Subgraphs with restricted degrees of their vertices in planar graphs

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Abstract

We prove that every 3-connected planar graph $G$ of order at least $k$ contains a connected subgraph $H$ on $k$ vertices each of which has degree (in $G$) at most $4k + 3$, the bound $4k + 3$ being best possible. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

Throughout this note we consider connected graphs with neither loops nor multiple edges. We use standard terminology and notation, see, e.g. [2]. We recall, however, more specialized notions. By a plane graph or, equivalently, by a plane map we mean an embedding of a connected planar graph in the plane. The degree of a face $\alpha$ of a plane graph is the number of edges incident with $\alpha$ where each cut-edge is counted twice. Vertices and faces of degree $i$ are called $i$-vertices and $i$-faces, respectively. For a plane graph $G$ let $V(G)$, $E(G)$ and $F(G)$ be the vertex-set, the edge-set and the face-set of $G$, respectively. The degree of a vertex $A$ (a face $\alpha$) in $G$ is denoted by $\deg_G(A)$ or $\deg(A)$ ($\deg_G(\alpha)$ or $\deg(\alpha)$) if $G$ is known from the context. A path and a cycle on $k$ vertices are called a $k$-path and a $k$-cycle, respectively. A $k$-path passing through vertices $A_1, A_2, \ldots, A_k$ is denoted by $[A_1, A_2, \ldots, A_k]$ provided that $A_iA_{i+1} \in E(G)$ for any $i = 1, 2, \ldots, k - 1$. A $k$-path is denoted by $P_k$. A kletepe $K(G)$ of a 3-connected plane graph $G$ is defined to be a triangulation obtained from $G$ by placing a new vertex into each face $\alpha$ of $G$ and joining it with all vertices incident with $\alpha$. Let in
the sequel $K^{(n)}(G) = K(K^{(n-1)}(G))$, $n = 1, 2, \ldots$, be a kleetope of the graph $K^{(n-1)}(G)$, where $K^{(0)}(G) = G$.

It is an old classical consequence of Euler's famous formula that each planar graph contains a vertex of degree at most 5. A beautiful theorem of Kotzig [13,14] (see e.g. [3,6,7,9,12,15]) states that every 3-connected planar graph contains an edge with degree-sum of its endvertices being at most 13. The bound is best possible. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs, see e.g. [8,9,15]. Jendrol' [10,11] has proved results analogous to the one due to Kotzig for $k$-paths, $k = 3, 4$ and 5, in 3-connected plane graphs. On the other hand Jendrol' proved in [10] that for every pair of integers $k$ and $m$, $k \geq 3$, $m \geq 3$, among all 2-connected planar graphs having a $k$-path there exists a graph $G$ in which every $k$-path contains a vertex $A$ such that $\deg_G(A) \geq m$.

The above considerations suggest the following problems.

**Problem 1.** For a connected planar graph $H$ let $\mathcal{B}(H)$ be the family of all 3-connected planar graphs having subgraphs isomorphic to $H$. What is the minimum integer $\phi(H)$ such that every graph $G \in \mathcal{B}(H)$ contains a subgraph $K$ isomorphic to $H$ for which
$$\deg_G(A) \leq \phi(H)$$
holds for every vertex $A \in V(K)$?

The answer to this question is rather surprising: the required minimum exists only for $k$-paths. Namely, Fabrici and Jendrol' [5] have proved

**Theorem 1.** (i) $\phi(P_k) = 5k$, $k \geq 1$.
(ii) $\phi(H) = \infty$ for any $H \neq P_k$, $k \geq 1$.

If we weaken the requirement on the subgraph $H$ in Problem 1, we can formulate

**Problem 2.** Let $\mathcal{H}(k)$ be a family of all 3-connected planar graphs of order at least $k$. What is the minimum integer $\tau(k)$ such that every graph $G \in \mathcal{H}(k)$ contains a connected subgraph $H$ of order $k$ such that
$$\deg_G(A) \leq \tau(k)$$
holds for every vertex $A \in V(H)$?

Euler's formula gives $\tau(1) = 5$. Kotzig's result [13] yields $\tau(2) = 10$. By Jendrol' [10] or Theorem 1 above we have $\tau(3) = 15$.

The main result of this paper is the following

**Theorem 2.** Let $k$ be an integer, $k \geq 1$. Then
(i) $\tau(1) = 5$. 

\( \tau(2) = 10, \)
\( \tau(k) = 4k + 3 \) \text{ for any } k \geq 3. 

The proof of Theorem 2 for \( k = 1 \) and \( k = 2 \) is a consequence of Euler's formula and the Kotzig's result, respectively. In Section 2 we prove that \( \tau(k) \leq 4k + 3, \) \( k \geq 3, \) and give a construction of a graph which has property that each subgraph of order \( k \) contains a vertex of degree at least \( 4k + 3. \) The problems related to Kotzig's theorem, Theorems 1 and 2 are considered in Sections 3 and 4.

2. Proof of Theorem 2

Suppose there is \( k \geq 3 \) such that \( \tau(k) > 4k + 3. \) Let a plane graph \( G \) be a counterexample on minimum number of vertices, say \( n, \) and maximum number of edges, say \( m, \) among all counterexamples on \( n \) vertices. A vertex \( A \) is a minor vertex if \( \deg(A) < 4k + 3 \) and is a major vertex if \( \deg(A) > 4k + 3. \) Now we shall investigate properties of \( G. \) The first property is easy to see.

Property 1. \( G \) is a 3-connected plane graph such that each connected subgraph of order \( k \) in it contains a major vertex.

By a minor subgraph we mean a connected subgraph all vertices of which are minor vertices.

Property 2. \( G \) is a triangulation.

Proof. Let \( G \) contain an \( r \)-face \( \alpha, r \geq 4. \) If \( \alpha \) is incident with a major vertex \( A \) we insert a diagonal \( AB \) into \( \alpha \) where \( B \) is a vertex incident with \( \alpha \) and not adjacent with \( A. \) Because the diagonal \( AB \) cannot create a minor subgraph of order at least \( k, \) we get a counterexample with \( m + 1 \) edges, a contradiction. If \( \alpha \) is incident only with minor vertices, all these vertices belong to the same minor component and we can again add a diagonal into \( \alpha \) without loss of Property 1; a contradiction. \( \square \)

Let \( M = M(G) \) be a subgraph of \( G \) induced on all major vertices of \( G. \)

Property 3. \( M \) is a connected subgraph of \( G. \)

Proof. Suppose that \( M \) has two distinct components \( M_1 \) and \( M_2. \) Let \( G_1 \) be a component of a plane subgraph of \( G - M_1 \) which contains as a subgraph the component \( M_2. \) All faces of \( G_1 \) are 3-faces except for at most one, say \( \beta. \) Because of Property 1, \( \beta \) is an \( s \)-face with \( s \leq k - 1. \) If we split the face \( \beta \) into triangles by inserting diagonals we obtain a plane graph \( G^* \) without minor subgraphs of order at least \( k \) but with less than \( n \) vertices, a contradiction. \( \square \)
Let $V_M = V(M)$, $F_M = F(M)$ and $E_M = E(M)$ be the set of vertices, faces and edges of the plane graph $M$, respectively, and let $v = |V_M|$, $f = |F_M|$ and $e = |E_M|$ be cardinalities of these sets. Clearly

$$\sum_{x \in F_M} \deg_M(x) = 2e = \sum_{x \in V_M} \deg_M(x).$$

(1)

Euler's polyhedral formula provides

$$f \leq 2v - 4, \quad e \leq 3v - 6.$$ 

(2)

Since each face $x$ of $M$ contains in its interior $r \leq k - 1$ minor vertices of $G$, for $m_x$, the number of edges between major vertices of $x$ and minor vertices lying in $x$, we have

$$m_x \leq 2k - 4 + \deg_M(x).$$

(3)

To see this consider a triangulation obtained when taking the face $x$ together with its inside in $G$. Add outside of $x$ a new vertex $C$ and join it with all vertices of $x$. (This is possible because for our purposes we can assume that the boundary of $x$ is a cycle). The obtained triangulation has at most $r + \deg_M(x) + 1$ vertices and at most $3r + 3\deg_M(x) - 6$ edges. Hence, $m_x \leq 3r + 3\deg_M(x) - 3 - 2\deg_M(x) - (r - 1) = 2r - 2 + \deg_M(x) \leq 2k - 4 + \deg_M(x)$.

Using (1)–(3) we have

$$\sum_{X \in F_M} \deg_G(X) = \sum_{x \in F_M} \deg_M(x) + \sum_{x \in F_M} m_x = \sum_{x \in F_M} \deg_M(x) + \sum_{x \in F_M} m_x$$

$$\leq 2e + \sum_{x \in F_M} (2k - 4 + \deg_M(x)) = 2e + \sum_{x \in F_M} \deg_M(x)$$

$$+ f(2k - 4) = 2e + 2e + f(2k - 4) = 4e + f(2k - 4)$$

$$\leq 4(3v - 6) + (2v - 4)(2k - 4) = 4(v - 2)(k + 1).$$

This implies that there is a vertex $A \in V_M$ such that $\deg_G(A) \leq 4(v - 2)(k + 1)/v = 4k + 4 - 8(k + 1)/v$. Thus $\deg_G(A) \leq 4k + 3$, a contradiction because $A$ is a major vertex.

To prove the lower bound it is enough to show an example of a 3-connected plane graph in which each connected subgraph of order $k$ has a vertex of degree at least $4k + 3$. We show that a graph of Enomoto and Ota [4] constructed for a different purpose has the required property. For the sake of completeness we recall the construction here.

Suppose that $m \gg k \geq 4$. Let $W$ be a triangulation obtained from two copies of the wheel of order $m + 1$ by joining edges from each vertex on the rim of one wheel to the corresponding vertex of the other, and to its successor. Put one vertex into each edge of $W$ and join two new vertices whenever the corresponding edges share a face of $W$. Let $\tilde{W}$ be the resulting graph.
Now we embed into each face of $\tilde{W}$ one of two subgraphs on $k - 1$ vertices

1. Let $P = [A_1, A_2, \ldots, A_{k-1}]$ be a path. Embed $P$ into a face $XYZ$, and join the edges $XA_i, YA_i, ZA_i, 1 \leq i \leq k - 1$. We say that $X$ is the head and $Y$ and $Z$ are sides of $P$.

2. Let $Q$ be a path $[A_1, A_2, \ldots, A_{k-1}]$ with an extra edge $A_1A_3$. Embed $Q$ into a face $XYZ$ and join the edges $XA_1, XA_2, YA_i, 1 \leq i \leq k - 1, i \neq 2$, and $ZA_i, 2 \leq i \leq k - 1$.

We say that $X$ is head and $Y$ and $Z$ are sides of $Q$.

It is easy to see that we can embed $P$ or $Q$ into each face of $\tilde{W}$ subject to the following conditions:

(i) Each vertex $A$ of $\tilde{W}$ with $\deg_{\tilde{W}}(A) = 5$ is the head of one $Q$ and a side of four $P$'s.

(ii) Each vertex $A$ of $\tilde{W}$ with $\deg_{\tilde{W}}(A) = 6$ is the head of two $P$'s and a side of four $P$'s, or a side of three $P$'s and one $Q$.

(iii) The centre of each of the wheels is always a head of $P$.

Let $G$ be the resulting graph. Obviously each connected subgraph $H$ of $G$ of order $k$ must contain a vertex $B$ of $\tilde{W}$. It is easy to see that $\deg_G(B) \geq 4k + 3$.

For $k = 3$ a kleetope $K$ of the dodecahedron is taken instead of $\tilde{W}$ and then one $P$ is embedded into each face of $K$ in such a way that each vertex $A$ of $K$ with $\deg_K(A) = 5$ is a side of five $P$'s and each vertex $B$ of $K$ with $\deg_K(B) = 6$ is a head of three $P$'s and a side of three $P$'s. The resulting graph has required properties. 

3. The degree sum problems

As we have mentioned in the Introduction, Kotzig’s theorem states that each 3-connected planar graph contains an edge with degree sum of its endvertices being at most 13. This leads to the following problem.

**Problem 3.** What is the minimum integer $w(k)$ such that every graph $G \in \mathcal{G}(P_k)$ contains a $k$-path $[A_1, A_2, \ldots, A_k]$ with

$$\sum_{i=1}^{k} \deg_G(A_i) \leq w(k)?$$

By Euler [2] and Kotzig [13] we have $w(1) = 5$ and $w(2) = 13$, respectively. In [1], it is shown that $w(3) = 21$.

We are able to prove the following.

**Theorem 3.** Let $k \geq 4$ be an integer. Then

$$f(k) \leq w(k) \leq 5k^2,$$
where

\[ f(k) = \begin{cases} 9k - 4 & \text{for } k = 5, 7, 10 \text{ and } 11 \text{ or} \\ 3k - 6 + \left( 3 - 3t + 5 \left\lfloor \frac{k}{2^t} \right\rfloor \right) 2^t + 3 \sum_{i=0}^{t-1} \left( \left\lfloor \frac{k}{2^{i+1}} \right\rfloor + i \right) 2^i, & \text{with } t = \left\lfloor \log_2 k \right\rfloor - 1 \text{ for other } k \geq 4, \text{ respectively.} \end{cases} \]

**Proof.** The upper bound is a corollary of Theorem 1.

The lower bound is obtained by constructing several series of graphs each of which has a \( k \)-path with the degree sum of their vertices \( f(k) \). For \( k = 5, 7, 10 \text{ and } 11 \) the graph \( \overline{G} \) used in the proof of the lower bound in Theorem 1 of [5] has the required property, that is \( f(k) = 9k - 4 \).

Let \( M_0 \) be the icosahedron and \( M_1 \) be the graph obtained from the dodecahedron by inserting a configuration in Fig. 1 into each of its 12 pentagons.

It is a routine matter to check that in the graph \( K^{(r)}(M_r) \), with \( t = \left\lfloor \log_2 k \right\rfloor - 1 \) and \( r = \left\lfloor k/2^t \right\rfloor - 2 \), each \( k \)-path has the degree sum of its vertices at least \( f(k) \). \( \square \)

**Remark 1.** It is not difficult to show that

\[ w(k) \geq f(k) \geq k \log_2 k. \]

In [1], the following problem was posed.

**Problem 4.** Let \( k \geq 1 \) be an integer. What is the minimum integer \( \sigma(k) \) such that every graph \( G \in \mathscr{K}(k) \) contains a connected subgraph \( H \) with \( V(H) = \{A_1, A_2, \ldots, A_k\} \) such that

\[ \sum_{i=1}^{k} \deg_G(A_i) \leq \sigma(k) ? \]

Enomoto and Ota [4] have proved the following

**Theorem 4.** Let \( k \geq 4 \) be an integer. Then

\[ 8k - 5 \leq \sigma(k) \leq 8k - 1. \]
4. Remarks

If we define the integer \( \varphi(k) = \varphi(P_k) \) analogously as in Problem 1 also for projective plane polyhedral graphs and for toroidal polyhedral graphs we can prove by the analogues arguments as in [5] the following.

**Theorem 5.** Let \( k \geq 1 \) be an integer. Then
(i) for projective plane polyhedral graphs there holds
\[
\varphi(k) = 5k;
\]
(ii) for toroidal polyhedral graphs there holds
\[
\varphi(k) = 6k \text{ if } k \text{ is even}
\]
and
\[
6k - 3 \leq \varphi(k) \leq 6k \text{ if } k \text{ is odd}.
\]

**Problem 5.** Find analogues of Theorems 1–4 of this paper for polyhedral maps on orientable and nonorientable surfaces.

For analogues of Kotzig’s theorem for orientable surfaces see Ivančo [9] and Zaks [15]. The bounds in theorems of this paper can probably be improved also for planar cases if we add the requirements on minimum degree of vertices or faces of graphs in the families \( \mathcal{G}(P_k) \) and \( \mathcal{H}(k) \). For partial results see [1,4,10,11].

**Remark 2.** The results of this paper can be interpreted as those concerning convex three-dimensional polytopes. Due to Steinitz’s theorem, see e.g. [7], each 3-connected plane map is isomorphic to the graph of some convex three-dimensional polytope.

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**References**