Exposed Points in Lebesgue–Bochner and Hardy–Bochner Spaces

Wolfgang Hensgen

Universität Regensburg, NWF I-Mathematik, 93040 Regensburg, Germany

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INTRODUCTION AND SUMMARY

Extreme points of the unit ball of Lebesgue–Bochner spaces have been studied by Sundaresan [39], Johnson [24], and Greim [14]. After the work of Johnson [25] and Greim [15, 16], recently Hu and Lin [21] have succeeded in characterizing strongly exposed points of the ball. I study here the same problem for the intermediate notion of an exposed point which goes back to Straszewicz [38] (see Section 1 for definitions), and I do so also for Hardy–Bochner spaces.

Let $X$ be a Banach space and $\text{exp}_X$ the set of exposed points of the unit ball $B_X$, obviously contained in the unit sphere $S_X$. Let $\mu$ be an $\sigma$-finite measure. Adopting the terminology of Smith [37, p. 157], for $f \in S_{L^p(\mu; X)}$ to belong to $\text{exp}_{L^p(\mu; X)}$ the “natural condition” would be $f(t) \in \{f(t)\} \text{exp}_{B_X}$ a.e. $1 < p < \infty$, resp. $f(t) \in \text{exp}_{B_X}$ a.e. $p = \infty$ (the case $p = 1$ is trivial; see 1.12).

The natural condition is sufficient if $X$ is separable and reflexive (2.3). (Note that in the realm of separable reflexive spaces $X$ the notions of exposed and strongly exposed points of $B_X$ do not coincide [30, p. 145].) The proof is an application of the Jankov–von Neumann measurable set(ation) theorem 1.5. Sufficiency also holds if $X$ is an AL space (2.1), or an AM space, or smooth (2.3). If in the “natural condition” the set $\text{exp}_{B_X}$ is replaced by the smaller set $(x \in \text{exp}_{B_X} : x$ is exposed by every support functional) then the resulting stronger condition is sufficient for arbitrary $X$ (2.4). Geometrically, the exposed points of this special type can be characterized as those points of $S_X$ which are not the end point of a line.
segment contained in \( S_X \). This time, the proof utilizes the Graf–Talagrand measurable selection theorem 1.6 which seems to have gone unnoticed in this context.

The natural condition is necessary (if \( p = \infty \), only for \( f \) to be exposed by an integral functional; see 1.9) if \( X \) is separable and reflexive (2.7), or if \( X \) is only separable but \( \mu \) is a regular Borel measure on a locally compact space (2.7'). Obviously, the proof of this is influenced by Johnson’s paper [24].

In particular, for \( 1 < p < \infty \) and \( X \) separable and reflexive, the natural condition characterizes \( \exp B_{L^p(\mu; X)} \). I was informed by the referee that this result is also contained in Theorem 11 of the forthcoming paper by Hu and Lin [22]. Their proof is different. In a very recent note [20] Hu modified this proof, obtaining the necessity of the natural condition for separable \( X \) and arbitrary \( \mu \).

Turning to the case of Hardy–Bochner spaces \( H^p(X) \) in Section 3, the analogous theorems about sufficiency hold, where \( f \) is only supposed to satisfy the relevant condition on a set of positive measure (3.3), (3.4). The reason for this behaviour, as well as for the fact that no exposedness of the values of \( f \) is necessary for \( f \) to be exposed (3.7), is the identity theorem 3.1. For \( p = \infty \), the material of 3.4, 3.5, and 3.7 is contained in the author’s habilitation thesis [18].

In the appendix, independent of the rest of the paper, I give a complete proof of the result that \( B_{H^p(X)} \) has no strongly exposed point. If \( X = \mathbb{C} \), a proof has been sketched by Pelczyński [32, p. 42] working with peak sets in the spectrum of \( L^\infty \). For the vector case, I had to transfer the idea to the circle, simultaneously filling in the details which I found quite appealing.

1. PRELIMINARIES

(a) Geometry of the Unit Sphere

Let \( X \neq \{0\} \) be a real or complex Banach space with unit ball \( B_X \), sphere \( S_X \), dual \( X' \). \( \text{ext} B_X \) denotes the set of extreme points of \( B_X \).

1.1. Definitions. Let \( x \in S_X \).

1. \( x' \in S_{X'} \) with \( \langle x, x' \rangle = 1 \) is called a support functional of \( x \); \( \text{Supp} x := \{ x' \in S_{X'} : \langle x, x' \rangle = 1 \} \) (\( \neq \emptyset \) by Hahn–Banach).

2. \( x \) is called a smooth point of \( B_X \) (\( x \in \text{sm} B_X \)) if \( \text{Supp} x \) consists only of one point; if \( \text{sm} B_X = S_X \) then \( X \) is smooth.

3. If \( x \) admits of a support functional \( x' \) with \( \Re \langle y, x' \rangle < 1 \) \( \forall y \in B_X \setminus \{x\} \) then \( x \) is an exposed point of \( B_X \) (\( x \in \exp B_X \)), exposed by \( x' \); \( \text{Exp} x := \{ x' \in \text{Supp} x : x' \text{ exposes } x \} \).
4. In 3, \( x' \) strongly exposes \( x \) if \( x_n \in B_x, \langle x_n, x' \rangle \to 1 \) implies \( x_n \to x \) in norm.

5. \( x \) is a point of local uniform rotundity of \( B_x(x \in \text{lur } B_x) \) if \( x_n \in X, \| x_n \| \to 1 \), and \( \| (x + x_n)/2 \| \to 1 \) imply \( x_n \to x \).

1.2. Remarks. 1. \( \text{exp } B_x \subset \text{ext } B_x \).

2. \( X \) strictly convex (i.e., \( \text{ext } B_x = S_X \)) \( \Rightarrow \text{exp } B_x = S_X \) and \( \text{Exp } x = \text{Supp } x \forall x \in S_X \) [8, p. 23].

3. For \( x \in \text{exp } B_x \), \( \text{Exp } x \) is dense in \( \text{Supp } x \) (if \( x_0 \in \text{Exp } x, x_1 \in \text{Supp } x \) then \( \alpha x_0 + (1 - \alpha)x_1 \in \text{Exp } x, 0 < \alpha \leq 1 \)).

4. \( \text{Supp } x \) is weak\(^*\) compact \( \forall x \in S_X \).

5. By 3 and 4, for \( x \in \text{exp } B_x \), \( \text{Exp } x = \text{Supp } x \) iff \( \text{Exp } x \) is weak\(^*\) compact.

Although not needed later, it is interesting to compare the following geometric characterization with the definition of "extreme point": For \( x \in S_X \), \( \text{Exp } x = \text{Supp } x \) iff \( x \) is not the end point of a (non-trivial) line segment contained in \( S_X \). Paya observed that this is tantamount to \( x \in \text{lur } B_F \) for every finite-dimensional subspace \( F \subset X \) with \( x \in F \). Also, \( \text{Exp } x = \text{Supp } x \) does not imply \( x \in \text{sm } B_X \) and is incomparable with \( x \) being strongly exposed: in Lindenstrauss' example [30, p. 145] quoted in the Introduction, the critical point \( x \) is actually in \( \text{exp } B_X \cap \text{sm } B_X \) so that \( \text{Exp } x = \text{Supp } x \), but \( x \) is not strongly exposed.

1.3. Lemma. \( X \) separable \( \Rightarrow \text{sm } B_X \) is a \( G_b \) subset of \( S_X \).

Proof. It is a classical result of Mazur [31, Satz 2] that \( \text{sm } B_X \) is even a dense \( G_b \). To see only the \( G_b \) nature of \( \text{sm } B_X \), it suffices to observe that \( S_X \setminus \text{sm } B_X = \bigcup_{n \in \mathbb{N}} \{ x \in S_X : \max_{(x', y') \in K}(\Re \langle x, x' \rangle + \Re \langle x, y' \rangle) = 2 \} \), where \( (K_n)_{n \in \mathbb{N}} \) is a \( w^* \times w^* \) compact exhaustion of \( B_X \times B_X \setminus \text{diagonal} \), and to use the continuity of the max functional on every \( C(K) \) space.

\( \mathcal{W}^*(S) \) denotes the collection of non-empty subsets of \( S \).

1.4. Lemma. The (weak\(^*\) compact valued) multifunction \( \text{Supp} \colon S_X \to \mathcal{W}^*(S_X) \) is norm-to-weak\(^*\) upper semi-continuous (u.s.c.). In particular, \( \text{Supp } | \text{sm } B_X \to S_X \) is norm-to-weak\(^*\) continuous.

Proof. The second statement is proved in [8, p. 22]. For the first one, following [13, 1.11] (and not [7, 4.2]), recall that if \( S, T \) are topological spaces, then \( F \colon S \to \mathcal{W}(T) \) is called u.s.c. if \( F^{-1}(A) := \{ s \in S : F(s) \cap A \neq 0 \} \) is closed in \( S \) for all closed sets \( A \subset T \). Let \( A \subset S_X \) be weak\(^*\) closed, and suppose \( (x_n) \) is a sequence in \( S_X \), norm convergent to \( x \in S_X \), such that \( \exists x_n' \in \text{Supp}(x_n) \cap A \). Let \( x' \in B_{X'} \) be a weak\(^*\) cluster point of \( (x_n') \); the computation in [8, p. 22] shows that \( \langle x, x' \rangle = 1 \), whence \( x' \in S_{X'} \).
whence \( x' \in A \). Altogether, \( x' \in \text{Supp}(x) \cap A \), and \( x \in \text{Supp}^{-1}(A) \). \( \text{Supp}^{-1}(A) \) is norm closed.

(b) **Measurable Selection Theorems**

I use here the excellent survey (with proofs) of Graf [13]. Recall that a Hausdorff space \( S \) is called analytic if it is the continuous image of a Polish space; \( \mathcal{B}_S \), resp. \( \mathcal{B}_S^\cdot \), denote the \( \sigma \)-algebras in \( S \) of Borel sets, resp. universally measurable sets (w.r.t. all finite measures).

1.5. **Theorem** (Jankov and von Neumann [13, 2.6]). Let \( p: R \to S \) be a continuous and surjective map between analytic spaces \( R, S \); then \( p \) admits a \( \mathcal{B}_S^u - \mathcal{B}_R \) measurable section \( s: S \to R \): \( p \circ s = \text{id}_S \).

**Corollary** [13, 2.7]. Let \( F: S \to \mathcal{B}_S^u(R) \) be a correspondence with analytic graph between analytic spaces \( S, R \); then \( F \) admits a \( \mathcal{B}_S^u - \mathcal{B}_R \) measurable selection \( f: S \to R \): \( f(s) \in F(s) \ \forall s \in S \).

1.6. **Theorem** (Graf and Talagrand [13, 4.16, 17]). Let \( (S, \mathcal{A}, \nu) \) be a complete finite measure space, \( \mathcal{F} \subset \mathcal{A} \) a topology on \( S \) such that \((S, \mathcal{A}, \nu, \mathcal{F})\) admits a strong lifting (this is the case if \( S, \mathcal{F} \) is 2nd countable and \( \nu(U) > 0 \ \forall U \in \mathcal{F} \setminus \{\emptyset\} \)). Let \( R \) be a regular Hausdorff space and \( F: S \to \mathcal{B}_S^u(R) \) be an u.s.c. compact valued correspondence; then \( F \) admits an \( \mathcal{A} - \mathcal{B}_R \) measurable selection \( f: S \to R \): \( f(s) \in F(s) \ \forall s \in S \).

(c) **Dual of \( L^p(\mu; X) \), \( 1 \leq p \leq \infty \)**

Let \((T, \Sigma, \mu)\) be a \( \sigma \)-finite measure space, \( 1 \leq q \leq \infty \), and \( X \) a Banach space over \( \mathbb{K} \).

1.7. **Definitions and Remarks.** 1. \( g: T \to X' \) is weak* \( \mu \)-measurable: \( \Leftrightarrow \forall x \in X: \langle x, g \rangle : T \to \mathbb{K} \) is \( \mu \)-measurable. In this case, there exists in the vector lattice \( L^q(\mu) \) of \( \mu \)-measurable functions modulo \( \mu \)-null functions the supremum \( |g| = \sup_{x \in B} |\langle x, g \rangle| \). In general, \( |g(t)| \leq \|g(t)\| \) a.e. and the inequality may be strict. However, if \( g \) is (strongly) \( \mu \)-measurable, or if \( X \) is separable, then \( |g(t)| = \|g(t)\| \) a.e. [18, 1.3].

2. \( \mathcal{L}^q(\mu; X', X) := T \to X' \) weak* \( \mu \)-measurable: \( |g| \in L^q(\mu) \) is equipped with the seminorm \( \|g\|_q := \|g\|_q \). Finally, let \( L^q(\mu; X', X) := \mathcal{L}^q(\mu; X', X) / \|\cdot\|_q^\cdot(0) \). From now on, the letter \( \mu \) will often be omitted from notation.

1.8. Now let \( 1 \leq p \leq \infty \), \( 1/p + 1/q = 1 \). For \( f \in L^p(X) \), \( g \in L^q(X', X) \), the expression \( \langle f(\cdot), g(\cdot) \rangle = \langle f, g \rangle \) well-defined a member of \( L^1 \) and \( \|f, g\| \leq \|f\|_p \|g\|_q \) a.e. [17, (0.55)].

1.9. **Theorem.** The mapping \( L^q(X', X) \to L^p(X) \), \( g \to \langle f(\cdot), g(\cdot) \rangle d\mu \) is an isometry, surjective if \( 1 \leq p < \infty \). Moreover, \( g \in L^q(X', X) \) admits of a representative, call it again \( g \in L^q(X', X) \) with \( \|g(\cdot)\| = \|g(t)\| \) a.e.
Proof. For the representation of $L^p(X')$ ($1 \leq p < \infty$), see [2, 10, 23, 36]. The isometric nature of the map in question ($p = \infty$) was noted in [29], cf. [19, 1.5]. The last assertion can e.g. be deduced from [10, Sect. 13, Theorem 5]; cf. [18, 1.4].

The functionals on $L^p(X)$ given by an $L^1(X', X)$ function are called integral.

(d) Miscellaneous

1.10. General Measure Spaces. I did not pursue systematically the case of not necessarily $\sigma$-finite measures $\mu$. For $p < \infty$ this is not a real restriction, because of the following argument in which $\mu$ is arbitrary. Let $f \in S_{L^p(T, \Sigma, \mu; X')}$, then $T_f := \{ t \in T : f(t) \neq 0 \} \in \Sigma$ (defined up to a $\mu$-null set) has $\sigma$-finite $\mu$-measure. Let $\Sigma_{T_f} := T_f \cap \Sigma$ and $\mu_0 := \mu|_{\Sigma_{T_f}}$, then $f \in L^p(T_f, \Sigma_{T_f}, \mu_0; X) \subset L^p(T, \Sigma, \mu; X)$ canonically.

Claim. $f \in \text{exp } B_{L^p(T, \mu; X)}$ $\iff$ $f \in \text{exp } B_{L^p(T_f, \mu_0; X')}$. This follows from a more general consideration:

1.11. Let $Z_0 \subset Z$ be an inclusion of Banach spaces, then always $Z_0 \cap \text{exp } B_Z \subset \text{exp } B_{Z_0}$. Equality holds if $Z_0$ is complemented in $Z$ by a strictly contractive projection $P : Z \rightarrow Z_0$ (i.e., $\|Pz\| < \|z\| \forall z \in Z \setminus Z_0$), e.g., an $L^p$-projection, $1 \leq p < \infty$. (If $x_0 \in S_X$ is exposed by $x'_0 \in S_{X'}$, then as a member of $S_X$, $x_0$ is exposed by $P'x'_0 \in S_{X'}$.) From now on let $(T, \Sigma, \mu)$ be a $\sigma$-finite measure space.

1.12. The case $p = 1$. This is easily settled: A function $f \in S_{L^1(X)}$ is in $\text{exp } B_{L^1(X)}$ iff there exists a $\mu$-atom $A \in \Sigma$ such that $f = (1/\mu A)1_A X$ where $x \in \text{exp } B_X$ ($\mu A < \infty$ since $\mu$ is $\sigma$-finite).

For the proof, note first that if $f \in \text{ext } B_{L^1(X)}$ then $f$ is of the form given with $x \in S_X$ [39, Proposition 1]. Now both implications follow from 1.11, applied to the range of the isometric embedding $X \rightarrow L^1(X)$, $x \mapsto (1/\mu A)1_A X$.

On the other hand, even in the scalar case “there is no good characterization of the exposed points of $B_H$” [12, p. 159] (see [40], however). From now on always $1 < p \leq \infty$.

2. EXPOSED POINTS IN $L^p(X)$

(a) Sufficient Conditions

The following basic lemma is the analogue of [25, Theorem 1], cf. also [6, Theorem 6]. Recall $T_f := \{ f \neq 0 \}$.
2.1. **Lemma.** (a) Let $1 < p < \infty$, $f \in S_{L^1(X)}$, $g : T_f \to X'$ weak$^*$-measurable with $g(t) \in \text{Exp}(f(t)/\|f(t)\|)$ a.e. on $T_f$. Then $g' : T \to X'$,

$$g'(t) := \begin{cases} \|f(t)\|^{p-1} g(t), & t \in T_f \\ 0 & t \notin T_f \end{cases}$$

defines a member of the unit sphere of $L^p(X', X) = L^p(X'Y)$ which exposes $f$ \((1/p + 1/q) = 1\).

(b) Let $f \in S_{L^1(X)}$ with $\|f(t)\| = 1$ a.e., $g : T \to X'$ weak$^*$-measurable with $g(t) \in \text{Exp}(f(t))$ a.e. Choose and fix a measurable, everywhere positive function $u$ on $T$ with $|u| \, d\mu = 1$. Then $g' : T \to X'$, $g' := ug$ defines a member of the unit sphere of $L^1(X', X) \subset L^1(X'Y)$ which exposes $f$.

**Proof.** (a(i)) $\|g(t)\| = 1$ a.e. on $T_f$, hence $\|g'(t)\| \leq \|f(t)\|^{p-1}$ a.e. on $T$, hence $|g'| \leq |f|^{p-1}$ since $|f| = \|f(t)\|$ is measurable (see 1.7). Thus $\int |g'|^q \, d\mu \leq \int |f|^q \, d\mu = 1$ so that $g' \in B_{L^1(X', X)}$.

(ii) $\langle f', g' \rangle \, d\mu = \int_{T_f} \|f(t)\|^{p-1} \langle f(t), g(t) \rangle \, d\mu = \int_{T_f} \|f(t)\| \, d\mu = 1$; hence $\|g'\|_q = 1$ (hence $|g'| = |f|^{p-1}$) and $g'$ supports $f$.

(iii) $g'$ exposes $f$: Suppose that also $f' \in S_{L^1(X)}$ with (see 1.8) $1 = \int |\langle f', g' \rangle| \, d\mu \leq \int |f'| \, |g'| \, d\mu \leq \|f'\|_p \|g'\|_q = 1$. It follows first that $\langle f', g' \rangle = |\langle f', g' \rangle|$ (\(*\)). Second, $\int |f'| \, |g'| \, d\mu = 1 = \int |f| \, |g'| \, d\mu$, so smoothness of $L^1(\mu)$ implies $|f'| = |f|$. For $t \in T_f = T_{f'}$, $\langle f'(t), g'(t) \rangle = 1$. Because $g(t)$ exposes $f(t)/\|f(t)\|$ this implies $f'(t)/\|f'(t)\| = f(t)/\|f(t)\|$, hence $f'(t) = f(t)$.

(b) Similar but simpler.

**Corollary 1.** Let $f \in S_{L^1(X)}$ satisfy the “natural condition” $f_1(t) := \langle f(t)/\|f(t)\| \rangle \in \text{Exp} B_{X}$ a.e. on $T_f$ (resp. $f(t) \in \text{Exp} B_X$ a.e. on $T$ if $p = \infty$). If $f_1$ is (a.e.) countably valued (in particular, if $\text{Exp} B_{X}$ is countable) then $f \in \text{Exp} B_{L^1(X)}$ (exposed by an integral functional if $p = \infty$).

**Corollary 2.** If $X = L^1(v)$, $v$ arbitrary, then the natural condition is sufficient.

**Proof.** If $X = L^1(v; \mathbb{R})$ and $v$ is $\sigma$-finite then $\text{Exp} B_{X}$ is countable (1.12) and Corollary 1 applies. If $X = L^1(v; \mathbb{C})$ ($v$ $\sigma$-finite) then $\text{Exp} B_{X}$ is “countable up to multiplication by a unimodular scalar” from which it is also easy to conclude. Finally, the case of an arbitrary $v$ can be reduced to the $\sigma$-finite case by means of 1.11.

2.2. **Lemma.** Let $X$ be a separable and reflexive Banach space. Then $\text{Exp} B_{X}$ is weakly analytic and there is a selection $s : \text{Exp} B_{X} \to S_{X'}$ of $\text{Exp}$, measurable $\mathcal{B}_{\text{Exp} B_{X}}$ to $\mathcal{B}_{S_{X}}$ (note that Borel (norm) = Borel (weak)).
Proof. The set of exposing functionals $\bigcup_{x \in S_X} \text{Exp} \ x$ is exactly $sm \ B_{X'}$, because $X$ is reflexive. By separability and Lemma 1.3, this set is a $G_\delta$ in $S_{X'}$, hence Polish. Since $\text{Exp} B_X$ is the range of the norm-to-weak continuous (1.4) map $\text{Supp} \sm \ B_{X'} \to S_{X'}$, the first assertion follows. The desired selection $s$ is simply a section of this last map which exists with the asserted measurability by 1.5.

The weak analytic nature of $\text{Exp} B_X$ is also a special case of [4, 1.12] (which, naturally is more difficult to prove). As noted in [26, p. 254], it follows by a formal argument that $\text{Exp} B_X$ is even strongly analytic ($X$ separable reflexive). For pathological examples of sets of exposed points see [28, 6.10; 26].

2.3. Theorem. Let $f \in S_{L^p(X)}$ satisfy the “natural condition” $f_\delta(t) := f(t)/|f(t)| \in \text{Exp} B_X$ a.e. on $T_f$ ($1 < p < \infty$), resp. $f(t) \in \text{Exp} B_X$ a.e. on $T$ ($p = \infty$). Then $f \in \text{Exp} B_{L^p(X)}$ (exposed by an integral functional if $p = \infty$) in each of the following cases:

(i) $X$ separable and reflexive

(ii) $X$ smooth

(iii) $X = C(K)$ (in particular, $X = L^p(\nu)$, $\nu$ arbitrary)

Proof. In each case there exists a selection $s$: $\text{Exp} B_X \to S_{X'}$ of $\text{Exp}$, measurable $\mathfrak{B}_\mu$ to $\mathfrak{B}_w$. See 2.2 in Case i and 1.4 in Case ii. In Case iii, if $K$ supports no (regular) probability then $\text{Exp} B_X = \emptyset$ [33, Proposition 2] and there is nothing to prove. Suppose that $K$ supports a probability $m$, then $s$: $\text{Exp} B_X = \{ x \in C(K): |x| = 1 \text{ on } K \} \to S_{X'}$, $s(x) := \hat{x}m$ is the desired selection [loc. cit.].

Now assume w.l.o.g. that $f_\delta(t) \in \text{Exp} B_X$ for all $t \in T_f$ (resp. $f(t) \in \text{Exp} B_X$ for all $t \in T$ if $p = \infty$). Regardless of $p \in [1, \infty]$, define $g: T_f \to X'$, $g(t) := s(f_\delta(t))$. The function $f_\delta$ is measurable $\Sigma_\mu$ to $\mathfrak{B}_{\text{Exp} B_X}$, hence also $\Sigma_\mu^\mu$ to $\mathfrak{B}_{\mu \text{Exp} B_X}$ (superscripts denote completion). As a $\sigma$-finite measure, $\mu$ is equivalent to a finite measure. The latter also holds for $\mu \circ f_\delta^{-1}$, so that $\mathfrak{B}_{\mu \circ f_\delta^{-1}} \supseteq \mathfrak{B}_\mu$, and the measurability of $s$ stated above is enough to conclude that $g$ is measurable $\Sigma_\mu$ to $\mathfrak{B}_{(L^p)^\mu}$, in particular $g$ is weak* $\mu$-measurable. By construction, $g(t) \in \text{Exp} f_\delta(t)$ on $T_f$ and Lemma 2.1 completes.

Since in an AM space $X$ without unit, $\text{ext} B_X = \emptyset$ ($e, x \in B_X \Rightarrow e \leq |x| = |e|$, $e \in B_X$), so that $|x| \leq |e|$ if $e \in \text{ext} B_X$, the case of an arbitrary AM space is settled.

2.4. Theorem. Let $X$ be an arbitrary Banach space and $f \in S_{L^p(X)}$. For $1 < p < \infty$, suppose that a.e. on $T_f$, $f_\delta(t) := f(t)/|f(t)| \in \text{Exp} B_X$ with $\text{Exp} f_\delta(t) = \text{Supp} f_\delta(t)$. For $p = \infty$, suppose that a.e. on $T$, $f(t) \in \text{Exp} B_X$ with $\text{Exp} f(t) = \text{Supp} f(t)$.
with \( \text{Exp } f(t) = \text{Supp } f(t) \). Then \( f \in \exp B_{L^p(X)} \) (exposed by an integral functional if \( p = \infty \)).

**Proof.** Choose the representative \( f \) so that \( f_1(T) \subset \{ x \in \exp B_X : \exp x = \text{Supp } x \} \) is separable. Let \( \mu_1 \) be a finite measure on \( \Sigma \) equivalent to \( \mu \). As a Borel measure on the second countable space \( f_1(T) \), the image \( v := \mu_1 \circ f_1^{-1} \) has a support \( S \) (complement of the union of all open \( \nu \)-null sets). Redefining \( f(t) \) to be 0 on the \( \mu \)-null set \( T_f \setminus f_1^{-1}(S) \), I can assume from the outset that \( S = f_1(T) \). For later application, I record that \( f_1 : (T, \Sigma^y_f) \to (S, \frak{B}_S^y) \) is measurable.

Theorem 1.6 can be applied to the measure space \((S, \frak{B}_S^y, \nu)\), the norm topology \( \frak{T} \) on \( S \), the regular Hausdorff space \((S_{\frak{T}}, \text{weak}^*)\), and the multifunction \( \text{Exp} | S = \text{Supp } | S \to \frak{B}(S_{\frak{T}}) \) which is u.s.c. and compact valued after 1.4. So there is a selection \( s : S \to S_{\frak{T}} \) of this multifunction, measurable \( \frak{B}_S^y \to \frak{B}(S_{\frak{T}}, \text{weak}^*) \). The composition \( g := s \circ f_1 ; T_f \to S_{\frak{T}} \) is measurable \( \Sigma^y_f \to \frak{B}(S_{\frak{T}}, \text{weak}^*) \) and the proof finishes as before.

**Remarks.** 1. If \( X \) is separable, then \((S_{\frak{T}}, \text{weak}^*)\) is Polish and the classical selection theorem of Castaing and Kuratowski–Ryll–Nardzewski [13, 2.1] is sufficient for the proof.

2. The additional assumption “\( \text{Exp } f_1(t) = \text{Supp } f_1(t) \) a.e. on \( T_f \)” is satisfied in the following two cases:
   (i) \( f_1(t) \in \text{sm } B_X \) a.e. on \( T_f \). Of course, no selection is needed in this case.
   (ii) \( X \) is strictly convex (by 1.2.2). For \( 1 < p < \infty \), this yields only the well known implication that \( X \) strictly convex \( \Rightarrow L^p(X) \) strictly convex.

**2.5. Lemma.** Let \( X \) be a separable Banach space and \( A \) be a \( \text{w}^* \)-analytic subset of \( \{ x' \in S_{\frak{T}} : x' \text{ supports more than one } x \in S_{\frak{T}} \} = B \). Then there exist two functions \( s_1, s_2 : A \to S_{\frak{T}} \), measurable \( \frak{B}_{(A, \text{w}^*)} \) to \( \frak{B}_{S_{\frak{T}}} \), with \( s_1(x') \neq s_2(x') \) and \( \langle s_1(x'), x' \rangle = \langle s_2(x'), x' \rangle \forall x' \in A \).

**Proof.** The correspondence \( F: (A, \text{w}^*) \to \frak{B}(S_X) \), \( x' \mapsto X \cap \text{Supp } x' \) has closed graph. By [24, Lemma 1], the correspondence \( (A, \text{w}^*) \to \frak{B}(S_X \times S_X) \), \( x' \mapsto F(x') \times F(x') \setminus \text{diagonal} \) has Borelian graph. After Corollary 1.5, there exists a selection \( s = (s_1, s_2) \) of this correspondence with the asserted measurability.

**2.6. Note.** \( f \in \exp B_{L^p(X)} \Rightarrow \| f(t) \| = 1 \) a.e. (this holds already if only \( f \in \text{ext } B_{L^p(X)} \) [14]).

**2.7. Theorem.** Let \( X \) be separable and reflexive and \( f \in \exp B_{L^p(X)} \), exposed by an integral functional if \( p = \infty \). Then \( f(t) \in \| f(t) \| \exp B_X \) a.e.
Thus by (2.3), (2.7), for $X$ separable and reflexive, $1 < p < \infty$, the “natural condition” [37, p. 157] \( f(t) \leq ||f(t)|| \exp B_X \) a.e. characterizes the exposed points of \( B_{L^p(X)} \).

2.7. Theorem. Let \( X \) be only separable, but \( \mu \) be a regular Borel measure on a locally compact space \( T \). Then the conclusion of 2.7 holds, too.

Proof. Simultaneous for 2.7 and 2.7. The assertion is that \( f_j(t) := \frac{f(t)}{||f(t)||} \in \exp B_X \) a.e. on \( T_1 \). Let \( g \in S_{l_{\infty}(X)} \), \((1/p + 1/q = 1)\) expose \( f \). It follows that \( g(t) \neq 0 \) a.e. on \( T_1 \). After 1.9, I can assume that \( |g(t)| = ||g(t)|| \) a.e. Let \( g_j(t) := g(t)/||g(t)|| \) defined (a.e.) on \( T_1 \).

The usual string of inequalities (see iii in the proof of 2.1) yields \( \langle f, g \rangle = |f||g| \) a.e., so \( g_j(t) \) supports \( f_j(t) \) a.e. on \( T_1 \). I claim that \( g_j(t) \) exposes \( f_j(t) \) a.e. on \( T_1 \). Suppose not, then there exists a set \( T_0 \subseteq T_1 \), \( 0 < \mu T_0 < \infty \), such that \( g_j(T_0) \subset B \), the set of Lemma 2.5. Let \( \mu_0 := \mu|T_0 \to S_X \) is measurable \( \Sigma_{T_1}^{\mu_0} \) to \( \Sigma_{(S_{l_{\infty}(X)}^{\mu_0})} \), since \( X \) is separable.

Now in the case of 2.7, \( B = S_X \setminus B_X \) is Borel (norm = weak*) after 1.3, so in 2.5 let \( A := B \) and set \( T_1 := T_0 \). In case of 2.7, by Luzin’s theorem [9, Sect. 15.8] there exists a compact set \( T_1 \subseteq T_0 \), \( \mu T_1 \geq 0 \), such that \( g_j|T_1 \to B, B_X^{\mu_0} \) are measurable \( \Sigma_{T_1}^{\mu_0} \) to \( \Sigma_{(S_{l_{\infty}(X)}^{\mu_0})} \). Putting \( \Sigma_{T_1}^{\mu_0} := \mu|T_1 \to A \) is measurable \( \Sigma_{T_1}^{\mu_0} \) to \( \Sigma_{(S_{l_{\infty}(X)}^{\mu_0})} \), since \( X \) is separable.

With the two functions \( s_j: A \to S_X \) of 2.5, \( j = 1, 2 \), define \( f_j(t) := \langle f(t), s_j(g(t)) \rangle \) for \( t \in T_1 \), and \( f_j(t) := f(t) \) for \( t \in T \setminus T_1 \). These functions are measurable \( \Sigma_{T_1}^{\mu_0} \) to \( \Sigma_{(S_{l_{\infty}(X)}^{\mu_0})} \) and \( ||f_j(t)|| = ||f(t)|| \) for all \( t \), hence (Pettis) \( f_j \in S_{L^p(X)} \). Moreover, \( \int_{T_1} \langle f_j(t), g(t) \rangle d\mu(t) = \int_{T_1} ||f(t)|| \cdot ||g(t)|| \cdot \langle s_j(g_j(t), g_j(t)) \rangle d\mu(t) = \int_{T_1} ||f|| \cdot ||g|| d\mu = \int_{T_1} f_j \langle f, g \rangle d\mu = \int_{T_1} \langle f_j, g \rangle d\mu = 1 \). Since \( f_j(t) \neq f(t) \) on \( T_1 \), \( g_j(t) \) cannot expose \( f_j \), a contradiction.

3. EXPOSED POINTS IN \( \mathbb{H}^p(X) \)

From now on, \( \mathbb{K} = \mathbb{C} \), \( T \) is the unit circle, \( \Sigma = \Sigma_{T} \), \( d\mu = d\theta/2\pi \) normalized Lebesgue measure. Let \( \mathbb{H}^p(X) := \{ f \in L^p(X) : \hat{f}(n) = 0 \ \forall n < 0 \} \) be the subspace of functions of “analytic type.” The theory of exposed points in this Hardy–Bochner space is governed by the following vector-valued identity theorem, a trivial consequence of its scalar counterpart [34, 17.18]:

3.1. Fact. If \( f \in \mathbb{H}^p(X) \), \( 1 \leq p \leq \infty \), vanishes on a set of positive measure then \( f = 0 \). (In other words, \( T_1 = T \) unless \( f = 0 \).)

Most proofs in this section are similar to those of Section 2 so their style is terse.
(a) **Sufficient Conditions**

3.2. **Lemma.** (a) Let \(1 < p < \infty\), \(f \in S_{\mathbb{H}^r(X)}\), \(T_0 \in \Sigma\), \(\mu T_0 > 0\), \(g': T_0 \to X'\) weak* \(\mu\)-measurable with \(g(t) \in \text{Exp}(f(t)/\|f(t)\|)\) a.e. on \(T_0\). Then \(g': T \to X'\),

\[
g'(t) := \begin{cases} 
\frac{\|f(t)\|^{p-1}}{\|f1_{T_0}\|_p^p} g(t), & t \in T_0 \\
0, & t \notin T_0 
\end{cases}
\]

defines a member of \(L^p(X',X)(1/p + 1/q = 1)\) which as a functional on \(\mathbb{H}^r(X)\) is in \(\text{Exp}\ f\).

(b) Let \(f \in S_{\mathbb{H}^r(X)}\) with \(\|f(t)\| = 1\) a.e. on \(T_0 \in \Sigma\), \(\mu T_0 > 0\), \(g': T_0 \to X'\) weak* \(\mu\)-measurable with \(g(t) \in \text{Exp} f(t)\) a.e. on \(T_0\). Then \(g': T \to X'\),

\[
g'(t) := \begin{cases} 
\frac{1}{\mu T_0} g(t), & t \in T_0 \\
0, & t \notin T_0 
\end{cases}
\]

defines a member of \(L^1(X',X)\) which as a functional on \(\mathbb{H}^r(X)\) is in \(\text{Exp}\ f\).

**Proof.** (a) Repeat the arguments of the proof of 2.1 to obtain step by step

(i) \(|g'| \leq |f|^{p-1} 1_{T_0}/\|f1_{T_0}\|_{L^p}^p\), hence \(g' \in B_{L^p(X',X)}\);
(ii) \(\langle f, g' \rangle d \mu = 1\) hence \(g' \in \text{Supp} f\), \(\|g'\|_{L^q} = 1\), and \(|g'| = |f|^{p-1} 1_{T_0}/\|f1_{T_0}\|_{L^p}^p\).
(iii) if also \(f' \in S_{\mathbb{H}^r(X)}\) with \(\langle f', g' \rangle d \mu = 1\) then \(\langle f', g' \rangle = \|f'\|g'\|_{L^p} = \|f\|g\|_{L^p}\) and \(|f'| = |f|\), hence a.e. on \(T_0\): \(\langle f'(t)/\|f'(t)\|, g(t) \rangle = 1\) a.e. on \(T_0\): \(f'(t) = f(t)\). The identity theorem 3.1 yields \(f' = f\).

(b) Similar but simpler.  

Corollaries analogous to those of 2.1 hold. In particular, for \(L^1(\nu)\) spaces, the “natural condition on a set of positive measure” is sufficient.

The functionals on \(\mathbb{H}^r(X)\) given by an \(L^1(X',X)\) function are again called integral.

3.3. **Theorem.** Let \(X\) be in one of the classes (i), (ii), (iii) of 2.3, and \(f \in S_{\mathbb{H}^r(X)}\). If, on a set of positive measure, \(f(t) \in \|f(t)\|\exp B_X (1 < p < \infty)\), resp. \(f(t) \in \exp B_X (p = \infty)\) then \(f \in \exp B_{\mathbb{H}^r(X)} (\text{exposed by an integral functional if } p = \infty)\).

**Proof.** Let \(T_0\) be such a set of positive measure. Then the proof is identical with that of 2.3, replacing \(T_f\) by \(T_0\) and 2.1 by 3.2.  

3.4. Theorem. Let \( X \) be an arbitrary Banach space and \( f \in S_{H^p(X)} \). For \( 1 < p < \infty \), suppose that, on a set of positive measure, \( f_1(t) := f(t)/\|f(t)\| \in \exp B_X \) with \( \exp f_1(t) = \text{Supp} f_1(t) \). For \( p = \infty \), suppose that, on a set of positive measure, \( f(t) \in \exp B_X \) with \( \exp f(t) = \text{Supp} f(t) \). Then \( f \in \exp B_{H^p(X)} \) (exposed by an integral functional if \( p = \infty \)).

Proof. As for 2.4, with the same modifications as above.

Of course, the Remarks of 2.4 apply mutatis mutandis. In particular, if \( X \) is strictly convex, \( f \in S_{H^p(X)} \), \( \|f(t)\| = 1 \) on a set of positive measure, then \( f \in \exp B_{H^p(X)} \), exposed by an integral functional. In the scalar case this is due to Fisher [11].

(b) Necessary Conditions; Counterexamples

3.5. Theorem (Amar and Lederer for \( X = \mathbb{C} \) [1]). If \( f \in \exp B_{H^p(X)} \) then \( \|f(t)\| = 1 \) on a set of positive measure.

Proof. The scalar proof given by Khavin [27, 13], working entirely on \( T \), can be used verbatim.

3.6. Corollary. Let \( X \) be strictly convex, \( f \in S_{H^p(X)} \). TFAE:

1. \( f \in \exp B_{H^p(X)} \), exposed by an integral functional
2. \( f \in \exp B_{H^p(X)} \)
3. \( f \) is supported by an integral functional
4. \( \|f(t)\| = 1 \) on a set of positive measure.

Proof. In view of 3.4 (remark) and 3.5, only \( 3 \rightarrow 4 \) remains to be shown. No strict convexity is needed for this implication. Let \( \varphi' \in S_{H^p(X)} \) be an integral support functional of \( f \). By a proximinality argument (F. and M. Riesz plus weak* compactness, see [18, 2.4 Remark 2]) one can prove that there exists a \( g \in L^1(X', X) \) of unit norm representing \( \varphi \). (Actually, every norm-preserving extension of \( \varphi \) over \( L^1(X) \) is again integral, by Gleason and Whitney, see [18, Theorem 2.4; 19, 2.6].) Then the relations \( \langle f, g \rangle \leq |f||g| \leq |g| \) a.e. and \( \int \langle f, g \rangle \, d\mu = 1 = \int |g| \, d\mu \) imply \( \langle f, g \rangle = |g| \) a.e. By 1.9 I assume w.l.o.g. that \( |g(t)| = \|g(t)\| \) a.e. Since \( g \neq 0 \), this entails \( \langle f(t), g(t)/\|g(t)\| \rangle = 1 \) on a set of positive measure, hence \( \|f(t)\| = 1 \) on this set.

3.7. No condition of the type of 2.7 (exposedness of the values of \( f \)) is necessary for \( f \in \mathbb{H}^p(X) \) to be exposed. In fact, I give examples of \( f \in \exp B_{H^p(X)} \) (exposed by an integral functional if \( p = \infty \)) such that for a.e. \( t \in T : \|f(t)\| = 1 \) but \( f(t) \) is not even an extreme point of \( B_X \).

Example 1. The easiest construction is to take \( X := l^2(2) := (C^2, \|\cdot\|_2) \), to decompose \( T = T_1 \cup T_2 \) with \( T_i \in \Sigma, \mu T_i > 0, i = 1, 2 \), and to
define $f_i$ as the outer function [34, 17.16] of modulus $|f_i| = 1_{T_i} + \frac{1}{2}1_{T\setminus T_i}$, $i = 1, 2$. Putting $f = (f_1, f_2)$, clearly $\|f(t)\| = 1$ and $f(t) \notin \text{ext } B_{\ell_1^\infty}$ a.e.

To see that $f$ is an exposed point of $B_{\mathcal{H}(X)}$ ($1 < p \leq \infty$), exposed by an integral functional if $p = \infty$, define $g_i := \int f_i 1_{T_i}$ and $g : T \to l^1(2) = X'$, $g = (g_1, g_2)$. Then $g(t) \in \text{Supp } f(t)$ a.e. on $T$. Let $g' : T \to X'$ be defined as in Lemma 3.2. An inspection of its proof reveals that this last Supp relation suffices to conclude that $g' \in \text{Supp } f$, and that if also $f' \in S_{\mathcal{H}(X)}$ with $\langle f', g' \rangle \,d\mu = 1$ then $|f'| = |f|$ (= 1 a.e.) and $\langle f'(t), g(t) \rangle = 1$ a.e. on $T$. This means $f_1^2g_1 + f_2^2g_2 = 1$ a.e. on $T$, so that a.e. on $T_i$: $f_i^2 = f_i$, $i = 1, 2$. The identity theorem 3.1 yields $f_i' = f_i$ a.e. on $T$ and $f' = f$. Thus $g'$ exposes $f$.

**Example 2.** A variant of Example 1, where now $X := c_0$, is also interesting, since a priori it is not even clear that $B_{\mathcal{H}(c_0)}$ possesses extreme points at all. (ex $B_{c_0} = \emptyset$; moreover $\mathcal{H}(c_0)$ is not a dual space since $c_0$ does not embed complementably into any dual space [35, 32].)

Write $T = \bigcup_{n \in \mathbb{N}} T_n$, where $T_n \in \Sigma$, $\mu T_n > 0$. Let $f_n \in H^\infty$ be the outer function of modulus on the boundary $|f_n| = 1_{T_n} + (1/n)1_{T \setminus T_n}$, then $f := (f_n)_{n \in \mathbb{N}} \in \mathcal{H}(c_0)$ (by Pettis' theorem), $\|f(e''t)\| = 1$ a.e., and $f(e''t) \notin \text{ext } B_{c_0} = \emptyset$. To see that $f$ is exposed, define $g_n := \int f_n 1_{T_n}$, $g := (g_n)_{n \in \mathbb{N}} \in L^p(T)$, and proceed as before to prove that $g$ exposes $f$ ($1 < p \leq \infty$).

**APPENDIX: STRONGLY EXPOSED POINTS IN $\mathcal{H}^p(X)$**

This section, independent of the rest of the paper, is devoted to the proof of

**A.1. Theorem.** $B_{\mathcal{H}(X)}$ has no strongly exposed point.

In the scalar case, a sketch of proof has been given by Pełczyński [32, p. 42], working via Gel'fand transform with peak sets in the spectrum $\Delta$ of $L^\infty(\mu)$. This technique is not available in the vector-valued situation, because $L^\infty(\mu; X)$ cannot be identified with $C(\Delta; X)$. So the proof of A.1 to follow consists of simultaneously filling in the details into Pełczyński's sketch and transferring it to the circle $T$.

**A.2. Definition-Theorem [29, 5, 3].** A functional $\varphi \in L^\infty(X)'$ is concentrated on a set $T_0 \in \Sigma : \varphi(f) = \varphi(1_{T_0} f)$ $\forall f \in L^\infty(X)$, and singular if concentrated on sets of arbitrarily small measure. Every $\varphi \in L^\infty(X)'$ is the sum of an integral (see 1.9) and a singular functional.
A.3. Lemma. Let $D$ be the open unit disc and $E := \{re^{i\theta} \in D : 0 \leq r \leq 1, |\theta| \leq (1 - r)^2\}$. Then

(i) $E$ is a compact subset of $D \cup \{1\}$

(ii) $\|z^n - 1\|_E \to 1(n \to \infty)$ (\$\|\cdot\|_E$: sup norm over $E$).

Remark. (ii) does not follow from (i) alone, even if $E$ is contained in a non-tangential approach region at 1 given by $|\theta| \leq c(1 - r)$, $c > 0$ fixed. To see this, consider the set $E := \{r_n e^{i\theta_n} : n \in \mathbb{N}\} \cup \{1\}$ where $r_n = 1 - 1/n$, $\theta_n = \pi/n$.

Proof (of the Lemma). (i) Clear. (ii) $0 \in E \Rightarrow \lim_{n \to \infty} z^n - 1 \|_E \geq 1$. To establish $\lim_{n \to \infty} z^n - 1 \|_E = 1$, note first that by a simple geometric consideration, if $z = re^{i\theta} \in \overline{D}$ with $|\theta| \leq \pi/3$ then $|z^n - 1| \leq 1$. For $z = re^{i\theta} \in E$ with $|\theta| \geq \pi/3$ we have $\pi/3/n \leq |\theta| \leq (1 - r)^2 \Rightarrow r \leq 1 - \sqrt{\frac{\pi}{3}} / \sqrt{n} \Rightarrow |z^n - 1| = r^n \leq (1 - \sqrt{\frac{\pi}{3}} / \sqrt{n})^n \to 0 (n \to \infty)$. The assertion follows. \qed

A.4. Lemma. Given a Borel set $F \subset T$, $\mu F > 0$ there exists a function $f_F \in B_{H^\infty}$ such that

(a) $\|f_F(e^{i\theta})\|_\infty < 1$ a.e. on $T$

(b) $\|f_F 1_T\|_\infty < 1$

(c) $\|f_F 1_F\|_\infty = 1$

(d) $\|f_F^n - 1\|_\infty \to 1 (n \to \infty)$.

Proof. I use freely the identification, via radial boundary values, of $H^\infty$ ($= \mathcal{H}^\infty(\mathbb{C})$) with the space of bounded holomorphic functions on $D$ [34]. I can obviously assume $\mu F < 1$.

Let $h \in B_{H^\infty}$ be the outer function [34, 17.16] of modulus $|h| = 1_F + \frac{1}{2} 1_{T \setminus F}$ a.e. on $T$. Choose any point $e^{i\theta_0}$ (of $F$) where $\lim_{r \to 1} |h(re^{i\theta_0})| = 1$; after multiplication of $h$ with a unimodular constant I can assume $\lim_{r \to 1} h(re^{i\theta_0}) = 1$. Let $E$ be the set of A.3. By the Riemann–Caratheodory mapping theorem (the relatively easy statement [34, 14.19] suffices), there exists a homeomorphism $g: \overline{D} \to E$, $g|D \to E$ biholomorphic, $g(1) = 1$. Let $f_F := g \circ h \in B_{H^\infty}$, then a.e. on $T$: $\lim_{r \to 1} f_F(re^{i\theta}) = g(\lim_{r \to 1} h(re^{i\theta}))$ or $f_F(e^{i\theta}) = g(h(e^{i\theta}))$ for short.

(a) Thus, a.e. on $T$, $f_F(e^{i\theta}) \in E$, hence $= 1$ or of modulus $< 1$. By the identity theorem [34, 17.18], the first event cannot happen on a set of positive measure.

(b) Almost everywhere on $T \setminus F$ we have $|h(e^{i\theta})| = \frac{1}{2}$; since $g(\frac{1}{2} T) \subset D$ is compact, b follows.

(c) $\lim_{r \to 1} f_F(re^{i\theta}) = g(1) = 1$, hence, computing $\|f_F\|_\infty$ on $D$, $\|f_F\|_\infty = 1$ and $\|f_F 1_F\|_\infty = 1$ in view of b.
The range of \( f_F \) is contained in \( E \), hence \( \lim_{n \to \infty} \| f_F^n - 1 \|= 1 \). Since e.g. \( |f_F(0)| < 1 \), the relation \( \lim \geq 1 \) is trivial.

**Proof of A.1.** Let \( f \in S_{\text{H}(X)} \) be given, along with \( \varphi \in S_{\text{H}(X')}, \varphi(f) = 1 \). Use the same letter \( f \) for a fixed Hahn–Banach extension of \( \varphi \) over \( L^A(X) \), and decompose it into integral part (given by \( g \in L^A(X', X) \)) and singular part \( \varphi_s \) (see A.2). By definition of singularity, there exists a Borel set \( F \subseteq T, \mu F > 0 \), such that \( \varphi_s \) is concentrated on \( T \setminus F \) and \( \| f(e^{it}) \| \geq \frac{1}{2} \) a.e. on \( F \) (actually, I could assume \( \| f(e^{it}) \| = 1 \) in view of 3.5 but I do not need this). With \( f_F \) from A.4 put \( \gamma_n := \|1 - f_F^n\|_\infty \to 1 \), \( n \to \infty \) and \( f_n := (1/\gamma_n)f(1 - f_F^n) \in B_{\text{H}(X)} \). Then on the one hand, \( \varphi(f_n) = \langle f, f_n, g \rangle \ d\mu + \varphi_s(f_n) = (1/\gamma_n)\langle f, f_F^n, g \rangle \ d\mu + (1/\gamma_n)\varphi_s(f) \). Here the first summand tends to \( \langle f, g \rangle \ d\mu \) (A.4a) and dominated convergence. The second summand equals \( (1/\gamma_n)\varphi_s(1_{T \setminus F}(1 - f_F^n)) \to \varphi_s(1_{T \setminus F}f) = \varphi_s(f) \) (A.4b). Thus \( \varphi(f_n) \to \varphi(f) \). On the other hand, \( \| f - f^n \|= (1/\gamma_n)\| f(1 - f_F^n) - \gamma_n f \|_\infty = (1/\gamma_n)\| f_F^n - (1 - \gamma_n)f \|_\infty \geq (1/\gamma_n)\| f \|_\infty \). Since \( \| f \|_\infty \leq 1 \), the relation \( \lim \geq 1 \) is trivial. Thus \( \varphi(f_n) \to \varphi(f) \). Then on the other hand, \( \| f - f^n \|= (1/\gamma_n)\| f(1 - f_F^n) - \gamma_n f \|_\infty = (1/\gamma_n)\| f_F^n - (1 - \gamma_n)f \|_\infty \geq (1/\gamma_n)\| f \|_\infty \). Since \( \| f \|_\infty \leq 1 \), the relation \( \lim \geq 1 \) does not tend to \( f \), and \( f \) is not strongly exposed by \( \varphi \).

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