q} dx. \end{aligned}$$

Then using the formula

$$\partial^2 P_t * f(x) = - \sum_{i=1}^n \partial_i^2 P_t * f(x),$$

we get

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+} \|t^2 \partial^2 P_t * f(x)\|^q \frac{dt}{t} \right)^{p/q} dx \leq C^p \|\mathcal{G}_q^2(f)\|_{L^p(\mathbb{R}^n)}^p.$$

Finally, using the arguments in the second part of the above proof, we can go down to  $\partial P_t * f(x)$  to have

$$\|\mathcal{G}_q^1(f)\|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n} \|\mathcal{G}_q^2(f)\|_{L^p(\mathbb{R}^n)}.$$

This completes the proof of the proposition.  $\square$

**Remark.** The equivalence in Proposition 4.3 still holds when  $L^p$  is replaced by  $L^{1,\infty}$ .

The last result of this section is a duality theorem for the boundedness of the  $g$ -functions (Theorem 4.5 below). This theorem is a particular case of Theorem 3.1. It is also the analogue for  $\mathbb{R}^n$  of [Xu, Theorem 2.4] (for the torus). As in such a situation the key is again the existence of a certain projection. For the reader interested only in the  $\mathbb{R}^n$  case, we include a proof for this latter fact, which is much simpler than that of Theorem 3.2. Fix a Banach space  $\mathbb{B}$  and  $q \in (1, \infty)$ , and we keep the notation used in the previous proof with  $\mathbb{A} = L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})$ . The projection in question is defined as (recalling that  $\Phi_t = t\partial P_t$ )

$$Q(h)(x) = \int_0^\infty \Phi_t * h(\cdot, t)(x) \frac{dt}{t}, \quad x \in \mathbb{R}^n.$$

Note that  $Q(h)$  is well defined for functions  $h$  in a dense family of  $L^p_{\mathbb{A}}(\mathbb{R}^n)$ , for instance, for those which are compactly supported continuous functions on  $\mathbb{R}^n \times \mathbb{R}_+$ .

The proof of the following lemma is an adaption for  $\mathbb{R}^n$  of [Xu, Lemma 2.3], so we are rather sketchy.

**Lemma 4.4.** *Let  $\mathbb{B}$  be a Banach space and  $p, q \in (1, \infty)$ . Let  $Q$  be defined as before. Then for any  $\mathbb{B}$ -valued continuous function  $h$  with compact support in  $\mathbb{R}^n \times \mathbb{R}_+$*

$$\|\mathcal{G}_q^1(Q(h))\|_{L^p(\mathbb{R}^n)} \leq C_{p,q} \|h\|_{L^p_{\mathbb{A}}(\mathbb{R}^n)}.$$

Consequently, the map  $h \mapsto \mathcal{G}_q^1(Q(h))$  extends to a bounded map from  $L^p_{\mathbb{A}}(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$ .

**Proof.** Set  $f = Q(h)$ . Then

$$\begin{aligned} s\Phi_s * f &= s \int_0^\infty \Phi_s * \Phi_t * h(\cdot, t)(x) \frac{dt}{t} \\ &= st \int_0^\infty \partial^2 P_{s+t} * h(\cdot, t)(x) \frac{dt}{t} = \int_0^\infty k_{s,t} * h(\cdot, t)(x) \frac{dt}{t}, \end{aligned}$$

where

$$k_{s,t} = st \partial^2 P_{s+t} = st \frac{\partial^2}{\partial u^2} P_u \Big|_{u=s+t} .$$

Now consider the operator  $K(x) : \mathbb{A} \rightarrow \mathbb{A}$  defined by

$$K(x)(\varphi)(s) = \int_0^\infty k_{s,t}(x) \varphi(t) \frac{dt}{t}$$

for every  $\varphi \in \mathbb{A}$ . Using the inequality

$$|k_{s,t}(x)| \leq C \frac{st}{(|x| + s + t)^{(n+2)}}$$

and a similar one for the derivative of  $k_{s,t}(x)$  in  $x$ , one can easily check that for any  $x \in \mathbb{R}^n \setminus \{0\}$ ,  $K(x)$  is bounded and

$$\|K(x)\| \leq \frac{C}{|x|^n}, \quad \|\nabla K(x)\| \leq \frac{C}{|x|^{n+1}} .$$

Thus  $(x, y) \mapsto K(x - y)$  is a Calderón–Zygmund kernel. Hence to prove the lemma it suffices to show that the singular integral operator  $h \mapsto K * h$  is bounded on  $L^q_{\mathbb{A}}(\mathbb{R}^n)$ , in virtue of Theorem 4.1. This is easily done as follows. For  $x \in \mathbb{R}^n$  and  $s \in (0, \infty)$  we have

$$\begin{aligned} \|K * h(x, s)\| &\leq \left( \int_0^\infty \int_{\mathbb{R}^n} |k_{s,t}(x - y)| dy \frac{dt}{t} \right)^{1/q'} \\ &\quad \times \left( \int_0^\infty \int_{\mathbb{R}^n} |k_{s,t}(x - y)| \|h(y, t)\|^q dy \frac{dt}{t} \right)^{1/q} \\ &\leq C \left( \int_0^\infty \int_{\mathbb{R}^n} |k_{s,t}(x - y)| \|h(y, t)\|^q dy \frac{dt}{t} \right)^{1/q} . \end{aligned}$$

Therefore,

$$\begin{aligned} \|K * h\|^q_{L^q_{\mathbb{A}}(\mathbb{R}^n)} &\leq C^q \int_{\mathbb{R}^n} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} |k_{s,t}(x - y)| \|h(y, t)\|^q dy \frac{dt}{t} \frac{ds}{s} dx \\ &\leq C^q \int_{\mathbb{R}^n} \int_0^\infty \left[ \int_{\mathbb{R}^n} \int_0^\infty \frac{st}{(|x - y| + s + t)^{(n+2)} s} ds dx \right] \\ &\quad \times \|h(y, t)\|^q \frac{dt}{t} dy \\ &\leq C^q_{q,n} \|h\|^q_{L^q_{\mathbb{A}}(\mathbb{R}^n)} . \end{aligned}$$

This implies the desired boundedness of the singular integral on  $L^q_{\mathbb{A}}(\mathbb{R}^n)$ .  $\square$

**Remark.** Lemma 4.4 holds as well for  $\mathcal{G}_q$  and  $\mathcal{G}_q^2$  instead of  $\mathcal{G}_q^1$ . Moreover, the weak type (1, 1) inequality is true too.

From Lemma 4.4 and using the arguments in the proof of Theorem 3.1, we deduce the following

**Theorem 4.5.** *Let  $\mathbb{B}$  be a Banach space and  $q \in (1, \infty)$ . Then the following statements are equivalent:*

- (i) *One of the statements in Corollary 4.2 holds.*
- (ii) *For every  $p \in (1, \infty)$  (equivalently for some  $p \in (1, \infty)$ ) there is a constant  $C$  such that*

$$\|f\|_{L^p_{\mathbb{B}^*}(\mathbb{R}^n)} \leq C \|\mathcal{G}_q^1\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p_{\mathbb{B}^*}(\mathbb{R}^n).$$

### 5. Poisson semigroup on $\mathbb{R}^n$ continued

Our aim in this section is proving that in the definition of the Lusin type or cotype the  $G_q$ -function on the torus can be replaced by that on  $\mathbb{R}^n$ . This, together with [Xu], will imply the validity of (ii)  $\Rightarrow$  (i) in both Theorems 2.1 and 2.2 for the particular case of the Poisson semigroup on  $\mathbb{R}^n$ . This is done by a careful analysis of the Poisson kernels on  $\mathbb{T}$  and on  $\mathbb{R}$  and a comparison of its essential parts.

We shall also need a lemma, similar to Lemma 2.6, for the Poisson kernel on  $\mathbb{R}$ . We leave its elementary proof to the reader.

**Lemma 5.1.** *Let  $\mathbb{B}$ ,  $p$ ,  $q$  and  $\delta$  be as in the previous lemma. Then for any  $f \in L^p_{\mathbb{B}}(\mathbb{R})$*

$$\left\| \left[ t \frac{\partial P_t}{\partial t} \chi_{(0,\delta)}(t) \chi_{(\delta,\infty)}(|x|) \right] * f \right\|_{L^p_{L^q_{\mathbb{B}}((0,\infty), \frac{dt}{t})}(\mathbb{R})} \leq C_{\delta} \|f\|_{L^p_{\mathbb{B}}(\mathbb{R})}.$$

Now we state our result on the Lusin cotype for the Poisson semigroup on  $\mathbb{R}^n$ .

**Theorem 5.2.** *Let  $\mathbb{B}$  be a Banach space and  $2 \leq q < \infty$ . Then the following statements are equivalent:*

- (i)  *$\mathbb{B}$  is of Lusin cotype  $q$ .*
- (ii) *For every (or equivalently, for some) positive integer  $n$  and for every (or equivalently, for some)  $p \in (1, \infty)$  there is a constant  $C > 0$*

$$\|\mathcal{G}_q(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n).$$

(iii) For every (or equivalently, for some) positive integer  $n$  there is a constant  $C > 0$  such that

$$\|\mathcal{G}_q(f)\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)} \quad \forall f \in L^1_{\mathbb{B}}(\mathbb{R}^n).$$

The same equivalences hold with  $\mathcal{G}_q^1$  or  $\mathcal{G}_q^2$  instead of  $\mathcal{G}_q$  in (ii) and (iii).

**Proof.** In virtue of Proposition 4.3, we need only to prove the theorem for  $\mathcal{G}_q^1$ .

(i)  $\Rightarrow$  (ii). This is a particular case of Theorem 2.1.

(ii)  $\Leftrightarrow$  (iii). This equivalence for a given integer  $n$  is already contained in Corollary 4.2.

(ii) for  $n > 1 \Rightarrow$  (ii) for  $n = 1$ . By Corollary 4.2, it is enough to get the boundedness from  $L^\infty_{c,\mathbb{B}}$  into BMO of  $\mathcal{G}_q^1$  on  $\mathbb{R}$  from the same boundedness property of  $\mathcal{G}_q^1$  on  $\mathbb{R}^n$ . To this end, consider  $\tilde{x} = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ , and  $h \in L^\infty_{c,\mathbb{B}}(\mathbb{R})$ , and define  $f(x) = h(x_1)\chi_{[0,1]^{n-1}}(\tilde{x})$ , where  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ . The symmetric diffusion semigroup generated by the Laplacian on  $\mathbb{R}^n$  is given by convolution with the Gaussian density. Then we have

$$\begin{aligned} \mathcal{T}_t f(x) &= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} f(y) dy \\ &= C_0 \int_{\mathbb{R}} \frac{1}{(4\pi t)^{1/2}} e^{-\frac{|x_1-y_1|^2}{4t}} h(y_1) dy_1 = C_0 \mathcal{T}_t^1 h(x_1), \end{aligned}$$

where  $\mathcal{T}_t^1$  is the heat kernel in  $\mathbb{R}$ . If we denote by  $\mathcal{P}_t^1$  the Poisson semigroup subordinated to  $\mathcal{T}_t^1$  on  $\mathbb{R}$  and by  $P_t^1$  the Poisson kernel on  $\mathbb{R}$ , the formula (1.6) implies that  $P_t * f(x) = \mathcal{P}_t f(x) = C_0 \mathcal{P}_t^1 h(x_1) = C_0 P_t^1 * h(x_1)$ , and therefore  $\mathcal{G}_q^1 f(x) = C_0 \mathcal{G}_q^1 h(x_1)$ . Now, for every interval  $I \subset \mathbb{R}$  consider  $Q = I^n$  the cube in  $\mathbb{R}^n$  whose sides are the interval  $I$ . Then,

$$\frac{1}{|Q|} \int_Q \mathcal{G}_q^1 f(x) dx = \frac{1}{|I|^n} \int_{I^n} C_0 \mathcal{G}_q^1 h(x_1) dx_1 \dots dx_n = \frac{C_0}{|I|} \int_I \mathcal{G}_q^1 h(x_1) dx_1.$$

Therefore, and also by using similar arguments,

$$\frac{1}{|Q|} \int_Q \left| \mathcal{G}_q^1 f(x) - (\mathcal{G}_q^1 f)_Q \right| dx = \frac{C_0}{|I|} \int_I \left| \mathcal{G}_q^1 h(x_1) - (\mathcal{G}_q^1 h)_I \right| dx_1.$$

Hence,

$$\|\mathcal{G}_q^1 h\|_{\mathbf{BMO}(\mathbb{R})} = \frac{1}{C_0} \|\mathcal{G}_q^1 f\|_{\mathbf{BMO}(\mathbb{R}^n)} \leq C \|f\|_{L^\infty_{\mathbb{B}}(\mathbb{R}^n)} = C \|h\|_{L^\infty_{\mathbb{B}}(\mathbb{R})}.$$

(ii) for  $n = 1 \Rightarrow$  (i). By Corollary 4.2 and the corresponding result in [Xu] for the torus, we need to prove that for every  $f \in L^q_{\mathbb{B}}(\mathbb{T})$

$$\|G_q^1(f)\|_{L^q(\mathbb{T})} \leq C \|f\|_{L^q_{\mathbb{B}}(\mathbb{T})}.$$

(Note that we take  $p = q$  here; also recall that  $G_q^1$  is the  $G_q$ -function on the torus relative to the derivative in the radius.) But by Lemma 2.6 it is enough to show that for  $\delta > 0$  very close to 1

$$\left\| \left[ (1-r) \frac{\partial P_r(\varphi)}{\partial r} \chi_{(1-\delta,1)}(r) \chi_{(-\delta,\delta)}(\varphi) \right] * f \right\|_{L^q_{L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})}(\mathbb{T})} \leq C \|f\|_{L^q_{\mathbb{B}}(\mathbb{T})}. \tag{5.1}$$

By the change of variables  $r = e^{-t}$ , we have

$$\frac{\partial P_r(\varphi)}{\partial r} \chi_{(1-\delta,1)}(r) = \frac{2(1 - e^{-t})^2 - 4(1 + e^{-2t}) \sin^2(\varphi/2)}{[(1 - e^{-t})^2 + 4e^{-t} \sin^2(\varphi/2)]^2} \equiv k_t(\varphi).$$

Thus (5.1) is reduced to

$$\int_{\mathbb{T}} \int_0^\varepsilon \left\| t \int_{-\delta}^\delta k_t(\varphi) f(\theta - \varphi) d\varphi \right\|^q \frac{dt}{t} d\theta \leq C^q \|f\|_{L^q_{\mathbb{B}}(\mathbb{T})}^q, \tag{5.2}$$

where  $\varepsilon = \log \frac{1}{1-\delta}$ . It is elementary to decompose  $k_t(\varphi)$  as follows

$$k_t(\varphi) = k_t^0(\varphi) + k_t^1(\varphi) + k_t^2(\varphi),$$

where

$$k_t^0(\varphi) = 2 \frac{t^2 - \varphi^2}{(t^2 + \varphi^2)^2} \chi_{(0,\varepsilon)}(t) \chi_{(-\delta,\delta)}(\varphi) \tag{5.3}$$

and where  $k_t^1(\varphi)$  and  $k_t^2(\varphi)$  are supported on  $(0, \varepsilon) \times (-\delta, \delta)$  and satisfy

$$|k_t^1(\varphi)| \leq C_\delta \frac{t}{t^2 + \varphi^2}, \quad |k_t^2(\varphi)| \leq C_\delta.$$

The verification of this decomposition, though entirely elementary, could be tedious. One way to do this is to replace each time only one term of  $e^{-t}$  and  $\sin(\varphi/2)$  by their respective equivalents  $1 - t$  and  $\varphi/2$  in  $k_t(\varphi)$  and in the functions so successively

obtained. At each stage the difference between the old and new functions is of type  $k_t^1(\varphi)$  when  $e^{-t}$  is replaced or of type  $k_t^2(\varphi)$  when  $\sin(\varphi/2)$  is replaced.

It is evident that

$$\int_{\mathbb{T}} \int_0^\varepsilon \left\| t \int_{-\delta}^\delta k_t^2(\varphi) f(\theta - \varphi) d\varphi \right\|^q \frac{dt}{t} d\theta \leq C_\delta^q \|f\|_{L_{\mathbb{B}}^q(\mathbb{T})}^q.$$

It is also easy to get such an inequality for  $k_t^1(\varphi)$ . Indeed, we have

$$\begin{aligned} & \int_{\mathbb{T}} \int_0^\varepsilon \left\| t \int_{-\delta}^\delta k_t^1(\varphi) f(\theta - \varphi) d\varphi \right\|^q \frac{dt}{t} d\theta \\ & \leq C_\delta^q \int_0^\varepsilon t^q \left[ \int_{-\delta}^\delta \frac{t}{t^2 + \varphi^2} \|f\|_{L_{\mathbb{B}}^q(\mathbb{T})} d\varphi \right]^q \frac{dt}{t} \\ & \leq C_{q,\delta}^q \|f\|_{L_{\mathbb{B}}^q(\mathbb{T})}^q. \end{aligned}$$

Therefore, (5.2) is reduced to

$$\int_{\mathbb{T}} \int_0^\varepsilon \left\| t \int_{-\delta}^\delta k_t^0(\varphi) f(\theta - \varphi) d\varphi \right\|^q \frac{dt}{t} d\theta \leq C^q \|f\|_{L_{\mathbb{B}}^q(\mathbb{T})}^q. \tag{5.4}$$

Now we use the Poisson kernel  $P_t$  on  $\mathbb{R}$ . By the definition of  $k_t^0$  in (5.3),

$$k_t^0(x) = \frac{1}{2} \frac{\partial P_t(x)}{\partial t} \chi_{(0,\varepsilon)}(t) \chi_{(-\delta,\delta)}(x).$$

Put  $\tilde{f}(x) = f(x) \chi_{(-\pi,\pi)}(x)$  for  $x \in \mathbb{R}$ . Then by Lemma 5.1, we see that (5.4) is further reduced to

$$\int_{\mathbb{R}} \int_0^\varepsilon \left\| t \frac{\partial P_t}{\partial t} * \tilde{f}(x) \right\|^q \frac{dt}{t} dx \leq C^q \|\tilde{f}\|_{L_{\mathbb{B}}^q(\mathbb{R})}^q.$$

This last inequality follows from hypothesis (iii). Therefore,  $\mathbb{B}$  is of Lusin cotype  $q$ , and thus the theorem is proved.  $\square$

The following is the dual version of Theorem 5.2.

**Theorem 5.3.** *Let  $\mathbb{B}$  be a Banach space and  $1 < q \leq 2$ . Then the following statements are equivalent:*

- (i)  $\mathbb{B}$  is of Lusin type  $q$ .

(ii) For every (or equivalently, for some)  $n \geq 1$  and for every (or equivalently, for some)  $p \in (1, \infty)$  there is a constant  $C > 0$

$$\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leq C \|\mathcal{G}_q(f)\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n).$$

The same equivalence holds with  $\mathcal{G}_q^1$  or  $\mathcal{G}_q^2$  instead of  $\mathcal{G}_q$  in (ii).

**Proof.** (i)  $\Rightarrow$  (ii) is a particular case of Theorem 2.2. (ii)  $\Rightarrow$  (i) is done by duality in virtue of Theorems 4.5 and 5.3.  $\square$

### 6. Ornstein–Uhlenbeck semigroup

Our purpose of this section is to extend the results in the previous one to the Poisson semigroup subordinated to the Ornstein–Uhlenbeck semigroup on  $\mathbb{R}^n$ . Recall that this latter semigroup is defined by

$$\mathcal{O}_t f(x) = \frac{1}{(\pi(1 - e^{-2t}))^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|e^{-t}x - y|}{1 - e^{-2t}}\right] f(y) dy.$$

We denote by  $\{\mathcal{O}_t\}_{t \geq 0}$  the Poisson semigroup subordinated to  $\{\mathcal{O}_t\}_{t \geq 0}$  as defined in (1.6).

Let  $\mathbb{B}$  be a Banach space and  $1 < q < \infty$ . As for the usual Poisson semigroup on  $\mathbb{R}^n$ , we introduce the Littlewood–Paley  $g$ -function associated to  $\{\mathcal{O}_t\}_{t \geq 0}$ :

$$g_q(f)(x) = \left( \int_0^\infty t^q \|\nabla \mathcal{O}_t f(x)\|_{\ell^2_{\mathbb{B}}}^q \frac{dt}{t} \right)^{1/q}, \quad x \in \mathbb{R}^n.$$

Here  $\nabla$  still denotes the gradient in  $\mathbb{R}^n \times \mathbb{R}_+$ . We shall also consider its two variants corresponding to the time derivative and the space variable gradient, respectively:

$$g_q^1(f)(x) = \left( \int_0^\infty t^q \left\| \frac{\partial \mathcal{O}_t f}{\partial t}(x) \right\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q}$$

and

$$g_q^2(f)(x) = \left( \int_0^\infty t^q \|\nabla_x \mathcal{O}_t f(x)\|_{\ell^2_{\mathbb{B}}}^q \frac{dt}{t} \right)^{1/q}.$$

The following is the analogue of Theorem 5.2 for the Ornstein–Uhlenbeck semigroup.  $\gamma_n$  stands for the Gaussian measure on  $\mathbb{R}^n$ , i.e.,  $\gamma_n = \exp(-|x|^2) dx$ .

**Theorem 6.1.** Let  $\mathbb{B}$  be a Banach space and  $2 \leq q < \infty$ . Then the following statements are equivalent:



- (i)  $\mathbb{B}$  is of *Lusin cotype*  $q$ .
- (ii) For every (or equivalently, for some) positive integer  $n$  and for every (or equivalently, for some)  $p \in (1, \infty)$  there is a constant  $C > 0$

$$\|g_q(f)\|_{L^p(\mathbb{R}^n, \gamma_n)} \leq C \|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n, \gamma_n)} \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n, \gamma_n).$$

- (iii) For every (or equivalently, for some) positive integer  $n$  there is a constant  $C > 0$  such that

$$\|g_q(f)\|_{L^{1,\infty}(\mathbb{R}^n, \gamma_n)} \leq C \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n, \gamma_n)} \quad \forall f \in L^1_{\mathbb{B}}(\mathbb{R}^n, \gamma_n).$$

The same equivalences hold with  $g_q^1$  or  $g_q^2$  instead of  $g_q$  in (ii) and (iii).

We also have a similar result for *Lusin type*.

**Theorem 6.2.** *Let  $\mathbb{B}$  be a Banach space and  $1 < q \leq 2$ . Then the following statements are equivalent:*

- (i)  $\mathbb{B}$  is of *Lusin type*  $q$ .
- (ii) For every (or equivalently, for some)  $n \geq 1$  and for every (or equivalently, for some)  $p \in (1, \infty)$  there is a constant  $C > 0$

$$\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n, \gamma_n)} \leq C \left( \left\| \int_{\mathbb{R}^n} f \, d\gamma_n \right\|_{\mathbb{B}} + \|g_q(f)\|_{L^p(\mathbb{R}^n, \gamma_n)} \right) \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n, \gamma_n).$$

The same equivalence holds with  $g_q^1$  or  $g_q^2$  instead of  $g_q$  in (ii).

The proofs of the theorems above can be reduced to those on the usual Poisson semigroup on  $\mathbb{R}^n$  already considered in the previous section. The usual technique dealing with operators related to the Ornstein–Uhlenbeck semigroup consists in decomposing  $\mathbb{R}^n$  into two regions: one where the Gaussian and Lebesgue’s measure are equivalent, and the corresponding operators comparable, and the other where the kernels of the operators can be estimated by a well-behaved positive kernel. This technique was invented by Muckenhoupt in the one-dimensional case, and extended by Sjögren to higher dimensions, for the maximal operator. For vector-valued functions, the technique has been developed in [HTV], see also the references therein. Following this, for the  $g$ -function operator, define the domains in  $\mathbb{R}^n \times \mathbb{R}^n$ :

$$D_1 = \left\{ (x, y) : |x - y| < \frac{n(n + 3)}{1 + |x| + |y|} \right\}$$

and

$$D_2 = \left\{ (x, y) : |x - y| < \frac{2n(n + 3)}{1 + |x| + |y|} \right\}.$$

Let  $\varphi$  be a smooth function on  $\mathbb{R}^n \times \mathbb{R}^n$  which is supported on  $D_2$ , equal to 1 on  $D_1$  and satisfies

$$\|\nabla_x \varphi(x, y)\| + \|\nabla_y \varphi(x, y)\| \leq C|x - y|^{-1}.$$

Let  $T$  be a Calderón–Zygmund singular integral operator on  $\mathbb{R}^n$  with kernel  $k(x, y)$  as described at the beginning of Section 4 (and satisfying conditions (a) and (b) there). We decompose  $T$  into its local and global parts

$$T_{\text{glob}}f(x) = \int k(x, y)[1 - \varphi(x, y)]f(y)dy \quad \text{and} \quad T_{\text{loc}} = T - T_{\text{glob}}.$$

Now we can apply this decomposition to our favorite Littlewood–Paley  $g$ -functions. We get the corresponding operators  $g_{q,\text{loc}}, g_{q,\text{glob}} \dots$  for the subordinated Poisson Ornstein–Uhlenbeck semigroup, and  $\mathcal{G}_{q,\text{loc}}, \mathcal{G}_{q,\text{glob}} \dots$  for the usual Poisson semigroup. The proofs of Theorems 6.1 and 6.2 are sketchy since the estimates needed are rather technical and can be obtained in a parallel way as done in [HTV].

**Proofs of Theorems 6.1 and 6.2.** We shall use the following known facts from [HTV]

- (a)  $g_{q,\text{glob}}f(x) \leq \int_{\mathbb{R}^n} Q_1(x, y)\|f(y)\|_{\mathbb{B}} dy$ , where  $Q_1$  is a non-negative kernel supported on  $D_1^c$  such that the associated integral operator is of weak type  $(1, 1)$  and of strong type  $(p, p)$  for every  $p \in (1, \infty)$  with respect to the Gaussian measure;
- (b)  $|g_{q,\text{loc}}f(x) - \mathcal{G}_{q,\text{loc}}f(x)| \leq \int_{\mathbb{R}^n} Q_2(x, y)\|f(y)\|_{\mathbb{B}} dy$ , where  $Q_2$  is a non-negative kernel supported on  $D_2$  such that

$$\sup_x \int_{\mathbb{R}^n} Q_2(x, y)dy < \infty \quad \text{and} \quad \sup_y \int_{\mathbb{R}^n} Q_2(x, y)dx < \infty.$$

Consequently, the integral operator associated to  $Q_2$  is of strong type  $(p, p)$  for every  $p \in [1, \infty)$  with respect to both Lebesgue and Gaussian measures;

- (c) similar statements hold for  $g_q^1$  and  $g_q^2$  in place of  $g_q$ .

Then, using Theorem 5.2, we can show Theorem 6.1 as in [HTV]. We omit the details.

Theorem 6.2 is dual to Theorem 6.1 in the case of  $g_q^1$ , because of the general Theorem 3.1. Similar duality results hold for  $g_q$  and  $g_q^2$  too. Indeed, using the facts above, we get a projection result (concerning  $g_q$  and  $g_q^2$ ) for the subordinated Ornstein–Uhlenbeck Poisson semigroup similar to Lemma 4.4. Then we deduce the desired duality result on  $g_q$  and  $g_q^2$ . We leave again the details to the interested reader.  $\square$

### 7. Almost sure finiteness

We have seen in the previous sections (and also in [Xu]) that the Lusin cotype property is equivalent to the boundedness of the various generalized Littlewood–Paley

$g$ -functions on  $L^p$ -spaces. The following result shows that this is still equivalent to an apparently much weaker condition on the  $g$ -functions

**Theorem 7.1.** *Given a Banach space  $\mathbb{B}$  and  $q \in [2, \infty)$ , the following statements are equivalent:*

- (i)  $\mathbb{B}$  is of Lusin cotype  $q$ .
- (ii) For any  $f \in L^1_{\mathbb{B}}(\mathbb{T})$ ,  $G_q^1 f(z) < \infty$  for almost every  $z \in \mathbb{T}$ .
- (iii) For any  $f \in L^1_{\mathbb{B}}(\mathbb{R}^n)$ ,  $\mathcal{G}_q^1 f(x) < \infty$  for almost every  $x \in \mathbb{R}^n$ .

The equivalences also hold when in statement (ii)  $G_q^1$  is replaced by  $G_q^2$  or  $G_q$ , and also in statement (iii)  $\mathcal{G}_q^1$  by  $\mathcal{G}_q^2$  or  $\mathcal{G}_q$ .

**Proof.** By [Xu], Theorem 5.2 and Corollary 4.2, we have (i)  $\Rightarrow$  (ii) and (i)  $\Rightarrow$  (iii). The two converse implications are implicitly contained in [GR, VI.2]. Let us first prove (ii) implies (i) (for  $G_q^1$ ). To this end, observe that

$$G_q^1(f)(z) = \|Tf(z)\|_{L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})} = \sup_{\varepsilon > 0} \|T^\varepsilon f(z)\|_{L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})}, \tag{7.1}$$

where  $T^\varepsilon$  is the operator that sends  $\mathbb{B}$ -valued functions to  $L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})$ -valued functions given by

$$T^\varepsilon f(z) = \left[ (1-r)\chi_{(\varepsilon, 1-\varepsilon)}(r) \frac{\partial P_r}{\partial r} \right] * f(z).$$

It is clear that  $T^\varepsilon$  is bounded from  $L^1_{\mathbb{B}}(\mathbb{T})$  to  $L^1_{L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})}(\mathbb{T})$ . Consequently, the sublinear operator  $f \mapsto \|T^\varepsilon(f)\|_{L^q_{\mathbb{B}}((0,1), \frac{dr}{1-r})}$  is continuous from  $L^1_{\mathbb{B}}(\mathbb{T})$  to  $L^0(\mathbb{T})$  (the latter space being equipped with the measure topology). By (7.1),  $G_q^1$  is the supremum of these sublinear operators, and by (ii),  $G_q^1(f) \in L^0(\mathbb{T})$  for all  $f \in L^1_{\mathbb{B}}(\mathbb{T})$ . Therefore it follows from the Banach–Steinhaus uniform continuity principle,  $G_q$  is continuous from  $L^1_{\mathbb{B}}(\mathbb{T})$  to  $L^0(\mathbb{T})$ . Next, we apply Stein’s theorem. A proof of the scalar version can be found in [GR, VI.2], and by using the ideas there, one can prove the following vector-valued version.

**Lemma 7.2.** *Let  $G$  be a locally compact group with Haar measure  $\mu$ ,  $\mathbb{B}$  be a Banach space of Rademacher type  $p_0$  and let  $T : L^p_{\mathbb{B}}(G) \rightarrow L^0(G)$  be a continuous sublinear operator invariant under left translations. Then for every compact subset  $K$  of  $G$  there exists a constant  $C_K$  such that*

$$\mu(\{x \in K : |Tf(x)| > \lambda\}) \leq C_K \left( \frac{\|f\|_{L^p_{\mathbb{B}}}}{\lambda} \right)^q$$

with  $q = \inf\{p, p_0\}$ . In particular, if the group  $G$  is compact,  $T$  is of weak type  $(p, q)$ .

Let us recall that every Banach space is of Rademacher type 1. Then,  $G_q^1$  is of weak type (1, 1), because it is clearly sublinear and it is given by a convolution, which is invariant under translations.

The proof for the implication (iii)  $\Rightarrow$  (i) is similar. Again the sublinear operator  $f \mapsto \mathcal{G}_q^1(f)$  is continuous from  $L_{\mathbb{B}}^1(\mathbb{R}^n)$  to  $L^0(\mathbb{R}^n)$ . To infer as above that it is of weak type (1,1), we use, instead of Lemma 7.2, the following

**Lemma 7.3.** *Let  $\mathbb{B}$  be a Banach space of Rademacher type  $p_0$  and  $0 < p \leq p_0$ . Then every translation and dilation invariant continuous sublinear operator  $T : L_{\mathbb{B}}^p(\mathbb{R}^n) \rightarrow L^0(\mathbb{R}^n)$  is of weak type  $(p, p)$ .*

This lemma can be proved in the same way as the corresponding result in the scalar valued case in [GR, VI.2]. We omit the details. Thus the proof of the theorem is finished.  $\square$

**Remark.** Theorem 7.1 holds also for the  $g$ -function associated to the subordinated Poisson Ornstein–Uhlenbeck semigroup.

In the same spirit, we also have a result similar to Theorem 7.1 in the case of martingales.

**Theorem 7.4.** *Given a Banach space  $\mathbb{B}$  and  $2 \leq q < \infty$ , the following statements are equivalent:*

- (i)  $\mathbb{B}$  is of martingale cotype  $q$ .
- (ii) If  $f$  is a martingale bounded in  $L_{\mathbb{B}}^1$ , then  $S_q(f) < \infty$  almost everywhere.

For the proof of this theorem, we will use martingale transform operators. Let  $\mathbb{B}_1$  and  $\mathbb{B}_2$  be two Banach spaces,  $(\Omega, \mathcal{F}, P)$  be a probability space, and  $\{\mathcal{F}_n\}_{n \geq 1}$  be an increasing filtration of  $\sigma$ -subalgebras of  $\mathcal{F}$ . A *multiplying sequence*  $v = \{v_n\}_{n \geq 1}$  is a sequence of random variables on  $\Omega$  with values in  $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$  such that each  $v_n$  is  $\mathcal{F}_{n-1}$ -measurable and  $\sup_{n \geq 1} \|v_n\|_{L_{\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)}^\infty} < \infty$ . Given such a multiplying sequence, define the martingale transform operator  $T$  given by  $v$  by  $(Tf)_n = \sum_{k=1}^n v_k d_k f$  for every martingale  $f$ . It is proved in [MT] that a martingale transform operator  $T$  is of weak type (1, 1) iff it is of type  $(p, p)$  for  $1 < p < \infty$ :

$$\sup_{\lambda > 0} \lambda P\{(Tf)^* > \lambda\} \leq C \|f\|_{L_{\mathbb{B}_1}^1} \Leftrightarrow \|(Tf)^*\|_{L^p} \leq C_p \|f\|_{L_{\mathbb{B}_1}^p}, \tag{7.2}$$

where it is understood that each side must hold for all martingales  $\{f_n\}$  with respect to the fixed filtration  $\{\mathcal{F}_n\}$ . It is also proved there that, if  $T$  is a translation invariant martingale transform operator such that each term of its multiplying sequence  $\{v_k\}$  is a constant (in  $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ ) and such that

$$f^* \in L^1 \implies Tf \text{ converges a.e.}, \tag{7.3}$$

then  $T$  also verifies the inequalities in (7.2). By translation invariance of  $T$  we mean that for any  $k_0 \in \mathbb{N}$ , the sequence  $\{v_k^{k_0}\}_{k \geq 1}$ ,  $v_k^{k_0} = v_{k_0+k}$ , defines a martingale transform operator  $T_{k_0}$  such that for any martingale  $f$  bounded in  $L^1_{\mathbb{B}_1}$ ,

$$\begin{aligned} \|(Tf)_n\|_{\mathbb{B}_2} &= \left\| \sum_{k=1}^n v_k d_k f \right\|_{\mathbb{B}_2} = \left\| \sum_{k=1}^n v_{k_0+k} d_k f \right\|_{\mathbb{B}_2} \\ &= \|(T_{k_0}f)_n\|_{\mathbb{B}_2} \quad \forall n \geq 1. \end{aligned}$$

Now, let  $Q_q$  be the martingale transform operator mapping  $\mathbb{B}$ -valued martingales into  $\ell^q_{\mathbb{B}}$ -valued martingales defined by the multiplying sequence  $\{v_k\}_{k \geq 1}$  such that each  $v_k$  is the constant given by  $v_k(b) = (0, \binom{k-1}{\cdot}, 0, b, 0, \dots)$  for any  $b \in \mathbb{B}$ . Then for a  $\mathbb{B}$ -valued martingale  $f$

$$(Q_q f)_n = \sum_{k=1}^n v_k d_k f = (d_1 f, d_2 f, \dots, d_n f, 0, \dots) \in \ell^q_{\mathbb{B}},$$

$$\|(Q_q f)_n\|_{\ell^q_{\mathbb{B}}} = \left( \sum_{k=1}^n \|d_k f\|_{\mathbb{B}}^q \right)^{1/q}, \quad (Q_q f)^* = \sup_n \|(Q_q f)_n\|_{\ell^q_{\mathbb{B}}} = S_q f.$$

**Proof of Theorem 7.4.** (i)  $\Rightarrow$  (ii) is obvious. To prove the inverse, we use that (7.3) implies the inequalities (7.2), applied to  $Tf = Q_q f$ .  $Q_q$  is translation invariant and for  $f^* \in L^1$ ,

$$\|(Q_q f)_n - (Q_q f)_m\|_{\ell^q_{\mathbb{B}}} = \left( \sum_{k=m+1}^n \|d_k f\|_{\mathbb{B}}^q \right)^{1/q} \rightarrow 0 \quad \text{a.e. as } n, m \rightarrow \infty,$$

since it is the remaining of a convergent series (by (ii)).  $\square$

We end with a final remark.

**Remark.** As in [Xu] for the torus, besides the Littlewood–Paley  $g$ -function we can also consider the Lusin area function on  $\mathbb{R}^n$ . In our vector-valued setting this function is defined by

$$\mathcal{A}_q(f)(x) = \left( \int \int_{\Gamma(x)} t^q \|\nabla P_t * f(y)\|_{\ell^2_{\mathbb{B}}}^q \frac{dy dt}{t^{n+1}} \right)^{1/q},$$

where  $\Gamma(x)$  is the cone with vertex  $x$  and width 1:

$$\Gamma(x) = \{(t, y) \in \mathbb{R}_+^{n+1} : t > 0, |x - y| \leq t\}.$$

Similarly, we can as well introduce the two variants  $\mathcal{A}_q^1$  involving only the derivative in time and  $\mathcal{A}_q^2$  relative to the gradient in the space variable. As in [Xu] for the torus, all the preceding results (in Sections 4–6) are still valid with  $\mathcal{G}_q$  replaced by  $\mathcal{A}_q$ . For instance,  $\mathbb{B}$  is of Lusin cotype  $q$  iff  $\|\mathcal{A}_q(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)}$  for some (or all)  $p \in (1, \infty)$ , and iff  $f \in L^1_{\mathbb{B}}(\mathbb{R}^n) \Rightarrow \mathcal{A}_q(f) < \infty$  a.e. on  $\mathbb{R}^n$ .

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