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An isoperimetric inequality for uniformly log-concave measures and uniformly convex bodies

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Abstract

We prove an isoperimetric inequality for the uniform measure on a uniformly convex body and for a class of uniformly log-concave measures (that we introduce). These inequalities imply (up to universal constants) the log-Sobolev inequalities proved by Bobkov, Ledoux [S.G. Bobkov, M. Ledoux, From Brunn–Minkowski to Brascamp–Lieb and to logarithmic Sobolev inequalities, Geom. Funct. Anal. 10 (5) (2000) 1028–1052] and the isoperimetric inequalities due to Bakry, Ledoux [D. Bakry, M. Ledoux, Lévy–Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator, Invent. Math. 123 (2) (1996) 259–281] and Bobkov, Zegarliński [S.G. Bobkov, B. Zegarliński, Entropy bounds and isoperimetry, Mem. Amer. Math. Soc. 176 (829) (2005), x+69]. We also recover a concentration inequality for uniformly convex bodies, similar to that proved by Gromov, Milman [M. Gromov, V.D. Milman, Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces, Compos. Math. 62 (3) (1987) 263–282].

Keywords: Isoperimetric inequalities; Uniformly convex

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1. Introduction

Let $V = (\mathbb{R}^n, \|\cdot\|)$ be a normed space, and let μ be a probability measure on V with density $f = \exp(-g)$, $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. If g is convex, the function f and the measure μ are called log-concave. Log-concave functions and measures boast many important properties (cf. Borell [16], Bobkov [11], etc.)

In this note, we study more restricted classes of measures. Let

$$\delta : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\},$$

and consider the following condition:

$$\frac{g(x) + g(y)}{2} - g\left(\frac{x+y}{2}\right) \geqslant \delta\big(\|x - y\|\big). \tag{1.1}$$

Example 1.1. The log-concavity condition corresponds to $\delta \equiv 0$.

By analogy with uniformly convex bodies (cf. Section 1.2.2), we define the modulus of convexity $\delta_{g,\|\cdot\|}$ of g with respect to the norm $\|\cdot\|$ as

$$\delta_{g,\|\cdot\|}(t) := \inf\left\{\frac{g(x) + g(y)}{2} - \left(\frac{x+y}{2}\right);\ \|x - y\| \geqslant t \text{ and } g(x), g(y) < \infty\right\}.$$

If $\delta_{g,\|\cdot\|}(t) > 0$ for all $t > 0$, we say that f and μ are uniformly log-concave, and that g is uniformly convex. Obviously, this notion does not depend on the choice of the norm $\|\cdot\|$.

It is easy to check that $\delta_{g,\|\cdot\|}(t)/t$ is always a non-decreasing function of t; therefore in the sequel we consider measures μ satisfying (1.1) with respect to a function δ such that

$$\begin{cases} \delta(t) > 0, & t > 0, \\ t \mapsto \delta(t)/t & \text{is non-decreasing.} \end{cases} \tag{1.2}$$

Example 1.2. Let $\|\cdot\| = |\cdot|$ be the Euclidean norm, and let $\delta(t) = t^2/8$. Then (1.1) holds iff μ has log-concave density with respect to the standard Gaussian measure; in other words, if μ satisfies the Bakry–Émery curvature–dimension condition $CD(1, +\infty)$ (cf. Bakry and Émery [3]; recall that the usual log-concavity of μ is equivalent to $CD(0, +\infty)$).

Remark 1.3. The condition (1.1) is translation invariant. Therefore one may extend it to measures on an affine space $\mathbb{A}^n$ on which V acts by translations; note that both sides of (1.1) are still defined. This point of view will be convenient in Section 2.

1.0. Assumptions and notation

Unless mentioned otherwise, the sets in this note are Borel subsets of $\mathbb{R}^n$, and the measures are Borel measures on $\mathbb{R}^n$.

The Lipschitz norm of a map $T : V_1 \to V_2$ between two normed spaces $V_i = (X_i, \|\cdot\|_i)$, $i = 1, 2$, is defined as

$$\|T\|_{\mathrm{Lip}} = \sup_{x,y\in X_1, x\neq y} \frac{\|T(x) - T(y)\|_2}{\|x - y\|_1}. \tag{1.3}$$

T is called Lipschitz if $\|T\|_{\mathrm{Lip}} < \infty$. If

$$\sup_{x,y\in K, x\neq y} \frac{\|T(x)-T(y)\|_2}{\|x-y\|_1} < +\infty$$

for any compact subset $K \subset X_1$, T is called locally Lipschitz.

A Borel map $T: V_1 \to V_2$ is said to push a measure μ on V_1 forward to a measure λ on V_2 (notation: $T_*\mu = \lambda$) if $\mu(T^{-1}(B)) = \lambda(B)$ for every $B \subset X_2$.

If μ is a probability measure on $V = (X, \|\cdot\|)$, the Minkowski boundary measure associated with μ (and $\|\cdot\|$) is defined by

$$\mu^+_{\|\cdot\|}(A) = \liminf_{\varepsilon\to 0} \frac{\mu(A_{\varepsilon,\|\cdot\|}) - \mu(A)}{\varepsilon}, \quad A \subset X, \tag{1.4}$$

where

$$A_{\varepsilon,\|\cdot\|} = \bigl\{x \in X \bigm| \exists y \in A,\ \|x-y\| < \varepsilon\bigr\}$$

is the ε-extension of A in the metric induced by $\|\cdot\|$. In addition, we denote:

$$\widetilde{\mu(A)} = \min\bigl(\mu(A), 1-\mu(A)\bigr)$$

for all $A \subset X$. Lastly, we denote the Lebesgue measure on $\mathbb{R}^n$ by mes_n.

1.1. Isoperimetric inequalities

The first topic of this note is an isoperimetric inequality for μ. In the setting of Example 1.2 (and actually in a much more abstract one), Bakry and Ledoux proved [4] the following isoperimetric inequality.

Theorem *(Bakry–Ledoux). If the measure μ satisfies* (1.1) *with $\|\cdot\| = |\cdot|$ and $\delta(t) = t^2/8$, then for any $A \subset \mathbb{R}^n$*

$$\mu^+_{|\cdot|}(A) \geqslant \phi\bigl(\Phi^{-1}\bigl(\widetilde{\mu(A)}\bigr)\bigr). \tag{1.5}$$

Here as usual $\phi(t) = \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)$ and $\Phi(t) = \int_{-\infty}^t \phi(s)\,ds$.

This theorem is a generalisation of the isoperimetric inequality for the Gaussian measure, proved by Sudakov, Tsirelson, and Borell [17,36]. In [12], Bobkov gave a proof of the Bakry–Ledoux inequality using the localisation technique; the latter was introduced by Gromov and Milman [26] and developed by Kannan, Lovász and Simonovits [28,32] (see also Gromov [25, Section $3\frac{1}{2}$.27]). We extend Bobkov's approach to the general case (1.1) and prove:

Theorem 1.1. *Suppose μ satisfies* (1.1) *and* (1.2)*. Then*

$$\mu^+_{\|\cdot\|}(A) \geqslant C_\delta \widetilde{\mu(A)}\, \gamma\left(\log\frac{1}{\widetilde{\mu(A)}}\right) \quad \textit{for all } A \subset \mathbb{R}^n, \tag{1.6}$$

where

$$C_\delta = \frac{e-1}{2e\max(2\delta(\int_0^{+\infty}\exp(-2\delta(t))\,dt), 1)},$$
$$\gamma(t) = \frac{t}{\delta^{-1}(t/2)}, \quad \textit{and} \quad \widetilde{\mu(A)} = \min\bigl(\mu(A), 1-\mu(A)\bigr).$$

Corollary 1.2. *Let $\delta(t) = \alpha t^p$ for $p \geqslant 2$ and $\alpha > 0$ in the setting of the previous theorem. Then*

$$\mu^+_{\|\cdot\|}(A) \geqslant c\alpha^{1/p}\widetilde{\mu(A)}\log^{1-1/p}\frac{1}{\widetilde{\mu(A)}}, \tag{1.7}$$

where $c > 0$ is a universal constant (*independent of p*).

Remark 1.4. Note that p cannot be less than 2; this follows from a second-order Taylor expansion of g in (1.1).

Remark 1.5. For $p = 2$, Corollary 1.2 recovers the Bakry–Ledoux theorem up to a universal constant: indeed,

$$\phi\bigl(\Phi^{-1}(t)\bigr) \leqslant C' t\sqrt{\log 1/t}, \quad 0 \leqslant t \leqslant 1/2.$$

Remark 1.6. In [11], Bobkov proved that the following inequality holds for any log-concave measure μ and any $r > 0$:

$$\mu^+_{\|\cdot\|}(A) \geqslant \frac{1}{2r}\left\{\mu(A)\log\frac{1}{\mu(A)} + \bigl(1-\mu(A)\bigr)\log\frac{1}{1-\mu(A)} + \log\mu\bigl\{\|x\| \leqslant r\bigr\}\right\}. \tag{1.8}$$

In particular, (1.8) implies a non-trivial isoperimetric inequality for measures satisfying (1.1), (1.2). However, this inequality would become weaker in higher dimension, whereas our results are dimension-free.

1.2. Application: uniformly convex bodies

As before, let $V = (\mathbb{R}^n, \|\cdot\|)$ be a normed space. The volume measure $\lambda = \lambda_V$ on the unit ball of V is defined by

$$\lambda = \frac{\mathrm{mes}_n|_{\{\|x\| \leqslant 1\}}}{\mathrm{mes}_n(\{\|x\| \leqslant 1\})}; \tag{1.9}$$

it arises naturally in geometric applications.

We would like to prove an isoperimetric inequality for λ, with respect to the norm $\|\cdot\|$. It is easy to see that λ never satisfies the condition (1.1) with $\delta > 0$. Therefore we follow the approach introduced by Bobkov and Ledoux [14] and define an auxiliary measure μ that satisfies (1.1).

1.2.1. p-Uniformly convex bodies

Choose $p \geqslant 2$, and let μ be the measure with density

$$\frac{\exp(-\|x\|^p)}{\Gamma(1+n/p)\mathrm{mes}_n(\{\|x\| \leqslant 1\})}, \tag{1.10}$$

with respect to the Lebesgue measure.

Proposition *(Bobkov–Ledoux). There exists a map $S: V \to V$ such that $S_*\mu = \lambda$ and $\|S\|_{\mathrm{Lip}} \leqslant C(\Gamma(1+n/p))^{-1/n}$, where $C > 0$ is a universal constant.*

It is clear that Lipschitz maps preserve isoperimetric inequalities, so we may first establish one for μ. The condition (1.1) for μ, with $\delta(t) = \alpha t^p$, reads as

$$\frac{\|x\|^p + \|y\|^p}{2} - \left\|\frac{x+y}{2}\right\|^p \geqslant \alpha\|x-y\|^p \quad \text{for all } x, y \in \mathbb{R}^n. \tag{1.11}$$

This is one of the definitions of a p-uniformly convex norm (cf. Pisier [35]).

Example 1.7. The ℓ_q norm $\|\cdot\|_q$, $1 < q < \infty$, satisfies (1.11) with

$$p = \begin{cases} 2, & q < 2, \\ q, & q \geqslant 2, \end{cases} \qquad \alpha = \begin{cases} \frac{q-1}{4}, & q < 2, \\ 2^{-q}, & q \geqslant 2. \end{cases}$$

In fact, the same estimates holds for the space L_q. The case $q \geqslant 2$ is due to Clarkson [18] (see also Hanner [27]), while the case $q < 2$ follows from an unpublished argument of Ball and Pisier (see Ball, Carlen and Lieb [7]).

Therefore, if $\|\cdot\|$ is p-uniformly convex with coefficient α (that is, if (1.11) holds), we can apply Corollary 1.2 and deduce (1.7). Combining with the Bobkov–Ledoux proposition above, we obtain the following.

Theorem 1.3. *Suppose the space V is p-uniformly convex with constant α (that is, satisfies (1.11)); let λ be the uniform measure on the unit ball of $\|\cdot\|$ (as in (1.9)). Then for any $A \subset \mathbb{R}^n$:*

$$\lambda^+_{\|\cdot\|}(A) \geqslant C\alpha^{1/p} n^{1/p} \widetilde{\lambda(A)} \log^{1-1/p} \frac{1}{\widetilde{\lambda(A)}}, \tag{1.12}$$

where $C > 0$ is a universal constant.

This theorem continues the study of isoperimetric properties of p-uniformly convex bodies by Bobkov and Zegarliński [15, Chapter 14]. In particular, when $\lambda(A)$ is not exponentially small in the dimension, the inequality in Theorem 1.3 improves the bound in [15, Theorem 14.6]. Under the same restriction, (1.12) improves (1.8) with $r = 1$ (which is however best possible in the class of all convex bodies).

Remark 1.8. Here, as well as in Theorem 1.6 below, one may use an isoperimetric inequality due to Barthe [8] (which extends (1.8)) and get a better bound for exponentially small sets. We do not pursue this point.

1.2.2. General uniformly convex bodies

We also generalise the above results to arbitrary uniformly convex spaces. Recall that the modulus of convexity $\delta_V : [0,2] \to [0,1]$ of a normed space $V = (X, \|\cdot\|)$ is defined as

$$\delta_V(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|;\ \|x\|, \|y\| \leqslant 1,\ \|x-y\| \geqslant \varepsilon\right\}.$$

The space is called uniformly convex if $\delta_V(\varepsilon) > 0$ for all $\varepsilon > 0$. From the works of Figiel [20], Figiel, Pisier [21] and Pisier [35], it is known that if

$$\delta_V(\varepsilon) \geqslant \alpha' \varepsilon^p \quad \text{for all } \varepsilon \in [0,2], \tag{1.13}$$

then (1.11) holds with $\alpha = \min(c, \alpha'/2^p)$, and that if (1.11) holds then (1.13) holds with $\alpha' = \alpha/p$ (here $c > 0$ is a universal constant). A space is therefore p-uniformly convex if either (1.11) or (1.13) hold, it is however important to specify which definition one uses if the dependence on p is of interest.

In Section 4 we derive the following proposition from the results of Figiel, Pisier [21].

Proposition 1.4. *For all $x, y \in X$ such that $\|x\|^2 + \|y\|^2 \leqslant 2$, one has*:

$$\frac{\|x\|^2 + \|y\|^2}{2} - \left\|\frac{x+y}{2}\right\|^2 \geqslant c\delta_V\left(\frac{\|x-y\|}{4}\right),$$

where $c > 0$ is a universal constant.

Returning to the case $X = \mathbb{R}^n$, choose μ to be the probability measure with density

$$f(x) = \frac{1}{Z}\exp\left(-\frac{n}{c}\|4x\|^2\right)\mathbf{1}\left\{\|x\| \leqslant \frac{1}{4}\right\} \tag{1.14}$$

with respect to the Lebesgue measure, where $Z > 0$ is a scaling factor. Proposition 1.4 clearly implies that μ is uniformly log-concave, so we can apply Theorem 1.1 and deduce an isoperimetric inequality for μ. To transfer this inequality to the measure λ_V, we need to extend the Bobkov–Ledoux proposition of the previous subsection. Our next observation, which may be of independent interest, does precisely that.

Definition. A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called radial if it maps every ray to itself in a monotone way; that is, if for every $x \neq 0$

$$\begin{cases} T(\mathbb{R}_+ x) \subset \mathbb{R}_+ x & \text{and} \\ T|_{\mathbb{R}_+ x} : \mathbb{R}_+ x \to \mathbb{R}_+ x \text{ preserves the order on } \mathbb{R}_+ x. \end{cases}$$

Let $d\mu = f\, d\mathrm{mes}_n$ be an even log-concave probability measure (with log-concave density f). Denote

$$K_f = \left\{ x \in \mathbb{R}^n;\ n \int_0^{+\infty} f(rx) r^{n-1}\, dr \geqslant 1 \right\}. \tag{1.15}$$

It is not hard to see (cf. Proposition 3.1) that there exists a canonical radial map T_f pushing forward μ to the restriction λ of the Lebesgue measure to K_f.

K. Ball showed [6] that K_f is a symmetric convex body; in other words, the unit ball of a norm $\|\cdot\|_{K_f}$. In Section 3 we prove the following result (in a slightly more general form).

Theorem 1.5. *Let $d\mu = f\, d\mathrm{mes}_n$ be an even log-concave probability measure (with log-concave density f); let λ denote the restriction of the Lebesgue measure on K_f, and let $T = T_f$ denote the canonical radial map such that $T_*\mu = \lambda$. Then as a map $T : V \to V$ where $V = (\mathbb{R}^n, \|\cdot\|_{K_f})$, we have $\|T\|_{\mathrm{Lip}} \leqslant C f(0)^{1/n}$, where $C > 0$ is a universal constant.*

Remark 1.9. The Bobkov–Ledoux proposition above is a particular case of the last theorem (up to another universal constant). We provide the details at the end of Section 3.2.

In Section 4 we apply Theorems 1.1 and 1.5 to deduce the following.

Theorem 1.6. *Let $V = (\mathbb{R}^n, \|\cdot\|)$ be a uniformly convex space, and let $\delta = \delta_V$ denote its modulus of convexity. Let $\lambda = \lambda_V$ denote the uniform measure on the unit-ball of V (as in (1.9)) and let $A \subset \mathbb{R}^n$. Then*

$$\lambda^+_{\|\cdot\|}(A) \geqslant c' C_{n,\delta} \frac{\widetilde{\lambda(A)} \log \frac{1}{\widetilde{\lambda(A)}}}{\delta^{-1}(\frac{1}{2n} \log \frac{1}{\widetilde{\lambda(A)}})},$$

where

$$C_{n,\delta} = \frac{e-1}{2e \max(n\delta(\int_0^{1/4} \exp(-2n\delta(t))\, dt), 1)}, \tag{1.16}$$

and $c' > 0$ is a universal constant.

Note that when $\delta(t) = \alpha t^p$ ($p \geqslant 2$), Theorem 1.6 recovers Theorem 1.3 up to a universal constant.

1.3. Connection to functional inequalities and concentration

In this subsection we study some corollaries of the isoperimetric inequalities of the form (1.6) and (1.7).

1.3.1. Concentration

It is well known that an isoperimetric inequality can be equivalently rewritten in global form. It will be convenient to use this in the following formulation (see Bobkov and Zegarliński [15, p. 46] for an equivalent form).

Proposition 1.7. *Let μ be a probability measure on $\mathbb{R}^n$ satisfying*

$$\mu^+_{\|\cdot\|}(A) \geqslant \widetilde{\mu(A)}\gamma\left(\log\frac{1}{\widetilde{\mu(A)}}\right) \tag{1.17}$$

for every Borel set $A \subset \mathbb{R}^n$ and some continuous function $\gamma : [\log 2, +\infty) \to \mathbb{R}_+$. Then for any Borel set $B \subset \mathbb{R}^n$ and any $\varepsilon > 0$

$$1 - \mu(B_{\varepsilon,\|\cdot\|}) \leqslant \exp\bigl(-h^{-1}_{1-\mu(B)}(\varepsilon)\bigr), \tag{1.18}$$

where

$$h_a(x) = \int_{\log 1/a}^{x} \frac{dy}{\gamma(y)}; \tag{1.19}$$

for $y < \log 2$, $\gamma(y)$ should be interpreted as $\gamma(\log\frac{1}{1-\exp(-y)})$.

Conversely, if μ satisfies (1.18) *for any Borel set $B \subset \mathbb{R}^n$, then* (1.17) *holds.*

Corollary 1.8. *Let μ be a measure on $\mathbb{R}^n$ such that for all $A \subset \mathbb{R}^n$*

$$\mu^+_{\|\cdot\|}(A) \geqslant \mathbf{c}_0\widetilde{\mu(A)}\log^{1-1/p}\frac{1}{\widetilde{\mu(A)}}. \tag{1.20}$$

Then for every $B \subset \mathbb{R}^n$, $\mu(B) \geqslant 1/2$, and every $\varepsilon > 0$,

$$1 - \mu(B_{\varepsilon,\|\cdot\|}) \leqslant \exp\left\{-\left[\log^{1/p}\frac{1}{1-\mu(B)} + \frac{\mathbf{c}_0\varepsilon}{p}\right]^p\right\}. \tag{1.21}$$

In Section 5.1 we combine Proposition 1.7 and Corollary 1.8 with the results of the previous subsections, to deduce a concentration inequality for uniformly convex bodies. Then we compare this inequality with the Gromov–Milman theorem [26].

For completeness, we prove Proposition 1.7 in Section 5.2.

1.3.2. Functional inequalities

An isoperimetric inequality can be written in a functional form; this was brought forth by Maz'ya, Federer and Fleming [19,33] in the early 1960s and later adapted by Bobkov and Houdré [13] to the context of probability measures.

Proposition *(Bobkov–Houdré). Let μ be a probability measure on a normed space $(\mathbb{R}^n, \|\cdot\|)$, and let $I : [0, 1/2] \to \mathbb{R}_+$ be an increasing continuous function such that $I(0) = 0$. The following are equivalent*:

1. *For any Borel set* $A \subset \mathbb{R}^n$,

$$\mu^+_{\|\cdot\|}(A) \geqslant I\big(\widetilde{\mu(A)}\big). \tag{1.22}$$

2. *For any locally Lipschitz function* $F : \mathbb{R}^n \to [0, 1]$ *such that*

$$\mu\{F = 1\} \geqslant t \in (0, 1/2) \quad \text{and} \quad \mu\{F = 0\} \geqslant 1/2, \tag{1.23}$$

we have

$$\int \|\nabla F\|_* \, d\mu \geqslant I(t), \tag{1.24}$$

where

$$\|\nabla F\|_* = \limsup_{y \to x} \frac{|F(y) - F(x)|}{\|y - x\|}.$$

Let us focus on the case $I(t) = \mathbf{c}_0 t \log^{1/q} 1/t$, where $1/q = 1 - 1/p$. We have the following:

Proposition 1.9. *Suppose a probability measure* μ *on* $(\mathbb{R}^n, \|\cdot\|)$ *satisfies*

$$\mu^+_{\|\cdot\|}(A) \geqslant \mathbf{c}_0 \widetilde{\mu(A)} \log^{1/q} \frac{1}{\widetilde{\mu(A)}} \tag{1.25}$$

for all $A \subset \mathbb{R}^n$. *Then*:

1. *For any locally Lipschitz function* $F : \mathbb{R}^n \to [0, 1]$ *satisfying* (1.23)*, we have*:

$$\int \|\nabla F\|_* \, d\mu \geqslant \mathbf{c}_0 t \log^{1/q} 1/t. \tag{1.26}$$

2. *For any locally Lipschitz function* $F : \mathbb{R}^n \to [0, 1]$ *satisfying* (1.23)*, we have*:

$$\int \|\nabla F\|_*^q \, d\mu \geqslant c\mathbf{c}_0^q t \log 1/t, \tag{1.27}$$

where $c > 0$ *is a universal constant.*

3. *For any locally Lipschitz function* $F : \mathbb{R}^n \to \mathbb{R}_+$,

$$\int \|\nabla F\|_*^q \, d\mu \geqslant c'\mathbf{c}_0^q \int F^q \log \frac{F^q}{\int F^q d\mu} \, d\mu, \tag{1.28}$$

where $c' > 0$ *is a universal constant.*

Of course, part 1 follows from the previous proposition (and in fact, (1.26) is equivalent to (1.17)). Then, part 1 implies part 2 via standard arguments that we reproduce for completeness in Section 5. Finally, part 2 is equivalent to part 3 (up to universal constants); this is a reformulation of the arguments developed by Bobkov and Zegarliński [15, Chapter 5] in the language of capacities put forth by Barthe and Roberto [9].

The inequality (1.28), called a q-log-Sobolev inequality, was studied by Bobkov and Ledoux [14] and Bobkov and Zegarliński [15]. In particular, part 3 of the last proposition extends Theorem 16.3 in [15]. Combining it with Theorems 1.1 and 1.3, we recover the q-log-Sobolev inequalities proved by Bobkov and Ledoux in [14], up to universal constants.

2. An isoperimetric inequality

2.1. Reduction to one dimension

This subsection is based on an argument that was introduced by Gromov and Milman [26] to reduce the spherical isoperimetric inequality to a certain one-dimensional fact; see also Gromov [25, Section $3\frac{1}{2}$.27] and Alesker [1]. The corresponding argument in the affine case was developed by Kannan, Lovász and Simonovits [28,32], who also coined the term ‘localisation lemma’; a different approach was put forth by Fradelizi and Guédon [23,24].

We formulate the localisation lemma in terms of μ-*needles*, as put forth by S. Bobkov; this corresponds to *convex descendants* in [25]. It will be natural to work in an n-dimensional affine space $\mathbb{A}^n$ (cf. Remark 1.3).

Let $V = (\mathbb{R}^n, \|\cdot\|)$ be a normed space acting by translations on an affine space $\mathbb{A}^n$. Let μ be a probability measure on $\mathbb{A}^n$ such that $\mu(H) = 0$ for every affine hyperplane $H \subset \mathbb{A}^n$.

Definition. A (probability) measure σ supported on an affine line $L \subset \mathbb{A}^n$ (and not on any point) is called a μ-needle if

$$\sigma = \lim_{k \to +\infty} \mu|_{C_k} / \mu(C_k)$$

is the weak limit of the scaled restrictions of μ to convex sets

$$C_1 \supset C_2 \supset C_3 \supset \cdots, \quad \mu(C_k) > 0.$$

If the measure μ admits a lower semicontinuous density f with respect to the Lebesgue measure, the definition can be made more explicit (see [24,32]). We will only use the following property (see e.g. [32, Lemma 2.5]).

Description of μ-needles. If ν is a μ-needle supported on L, then μ is absolutely continuous with respect to the Lebesgue measure on L, and its density is equal to $f|_L \phi$ for some log-concave function ϕ on L.

Localisation principle: global form. Let μ be a probability measure on $\mathbb{A}^n$ such that $\mu(H) = 0$ for every affine hyperplane $H \subset \mathbb{R}^n$; let $a, b \in (0, 1)$, $\varepsilon > 0$. If every μ-needle σ supported on an affine line L_σ satisfies

$$\sigma(A'_\varepsilon) \geqslant b \quad \text{for every } A' \subset L_\sigma \text{ such that } \sigma(A') = a, \tag{2.1}$$

then also

$$\mu(A_\varepsilon) \geqslant b \quad \text{for every } A \subset \mathbb{A}^n \text{ such that } \mu(A) = a. \tag{2.2}$$

This is essentially the first step in [32]. It will be more convenient to obtain an infinitesimal form of this localisation principle. Given an isoperimetric inequality in the general form

$$\mu^+(A) \geqslant I\big(\widetilde{\mu(A)}\big),$$

where $I : [0, 1/2] \to \mathbb{R}_+$ is a continuous function, we may of course write $I(a) = a\gamma(\log 1/a)$ for some continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, obtaining the form in (1.17). By Proposition 1.7, a local isoperimetric inequality of the form (1.17) is equivalent to the global inequality (1.18). Applying this twice, we deduce the following.

Localisation principle: local form. Let μ be a probability measure on $\mathbb{A}^n$ such that $\mu(H) = 0$ for every affine hyperplane $H \subset \mathbb{A}^n$, and let $I : [0, 1/2] \to \mathbb{R}_+$ denote a continuous function. If every μ-needle σ supported on an affine line L_σ satisfies

$$\sigma^+_{\|\cdot\|}(A') \geqslant I\big(\widetilde{\sigma(A')}\big) \quad \text{for every } A' \subset L_\sigma, \tag{2.3}$$

then also

$$\mu^+_{\|\cdot\|}(A) \geqslant I\big(\widetilde{\mu(A)}\big) \quad \text{for every } A \subset \mathbb{A}^n. \tag{2.4}$$

To complete the reduction to one dimension, let us show that "if μ is uniformly log-concave, its needles are also uniformly log-concave." The following lemma extends [12], [25, Section $3\frac{1}{2}$.27, Example (e)].

Lemma 2.1. *Let $V = (\mathbb{R}^n, \|\cdot\|)$ be a normed space acting by translations on an affine space $\mathbb{A}^n$, and let $\delta : \mathbb{R}_+ \to \mathbb{R}_+$. If a measure μ on $\mathbb{A}^n$ satisfies the uniform log-concavity condition* (1.1) *with respect to δ and $\|\cdot\|$, then every μ-needle σ supported on an affine line $L \subset \mathbb{A}^n$ satisfies* (1.1) *with respect to δ and the restriction of $\|\cdot\|$ to the tangent space $L - L$.*

Sketch of proof. Let f denote the density of μ with respect to the Lebesgue measure on $\mathbb{A}^n$. μ satisfies (1.1), hence f is in particular log-concave. The super-level sets of f are convex, hence f is equivalent to a lower semi-continuous density. Therefore the description of needles formulated above is valid.

Now, $f|_L$ satisfies (1.1) with respect to δ and $\|\cdot\|_{L-L}$. Since ϕ satisfies (1.1) with respect to $\delta' \equiv 0$, it follows that $f|_L\phi$ satisfies (1.1) with respect to $\delta + \delta' = \delta$ and $\|\cdot\|_{L-L}$. □

By the lemma, it is sufficient to prove Theorem 1.1 for $n = 1$. In this case, we only need the following property of one-dimensional uniformly log-concave measures.

Lemma 2.2. *Let $V = (\mathbb{R}^n, \|\cdot\|)$, and assume that $g : V \to \mathbb{R} \cup \{+\infty\}$ satisfies* (1.1). *Assume in addition that a is a minimum point of g. Then*

$$g(x) - g(a) \geqslant 2\delta\big(\|x - a\|\big), \tag{2.5}$$

for all $x \in \mathbb{R}^n$.

Proof. If $g(x) = +\infty$, the claim is trivial. Otherwise, apply (1.1) with $y = a$. Then

$$\delta\big(\|x-a\|\big) \leqslant \frac{g(x)-g(a)}{2} + g(a) - g\left(\frac{x+a}{2}\right) \leqslant \frac{g(x)-g(a)}{2},$$

where we used the fact that a is a minimum point of g in the last inequality. □

We will prove the isoperimetric inequality for one-dimensional measures μ with density $f = \exp(-g)$, where g satisfies (2.5). Any norm on $\mathbb{R}^1$ is Euclidean, hence without loss of generality $\|\cdot\| = |\cdot|$. Therefore Theorem 1.1 is reduced to the following proposition (note the factor 2 that we drop between (2.5) and (2.6) to simplify the notation).

Proposition 2.3. *Let σ denote a probability measure on $\mathbb{R}$ with density f. Assume that $f = \exp(-g)$, where $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a convex function with minimum at 0 and such that*

$$g(x) - g(0) \geqslant \delta\big(|x|\big) \tag{2.6}$$

for all $x \in \mathbb{R}$, and $\delta : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies (1.2). Then

$$\sigma^+(A) \geqslant C_\delta \widetilde{\sigma(A)} \gamma\left(\log \frac{1}{\widetilde{\sigma(A)}}\right) \tag{2.7}$$

for any $A \subset \mathbb{R}$, where

$$C_\delta = \frac{e-1}{2e \max(\delta(\int_0^{+\infty} \exp(-\delta(t))\, dt), 1)}, \qquad \gamma(t) = \frac{t}{\delta^{-1}(t)}.$$

2.2. Proof of the one-dimensional inequality

Before proceeding to the proof of Proposition 2.3, we collect several easy observations, using the same notation as in the proposition.

Lemma 2.4. *The function γ is non-decreasing. The function $x\gamma(\log \frac{1}{x})$ is strictly increasing on $[0, 1/e]$.*

Proof. The first part follows since $\delta(x)/x$ is non-decreasing by our assumption (1.2). For the second part, write

$$x\gamma\left(\log \frac{1}{x}\right) = \frac{x \log \frac{1}{x}}{\delta^{-1}(\log \frac{1}{x})},$$

so the claim follows since δ (and hence δ^{-1}) is non-decreasing, whereas $x \log \frac{1}{x}$ is increasing on $[0, 1/e]$. □

Now denote:

$$M_\delta = \int_0^\infty \exp\big(-\delta(x)\big)\, dx.$$

Lemma 2.5.

$$\exp\bigl(-g(0)\bigr) \geqslant (2M_\delta)^{-1}.$$

Proof. Using $\int f(x)\,dx = 1$ and (2.6) we have:

$$1 = \int_{\mathbb{R}} \exp\bigl(-g(x)\bigr)\,dx \leqslant \exp\bigl(-g(0)\bigr) \int_{\mathbb{R}} \exp\bigl(-\delta\bigl(|x|\bigr)\,dx. \qquad \square$$

Lemma 2.6. *If g is a convex function on $\mathbb{R}$ with minimum at 0, then for all $x > 0$,*

$$\int_x^\infty \exp\bigl(-g(y)\bigr)\,dy \leqslant \frac{x}{g(x) - g(0)} \exp\bigl(-g(x)\bigr).$$

Proof. By convexity, it follows that for all $y \geqslant x$,

$$g(y) \geqslant \frac{g(x) - g(0)}{x}(y - x) + g(x).$$

Using this to bound $\int_x^\infty \exp(-g(y))\,dy$ from above, the claim follows. $\square$

Given a finite measure μ on $\mathbb{R}$, we denote by $m(\mu)$ its median, i.e. (any) number m for which $\mu((-\infty, m]) \geqslant \mu(\mathbb{R})/2$ and $\mu([m, \infty)) \geqslant \mu(\mathbb{R})/2$.

Lemma 2.7. *For any finite log-concave measure $d\mu = f\,dx$ on $\mathbb{R}$,*

$$f\bigl(m(\mu)\bigr) \geqslant \frac{1}{2} \max_{x \in \mathbb{R}} f(x). \tag{2.8}$$

Proof. Without loss of generality, assume $m = m(\mu) > 0$, $f(0) = \max f$ and $f(m) < f(0)$. Then f is non-increasing on $\mathbb{R}_+$. Replace μ with $\mu|_{\mathbb{R}_+}$. Then the left-hand side of (2.8) may only decrease, whereas the right-hand side retains its value.

Now replace f by a log-affine function f_1 on $\mathbb{R}_+$ such that $f_1(0) = f(0)$ and $f_1(m) = f(m)$. In other words $f_1(x) = \exp(-ax + b)|_{\mathbb{R}_+}$, and our assumptions imply that $a > 0$. Setting $d\mu_1 = f_1 dx$, μ_1 is a finite measure. Then $f_1 \leqslant f$ on $[0, m]$ and $f_1 \geqslant f$ on $[m, +\infty)$; hence $m(\mu_1) \geqslant m(\mu)$ and $f(m(\mu)) = f_1(m(\mu)) \geqslant f_1(m(\mu_1))$.

Finally,

$$f_1\bigl(m(\mu_1)\bigr) = \frac{1}{2} \max_{x \in \mathbb{R}_+} f_1(x);$$

this concludes the proof. $\square$

Proof of Proposition 2.3. By a general result of Bobkov [10, Proposition 2.1] on extremal isoperimetric sets of log-concave densities, it is enough to verify (2.7) on sets A of the form

$(-\infty, a]$ and $[b, \infty)$. Given a point $x \in \mathbb{R}$, denote $A = [x, \infty)$ if $x \geqslant 0$ and $A = (-\infty, x]$ if $x < 0$. We will show that the set A satisfies

$$\sigma^+(A) \geqslant C_\delta \widetilde{\sigma(A)} \gamma\left(\log \frac{1}{\widetilde{\sigma(A)}}\right),$$

and this will conclude the proof. Assume without loss of generality that $x \geqslant 0$, since our hypotheses are symmetric about the origin.

First, recall that by another result of Bobkov [11, Proposition 4.1], a log-concave probability measure μ with density f on $\mathbb{R}$ always satisfies the following Cheeger-type isoperimetric inequality:

$$\mu^+(A) \geqslant 2f(m) \min\bigl(\mu(A), 1 - \mu(A)\bigr),$$

where m is the median of μ. Together with Lemma 2.7, this implies

$$\sigma^+(A) \geqslant \exp\bigl(-g(0)\bigr)\widetilde{\sigma(A)}. \tag{2.9}$$

Loosely speaking, this Cheeger-type inequality will take care of the case when $\widetilde{\sigma(A)}$ is large. The case when $\widetilde{\sigma(A)}$ is small will be handled by Lemma 2.6, which, together with the assumption (2.6) and the fact that δ is increasing, imply that for any $x > 0$,

$$\sigma(A) = \int_x^\infty \exp\bigl(-g(y)\bigr)\, dy \leqslant \frac{\delta^{-1}(g(x) - g(0))}{g(x) - g(0)} \exp\bigl(-g(x)\bigr).$$

Recalling the definition of γ and denoting $\sigma^+_{\max} = \exp(-g(0))$, this means

$$\sigma(A) \leqslant \frac{\sigma^+(A)}{\gamma(g(x) - g(0))} = \frac{\sigma^+(A)}{\gamma(\log \frac{\sigma^+_{\max}}{\sigma^+(A)})}. \tag{2.10}$$

This inequality is almost what we need, and the rest of the proof will be dedicated to replacing σ^+ with σ inside the γ function.

More formally, we distinguish between five cases.

1. $\widetilde{\sigma(A)} \geqslant c_\delta$, where $c_\delta \leqslant 1/e$ depends solely on δ and will be determined later. In this case, by (2.9) and Lemma 2.5,

$$\sigma^+(A) \geqslant \exp\bigl(-g(0)\bigr)\widetilde{\sigma(A)} \geqslant \frac{1}{2M_\delta}\widetilde{\sigma(A)}.$$

The function γ is non-decreasing by Lemma 2.4, therefore

$$\sigma^+(A) \geqslant \frac{1}{2M_\delta \gamma(\log \frac{1}{c_\delta})} \widetilde{\sigma(A)} \gamma\left(\log \frac{1}{\widetilde{\sigma(A)}}\right).$$

2. $1-\sigma(A)=\widetilde{\sigma(A)}<c_\delta$ and $g(x)-g(0)<\log\frac{1}{c_\delta}$. Using (2.6):

$$\sigma(A)\leqslant\int_0^\infty\exp(-g(y))\,dy\leqslant\exp(-g(0))\int_0^\infty\exp(-\delta(y))\,dy,$$

and since $g(x)-g(0)<\log\frac{1}{c_\delta}$ we conclude that

$$1-c_\delta<\sigma(A)\leqslant\frac{1}{c_\delta}\exp(-g(x))M_\delta=\frac{M_\delta}{c_\delta}\sigma^+(A).$$

By Lemma 2.4, $x\gamma(\log\frac{1}{x})$ is monotone increasing on $[0,1/e]$. Since $\widetilde{\sigma(A)}<c_\delta\leqslant 1/e$, we conclude that

$$\sigma^+(A)\geqslant\frac{(1-c_\delta)c_\delta\gamma(\log\frac{1}{c_\delta})}{M_\delta\gamma(\log\frac{1}{c_\delta})}\geqslant\frac{(1-c_\delta)}{M_\delta\gamma(\log\frac{1}{c_\delta})}\widetilde{\sigma(A)}\gamma\left(\log\frac{1}{\widetilde{\sigma(A)}}\right).$$

3. $\sigma(A)=\widetilde{\sigma(A)}<c_\delta$ and $g(x)-g(0)<\log\frac{1}{c_\delta}$. As in part 2,

$$\begin{aligned}1-\sigma(A)&=\int_{-\infty}^0\exp(-g(y))\,dy+\int_0^x\exp(-g(y))\,dy\\&\leqslant\exp(-g(0))M_\delta+\exp(-g(0))x\leqslant\frac{1}{c_\delta}\exp(-g(x))(M_\delta+x).\end{aligned}$$

Using (2.6) and the inequality $g(x)-g(0)<\log\frac{1}{c_\delta}$,

$$x\leqslant\delta^{-1}(g(x)-g(0))\leqslant\delta^{-1}\left(\log\frac{1}{c_\delta}\right).$$

Hence

$$1-c_\delta\leqslant 1-\sigma(A)\leqslant\frac{M_\delta+\delta^{-1}(\log\frac{1}{c_\delta})}{c_\delta}\sigma^+(A).$$

Now choose

$$c_\delta:=\min\bigl(1/e,\exp(-\delta(M_\delta))\bigr),\tag{2.11}$$

which yields

$$\sigma^+(A)\geqslant\frac{(1-c_\delta)c_\delta}{2\delta^{-1}(\log\frac{1}{c_\delta})}=\frac{(1-c_\delta)c_\delta\gamma(\log\frac{1}{c_\delta})}{2\log\frac{1}{c_\delta}}.$$

By the monotonicity of $x\gamma(\log\frac{1}{x})$ as in part 2, we conclude that

$$\sigma^+(A) \geqslant \frac{(1-c_\delta)}{2\log\frac{1}{c_\delta}}\widetilde{\sigma(A)}\gamma\left(\log\frac{1}{\widetilde{\sigma(A)}}\right).$$

4. $\widetilde{\sigma(A)} < c_\delta$, $g(x) - g(0) \geqslant \log\frac{1}{c_\delta}$ and $\frac{\sigma^+(A)}{\gamma(g(x)-g(0))} \geqslant 1/e$. Since γ is non-decreasing,

$$\sigma^+(A) \geqslant \frac{1}{e}\gamma\big(g(x)-g(0)\big) \geqslant \frac{1}{ec_\delta}c_\delta\gamma\left(\log\frac{1}{c_\delta}\right).$$

Using the monotonicity of $x\gamma(\log\frac{1}{x})$ as in part 2, we conclude that

$$\sigma^+(A) \geqslant \frac{1}{ec_\delta}\widetilde{\sigma(A)}\gamma\left(\log\frac{1}{\widetilde{\sigma(A)}}\right).$$

5. $\widetilde{\sigma(A)} < c_\delta$, $g(x) - g(0) \geqslant \log\frac{1}{c_\delta}$ and $\frac{\sigma^+(A)}{\gamma(g(x)-g(0))} < 1/e$. Recall that by (2.10):

$$\sigma(A) \leqslant \frac{\sigma^+(A)}{\gamma(g(x)-g(0))} < \frac{1}{e},$$

implying in particular that $\widetilde{\sigma(A)} = \sigma(A)$. We will show

$$\sigma^+(A) \geqslant D_\delta\frac{\sigma^+(A)}{\gamma(g(x)-g(0))}\gamma\left(\log\frac{\gamma(g(x)-g(0))}{\sigma^+(A)}\right), \tag{2.12}$$

which by the monotonicity of $x\gamma(\log\frac{1}{x})$ on $[0, 1/e]$ will imply

$$\sigma^+(A) \geqslant D_\delta\widetilde{\sigma(A)}\gamma\left(\log\frac{1}{\widetilde{\sigma(A)}}\right). \tag{2.13}$$

Denote $V_x = g(x) - g(0)$. Then (2.12) is equivalent to showing

$$\frac{\gamma(V_x(1 + \frac{\log\frac{\gamma(V_x)}{\exp(-g(0))}}{V_x}))}{\gamma(V_x)} \leqslant 1/D_\delta.$$

Recall that γ is non-decreasing and note that $\frac{\gamma(x)}{x} = \frac{1}{\delta^{-1}(x)}$ is non-increasing. Requiring that $D_\delta \leqslant 1$, it is therefore enough to show

$$1 + \frac{\log\frac{\gamma(V_x)}{\exp(-g(0))}}{V_x} \leqslant 1/D_\delta.$$

Denoting $B_\delta := 1/D_\delta - 1$, the latter is equivalent to

$$\gamma(V_x) \leqslant \exp(B_\delta V_x)\exp\big(-g(0)\big),$$

which from the definition of γ is equivalent to

$$\delta\big(V_x \exp(-B_\delta V_x)\exp\big(g(0)\big)\big) \leqslant V_x.$$

The maximum of the function $z \mapsto z\exp(-B_\delta z)$ is equal to $1/(eB_\delta)$, hence it is enough to require that

$$\delta\left(\frac{\exp(g(0))}{eB_\delta}\right) \leqslant V_x.$$

We have assumed that $V_x = g(x) - g(0) \geqslant \log\frac{1}{c_\delta}$; therefore by the definition (2.11) of c_δ the following condition will suffice:

$$\frac{\exp(g(0))}{eB_\delta} \leqslant M_\delta. \tag{2.14}$$

By Lemma 2.5, (2.14) holds for $B_\delta = 2/e$ (independent of δ in fact!). To conclude, (2.13) is satisfied with $D_\delta = \frac{e}{e+2}$.

Summing up all the five requirements for the constant C_δ in the conclusion of the proposition, we see that we can choose:

$$C_\delta \leqslant \min\left(\frac{1}{2M_\delta\gamma(\log\frac{1}{c_\delta})}, \frac{(1-c_\delta)}{M_\delta\gamma(\log\frac{1}{c_\delta})}, \frac{(1-c_\delta)}{2\log\frac{1}{c_\delta}}, \frac{1}{ec_\delta}, \frac{e}{e+2}\right).$$

From the definition (2.11) of c_δ, we see that $\log\frac{1}{c_\delta} = \max(\delta(M_\delta), 1)$ and that $\gamma(\log\frac{1}{c_\delta}) \leqslant \max(\delta(M_\delta), 1)/M_\delta$. It is then not hard to check that we can choose:

$$C_\delta := \frac{e-1}{2e\max(\delta(M_\delta), 1)},$$

as claimed. □

2.3. *A simpler proof with further assumptions*

Note that the uniform convexity (1.1) of g was not used in the statement and proof of Proposition 2.3. We remark here that by using this property, we obtain a simpler proof of a one-dimensional isoperimetric inequality, which may be used to complete the proof of Theorem 1.1 in place of Proposition 2.3. The key observation is the following.

Lemma 2.8. *Suppose* $g : (\mathbb{R}, |\cdot|) \to \mathbb{R} \cup \{+\infty\}$ *satisfies* (1.1)*, i.e.*:

$$\frac{g(x)+g(y)}{2} - g\left(\frac{x+y}{2}\right) \geqslant \delta\big(|x-y|\big) \geqslant 0, \quad x, y \in \mathbb{R}. \tag{2.15}$$

Then for any $x_0 \in \mathbb{R}$,

$$g(x) \geqslant g(x_0) + g'(x_0)(x - x_0) + 2\delta\big(|x - x_0|\big), \tag{2.16}$$

where $g'(x_0)$ is any value between $g'_l(x_0)$ and $g'_r(x_0)$, the left and right derivatives at x_0, respectively.

Proof. Immediate by applying Lemma 2.2 to the function $g - g'(x_0)(x - x_0)$, which attains its minimum at x_0. □

Proposition 2.9. *Let σ be a probability measure on $\mathbb{R}$ such that*

$$d\sigma(x) = \exp\bigl(-g(x)\bigr)\,dx,$$

where g satisfies (2.15). *Then*

$$\sigma^+(A) \geqslant \widetilde{\sigma(A)}\,\psi^{-1}\left(\frac{1}{2\widetilde{\sigma(A)}}\right), \quad A \subset \mathbb{R}, \tag{2.17}$$

where

$$\psi(t) = t\phi(t), \quad \phi(t) = \int_0^{+\infty} \exp\bigl(tx - 2\delta(x)\bigr)\,dx. \tag{2.18}$$

Proof. As before, by a general result of Bobkov [10, Proposition 2.1] on extremal isoperimetric sets of log-concave densities, it is enough to verify (2.17) on sets A of the form $(-\infty, x_0]$ and $[x_0, \infty)$. By symmetry, we may restrict ourselves to sets $[x_0, +\infty)$, $\sigma([x_0, \infty)) = a \leqslant 1/2$.

Denote $a^+ = \exp(-g(x_0))$. By (2.16),

$$a = \int_{x_0}^{\infty} \exp\bigl(-g(x)\bigr)\,dx \leqslant a^+\phi\bigl(-g'(x_0)\bigr), \tag{2.19}$$

and similarly

$$1/2 \leqslant 1 - a \leqslant a^+\phi\bigl(g'(x_0)\bigr). \tag{2.20}$$

Now consider two cases.

Case 1: $g'(x_0) > 0$. By (2.18), $\phi(-g'(x_0)) \leqslant 1/g'(x_0)$; hence $g'(x_0) \leqslant a^+/a$ using (2.19) and $a^+\phi(a^+/a) \geqslant a^+\phi(g'(x_0)) \geqslant 1/2$ using (2.20). Therefore

$$\psi\bigl(a^+/a\bigr) = \bigl(a^+/a\bigr)\phi\bigl(a^+/a\bigr) \geqslant 1/2a,$$

which implies (2.17).

Case 2: $g'(x_0) \leqslant 0$. By (2.20), $a^+\phi(0) \geqslant a^+\phi(g'(x_0)) \geqslant 1/2$, hence

$$a^+ \geqslant \frac{1}{2\phi(0)}. \tag{2.21}$$

Next, since ϕ is monotone, $\phi(\frac{1}{2a\phi(0)}) \geqslant \phi(0)$, hence $\psi(\frac{1}{2a\phi(0)}) \geqslant \frac{1}{2a}$, and we conclude by (2.21) that

$$a^+ \geqslant \frac{1}{2\phi(0)} \geqslant a\psi^{-1}\left(\frac{1}{2a}\right). \qquad \square$$

Remark 2.1. It is easy to verify that the function ϕ defined in (2.18) is log-convex, i.e. $\log \phi$ is convex.

Remark 2.2. Note that when $\delta(t) = ct^p$ $(p \geqslant 2)$, the inequalities obtained in Propositions 2.3 and 2.9 are equivalent, up to universal constants.

3. Lipschitz maps

This section is dedicated to the proof of an extended form of Theorem 1.5.

Proposition 3.1. *Let μ be a finite absolutely continuous measure on $\mathbb{R}^n$. There exists a μ-a.e. unique radial map T that pushes μ forward to the restriction of the Lebesgue measure to some star-shaped set $K \subset \mathbb{R}^n$.*

If $d\mu = f\, d\mathrm{mes}_n$, we may choose $K = K_f$ and $T = T_f$, where

$$K_f = \left\{x \in \mathbb{R}^n;\, v(x) \leqslant 1\right\},$$
$$v(x) = \left(n \int_0^{+\infty} f(rx)r^{n-1}\, dx\right)^{-\frac{1}{n}}, \tag{3.1}$$

and T_f is given by $T_f(0) = 0$ and

$$T_f(x) = \left(\frac{\int_0^1 f(rx)r^{n-1}\, dr}{\int_0^\infty f(rx)r^{n-1}\, dr}\right)^{\frac{1}{n}} \frac{x}{v(x)}, \quad x \neq 0. \tag{3.2}$$

Proof. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a radial map pushing μ forward to the Lebesgue measure restricted to a star-shaped body K. Define

$$w(x) = \inf\left\{t > 0;\ t^{-1}x \in K\right\}.$$

Then the restriction of T to a ray $\mathbb{R}_+x$, $w(x) = 1$, has the form:

$$rx \mapsto u(x,r)x, \quad r > 0.$$

Passing to polar coordinates and using the Fubini theorem, we see that $T_*\mu$ is equal to the restriction of mes_n to K iff, for almost every ray $\mathbb{R}_+x$, $w(x) = 1$, the map $u(x,\cdot)$ pushes $f(rx)r^{n-1}\, dr$ forward to $\mathbf{1}_{[0,1]}r^{n-1}\, dr$; that is, if

$$\int_0^1 \phi(r)r^{n-1}\, dr = \int_0^\infty \phi\big(u(x,r)\big) f(rx)r^{n-1}\, dr \tag{3.3}$$

for any test function $\phi \in C_0(\mathbb{R}_+)$. Setting $\phi = \mathbf{1}_{[0,T]}$ in (3.3) and letting $T \to \infty$, we see that

$$\frac{1}{n} = \int_0^\infty f(rx) r^{n-1}\, dr. \tag{3.4}$$

Hence $v(x) = 1$ for (almost) every x such that $w(x) = 1$. Both v and w are homogeneous functions, hence $v(x) = w(x)$ for μ-a.e. $x \in \mathbb{R}^n$.

Now use $\phi = \mathbf{1}_{[0,u(x,s)]}$ in (3.3). Since $u(x,\cdot)$ is monotone, we deduce:

$$u(x,s)^n = n \int_0^s f(rx) r^{n-1}\, dr = \frac{\int_0^s f(rx) r^{n-1}\, dr}{\int_0^\infty f(rx) r^{n-1}\, dr},$$

at every point of continuity s of $u(x,\cdot)$. Therefore

$$T(sx) = \left(\frac{\int_0^s f(rx) r^{n-1}\, dr}{\int_0^\infty f(rx) r^{n-1}\, dr} \right)^{1/n} x, \quad v(x) = 1, \tag{3.5}$$

which is equivalent to (3.2). □

Remark 3.1. Note that in particular, $\mathrm{mes}_n(K_f) = \mathrm{mes}_n(K) = \mu(\mathbb{R}^n)$.

The following proposition was proved by K. Ball [6] for even log-concave functions and extended by Klartag [29, Theorem 2.2] to general log-concave functions.

Proposition *(Ball). If f is a log-concave function on $\mathbb{R}^n$, then K_f is a convex body.*

Note that we do not assume at this stage that f is even. Therefore K_f may not necessarily be symmetric about the origin, so formally we can not identify it with the unit-ball of some norm $\|\cdot\|_{K_f}$. Nevertheless, we denote:

$$\|x\|_{K_f} = \left(n \int_0^\infty f(rx) r^{n-1}\, dr \right)^{-\frac{1}{n}}. \tag{3.6}$$

By the above proposition, this is a convex function on $\mathbb{R}^n$, which is in addition homogeneous. By definition (3.1), we have:

$$K_f = \left\{ x \in \mathbb{R}^n;\ \|x\|_{K_f} \leqslant 1 \right\}.$$

In addition, we denote

$$\widehat{K_f} = K_f \cap -K_f$$

which is now a convex body symmetric about the origin, and we associate with it the corresponding norm $\|\cdot\|_{\widehat{K_f}}$.

We can now state the following result, which extends Theorem 1.5.

Theorem 3.2. *Let f denote a log-concave function on $\mathbb{R}^n$ with barycenter at the origin such that $0 < \int f(x)\,dx < \infty$. Let μ denote the measure with density f, and let λ denote the restriction of the Lebesgue measure to K_f. Denote by $T = T_f$ the canonical radial map (given by (3.2)) such that $T_*\mu = \lambda$, and let $u : (\mathbb{R}^n, \|\cdot\|_{\widehat{K_f}}) \to [0,1]$ be defined by*

$$T(x) = u(x)\frac{x}{\|x\|_{K_f}}$$

for $x \neq 0$ and $u(0) = 0$. Then $\|u\|_{\mathrm{Lip}} \leqslant Cf(0)^{1/n}$, where $C > 0$ is a universal constant.

When f is in addition even, $\widehat{K_f} = K_f$ and $\|\cdot\|_{K_f}$ is indeed a norm. Theorem 1.5 is then deduced from Theorem 3.2 using the following lemma, which was essentially proved by Bobkov and Ledoux [14].

Lemma 3.3. *Let $V = (X, \|\cdot\|)$ denote a normed space, and let $T : V \to V$ be the map defined by $T(0) = 0$ and*

$$T(x) = u(x)\frac{x}{\|x\|}$$

for $x \neq 0$, where $u : X \to \mathbb{R}_+$ has a finite Lipschitz constant and satisfies $u(0) = 0$. Then

$$\|T\|_{\mathrm{Lip}} \leqslant 3\|u\|_{\mathrm{Lip}}.$$

Proof. Let $x, y \in X$. By continuity, we may assume that $x, y \neq 0$. Then

$$\begin{aligned}
\big\|T(x) - T(y)\big\| &= \left\| u(x)\frac{x}{\|x\|} - u(y)\frac{y}{\|y\|} \right\| \\
&\leqslant \left\| u(x)\frac{x}{\|x\|} - u(x)\frac{y}{\|y\|} \right\| + \left\| u(x)\frac{y}{\|y\|} - u(y)\frac{y}{\|y\|} \right\| \\
&= \big|u(x) - u(0)\big| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| + \big|u(x) - u(y)\big| \\
&\leqslant \|u\|_{\mathrm{Lip}}\|x\| \left(\left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\| + \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\| \right) + \|u\|_{\mathrm{Lip}}\|x - y\| \\
&= \|u\|_{\mathrm{Lip}}\|x\|\|y\| \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| + 2\|u\|_{\mathrm{Lip}}\|x - y\| \leqslant 3\|u\|_{\mathrm{Lip}}\|x - y\|. \qquad \square
\end{aligned}$$

For the proof of Theorem 3.2, we need to compile several known results about log-concave functions.

3.1. Additional preliminaries

Another convex body associated to a log-concave function f on $\mathbb{R}^n$ was put forth by B. Klartag and V. Milman [30]. Assume that $f(0) > 0$, we define the (convex) body K_f^0 as the set

$$K_f^0 = \big\{ x \in \mathbb{R}^n;\ f(x) \geqslant f(0)\exp(-n) \big\}. \tag{3.7}$$

We will use a relation between K_f and K_f^0 that was proved (under slightly different assumptions) by Klartag and Milman [30, Lemmata 2.1, 2.2].

Proposition 3.4 *(Klartag–Milman). Let f be a log-concave density on $\mathbb{R}^n$, and assume that $f(0) > 0$. Then*

$$K_f \subset C_n \Big(\sup_x f(x)\Big)^{\frac{1}{n}} K_f^0,$$

where $C_n > 1$ and $C_n \to 1$ as $n \to \infty$. Moreover, if f attains its maximum at 0, then

$$f(0)^{\frac{1}{n}} K_f^0 \subset D_n K_f,$$

where $D_n > 2$ and $D_n \to 2$ as $n \to \infty$.

The next lemma is a one dimensional computation for log-concave functions. For even functions, this fact goes back to Ball [5], and Milman and Pajor [34]. For arbitrary log-concave functions, this was extended by Klartag [29, Lemma 2.6] as follows.

Lemma 3.5. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ denote a non-constant log-concave function, and let $n \geqslant 1$. Assume that $f(0) = 1$ and that*

$$\sup_x f(x) \leqslant \exp(n). \tag{3.8}$$

Then

$$C_1 \leqslant \frac{n^{\frac{n+1}{n}}}{e(n+1)} \leqslant \frac{\int_0^\infty f(r) r^n\,dr}{(\int_0^\infty f(r) r^{n-1}\,dr)^{\frac{n+1}{n}}} \leqslant \frac{n!}{((n-1)!)^{\frac{n+1}{n}}} \leqslant C_2,$$

where $C_1, C_2 > 0$ are universal constants. In fact, the assumption (3.8) *is not needed for the right-hand side of the inequality.*

The last proposition we need is due to M. Fradelizi [22, Theorem 4].

Proposition 3.6 *(Fradelizi). Let f denote a log-concave density on $\mathbb{R}^n$ such that $0 < \int f(x)\,dx < +\infty$, and let x_0 denote its barycenter. Then*

$$g(x_0) \geqslant \exp(-n) \sup_{x \in \mathbb{R}^n} g(x).$$

3.2. Proof of Theorem 3.2

By (3.2), $T(x) = u(x) \frac{x}{\|x\|_{K_f}}$ for $x \neq 0$, where u is given by

$$u(x) = \left(\frac{\int_0^1 r^{n-1} f(rx)\,dr}{\int_0^\infty r^{n-1} f(rx)\,dr} \right)^{\frac{1}{n}} \tag{3.9}$$

for $x \neq 0$ and $u(0) = 0$. We thus verify that u is continuous at 0.

Step 1: Reduction to smooth f.

Define, for $\varepsilon > 0$, $f_\varepsilon := f * \varepsilon^{-n} G(x/\varepsilon)$, where G is the standard Gaussian density on $\mathbb{R}^n$ and $*$ denotes convolution. Clearly f_ε is a smooth function with barycenter at 0. By the Prékopa–Leindler theorem, f_ε is log-concave, as the convolution of two log-concave functions.

Let μ_ε denote the measure with density f_ε, λ_ε the Lebesgue measure on K_{f_ε}, and let T_ε denote the map radially pushing forward the measure μ_ε onto λ_ε. Let u_ε be defined by

$$T_\varepsilon(x) = u_\varepsilon(x) \frac{x}{\|x\|_{K_{f_\varepsilon}}},$$

with $u_\varepsilon(0) = 0$. Given $x, y \in \mathbb{R}^n$, it is clear from (3.9) and (3.6) that $u_\varepsilon(x) \to u(x)$, $u_\varepsilon(y) \to u(y)$, $\|x - y\|_{K_{f_\varepsilon}} \to \|x - y\|_{K_f}$ and $\|x - y\|_{\widehat{K_{f_\varepsilon}}} \to \|x - y\|_{\widehat{K_f}}$ as ε tends to 0. If we assume that $\|u_\varepsilon\|_{\mathrm{Lip}} \leqslant C f_\varepsilon(0)^{1/n}$, we have

$$\bigl|u_\varepsilon(x) - u_\varepsilon(y)\bigr| \leqslant C f_\varepsilon(0)^{1/n} \|x - y\|_{\widehat{K_{f_\varepsilon}}}.$$

Passing to the limit as $\varepsilon \to 0$, it follows that

$$\bigl|u(x) - u(y)\bigr| \leqslant C f(0)^{1/n} \|x - y\|_{\widehat{K_f}},$$

and we conclude that $\|u\|_{\mathrm{Lip}} \leqslant C f(0)^{1/n}$. It is therefore enough to restrict our discussion to smooth functions.

Step 2: Proof for smooth functions with $f(0) = 1$.

Assume that $f(0) = 1$.

Note that since f and thus u are assumed to be smooth,

$$\|u\|_{\mathrm{Lip}} = \sup_{x \in \mathbb{R}^n} \bigl\|\nabla u(x)\bigr\|^*_{\widehat{K_f}},$$

where $\|\cdot\|^*_{\widehat{K_f}} = \sup_{h \in \widehat{K_f}} \langle \cdot, h\rangle$ is the dual norm to $\|\cdot\|_{\widehat{K_f}}$.

Fixing $x \in \mathbb{R}^n$, $x \neq 0$, we will show that $\|\nabla u(x)\|^*_{\widehat{K_f}} \leqslant C$ for some universal constant $C > 0$. Write $f = \exp(-g)$, and denote for short,

$$A = \int_0^1 r^{n-1} f(rx)\, dr \quad \text{and} \quad B = \int_1^\infty r^{n-1} f(rx)\, dr.$$

Note that

$$u = \bigl(A/(A + B)\bigr)^{1/n},$$

$$\nabla A = -\int_0^1 r^n f(rx) \nabla g(rx)\, dr, \quad \text{and}$$

$$\nabla B = -\int_1^\infty r^n f(rx)\nabla g(rx)\,dr.$$

By Proposition 3.6, since $f(0)=1$ and 0 is the barycenter of f, then $\sup_x f(x) \leqslant \exp(n)$. This clearly implies that $A \leqslant \exp(n)/n$, and that $g(x) \geqslant -n$. Denote also

$$A^* = \int_0^1 r^n f(rx)\bigl\|\nabla g(rx)\bigr\|^*_{\widehat{K_f}}\,dr \quad \text{and} \quad B^* = \int_1^\infty r^n f(rx)\|\nabla g(rx)\|^*_{\widehat{K_f}}\,dr.$$

Then by (3.9)

$$\begin{aligned}
\bigl\|\nabla u(x)\bigr\|^*_{\widehat{K_f}} &= \frac{1}{n}\left(\frac{A}{A+B}\right)^{\frac{1}{n}-1} \frac{\|\nabla A(A+B) - A(\nabla A + \nabla B)\|^*_{\widehat{K_f}}}{(A+B)^2} \\
&\leqslant \frac{1}{n}\left(\frac{A}{A+B}\right)^{\frac{1}{n}} \frac{A^*B + AB^*}{A(A+B)} \\
&\leqslant \frac{1}{n}\frac{A^*}{A} + \frac{1}{n}\left(\frac{A}{A+B}\right)^{\frac{1}{n}} \frac{A^* + B^*}{A+B} \\
&\leqslant \frac{1}{n}\frac{A^*}{A} + \frac{e}{n}\frac{A^* + B^*}{(A+B)^{\frac{n+1}{n}}}.
\end{aligned} \tag{3.10}$$

Note that by the convexity of g, for all $x, y \in \mathbb{R}^n$

$$g(y) \geqslant g(x) + \bigl\langle \nabla g(x), y - x\bigr\rangle.$$

Recall the definition (3.7), stating that $y \in K_f^0$ iff $g(y) \leqslant n + g(0) = n$, and also recall that $g(x) \geqslant -n$. This implies that for $y \in K_f^0$,

$$\bigl\langle \nabla g(x), y\bigr\rangle \leqslant \bigl\langle \nabla g(x), x\bigr\rangle + g(y) - g(x) \leqslant \bigl\langle \nabla g(x), x\bigr\rangle + 2n.$$

By Proposition 3.4 $\widehat{K_f} \subset K_f \subset DK_f^0$, where $D = C(\sup_x f(x))^{1/n} \leqslant Ce$ for some universal $C > 1$; hence

$$\bigl\|\nabla g(x)\bigr\|^*_{\widehat{K_f}} \leqslant D\bigl(\bigl\langle \nabla g(x), x\bigr\rangle + 2n\bigr).$$

We will use this rough estimate to bound A^* and B^* from above. More generally, for $0 \leqslant a < b \leqslant \infty$,

$$\begin{aligned}\frac{1}{D}\int_a^b r^n f(rx)\left\|\nabla g(rx)\right\|_{\widehat{K_f}}^* dr &\leqslant \int_a^b r^n f(rx)\left(\left\langle \nabla g(rx), rx\right\rangle + 2n\right) dr \\ &= \frac{d}{dt}\bigg|_{t=1}\left(-\int_a^b r^n f(trx)\, dr\right) + 2n\int_a^b r^n f(rx)\, dr \\ &= \frac{d}{dt}\bigg|_{t=1}\left(-t^{-(n+1)}\int_{at}^{bt} r^n f(rx)\, dr\right) + 2n\int_a^b r^n f(rx)\, dr \\ &= (3n+1)\int_a^b r^n f(rx)\, dr + a^{n+1} f(ax) - b^{n+1} f(bx). \end{aligned} \tag{3.11}$$

Of course the last term is interpreted as 0 when $b = \infty$. With this bound in mind, let

$$A' = \int_0^1 r^n f(rx)\, dr \quad \text{and} \quad B' = \int_1^\infty r^n f(rx)\, dr.$$

Applying (3.11), we see that

$$A^*/D \leqslant (3n+1)A' - f(x);$$
$$(A^* + B^*)/D \leqslant (3n+1)(A' + B').$$

Hence by (3.10),

$$\left\|\nabla u(x)\right\|_{\widehat{K_f}}^* \leqslant \frac{(3n+1)D}{n}\left(\frac{A'}{A} + e\frac{A'+B'}{(A+B)^{\frac{n+1}{n}}}\right).$$

Obviously $A' \leqslant A$ since $r \leqslant 1$ in the integrand of A'. By Lemma 3.5 (that is applicable since $f(0) = 1$) we have:

$$A' + B' = \int_0^\infty r^n f(xr)\, dr \leqslant C\left(\int_0^\infty r^{n-1} f(xr)\, dr\right)^{\frac{n+1}{n}} = C(A+B)^{\frac{n+1}{n}},$$

where $C > 0$ is some universal constant. It follows that

$$\|u\|_{\mathrm{Lip}} = \sup_{x\in\mathbb{R}^n}\left\|\nabla u(x)\right\|_{\widehat{K_f}}^* \leqslant 4D(1 + eC).$$

Step 3: Proof for general smooth functions.

We have shown the assertion of the theorem for smooth functions f with $f(0)=1$. In the general case, obviously $f(0)>0$, since the barycenter of the log-concave f is at the origin. Let us push forward $f(x)\,dx$ by the map $S(x)=f(0)^{1/n}x$ to obtain $f'(x)\,dx$, where

$$f'(x)=f(0)^{-1}f\bigl(f(0)^{-1/n}x\bigr).$$

Clearly $K_{f'}$ is a homothetic copy of K_f, and since

$$\mathrm{mes}_n(K_{f'})=\int f'(x)\,dx=\int f(x)\,dx=\mathrm{mes}_n(K_f),$$

we see that $K_{f'}=K_f$. Let T denote the radial map pushing forward $f'(x)\,dx$ to the restriction of the Lebesgue measure on K_f, denoted λ. Let $u':(\mathbb{R}^n,\|\cdot\|_{K_f})\to[0,1]$ be defined by

$$T'(x)=u'(x)\frac{x}{\|x\|_{K_f}},$$

and $u'(0)=0$. Since $f'(0)=1$ and f' is smooth, Step 2 implies that $\|u'\|_{\mathrm{Lip}}\leqslant C$. Obviously $T=T'\circ S$ (e.g. by uniqueness of the radial map pushing forward $f(x)\,dx$ onto λ), and hence $u=u'\circ S$. This implies

$$\|u\|_{\mathrm{Lip}}=\|u'\|_{\mathrm{Lip}}f(0)^{1/n}\leqslant Cf(0)^{1/n},$$

and concludes the proof. □

Remark 3.2. Of course the proof uses the fact that the barycenter of f is at the origin in a very indirect way. In fact, it is clear from the proof that we may use any log-concave function f for which

$$f(0)\geqslant D^{-n}\sup_{x\in\mathbb{R}^n}f(x),$$

for some $D\geqslant 1$, yielding $\|u\|_{\mathrm{Lip}}\leqslant C(D)f(0)^{1/n}$, where $C(D)$ is a constant depending on D.

As an immediate corollary of Theorem 1.5, we obtain the Bobkov–Ledoux proposition from the introduction, although the direct route taken by Bobkov and Ledoux in [14] is simpler in this case and recovers a better universal constant in the bound.

Proof of the Bobkov–Ledoux proposition. It is easy to see that the Lipschitz constant of S as a map acting on $(\mathbb{R}^n,\|\cdot\|)$ is invariant to scaling of the Lebesgue measure, so we may assume that $\mathrm{mes}_n(K)=1$. By Theorem 1.5,

$$\|S\|_{\mathrm{Lip}}\leqslant Cf(0)^{1/n}=C\Gamma(1+n/p)^{-1/n}. \qquad □$$

We will see in the next section how Theorem 1.5 may be used to transfer isoperimetric inequalities from log-concave measures to uniform measures on convex bodies.

4. General uniformly convex bodies

In this section we give a proof of Proposition 1.4 and provide the details that lead to Theorem 1.6.

Let $\delta = \delta_V$ denote the modulus of convexity of a normed space $V = (X, \|\cdot\|)$. It is known that δ is not necessarily a convex function; we denote by $\tilde{\delta}$ the maximal convex function majorated by δ. We summarise several known facts about δ and $\tilde{\delta}$ (see Lindenstrauss and Tsafriri [31, Proposition 1.e.6, Lemmata 1.e.7, 1.e.8]).

Lemma 4.1.

1. *$\delta(t)/t$ is non-decreasing on* $[0, 2]$.
2. $\delta(t/2) \leqslant \tilde{\delta}(t) \leqslant \delta(t)$ *for all* $t \in [0, 2]$.
3. *There exists a constant $C \geqslant 1$ such that $\tilde{\delta}(t)/t^2 \leqslant C\tilde{\delta}(s)/s^2$, for all* $0 \leqslant t \leqslant s \leqslant 2$.

The following crucial fact is due to Figiel and Pisier [21] (see also [31, Lemma 1.e.10]).

Proposition 4.2 *(Figiel–Pisier). Let $x, y \in X$ such that $\|x\|^2 + \|y\|^2 = 2$. Then*

$$\|x + y\|^2 \leqslant 4 - 4\delta\big(\|x - y\|/2\big).$$

Proposition 1.4 is an easy corollary of these lemmata.

Proof of Proposition 1.4. Let $x, y \in X$ such that $\|x\|^2 + \|y\|^2 \leqslant 2$, and denote $s^2 := (\|x\|^2 + \|y\|^2)/2 \leqslant 1$. If $s = 0$ then $\|x\| = \|y\| = 0$ and the claim is trivial. Otherwise, denote $x' = x/s$ and $y' = y/s$, so that $\|x'\|^2 + \|y'\|^2 = 2$. Hence by Proposition 4.2:

$$\left\|\frac{x' + y'}{2}\right\|^2 \leqslant 1 - \delta\left(\frac{\|x' - y'\|}{2}\right),$$

or equivalently:

$$\left\|\frac{x + y}{2}\right\|^2 \leqslant s^2 - s^2\delta\left(\frac{\|x - y\|}{2s}\right).$$

Now, $s \leqslant 1$; hence by Lemma 4.1 we have for any $t \in [0, 2s]$:

$$s^2\delta(t/s) \geqslant s^2\tilde{\delta}(t/s) \geqslant c\tilde{\delta}(t) \geqslant c\delta(t/2),$$

where $c > 0$ is a universal constant. Applying this for

$$t = \frac{\|x - y\|}{2} \leqslant \frac{\|x\| + \|y\|}{2} \leqslant s,$$

we conclude that

$$\left\|\frac{x + y}{2}\right\|^2 \leqslant \frac{\|x\|^2 + \|y\|^2}{2} - c\delta\left(\frac{\|x - y\|}{4}\right), \tag{4.1}$$

as required. □

Remark 4.1. Using $\|x\| = \|y\| = 1$ in (4.1), we see that $\delta_V(\varepsilon) \geqslant \frac{c}{2}\delta(\varepsilon/4)$ for any function δ satisfying (4.1), so Proposition 1.4 is in fact a characterization (up to universal constants) of the modulus of convexity δ_V.

Now we can fill the details in the proof of Theorem 1.6. Assume that $V = (\mathbb{R}^n, \|\cdot\|)$ is a uniformly convex space, and let $\delta = \delta_V$ denote its modulus of convexity as before. Scale the Lebesgue measure on $\mathbb{R}^n$ so that $\text{mes}_n\{\|x\| \leqslant 1\} = 1$, since the statement of Theorem 1.6 is invariant to this scaling. Now denote by μ the probability measure with density

$$f(x) = \frac{1}{Z}\exp\bigl(-n/c\|4x\|^2\bigr)\mathbf{1}\bigl(\|x\| \leqslant 1/4\bigr)$$

with respect to the Lebesgue measure, where $c > 0$ is the constant from Proposition 1.4. Here $Z > 0$ is a scaling factor so that μ be indeed a probability measure. Integrating on level sets of $\|\cdot\|$, it is clear that

$$\begin{aligned} Z &= \int_{\mathbb{R}^n} \exp\left(-\frac{1}{c}n\|4x\|^2\right)\mathbf{1}\left(\|x\| \leqslant \frac{1}{4}\right)dx \\ &= n\int_0^{1/4} \exp\left(-\frac{16}{c}ns^2\right)s^{n-1}\,ds, \end{aligned}$$

and in particular $Z^{1/n} \geqslant c' > 0$.

Write $f = \exp(-g)$, with $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Proposition 1.4 then implies that g is uniformly convex, and satisfies

$$\frac{g(x) + g(y)}{2} - g\left(\frac{x+y}{2}\right) \geqslant n\delta_1\bigl(\|x - y\|\bigr),$$

where δ_1 coincides with δ on $[0, 1/4]$ and $\delta_1(t) = +\infty$ for $t > 1/4$. Since $\delta(t)/t$ is non-decreasing by Lemma 4.1, so is $\delta_1(t)/t$, and assumption (1.2) is fulfilled. We can therefore apply Theorem 1.1, and deduce an isoperimetric inequality for μ on V:

$$\mu^+_{\|\cdot\|}(A) \geqslant C_{n,\delta}\widetilde{\mu(A)}\gamma_n\left(\log\frac{1}{\widetilde{\mu(A)}}\right) \quad \text{for all } A \subset \mathbb{R}^n,$$

where $C_{n,\delta}$ is given by (1.16) and

$$\gamma_n(t) = \frac{t}{\delta_1^{-1}(t/(2n))}.$$

We would now like to transfer this isoperimetric inequality to λ_V, the uniform probability measure on $K_V = \{\|x\| \leqslant 1\}$, via a radial Lipschitz map. Clearly, K_f is a homothetic copy of K_V, and since

$$\text{mes}_n(K_f) = \int f(x)\,dx = 1 = \text{mes}_n(K_V),$$

it follows that $K_f = K_V$. Note also that

$$f(0)^{1/n} = Z^{-1/n} \leqslant (c')^{-1}.$$

Applying Theorem 1.5, it follows that the Lipschitz constant of the radial map pushing forward μ onto λ_V is bounded by a universal constant. Because of the truncation in the definition of δ_1, this only implies the statement of Theorem 1.6 for sets A such that

$$\widetilde{\lambda(A)} \geqslant \exp\bigl(-2\delta(1/4)n\bigr).$$

Now suppose

$$\widetilde{\lambda(A)} < \exp\bigl(-2\delta(1/4)n\bigr).$$

Then

$$\delta^{-1}\left(\frac{1}{2n}\log\frac{1}{\widetilde{\lambda(A)}}\right) \geqslant 1/4,$$

and hence by Bobkov's inequality (1.8) with $r = 1$

$$\lambda^+_{\|\cdot\|}(A) \geqslant \frac{1}{2}\widetilde{\lambda(A)}\log\frac{1}{\widetilde{\lambda(A)}} \geqslant c' C_{n,\delta}\frac{\widetilde{\lambda(A)}\log\frac{1}{\widetilde{\lambda(A)}}}{\delta^{-1}(\frac{1}{2n}\log\frac{1}{\widetilde{\lambda(A)}})}$$

with, say, $c' = e/(4(e-1))$.

This concludes the proof of Theorem 1.6.

5. Concentration and functional inequalities

5.1. Concentration of measure on uniformly convex bodies

In this subsection, we discuss the connection between our results and the following Gromov–Milman inequality [26], that we cite in the form of Arias-de-Reyna, Ball, and Villa [2].

Theorem *(Gromov–Milman). Let $V = (\mathbb{R}^n, \|\cdot\|)$ be a normed space, let $\delta = \delta_V$ be its modulus of convexity, and let λ be the uniform measure on the unit ball of V. Then*

$$1 - \lambda(B_{\varepsilon,\|\cdot\|}) \leqslant \frac{1}{\lambda(B)}\exp\bigl(-2n\delta(\varepsilon)\bigr) \quad \textit{for all } B \subset \mathbb{R}^n. \tag{5.1}$$

In particular, if $\delta(\varepsilon) \geqslant \alpha'\varepsilon^p$, then

$$1 - \lambda(B_{\varepsilon,\|\cdot\|}) \leqslant \frac{1}{\lambda(B)}\exp\bigl(-2\alpha' n\varepsilon^p\bigr) \quad \textit{for all } B \subset \mathbb{R}^n. \tag{5.2}$$

Let us compare this to our results. First assume $\delta(\varepsilon) \geqslant \alpha' \varepsilon^p$; then (1.11) holds with $\alpha = \alpha'/2^p$ (as mentioned in Section 1.2.2). Therefore by Theorem 1.3

$$\lambda^+_{\|\cdot\|}(A) \geqslant C'(\alpha')^{1/p} n^{1/p} \widetilde{\lambda(A)} \log^{1-1/p} \frac{1}{\widetilde{\lambda(A)}} \quad \text{for all } B \subset \mathbb{R}^n, \tag{5.3}$$

where C' is a universal constant. Hence by Corollary 1.8,

$$1 - \lambda(B_{\varepsilon,\|\cdot\|}) \leqslant \exp\left\{-\left[\log^{1/p} \frac{1}{1-\lambda(B)} + \frac{c(\alpha')^{1/p} n^{1/p} \varepsilon}{p}\right]^p\right\}. \tag{5.4}$$

The right-hand side in (5.4) is at most

$$\bigl(1 - \lambda(B)\bigr) \exp\left\{-C'(\alpha')^{1/p} n^{1/p} \log^{1-1/p} \frac{1}{1-\lambda(B)} \varepsilon\right\} < 1 - \lambda(B);$$

hence (5.4) yields a meaningful bound for any $\varepsilon > 0$, whereas (5.2) is meaningful for

$$\varepsilon \geqslant \left\{\frac{1}{2\alpha' n} \log \frac{1}{\lambda(B)(1-\lambda(B))}\right\}^{1/p}.$$

On the other hand, for larger ε the right-hand side of (5.4) behaves like

$$\exp\left\{-\frac{C'^p}{p^p} \alpha' n \varepsilon^p\right\};$$

that is, we lose a factor p^p in the exponent.

The preceding discussion can be extended to arbitrary moduli of convexity. In the general case, Theorem 1.6 yields

$$\lambda^+_{\|\cdot\|}(A) \geqslant C'_{n,\delta} \frac{\widetilde{\lambda(A)} \log \frac{1}{\widetilde{\lambda(A)}}}{\delta^{-1}(\frac{1}{2n} \log \frac{1}{\widetilde{\lambda(A)}})}; \tag{5.5}$$

hence by Proposition 1.7

$$1 - \lambda(B_{\varepsilon,\|\cdot\|}) \leqslant \exp\bigl\{-h^{-1}_{1-\lambda(B)}(\varepsilon)\bigr\}, \tag{5.6}$$

where

$$h_a(x) = \int_{\log 1/a}^{x} \frac{\delta^{-1}(y/2n)\, dy}{C'_{n,\delta} y}.$$

By Lemma 4.1 we can assume without loss of generality that δ is convex (and δ^{-1} is concave). Then,

$$h_a(x) = \int_{\log 1/a}^{x} \frac{dy}{C'_{n,\delta} y} \frac{\int_{\log 1/a}^{x} \frac{\delta^{-1}(y/2n)\,dy}{C'_{n,\delta} y}}{\int_{\log 1/a}^{x} \frac{dy}{C'_{n,\delta} y}}$$
$$\leqslant \int_{\log 1/a}^{x} \frac{dy}{C'_{n,\delta} y} \delta^{-1}\left\{ \frac{1}{2n} \frac{\int_{\log 1/a}^{x} \frac{dy}{C'_{n,\delta}}}{\int_{\log 1/a}^{x} \frac{dy}{C'_{n,\delta} y}} \right\}$$
$$= \frac{\log x - \log\log 1/a}{C'_{n,\delta}} \delta^{-1}\left\{ \frac{1}{2n} \frac{x - \log 1/a}{\log x - \log\log 1/a} \right\}.$$

Now, $t \mapsto \delta^{-1}(t)/t$ is decreasing, hence

$$h_a(x) \leqslant \frac{1}{C'_{n,\delta}} \delta^{-1}\left\{ \frac{1}{2n}(x - \log 1/a) \right\} \quad \text{if } x \leqslant e \log 1/a. \tag{5.7}$$

On the other hand,

$$h_a(e \log 1/a) = \int_{\log 1/a}^{e \log 1/a} \frac{\delta^{-1}(y/2n)\,dy}{C'_{n,\delta} y} \geqslant \frac{e-1}{e C'_{n,\delta}} \delta^{-1}\left\{ \frac{e \log 1/a}{2n} \right\};$$

hence for $\varepsilon \leqslant \frac{e-1}{eC'_{n,\delta}} \delta^{-1}\{\frac{e \log 1/a}{2n}\}$, $x = h_a^{-1}(\varepsilon) \leqslant e \log 1/a$, and (5.7) implies

$$h_a^{-1}(\varepsilon) \geqslant 2n\delta\big(C'_{n,\delta}\varepsilon\big) + \log 1/a.$$

We conclude by (5.6) that

$$1 - \lambda(B_{\varepsilon,\|\cdot\|}) \leqslant \big(1 - \lambda(B)\big) \exp\big\{-2n\delta\big(C'_{n,\delta}\varepsilon\big)\big\}. \tag{5.8}$$

Again, (5.8) is better than (5.1) for small ε; if

$$\varepsilon \leqslant \frac{e-1}{eC'_{n,\delta}} \delta^{-1}\left\{ \frac{e \log 1/a}{2n} \right\}$$

the inequalities (5.1) and (5.8) are similar, whereas for larger ε an inequality of type (5.8) can only be deduced from (5.5) under additional regularity assumptions on δ.

5.2. Proofs

It remains to prove Propositions 1.7 and 1.9.

Proof of Proposition 1.7. Let $B \subset \mathbb{R}^n$ be a Borel set such that

$$a = 1 - \mu(B) \leqslant 1/2;$$

the proof easily extends to the complementary case $a > 1/2$.

Denote $f(t) = 1 - \mu(B_t)$. Our assumptions then read:

$$f(0) = a; \qquad df/dt(t) \leqslant -f(t)\gamma\bigl(-\log f(t)\bigr)$$

(where strictly speaking df/dt should be the upper left derivative). Setting $g = -\log f$,

$$g(0) = \log 1/a; \qquad dg/dt \geqslant \gamma \circ g,$$

and if $h = g^{-1}$,

$$h(\log 1/a) = 0 \quad \text{and} \quad dh/dt \leqslant 1/(\gamma).$$

Therefore

$$h(x) \leqslant \int_{\log 1/a}^{x} \frac{dy}{\gamma(y)} = h_a(x),$$

and

$$f(t) = \exp\bigl(-h^{-1}(t)\bigr) \leqslant \exp\bigl(-h_a^{-1}(t)\bigr),$$

as required.

The converse direction is obvious. □

Proof of Proposition 1.9. Let us show that part 1 implies part 2. Let F be a function satisfying (1.23). Assume for simplicity that the distribution of F has no atoms except for 0 and 1 and that $\mu\{F = 0\} = 1/2$, $\mu\{F = 1\} = t = 1/2^k$. Choose

$$0 = u_1 < u_2 < \cdots < u_k = 1$$

so that

$$\mu\{u_i < F < u_{i+1}\} = 1/2^{i+1}.$$

Then

$$\int \|\nabla F\|_*^q \, d\mu = \sum \int_{u_i < F \leqslant u_{i+1}} \|\nabla F\|_*^q \, d\mu$$
$$\geqslant \sum \frac{1}{2^{i+1}} \left\{ 2^{i+1} \int_{u_i < F < u_{i+1}} \|\nabla F\|_* \, d\mu \right\}^q$$

by Jensen's inequality. Now, let us apply part 1 to the function

$$F_i = \max\left(0, \min\left(1, \frac{F - u_i}{u_{i+1} - u_i}\right)\right).$$

Since $\mu\{F_i = 1\} = \mu\{F \geqslant u_{i+1}\} = 1/2^{i+1}$, we obtain

$$\int_{u_i < F < u_{i+1}} \|\nabla F\|_* \, d\mu \geqslant c\mathbf{c}_0 (u_{i+1} - u_i) \frac{\log^{1/q} 2^{i+1}}{2^{i+1}}.$$

Therefore

$$\begin{aligned}
\int \|\nabla F\|_*^q \, d\mu &\geqslant \sum \frac{1}{2^{i+1}} \bigl\{ c\mathbf{c}_0 (u_{i+1} - u_i) \log^{1/q} 2^{i+1} \bigr\}^q \\
&\geqslant c'' \mathbf{c}_0^q \sum (u_{i+1} - u_i)^q \frac{i+1}{2^{i+1}} \\
&\geqslant c'' \mathbf{c}_0^q \Bigl(\sum (u_{i+1} - u_i) \Bigr)^q \Big/ \Bigl[\sum \Bigl(\frac{2^{i+1}}{i+1} \Bigr)^{p/q} \Bigr]^{q/p}
\end{aligned}$$

according to Hölder's inequality. Finally,

$$\sum_{i=1}^{k} \left(\frac{2^{i+1}}{i+1} \right)^{p/q} \leqslant C \bigl(2^k / k \bigr)^{p/q}$$

and thence

$$\int \|\nabla F\|_*^q \, d\mu \geqslant c''' \mathbf{c}_0^q \frac{k}{2^k} \geqslant c' \mathbf{c}_0^q t \log 1/t. \qquad \square$$

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