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On Quasi-Chebyshev Subspaces of Banach Spaces

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A quasi-Chebyshev subspace of a Banach space X has been defined as one in

paper we shall give various characterizations of quasi-Chebyshev subspaces in Banach spaces. Moreover, we present a characterization of the spaces in which all closed linear subspaces are quasi-Chebyshev.

Key Words: proximinality; pseudo-Chebyshev subspace; quasi-Chebyshev subspace; reflexive Banach space; duality.

1. INTRODUCTION AND PRELIMINARIES

Let X be a (complex or real) Banach space and let W be a linear subspace of X. A point $y_0 \in W$ is said to be a best approximation for $x \in X$ if

$$\|x - y_0\| = d(x, W) = \inf\{\|x - y\| : y \in W\}$$

If each $x \in X$ has at least one best approximation in W, then W is called a proximinal subspace of X. If each $x \in X$ has a unique best approximation in W, then W is called a Chebyshev subspace of X. For $x \in X$, put

$$P_W(x) = \{y \in W : \|x - y\| = d(x, W)\}$$

It is clear that $P_W(x)$ is a bounded, closed and convex subset of X. For an arbitrary non-empty covex set A in X, we shall denote by

$$\ell(A) = \{\alpha x + (1 - \alpha)\, y : x, y \in A; \alpha \text{ is scalar}\}$$

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the linear manifold spanned by A. For every fixed $y \in A$ the set $\ell(A) - y = \{x - y : x \in \ell(A)\}$ is a linear subspace of X, satisfying $\ell(A - y) = \ell(A) - y$. The dimension of A is defined by $\dim A = \dim \ell(A)$. Then, for every $y \in A$ we have

$$\dim A = \dim \ell(A) = \dim[\ell(A) - y] = \dim \ell(A - y) = \dim(A - y)$$

(For more details see [8].)

We say that W is a pseudo-Chebyshev subspace of X if $P_W(x)$ is a non-empty and finite-dimensional set in X for each $x \in X$.

In particular, every finite-dimensional linear subspace and every k-Chebyshev subspace ($k = 0, 1, 2, ...$) is pseudo-Chebyshev(for more details see [4, 8]). In [1] P. D. Morris has constructed examples of pseudo-Chebyshev subspaces of finite-codimensional of $\ell^\infty_{\mathbf{R}}$ which are not Chebyshev subspaces. In [7] there is a characterization of the spaces in which all closed linear subspaces are pseudo-Chebyshev.

A linear subspace W of a Banach space X is called quasi-Chebyshev if $P_W(x)$ is a non-empty and compact set in X for every $x \in X$ (see [2]). In [2] it is shown that every pseudo-Chebyshev subspace is quasi-Chebyshev, and given an example in which the converse is not true. For more details about quasi-Chebyshev subspaces see [2].

Let X^* be the dual space of the Banach space X. For $f \in X^*$, put

$$M_f = \{x \in X : f(x) = \|f\|, \|x\| = 1\}$$

It is clear that M_f is a bounded and closed subset of X.

We conclude this section by a list of known lemmas needed in the proof of the main results.

LEMMA 1.1 [8, Theorem 1.1]. *Let X be a normed linear space, W a linear subspace of X, $x \in X \backslash \overline{W}$ and $y_0 \in W$. Then $y_0 \in P_W(x)$ if and only if there exists $f \in X^*$ such that $\|f\| = 1$, $f \mid W = 0$ and $f(x - y_0) = \|x - y_0\|$.*

LEMMA 1.2 [5; 8, Theorems 1 and 4]. *Let X be a normed linear space, W a linear subspace of X, $x \in X \backslash \overline{W}$ and $y_0 \in W$. Then $y_0 \in P_W(x)$ if and only if*

$$\|x - y_0\|_{W^\perp} = \|x - y_0\|$$

where $\|x\|_{W^\perp} = \sup\{|f(x)| : \|f\| \leqslant 1, f \in W^\perp\}$.

LEMMA 1.3 [6, 8]. *Let X be a normed linear space, W a linear subspace of X, $x \in X \backslash \overline{W}$ and F a subset of W. Then F is a subset of $P_W(x)$ if and only* *if there exists* $f \in X^*$ *such that* $\|f\| = 1$, $f | W = 0$ *and* $f(x - y) = \|x - y\|$ *for every* $y \in F$.

LEMMA 1.4 [2, Theorem 2.4]. *Let X be a Banach space and let W be a proximinal subspace of X. Then the following are equivalent*:

(1) *W is quasi-Chebyshev in X.*

(2) *There do not exist* $f \in X^*$, $x_0 \in X$ *and a sequence* $\{x_n\}_{n \geqslant 1}$ *in X without a convergent subsequence and with* $x_0 - x_n \in W$ $(n = 1, 2, ...)$ *such that* $\|f\| = 1$, $f | W = 0$ *and* $f(x_n) = \|x_n\|$ *for all* $n = 0, 1, 2, ...$

(3) *There do not exist* $f \in X^*$, $x_0 \in X$ *and a sequence* $\{g_n\}_{n \geqslant 1}$ *in W without a convergent subsequence such that* $\|f\| = 1$, $f | W = 0$ *and* $f(x_0) = \|x_0\| = \|x_0 - g_n\|$ *for all* $n = 1, 2, ...$

2. MAIN RESULTS

Now, we are ready to state and prove our main results. In the following we give various characterizations of quasi-Chebyshev subspaces in Banach spaces. First, we use Lemma 1.4 and give another proof of a result in [3].

THEOREM 2.1. *Let X be a Banach space and let W be a proximinal subspace of X with codimension one. Then the following are equivalent*:

(1) *W is quasi-Chebyshev in X.*

(2) *Each sequence* $\{y_n\}_{n \geqslant 1}$ *in X with* $\|y_n\| = 1$ *and* $0 \in P_W(y_n)$ $(n = 1, 2, ...)$ *has a convergent subsequence.*

Proof. See Theorem 2.5 below.

In the following, we need the following definitions.

DEFINITION 2.2. A linear subspace W of a Banach space X is said to have the property (***C***), if for every $f \in W^*$ the set

$$\mathbf{E}_f = \{\tilde{f} \in X^* : \tilde{f} \mid W = f, \|\tilde{f}\| = \|f\|\}$$

is non-empty and compact in X^*. (Note that $\mathbf{E}_f$ is convex for every $f \in W^*$.)

DEFINITION 2.3. Let X be a Banach space. A linear subspace M of X^* is said to have the property (C^*), if for every $x \in X \backslash {}^{\perp}M$ the set

$$\mathbf{D}_x = \{y \in X : f(y) = f(x) \text{ for all } f \in M;\ \|y\| = \|x\|_M\}$$

is non-empty and compact in X, where

$$^{\perp}M = \{x \in X : f(x) = 0 \text{ for all } f \in M\}$$

and

$$\|x\|_M = \sup\{|f(x)| : \|f\| \leqslant 1, f \in M\}$$

(Note that $\mathbf{D}_x$ is convex for every $x \in X \setminus {}^{\perp}M$.)

THEOREM 2.4. *Let X be a Banach space and let W be a proximinal subspace of X. Then the following are equivalent*:

(1) *W is quasi-Chebyshev in X.*
(2) *$W^{\perp}$ has the property $(\boldsymbol{C}^*)$.*

If the quotient space X/W is reflexive, then the above statements are equivalent to the following:

(3) *For every $\Lambda \in (W^{\perp})^*$ the set*

$$\mathbf{S}_{\Lambda} = \{y \in X : f(y) = \Lambda(f) \text{ for all } f \in W^{\perp};\ \|y\| = \|\Lambda\|\}$$

is non-empty and compact in X. (Note that $\mathbf{S}_{\Lambda}$ is convex for every $\Lambda \in (W^{\perp})^*$. Also, (3) implies (1) and (2) without the reflexivity of X/W.)

Proof. $(1) \Rightarrow (2)$. Suppose that (2) does not hold. Since W is proximinal in X, by [8; Theorem 2.1] $\mathbf{D}_x$ is non-empty for each $x \in X$. Then there exits $x_0 \in X \setminus W$ such that $\mathbf{D}_{x_0}$ is not compact. It follows that there exists a sequence $\{y_n\}_{n \geqslant 1}$ in X without a convergent subsequence such that

$$f(y_n) = f(x_0), \qquad n = 1, 2, \ldots;\ f \in W^{\perp},$$

and

$$\|y_n\| = \|x_0\|_{W^{\perp}}, \qquad n = 1, 2, \ldots.$$

Then we have $f(y_n - y_1) = 0$ for all $f \in W^{\perp}$ and all $n \geqslant 1$. Therefore, $y_n - y_1 \in {}^{\perp}(W^{\perp}) = W$ $(n = 1, 2, \ldots)$, because W is a closed subspace of X. Let $y_0 = y_1$ and $g_n = y_{n+1} - y_1$, $n = 1, 2, \ldots$. Thus, $y_0 \in X \setminus W$, $\{g_n\}_{n \geqslant 1}$ is a sequence in W without a convergent subsequence and

$$\begin{aligned}
\|y_0 - g_n\| = \|y_{n+1}\| &= \|x_0\|_{W^{\perp}} \\
&= \sup\{|f(x_0)| : \|f\| \leqslant 1, f \in W^{\perp}\} \\
&= \sup\{|f(y_1)| : \|f\| \leqslant 1, f \in W^{\perp}\} \\
&= \|y_0\|_{W^{\perp}} = \|y_0 - g_n\|_{W^{\perp}},
\end{aligned}$$

for all $n=1, 2, \ldots$. It follows from Lemma 1.2 that $g_n \in P_W(y_0)$ for $n=1, 2, \ldots$. Therefore, $P_W(y_0)$ is not compact and hence W is not quasi-Chebyshev in X. Thus, (1) implies (2).

(2) $\Rightarrow$ (1). Assume if possible that W is not quasi-Chebyshev in X. Since W is proximinal in X, by Lemma 1.4 (the implication (1) $\Rightarrow$ (3)) for a suitable $f_0 \in X^*$ and $x_0 \in X \backslash W$ there exists a sequence $\{g_n\}_{n \geqslant 1}$ in W without a convergent subsequence such that $\|f_0\|=1, f_0 \mid W=0$ *and* $f_0(x_0)=\|x_0\|=\|x_0-g_n\|$, $n=1, 2, \ldots$. Since $x_0 \in X \backslash W$ and

$$f_0(x_0-g_n)=f_0(x_0)=\|x_0-g_n\|,$$

for all $n=1, 2, \ldots$, it follows from Lemma 1.3 that $g_n \in P_W(x_0)$ $(n=1, 2, \ldots)$. Then, by Lemma 1.2, we have

$$\|x_0-g_n\|=\|x_0-g_n\|_{W^{\perp}},$$

for all $n=1, 2, \ldots$.

Let $y_n=x_0-g_n$, $n=1, 2, \ldots$. Therefore, $\{y_n\}_{n \geqslant 1}$ is a sequence in X without a convergent subsequence. Now, let $f \in W^{\perp}$ be arbitrary. Then we have

$$f(y_n)=f(x_0-g_n)=f(x_0), \qquad n=1, 2, \ldots,$$

and

$$\|y_n\|=\|x_0-g_n\|=\|x_0-g_n\|_{W^{\perp}}=\|x_0\|_{W^{\perp}},$$

for all $n=1, 2, \ldots$. It follows that $y_n \in \mathbf{D}_{x_0}$ for all $n \geqslant 1$. Thus, $W^{\perp}$ does not have the property $(\boldsymbol{C}^*)$. Hence, (2) implies (1).

(2) $\Rightarrow$ (3). Assume that we have (2) and that the quotient space X/W is reflexive. Now, suppose that (3) does not hold. Since W is proximinal and X/W is reflexive, by [8; Theorem 2.1] $\mathbf{S}_{\Lambda}$ is non-empty for every $\Lambda \in (W^{\perp})^*$. Then for a suitable $\Lambda_0 \in (W^{\perp})^*$, $\mathbf{S}_{\Lambda_0}$ is not compact. It follows that there exists a sequence $\{y_n\}_{n \geqslant 1}$ in X without a convergent subsequence such that

$$f(y_n)=\Lambda_0(f), \qquad n=1, 2, \ldots; f \in W^{\perp},$$

and

$$\|y_n\|=\|\Lambda_0\|, \qquad n=1, 2, \ldots.$$

Now, let $x_0=y_1 \in X$. Then we have

$$f(y_n)=f(x_0), \qquad n=1, 2, \ldots; f \in W^{\perp},$$

and

$$\begin{aligned}\|y_n\| = \|\Lambda_0\| &= \sup\{|\Lambda_0(f)| : \|f\| \leqslant 1, f \in W^{\perp}\}\\ &= \sup\{|f(y_n)| : \|f\| \leqslant 1, f \in W^{\perp}\}\\ &= \sup\{|f(x_0)| : \|f\| \leqslant 1, f \in W^{\perp}\}\\ &= \|x_0\|_{W^{\perp}},\end{aligned}$$

for all $n = 1, 2, \ldots$. It follows that $y_n \in \mathbf{D}_{x_0}$ for all $n \geqslant 1$. Then $W^{\perp}$ does not have the property $(\boldsymbol{C}^*)$. Hence, (2) implies (3).

(3) ⇒ (1). Suppose that W is not quasi-Chebyshev in X. Then by the proof (2) ⇒ (1) we can find $x_0 \in X \backslash W$ and a sequence $\{y_n\}_{n \geqslant 1}$ in X without a convergent subsequence such that

$$f(y_n) = f(x_0), \qquad n = 1, 2, \ldots; f \in W^{\perp},$$

and

$$\|y_n\| = \|x_0\|_{W^{\perp}}, \qquad n = 1, 2, \ldots.$$

Now, define $\Lambda_0 : W^{\perp} \to \mathbf{C}$ by $\Lambda_0(f) = f(x_0)$ for every $f \in W^{\perp}$. It follows that $\Lambda_0 \in (W^{\perp})^*$, $f(y_n) = \Lambda_0(f)$ $(n = 1, 2, \ldots; f \in W^{\perp})$ and $\|\Lambda_0\| = \|x_0\|_{W^{\perp}} = \|y_n\|$ fo all $n = 1, 2, \ldots$. Then $y_n \in \mathbf{S}_{\Lambda_0}$ for $n = 1, 2, \ldots$. Therefore, (3) does not hold. Hence (3) implies (1), which completes the proof. ∎

THEOREM 2.5. *Let X be a Banach space and let W be a proximinal subspace of X. Then the following are equivalent*:

(1) *W is quasi-Chebyshev in X.*

(2) *In every linear subspace $Y_x \subset X$ $(x \in X \backslash W)$ of the form $Y_x = W \oplus \langle x \rangle$ each sequence $\{y_n\}_{n \geqslant 1}$ in Y_x with $\|y_n\| = 1$ and $0 \in P_W(y_n)$ $(n = 1, 2, \ldots)$ has a convergent subsequence.*

(3) *For every linear functional $0 \neq \varphi \in (Y_x)^*$ $(x \in X \backslash W)$ with the property $W = \{y \in Y_x : \varphi(y) = 0\}$, the set $M_{\varphi} = \{y \in Y_x : \varphi(y) = \|\varphi\|, \|y\| = 1\}$ is non-empty and compact in Y_x.*

(4) *$W^{\perp}$ has the property $(\boldsymbol{C}^*)$.*

If the quotient space X/W is reflexive, then the above statements are equivalent to the following:

(5) *For every $\Lambda \in (W^{\perp})^*$ the set*

$$\mathbf{S}_{\Lambda} = \{y \in X : f(y) = \Lambda(f) \text{ for all } f \in W^{\perp}, \|y\| = \|\Lambda\|\}$$

is non-empty and compact in X.

Proof. (1)$\Rightarrow$(2). Suppose that W is quasi-Chebyshev in X. Then W is quasi-Chebyshev in every Y_x $(x \in X \backslash W)$. Since codim $W=1$ in each Y_x $(x \in X \backslash W)$, by Theorem 2.1 (the implication (1)$\Rightarrow$(2)) each sequence $\{y_n\}_{n \geqslant 1}$ in Y_x with $\|y_n\|=1$ and $0 \in P_W(y_n)$ $(n=1, 2, ...)$ has a convergent subsequence. Hence we have (2).

(2)$\Rightarrow$(1). Assume that we have (2). Then codim $W=1$ in each subspace $Y_x \subset X$ $(x \in X \backslash W)$ of the form $Y_x = W \oplus \langle x \rangle$. Since W is proximinal in each Y_x $(x \in X \backslash W)$, it follows from Theorem 2.1 (the implication (2)$\Rightarrow$(1)) that W is quasi-Chebyshev in each Y_x $(x \in X \backslash W)$. But, $X = \bigcup_{x \in X \backslash W} Y_x$. It is clear that W is quasi-Chebyshev in X, and hence we have (1).

(3)$\Rightarrow$(2). Suppose that (2) does not hold. Then for a suitable $Y_{x0} \subset X$ $(x_0 \in X \backslash W)$ of the form $Y_{x0} = W \oplus \langle x_0 \rangle$ there exists a sequence $\{y_n\}_{n \geqslant 1}$ in Y_{x0} without a convergent subsequence such that $\|y_n\|=1$ and $0 \in P_W(y_n)$ $(n=1, 2, ...)$. It follows that $y_n \in Y_{x0} \backslash W$ for all $n=1, 2, ...$ Therefore, by Lemma 1.1, for each $n=1, 2, ...$ there exists $o \neq \varphi_n \in (Y_{x0})^*$ such that $\|\varphi_n\|=1$, $\varphi_n \mid W=0$ and $\varphi_n(y_n)=\|y_n\|=1$. Let $\varphi_0=\varphi_1$ and

$$W_0=\{y \in Y_{x0} \colon \varphi_0(y)=0\}.$$

Since $\varphi_0(x_0) \neq 0$, it follows that $Y_{x0}=W_0 \oplus \langle x_0 \rangle$. But, we have $Y_{x0}=W \oplus \langle x_0 \rangle$ and W is a subset of W_0. Then $W=W_0$. Thus, we have $0 \neq \varphi_0 \in (Y_{x0})^*$ with the property $W=\{y \in Y_{x0} \colon \varphi_0(y)=0\}$. Now, since $\varphi_n \mid W=0$ $(n=1, 2, ...)$, there exists a non-zero scalar α_n such that $\varphi_n=\alpha_n \varphi_0$ $(n=1, 2, ...)$. But, we have $\|\varphi_n\|=1$ for all $n=1, 2, ...$ Then $|\alpha_n|=1$ $(n=1, 2, ...)$. We may assume without loss of generality that $\alpha_n \to \alpha_0$ for some scalar $\alpha_0 \neq 0$ $(|\alpha_0|=1)$.

Let $x_n=\alpha_n y_n$, $n=1, 2, ...$ Now, since $\alpha_n \to \alpha_0 \neq 0$, it follows that $\{x_n\}_{n \geqslant 1}$ is a sequence in Y_{x0} without a convergent subsequence, $\|x_n\|=1$ $(n=1, 2, ...)$, and

$$\varphi_0(x_n)=\varphi_0(\alpha_n y_n)=\alpha_n \varphi_0(y_n)=\varphi_n(y_n)=1=\|\varphi_0\|,$$

for all $n=1, 2, ...$

Therefore, $x_n \in M_{\varphi_0}$ $(n=1, 2, ...)$. Then M_{φ_0} is not compact and (3) does not hold. Hence, (3) implies (2).

(4)$\Rightarrow$(3). Suppose that (3) does not hold. Since W is proximinal in X, by [8; Theorem 2.1] for every $0 \neq \varphi \in (Y_x)^*$ $(x \in X \backslash W)$ with the property $W=\{y \in Y_x \colon \varphi(y)=0\}$, M_φ is non-empty. Then for a suitable $Y_{x0} \subset X$ $(x_0 \in X \backslash W)$ there exists $0 \neq \varphi_0 \in (Y_{x0})^*$ with the property $W=\{y \in Y_{x0} \colon \varphi_0(y)=0\}$, M_{φ_0} is not compact. It follows that there exists a sequence $\{y_n\}_{n \geqslant 1}$ in Y_{x0} without a convergent subsequence such that $\|y_n\|=1$ and $\varphi_0(y_n)=\|\varphi_0\|$ $(n=1, 2, ...)$.

Since $y_n \in Y_{x_0} \backslash W$ ($n=1, 2, \ldots$), for each $n=1, 2, \ldots$ there exist a non-zero scalar λ_n and an $w_n \in W$ such that $y_n = w_n + \lambda_n x_0$ (note that $Y_{x_0} = W \oplus \langle x_0 \rangle$). But, we have $\varphi_0(y_n) = \|\varphi_0\|$ ($n=1, 2, \ldots$) and $\varphi_0(x_0) \neq 0$. Then $\lambda_n = \|\varphi_0\| \, (\varphi_0(x_0))^{-1} := \lambda_0$ for all $n=1, 2, \ldots$. (Note that $\lambda_0 \neq 0$.)

Now, let $x_n = \lambda_0^{-1} y_n$, $n=1, 2, \ldots$. It follows that $\{x_n\}_{n \geqslant 1}$ is a sequence in Y_{x_0} without a convergent subsequence, $|\varphi_0(x_n)| = \|\varphi_0\| \, \|x_n\|$ and $x_n - x_0 \in W$ for all $n=1, 2, \ldots$. Let $f \in W^{\perp}$ be arbitrary. Since $x_n - x_0 \in W$ ($n=1, 2, \ldots$), $f(x_n) = f(x_0)$ and $\varphi_0(x_n) = \varphi_0(x_0)$ for all $n=1, 2, \ldots$ (note that $\varphi_0 \mid W = 0$).

Let $\psi_0 = (\|\varphi_0\|)^{-1} \varphi_0$. Then we have $\|\psi_0\| = 1$, $\psi_0 \mid W = 0$, $\psi_0(x_n) = \psi_0(x_0)$ and $|\psi_0(x_n)| = \|x_n\|$ ($n=1, 2, \ldots$). Therefore,

$$\begin{aligned} \|x_n\| &= |\psi_0(x_n)| = |\psi_0(x_0)| \\ &\leqslant \sup\{|f(x_0)| : \|f\| \leqslant 1, f \in W^{\perp}\} = \|x_0\|_{W^{\perp}}, \end{aligned}$$

for all $n=1, 2, \ldots$.

On the other hand,

$$\begin{aligned} \|x_0\|_{W^{\perp}} &= \sup\{|f(x_0)| : \|f\| \leqslant 1, f \in W^{\perp}\} \\ &= \sup\{|f(x_n)| : \|f\| \leqslant 1, f \in W^{\perp}\} \leqslant \|x_n\|, \end{aligned}$$

for all $n=1, 2, \ldots$. Then $\|x_n\| = \|x_0\|_{W^{\perp}}$, $n=1, 2, \ldots$. Since $f(x_n) = f(x_0)$, $n=1, 2, \ldots$; $f \in W^{\perp}$, it follows that $x_n \in \mathbf{D}_{x_0}$ ($n=1, 2, \ldots$) and hence $W^{\perp}$ does not have the property ($\boldsymbol{C}^*$). Thus, (4) implies (3).

The equivalences (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) have been proved in Theorem 2.4, which completes the proof. ∎

Now, we shall obtain from Theorem 2.5 the following corollary on quasi-Chebyshev subspaces of Banach spaces.

Corollary 2.6. *Let X be a reflexive Banach space. Then all closed linear subspaces of X are quasi-Chebyshev if and only if for every $f \in X^*$ and every closed linear subspace W of X with $f \mid W$ is non-zero, the set $M_g = \{y \in W : g(y) = \|g\|, \|y\| = 1\}$ is non-empty and compact, where $g = f \mid W$.*

Proof. Assume that all closed linear subspaces of X are quasi-Chebyshev. Let $f \in X^*$ be arbitrary and let W be an arbitrary closed linear subspace of X such that $f \mid W$ is non-zero. Let $g = f \mid W$. Since $g \neq 0$, there exists $x_0 \in W$ such that $g(x_0) \neq 0$.

Now, let

$$W_0 = \{y \in W : g(y) = 0\} \text{ and } Y_{x_0} = W_0 \oplus \langle x_0 \rangle.$$

Then we have $W = Y_{x_0}$ and by hypothesis W_0 is a quasi-Chebyshev subspace of X. Since $g \in W^* = (Y_{x_0})^*$ $(x_0 \in X \backslash W_0)$ with the property $W_0 = \{y \in Y_{x_0} : g(y) = 0\}$ and W_0 is quasi-Chebyshev in X, by Theorem 2.5 (the implication 1) $\Rightarrow$ 3)) the set M_g is non-empty and compact in $Y_{x_0} = W$.

Conversely, suppose that for every $f \in X^*$ and every closed linear subspace W of X with $f | W$ is non-zero, the set M_g is non-empty and compact, where $g = f | W$. Let W be an arbitrary closed linear subspace of X.

Now, let $x \in X \backslash W$ be arbitrary and $Y_x = W \oplus \langle x \rangle$. It is clear that Y_x is a closed linear subspace of X. Let $0 \neq \varphi \in (Y_x)^*$ be arbitrary with the property

$$W = \{y \in Y_x : \varphi(y) = o\}.$$

Therefore, by Hahn-Banach Theorem, there exists a linear functional $f \in X^*$ such that $0 \neq \varphi = f | Y_x$. It follows, by hypothesis, that M_φ is a non-empty and compact set in Y_x. But, we have X is reflexive. Then by [8; Corollary 2.4] W is proximinal in X. Hence by Theorem 2.5 (the implication 3) $\Rightarrow$ 1)) W is quasi-Chebyshev in X, which completes the proof. ∎

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