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PBW-deformations of N-Koszul algebras

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Abstract

For a quotient algebra U of the tensor algebra we give explicit conditions on its relations for U being a PBW-deformation of an N-Koszul algebra A. We show that there is a one-to-one correspondence between such deformations and a class of A_∞-structures on the Yoneda algebra $\mathrm{Ext}^*_A(k,k)$ of A. We compute the PBW-deformations of the algebra whose relations are the anti-symmetrizers of degree N and also of cubic Artin–Schelter algebras.

Introduction

In this paper we consider a class of deformations of N-Koszul algebras, the Poincaré–Birkhoff–Witt(PBW)-deformations. The class of N-Koszul algebras was introduced by R. Berger in [3] and is a generalization of ordinary Koszul algebras. Let R be a subspace of homogeneous forms of degree N in the tensor algebra $T(V)$. Denoting by (R) the two-sided ideal in the tensor algebra generated by R we get the quotient algebra $A = T(V)/(R)$. When certain nice homological conditions are imposed on the resolution of the residue field k this is called an N-Koszul algebra, the quadratic case $N = 2$ giving ordinary Koszul algebras.

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By deforming R to a space P of non-homogeneous forms of degree N but keeping the homogeneous part of P of degree N fixed equal to R, we get a filtered algebra $U = T(V)/(P)$ which is a *PBW-deformation* of A if its associated graded algebra is A. (Note that U is augmented if the constant term in P is zero.) In the quadratic case such deformations have been considered by Braverman and Gaitsgory in [2], a typical example being when A is the symmetric algebra on a vector space V and U the enveloping algebra of a Lie algebra on V. We generalize their results to the setting of N-Koszul algebras and give a simple set of equations relating the homogeneous parts of P which describes when U gives a PBW-deformation.

In [11], Positselski in the quadratic case showed that giving an augmented PBW-deformation of A is equivalent to giving the Koszul dual $A^!$ of A the structure of a differential graded algebra. The Koszul dual $A^!$ may be identified as the Yoneda algebra $\mathrm{Ext}^*_A(k,k)$. In general for any augmented algebra the Yoneda algebra has the structure of an A_∞-algebra. For an N-Koszul algebra A it is natural to consider its Yoneda algebra B together with this A_∞-structure as its Koszul dual and Keller [8] computed this. We show that when we specify a certain class of properties of A_∞-structures of B, among them being that those products which are non-zero in the case computed by Keller are kept fixed, there is a one-to-one correspondence between such A_∞-structures on B and augmented PBW-deformations of A. Our description is quite explicit. In fact the properties we can impose on B reduce the whole menagerie of equations to be verified for an A_∞-algebra to only two simple types. We also verify that the A_∞-structure on the Yoneda algebra of U corresponds to, more precisely is quasi-isomorphic to, the A_∞-structure on B which we find.

We then proceed to compute PBW-deformations of some classes of concrete algebras. The most well-known classical case is of course the PBW-deformations of the symmetric algebra $S(V)$. Such a deformation U corresponds to giving V the structure of a Lie algebra and U is then the enveloping algebra of V. Here we consider the class of algebras

$$T(V)/\left(\bigwedge^N V\right) \tag{1}$$

whose N-Koszulity was shown by Berger [3]. Note that when $N = 2$ we get the symmetric algebra $S(V)$. We compute the PBW-deformations of these algebras. The result separates into two cases according to N being even or odd. When N is even, an augmented PBW-deformation of (1) corresponds to giving a Lie algebra structure $L : \bigwedge^2 V \to V$ on V together with additional structure provided by even symplectic forms $\Phi_{2r} : \bigwedge^{2r} V \to k$ where $2 \leqslant 2r < N$, fulfilling analogues of the Jacobi identity, namely the composition (with some abuse of notation)

$$L \circ \Phi_{2r} \circ L : \bigwedge^{2r+3} V \to V$$

is zero. Note that if we take $\Phi_0 = 1$ this is the ordinary Jacobi identity.

When N is odd a PBW-deformation of (1) is equivalent to freely choosing a linear form $l : V \to k$ and symplectic forms Φ_{2r} as above. No relations among them is required.

We also consider the (generic forms of) cubic Artin–Schelter algebras and compute their PBW-deformations. These are algebras that have considerable interest in non-commutative algebraic geometry.

We would like to give a quick comment about an aspect we have not discussed in this paper. Deformations of relations can be thought of as changes to the Massey product structure. With this in mind, the results about the A_∞-structure can be seen as an explicit example of the interplay between Massey products and ∞-structures. Also, the space of deformations we construct will be the versal family of the moduli of such deformations. This means that we do not consider equivalences of deformations, though the quotient by this equivalence relation gives the actual moduli. The moduli is in general difficult to understand, even in seemingly simple situations. For the case of Lie algebras on a three-dimensional vector space (corresponding to PBW-deformations of $k[x, y, z]$), see [13].

The organization of the paper is as follows. Section 1 introduces N-Koszul algebras and their PBW-deformations. We give the PBW-theorem for N-Koszul algebras which shows which equations must be fulfilled for the relations P. In Section 2 we state how a PBW-deformation of A corresponds to an A_∞-structure on its Yoneda algebra and Section 3 contains the main bulk of the proof of this.

Sections 4, 5, 6, and 7 are concerned with the PBW-deformations of the algebra (1). In Section 4 we state the results. Sections 5 and 6 are of a technical nature and develop most of the ingredients of the proofs. The final proofs are given in Section 7.

Section 8 contains explicit computations of the PBW-deformations of cubic Artin–Schelter algebras.

1. N-Koszul algebras and the PBW-theorem

1.1. Definitions

Let V be a finite-dimensional vector space, whose dimension we shall denote by v, over a field k. Let R be a subspace of $V^{\otimes N}$ where N is some integer $\geqslant 2$, and (R) the two-sided ideal generated by R in the tensor algebra $T(V)$. We then get a homogeneous algebra $A = T(V)/(R)$. In [3], R. Berger introduced the notion of N-Koszul algebras which is a generalization of ordinary Koszul algebras (which is the case $N = 2$). The algebra A is an *N-Koszul algebra* if the residue field k has a right A-module resolution of the form

$$k \leftarrow A \leftarrow V_1 \otimes A(-1) \leftarrow V_2 \otimes A(-N) \leftarrow \cdots \leftarrow V_i \otimes A\bigl(-n(i)\bigr) \leftarrow \cdots$$

where $n(2q) = Nq$ and $n(2q+1) = Nq + 1$. The V_i are vector spaces and here and in the rest of the paper all tensor products are over k. We can also define this notion using a resolution of k as a left module; these two definitions are then equivalent for a given algebra, see [3, Definition 2.10], and the following discussion.

Instead of a homogeneous space of relations R we may consider a non-homogeneous space of relations P in $\bigoplus_{i \leqslant N} V^{\otimes i}$, and get a non-homogeneous algebra $U = T(V)/(P)$. We shall be interested in the case when this algebra is a deformation of A of a particular kind, a PBW-deformation. We then assume that P intersects $\bigoplus_{i \leqslant N-1} V^{\otimes i}$ trivially and that R is the image of P by the natural projection of $\bigoplus_{i\leqslant N} V^{\otimes i}$ to $V^{\otimes N}$. Then there are linear maps $R \xrightarrow{\alpha_i} V^{\otimes N-i}$ such that we may write

$$P = \bigl\{r + \alpha_1(r) + \cdots + \alpha_N(r) \mid r \in R\bigr\}. \tag{2}$$

There is a natural filtration on the tensor algebra by letting $F^l T(V)$ be $\bigoplus_{i\leqslant l} V^{\otimes i}$. This induces a natural filtration $F^l U$ on the quotient algebra U, and in the situation described there is a surjection

$$A \twoheadrightarrow \operatorname{gr} U.$$

We say that U is a *PBW-deformation* of A if this map is an isomorphism.

Notation. The symbol $\mathbf{1}$ will be used to denote the identity map on a vector space. If ϕ and ψ are maps defined on subspaces U, respectively W, of $V^{\otimes a}$, respectively $V^{\otimes b}$, denote

$$[\phi, \psi] = \phi \otimes \psi - \psi \otimes \phi$$

which is defined on the subspace $(U \otimes W) \cap (W \otimes U)$ of $V^{\otimes a+b}$.

1.2. The PBW-theorem

The topic of this paper is to investigate PBW-deformations of N-Koszul algebras. In the quadratic case this has been done in various papers [2,10,11]. The following is a generalization of the main theorem of [2] to the N-Koszul case.

Theorem 1.1. *Let A be an N-Koszul algebra. Then U defined by relations* (2) *is a PBW-deformation of A if and only if*

$$(P) \cap F^N T(V) = P. \tag{3}$$

This is more explicitly equivalent to the image of

$$(V \otimes R) \cap (R \otimes V) \xrightarrow{[\mathbf{1},\alpha_1]} V^{\otimes N} \tag{4}$$

being contained in R and (*let $\alpha_{N+1} = 0$*)

$$\alpha_i \circ [\mathbf{1}, \alpha_1] = [\mathbf{1}, \alpha_{i+1}], \quad i = 1, \ldots, N. \tag{5}$$

Example 1.2. When the relations R are $\bigwedge^2 V$ so α_1 is a map $\bigwedge^2 V \to V$, and U is augmented, i.e., α_2 is zero, there is only one Eq. (5) and written out this is just the Jacobi identity meaning that α_1 gives V the structure of a Lie algebra. Thus we have the classical PBW-theorem that U is a PBW-deformation of the symmetric algebra $S(V)$ iff U is the enveloping algebra of a Lie algebra.

Remark 1.3. When U is a PBW-deformation (3), (4) and (5) still hold without the assumption that A is N-Koszul, as we now show in the first part of the proof.

Remark 1.4. Berger and Ginzburg, [4], have independently proved this result.

Proof of Theorem 1.1 (*first part*). When U is a PBW-deformation of A, clearly $(P) \cap F^N T(V)$ is equal to P.

Let x be in $(V \otimes R) \cap (R \otimes V)$. We get in the ideal (P) two elements

$$\sum_{i=1}^{N} (\mathbf{1} \otimes \alpha_i)(x) + x, \qquad \sum_{i=1}^{N} (\alpha_i \otimes \mathbf{1})(x) + x.$$

The difference of these two elements is in $(P) \cap F^N T(V)$ which is P. This gives us the necessary relations degree by degree. For instance, in degree N we have

$$[\mathbf{1}, \alpha_1](x)$$

so this must be in R, and the degree $N - 1$ part must be α_1 of this element, which gives us

$$\alpha_1 \circ [\mathbf{1}, \alpha_1](x) = [\mathbf{1}, \alpha_2](x)$$

and so forth. □

The second and difficult part of the proof will show how the conditions (4) and (5) of Theorem 1.1 imply that U is a PBW-deformation of A. For this we need some deformation theory.

1.3. Deformations

We recall the setup from [2]. Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a graded associative algebra over a field k. An ith level graded deformation of A is a graded $k[t]/(t^{i+1})$-algebra A^i, free as a module over $k[t]/(t^{i+1})$, together with an isomorphism $A^i/tA^i \cong A$. By taking inverse limits, a graded deformation of A is a graded $k[\![t]\!]$-algebra A^t, free as a module over $k[\![t]\!]$, with an isomorphism $A^t/tA^t \cong A$. In all cases considered in this paper, the deformations we construct are naturally defined also over $k[t]$, with the same conditions. Thus we can, for instance, take fibres over the point $t = 1$.

More explicitly, an ith level deformation is given by a set of maps $f_j : A \otimes A \to A$ such that the rule

$$a \times b = ab + \sum_{j=1}^{i} f_j(a,b)t^j$$

defines an associative algebra, and similarly for the inverse limit, but with the sum extended to infinity.

Let $\mathcal{C}, \mathcal{C}^i$ denote the categories of graded deformations and graded ith level deformations of A, respectively (with morphisms only isomorphisms).

The connection between graded Hochschild cohomology groups and deformations is given by the next lemma, taken directly from [2].

Lemma 1.5.

(a) *The set of isomorphism classes of* $\mathcal{C}^1$ *is* $HH^2_{-1}(A, A)$.
(b) *For* $A^i \in \mathcal{C}^i$*, the obstruction to its extension to level* $i + 1$ *lies in* $HH^3_{-i-1}(A, A)$.
(c) *Furthermore, the set of isomorphism classes of such extensions is an* $HH^2_{-i-1}(A, A)$*-homogeneous space.*

For the proof, refer to [2].

We let $C^\bullet_{\mathrm{gr}}(A, M)$ be the complex of graded morphisms from the bar complex of A to the A-bimodule M, so that in particular the graded Hochschild cohomology groups appearing above are graded cohomology modules of this complex.

Following Berger [3], there is a complex associated to an N-homogeneous algebra A, called its *N-Koszul* complex. Its first few terms are

$$K_0 = A \otimes A,$$

$$K_1 = A \otimes V \otimes A,$$

$$K_2 = A \otimes R \otimes A,$$

$$K_3 = A \otimes \big((V \otimes R) \cap (R \otimes V)\big) \otimes A.$$

In general, the term K_i has lowest degree terms $n(i)$, where $n(2q) = Nq$ and $n(2q + 1) = Nq + 1$. For details, including the definition of the differential, consult [3].

The algebra A is N-Koszul iff this is a resolution of A as an A-bimodule. In other words, there is a quasi-isomorphism from the Koszul complex to the bar complex:

$$\sigma : K_\bullet \to B_\bullet.$$

It is injective since $K_\bullet$ is a minimal resolution, see [3]. Note that this allows us to compute the graded Hochschild groups using maps from $K_\bullet \to A$; σ induces a map from $C^\bullet_{\mathrm{gr}}(A, A) = \mathrm{Hom}(B_\bullet, A)$ to $\mathrm{Hom}(K_\bullet, A)$, which is, of course, also a quasi-isomorphism. The differential we will be most interested in is the map $\mathrm{Hom}(K_2, A) \to \mathrm{Hom}(K_3, A)$. Given a bimodule map $\alpha : K_2 \to A$, its boundary $d\alpha$ is given on $(V \otimes R) \cap (R \otimes V)$ as $[\mathbf{1}, \alpha]$.

Lemma 1.6. *Suppose A is N-Koszul. The graded Hochschild cohomology groups* $HH^i_j(A, M)$ *vanish whenever* $j < -n(i)$ *and M is a non-negatively graded A-bimodule.*

This can easily be deduced from [2,5]; in fact, in the spirit of [2], it might be reasonable to use this as the definition of N-Koszulity.

We are now ready for the second part of the PBW-theorem.

Proof of Theorem 1.1 (*second part*). The maps $\alpha_i : R \to V^{\otimes N-i}$ of vector spaces naturally determine $A \otimes A^o$-maps from K_2 to A, which are denoted by the same symbols.

To get the first level deformation, we need a class in $HH^2_{-1}(A, A)$. Condition (4) from Theorem 1.1 states that the image of $[\mathbf{1}, \alpha_1]$ is contained in R which is equivalent to $d\alpha_1 = 0$ in the complex $\mathrm{Hom}(K_\bullet, A)$. It thus determines a class $[\alpha_1]$ in $HH^2_{-1}(A, A)$. We want to choose a representative for this class, $-f_1$, in the Hochschild complex, such that $-f_1 \circ \sigma = \alpha_1$ as cochains. Here, and later, it will always be possible to choose maps with the given properties because the map σ is determined by an (injective) map of vector spaces. So let $-\tilde{f}_1$ be any representative; then $-\tilde{f}_1 \circ \sigma - \alpha_1$ is cohomologically trivial, whence there exists ω in $\mathrm{Hom}(K_1, A)_{-1}$ with $-\tilde{f}_1 \circ \sigma - \alpha_1 = d\omega$. Let ω' be any Hochschild cochain such that $\omega' \circ \sigma = \omega$, and let $f_1 = \tilde{f}_1 + d\omega'$. Then

$$-f_1 \circ \sigma = -\tilde{f}_1 \circ \sigma + d\omega = \alpha_1.$$

The next steps of the deformation are constructed inductively. So suppose the deformation has been constructed up to level i, meaning that we have elements $f_1, \dots, f_i$ in $C^2_{\mathrm{gr}}(A, A)$ such that, for $j \leqslant i$, $-f_j \circ \sigma = \alpha_j$ and

$$-df_j = f_{j-1} \circ [\mathbf{1}, f_1]$$

(analogous to the properties of the α's from Theorem 1.1). Here $f_0 = 0$.

We construct f_{i+1} with these properties as follows. Let

$$-\tilde{f}_{i+1} \in C^2_{\mathrm{gr}}(A, A)_{-i-1}$$

be any element such that $-\tilde{f}_{i+1} \circ \sigma = \alpha_{i+1}$. Define

$$\gamma = -d\tilde{f}_{i+1} - f_i \circ [\mathbf{1}, f_1].$$

Then $\gamma \circ \sigma = 0$, and since σ is a quasi-isomorphism, γ is cohomologically trivial. So there exists μ in $C^2_{\mathrm{gr}}(A, A)_{-i-1}$ with $d\mu = \gamma$. Choose also μ' (easy diagram chase) such that $\mu' \circ \sigma = \mu \circ \sigma$ and $d\mu' = 0$. Finally, set $f_{i+1} = \tilde{f}_{i+1} + \mu - \mu'$. Then

$$-df_{i+1} = -d\big(\tilde{f}_{i+1} - \mu + \mu'\big) = \gamma + f_i \circ [\mathbf{1}, f_1] - \gamma + 0 = f_i \circ [\mathbf{1}, f_1]$$

and

$$-f_{i+1} \circ \sigma = -\tilde{f}_{i+1} - \mu \circ \sigma + \mu' \circ \sigma = \alpha_{i+1}.$$

Compare this to the obstruction

$$\sum_{j=1}^{i} f_j \circ [\mathbf{1}, f_{i-j+1}]$$

for lifting the structure, coming from the general theory of deformations. All the terms with $j < i$ are cohomologically trivial, since the composition with the quasi-isomorphism σ is zero for degree reasons. The only remaining term is covered by $-df_{i+1}$, as shown above, so the deformation can be extended to level $i+1$.

This gives the deformation up to level $N+1$. From there on, we use Lemma 1.6, which states that the cohomology groups $HH^i_j(A, A)$ vanish for $j < -n(i)$. By Lemma 1.5, for $i = 3$ this says that the space where the obstruction to extending the deformation further than level $N+1$ lives is zero. For $i = 2$ it says that the space of choices for this extension is zero.

So we have a collection of f_i such that the rule

$$a \times b = ab - \sum_{i=1}^{\infty} f_i(a, b)t^i$$

defines an associative multiplication on an algebra fibred over $k[t]$ (this works since only finitely many terms are non-zero). Let $\tilde{U}$ be the fibre over $t = 1$. Since it is constructed as a deformation, we know that $\operatorname{gr} \tilde{U} = A$. We have a canonical map $\tilde{\phi} : V \to \tilde{U}$ which extends to a map $\phi : U \to \tilde{U}$ (by definition of the f_i). We thus have a sequence of maps

$$A \overset{p}{\twoheadrightarrow} \operatorname{gr} U \xrightarrow{\operatorname{gr} \phi} \operatorname{gr} \tilde{U} \xrightarrow{\cong} A$$

and the composition is the identity on A. Thus p is an isomorphism. □

Example 1.7. It is easy to construct examples where the theorem fails for non-Koszul algebras. For instance, let A be the commutative algebra

$$A = k[x, y, z]/\left(y^2 - xz, xy, z^2\right).$$

This algebra is not Koszul; in fact $HH^3_{-4}(A, A) \neq 0$. It was considered in [6]. Define α_1 by $z^2 \mapsto -x$, all other relations (including the commutators) being mapped to zero. Then $[\mathbf{1}, \alpha_1]$ is identically zero, so the conditions of the theorem are trivially fulfilled. But the algebra $U = k[x, y, z]/(y^2 - xz, xy, z^2 + x)$ is finite-dimensional as a vector space, so $\operatorname{gr} U$ cannot be isomorphic to the algebra A, which has Krull dimension 1.

2. The A_∞-structure on the Yoneda algebra

For any augmented algebra A the Yoneda algebra $\operatorname{Ext}^*_A(k, k)$ has a natural A_∞-structure, see [9]. Keller computed this structure when A is an N-Koszul algebra. To state this we let $W = V^*$ and S be the perpendicular space of R, i.e., the kernel of $W^{\otimes N} \to R^*$. Write $A^! = T(W)/(S)$ and let B be the algebra with

$$B_{2p} = \left(A^!\right)_{Np}, \qquad B_{2p+1} = \left(A^!\right)_{Np+1}.$$

Consider k as a right A-module. Then B is the Yoneda algebra $\mathrm{Ext}^*_A(k,k)$ of A as follows from the N-Koszul complex for A. (If k is considered as a left A-module the Yoneda algebra will be the opposite algebra of B.) Keller shows, [8], that m_2 is ordinary multiplication if some argument is even (in the B-grading). When all arguments are odd then $m_N(b_1,\dots,b_N)$ is the ordinary product $b_1\cdots b_N$ in B. In all other cases the m_p vanish.

The following states that an augmented PBW-deformation U of A corresponds to giving an A_∞-structure on B with certain explicitly given properties.

Theorem 2.1. *There is a one-to-one correspondence between PBW-deformations of A (with $\alpha_N = 0$) and A_∞-structures on B with the following properties*:

(i) *m_n vanishes for $n > N$,*
(ii) *m_N is the ordinary product of N elements when all arguments are odd,*
(iii) *m_n vanishes for $3 \leqslant n \leqslant N$ if some argument is even, and*
(iv) *m_2 is ordinary multiplication when some argument is even.*

Remark 2.2. We see that the products in B which are non-zero in the non-deformed case are unchanged in the deformed case. The only products that we deform are the m_p of odd arguments for $p \leqslant N-1$ and m_1 of even arguments.

The above A_∞-structure on B corresponds to the natural A_∞-structure on the Yoneda algebra of U as the following shows. Recall that the bar construction $BU = \bigoplus_{n\geqslant 0} \overline{U}^{\otimes n}$, where $\overline{U}$ is the kernel of $U \to k$, comes with a natural coalgebra structure, see [12].

Theorem 2.3. *Let C be the graded dual of B. There is a quasi-isomorphism $C \to BU$ of A_∞-coalgebras.*

Remark 2.4. It might be conceivable that there are other A_∞-structures on B corresponding to a given PBW-deformation, since B is not in general minimal. But the above Theorem 2.1 shows that there is a unique one with the above requirements.

Let $B^{\mathrm{ev}} = \bigoplus B_{2p}$ be the *even* part of B, and $B^{\mathrm{od}} = \bigoplus B_{2p+1}$ be the *odd* part of B. We shall reserve m_1 to denote only the differential $B^{\mathrm{od}} \to B^{\mathrm{ev}}$. The differential $B^{\mathrm{ev}} \to B^{\mathrm{od}}$ shall be denoted by d. Also, if some argument of m_2 is even, so m_2 is ordinary multiplication we simply write $\cdot$ for this multiplication or even drop it so we have the ordinary concatenation notation.

With the requirement that m_p vanishes when $p \geqslant 3$ and some argument is even, the axioms to be checked for B to be an A_∞-algebra, [7, 3.1] reduce to the following two types, where $p \geqslant 0$. When all arguments are odd ($m_0 = 0$),

$$d \circ m_{p+1} = (-1)^p(\mathbf{1}\cdot m_p - m_p \cdot \mathbf{1}). \tag{6}$$

Let u denote an even element and a_i odd elements.

$$m_p(\dots, a_{i-1}, ua_i, \dots) = m_p(\dots, a_{i-1}u, a_i, \dots) \\ + (-1)^{p+1} m_{p+1}\big(\dots, a_{i-1}, d(u), a_i, \dots\big). \tag{7}$$

(If $i = 1$ then in the middle term u gives ordinary multiplication to the left "outside" m_p, and similarly for the first term when $i = p + 1$.)

Remark 2.5. An interesting thing here is that one realizes that this may be abstracted to give the following construction of a class of A_∞-algebras. Let B^{ev} be a positively even graded algebra, and B^{od} a module graded in positive odd degrees, with a derivation d and maps m_p of degree $2 - p$

$$B^{\mathrm{ev}} \xrightarrow{d} B^{\mathrm{od}}, \qquad \big(B^{\mathrm{od}}\big)^{\otimes p} \xrightarrow{m_p} B^{\mathrm{ev}}$$

such that (6) and (7) hold. Then $B = B^{\mathrm{ev}} \oplus B^{\mathrm{od}}$ becomes an A_∞-algebra.

Remark 2.6. Note that when $p = 1$ both (6) and (7) are instances of the derivation rule $\delta(ab) = \delta(a)b + (-1)^{|a|}a\delta(b)$. Hence they generalize this equation in different directions.

We shall now show how the A_∞-structure on B determines a PBW-deformation.

Proof of Theorem 2.1 *(the easy direction).* The A_∞-structure on B provides maps

$$W^{\otimes p} = (B_1)^{\otimes p} \xrightarrow{m_p} B_2 = W^{\otimes N}/S. \tag{8}$$

We define $\alpha_p : R \to V^{\otimes N-p}$ to be the sign-adjusted dual given by

$$\alpha_p = \sigma(p) \cdot (m_{N-p})^* \tag{9}$$

where $\sigma(p)$ is a sign determined by $\sigma(p-2) = -\sigma(p)$, $\sigma(N) = 1$ and $\sigma(N-1) = (-1)^{N-1}$. Equation (6) for $p < N - 1$ translates to the condition (5) of Theorem 1.1. Equation (6) for $p = N - 1$ shows that the map

$$W^{\otimes N} \to W^{\otimes N}/S \xrightarrow{d} W^{\otimes N+1}/(S \otimes W + W \otimes S) \tag{10}$$

is the dual of the map

$$(V \otimes R) \cap (R \otimes V) \xrightarrow{[\mathbf{1}, \alpha_1]} R \subseteq V^{\otimes N}$$

showing condition (4) of Theorem 1.1. □

Considerably more work it is to show the other direction of Theorem 2.1. We shall do this through several steps in the next section. Here we shall explain how, given the α_i's, to define the maps d and m_p.

Definition of d. First we define

$$W^{\otimes N} = B_2 \xrightarrow{d} B_3$$

as the dual of

$$(V \otimes R) \cap (R \otimes V) \xrightarrow{[\mathbf{1},\alpha_1]} V^{\otimes N}.$$

This may be extended to a derivation from $T(W)^{\mathrm{ev}} = \bigoplus_i T(W)_{Ni}$ to B^{od}. We shall show that this descends to a map $B^{\mathrm{ev}} \to B^{\mathrm{od}}$.

Definition of m_p. When all arguments are linear define m_p to be the sign-adjusted dual

$$m_p = \sigma(N-p)(\alpha_{N-p})^*.$$

Supposing m_{p+1} has been defined, we may then define

$$\big(T(W)^{\mathrm{od}}\big)^{\otimes p} \xrightarrow{m_p} B^{\mathrm{ev}}$$

inductively by Eq. (7) on the smallest integer i such that $a_{i+1}, \ldots, a_p$ are all linear. Then let ua_i be a factorization with a_i linear. We shall show that this descends to a map $(B^{\mathrm{od}})^{\otimes p} \to B^{\mathrm{ev}}$.

In the next section we shall prove that these definitions provide B with an A_∞-structure. Assuming this we here provide the proof of Theorem 2.3.

Proof of Theorem 2.3. The map $C \xrightarrow{\tau} U$ which is the projection on C_1 composed with the inclusion $C_1 = V \hookrightarrow U$ is a (generalized) twisting cochain, i.e., we have

$$\sum_{p=1}^{\infty} \mu_U \circ \tau^{\otimes p} \circ (m_p)^* = 0$$

where μ_U is the iterated multiplication on U. In fact this reduces simply to the equation (note that τ has degree one)

$$r + \sum_i \alpha_i(r) = 0.$$

Hence we get an A_∞-coalgebra morphism $C \to BU$. As in [9], see also [8], in order to show that this is a quasi-isomorphism, it will be sufficient to show that the twisted tensor product

$$C \otimes_\tau U = C_0 \otimes U \xleftarrow{d_\tau} C_1 \otimes U \xleftarrow{d_\tau} \cdots$$

is a resolution of k, where the differential d_τ is given by

$$\sum_{p=1}^{\infty}(\mathbf{1}_C \otimes \mu_U)\big(\mathbf{1}_C \otimes \tau^{\otimes p-1} \otimes \mathbf{1}_U\big)\big((m_p)^* \otimes \mathbf{1}_U\big).$$

But the complex $C \otimes_\tau U$ comes with a natural filtration $(C \otimes_\tau U)_{\leqslant p}$ given by

$$\begin{aligned} C_0 \otimes U_{\leqslant p} \leftarrow C_1 \otimes U_{\leqslant p-1} \leftarrow C_2 \otimes U_{\leqslant p-N} \leftarrow C_3 \otimes U_{\leqslant p-N-1} \\ \leftarrow C_4 \otimes U_{\leqslant p-2N} \leftarrow \cdots. \end{aligned}$$

Now the "non-deformed" graded complex $C \otimes_{\tau_0} A$, where τ_0 is the twisting cochain when U is specialized to A, is the Koszul complex for the N-Koszul algebra A and is thus a resolution of k. It is easily worked out that, for each p, the quotient

$$(C \otimes_\tau U)_{\leqslant p}/(C \otimes_\tau U)_{\leqslant p-1}$$

identifies as the strand of $C \otimes_{\tau_0} A$ in degree p, and so is acyclic for $p \geqslant 1$. The upshot is that $(C \otimes_\tau U)_{\leqslant p}$ is quasi-isomorphic to k for all $p \geqslant 0$. Since $C \otimes_\tau U$ is the colimit of the $(C \otimes_\tau U)_{\leqslant p}$ and this colimit commutes with homology, we get that $C \otimes_\tau U$ is a resolution of k. $\square$

3. Proof of Theorem 2.1

Assuming the definitions of d and m_p, we now proceed to show that the maps d and m_p descend, and that Eqs. (6) and (7) hold. Our first lemma will in particular be of help in showing the descent of the maps d and m_p. Our second lemma shows the descent of d. Then we proceed through a sequence of Lemmas 3.3–3.7 concerning the m_p. The argument is here inductive in the sense that each time we prove a lemma concerning m_p we assume that *all* the lemmas have been proven for m_r when $r \geqslant p+1$.

Lemma 3.1. *For* $m = 2, 3, \ldots, N-1$

$$W^{\otimes m-1} \otimes S \otimes W \subseteq W^{\otimes m} \otimes S + \bigcap_{i=1}^{m}\big(W^{\otimes m-i} \otimes S \otimes W^{\otimes i}\big). \tag{11}$$

Proof. From [3] we know that

$$\big(V^{\otimes m} \otimes R\big) \cap \big(R \otimes V^{\otimes m} + V \otimes R \otimes V^{\otimes m-1} + \cdots + V^{\otimes m-1} \otimes R \otimes V\big) \tag{12}$$

is equal to

$$\big(V^{\otimes m} \otimes R\big) \cap \big(R \otimes V^{\otimes m}\big) \tag{13}$$

$$+ \big(V^{\otimes m} \otimes R\big) \cap \big(V \otimes R \otimes V^{\otimes m-1} + \cdots + V^{\otimes m-1} \otimes R \otimes V\big) \tag{14}$$

and also

$$\big(V^{\otimes m} \otimes R\big) \cap \big(R \otimes V^{\otimes m}\big) \subseteq V^{\otimes m-1} \otimes R \otimes V.$$

When $m = 2$ this gives that (12) is included in $V \otimes R \otimes V$. When $m > 2$ we may by induction assume that (14) is included in $V \otimes V^{\otimes m-2} \otimes R \otimes V$. Whence we get

$$\begin{aligned} &\big(V^{\otimes m} \otimes R\big) \cap \big(R \otimes V^{\otimes m} + V \otimes R \otimes V^{\otimes m-1} + \cdots + V^{\otimes m-1} \otimes R \otimes V\big) \\ &\quad \subseteq V^{\otimes m-1} \otimes R \otimes V. \end{aligned} \tag{15}$$

In general when V and W are dual vector spaces and I and $I^{\perp}$ and J and $J^{\perp}$ are orthogonal complements for the pairing, then $I^{\perp} + J^{\perp}$ will be the orthogonal complement of $I \cap J$. Taking the orthogonal complement of (15), gives the statement of the lemma. □

Lemma 3.2. *The map $T(W)^{\mathrm{ev}} \xrightarrow{d} B^{\mathrm{od}}$ descends to a map $B^{\mathrm{ev}} \to B^{\mathrm{od}}$.*

Proof. Note first that $d(S) = 0$. This is because $W^{\otimes N} \to B_3$ is dual to $(V \otimes R) \cap (R \otimes V) \xrightarrow{[\mathbf{1},\alpha_1]} V^{\otimes N}$ which factors through $R \subseteq V^{\otimes N}$. Since d is a derivation, it will be enough to show that $d(W^{\otimes a} \otimes S \otimes W^{\otimes b})$ becomes zero when $a + b = N$. Note that this vanishes by the derivation property if a or b is zero.

Suppose $b = 1$. By tensoring Eq. (11) to the left with W when $m = N - 1$ we get

$$W^{\otimes N-1} \otimes S \otimes W \subseteq W^{\otimes N} \otimes S + \bigcap_{i=1}^{N-1} \big(W^{\otimes N-i} \otimes S \otimes W^{\otimes i}\big). \tag{16}$$

It will therefore be enough to show that d applied to the last term in (16) is zero.

Before proceeding note the following. Since $W^{\otimes N} \xrightarrow{d} B_3$ is the dual of

$$(V \otimes R) \cap (R \otimes V) \xrightarrow{[\mathbf{1},\alpha_1]} V^{\otimes N}$$

we have the equation (when arguments are in W)

$$dm_N = (-1)^{N-1}(\mathbf{1} \cdot m_{N-1} - m_{N-1} \cdot \mathbf{1}).$$

Summing up, we get (with arguments in W)

$$\sum_{i=0}^{N} \mathbf{1}^{\otimes i} \cdot dm_N \cdot \mathbf{1}^{\otimes N-i} = (-1)^{N-1}\big(m_{N-1} \cdot \mathbf{1}^{\otimes N+1} - \mathbf{1}^{\otimes N+1} \cdot m_{N-1}\big). \tag{17}$$

Applying this to an argument in the last term of (16) the right-hand side of (17) becomes zero in $B_5 = (A^!)_{2N+1}$ and the left-hand side reduces to

$$dm_N \cdot m_N + m_N \cdot dm_N.$$

But this is equal to $d(m_N \otimes m_N)$ by the derivation property, and hence d vanishes on the last term of (16).

When $b = 2$ then tensoring Eq. (11) to the right with W when $m = N - 1$ we get

$$W^{\otimes N-2} \otimes S \otimes W^{\otimes 2} \subseteq W^{\otimes N-1} \otimes S \otimes W + S \otimes W^{\otimes N}$$

and hence $d(W^{\otimes N-2} \otimes S \otimes W^{\otimes 2})$ is zero. In this way we may continue tensoring (11) to the right with powers of W and get inductively that $d(V^{\otimes a} \otimes S \otimes W^{\otimes b})$ is zero. □

Lemma 3.3. *Equation* (7) *holds when* $i \leqslant p$.

Proof. For the induction start note that it holds when $p = N$. Now if the a_j are linear for $j \geqslant i$, Eq. (7) is just the definition. So let $j \geqslant i$ be the least index such that $a_{j+1}, \ldots, a_p$ are all linear, but a_j is not linear.

Suppose first that $j \geqslant i + 2$ and let $a_j = vl$ where v is even (i.e., in $W^{\otimes Nq}$ for some q), and l is linear. The first term in Eq. (7) is then by definition equal to

$$\begin{aligned} &m_p(\ldots, a_{i-1}, ua_i, \ldots, a_{j-1}v, l, \ldots) \\ &\quad + (-1)^{p+1} m_{p+1}\big(\ldots, a_{i-1}, ua_i, \ldots, a_{j-1}, d(v), l, \ldots\big). \end{aligned} \tag{18}$$

The second term in Eq. (7) is by definition

$$\begin{aligned} &m_p(\ldots, a_{i-1}u, a_i, \ldots, a_{j-1}v, l, \ldots) \\ &\quad + (-1)^{p+1} m_{p+1}\big(\ldots, a_{i-1}u, a_i, \ldots, a_{j-1}, d(v), l, \ldots\big) \end{aligned} \tag{19}$$

and the third term in Eq. (7) is

$$\begin{aligned} &(-1)^{p+1} m_{p+1}\big(\ldots, a_{i-1}, d(u), a_i, \ldots, a_{j-1}v, l, \ldots\big) \\ &\quad - m_{p+2}\big(\ldots, a_{i-1}, d(u), a_i, \ldots, a_{j-1}, d(v), l, \ldots\big). \end{aligned} \tag{20}$$

By ascending induction on j (we will take care of the initial cases $j = i, i + 1$ shortly) and the induction assumptions, we may assume that (18) is

$$\begin{aligned} &m_p(\ldots, a_{i-1}u, a_i, \ldots, a_{j-1}v, l, \ldots) \\ &\quad + (-1)^{p+1} m_{p+1}\big(\ldots, a_{i-1}, d(u), a_i, \ldots, a_{j-1}v, l, \ldots\big) \\ &\quad + (-1)^{p+1} m_{p+1}\big(\ldots, a_{i-1}u, a_i, \ldots, a_{j-1}, d(v), l, \ldots\big) \\ &\quad - m_{p+2}\big(\ldots, a_{i-1}, d(u), a_i, \ldots, a_{j-1}, d(v), l, \ldots\big) \end{aligned}$$

which is the sum of (19) and (20).

When $j = i + 1$ the same type of argument goes through, assuming it is shown for $j = i$, and when $j = i$, an analogous argument goes through, this time using $d(uv) = d(u)v + ud(v)$. □

Lemma 3.4. *Equation* (6) *holds.*

Proof. If all arguments are linear, this is simply the dual of the conditions (5) of Theorem 1.1. An odd argument which is not linear can be written (as a sum of) $m_N(a_1,\dots,a_N)\cdot a_{N+1}$ where the a_i are odd elements. Suppose the $(l+1)$th argument of Eq. (6) is of this form, where $l \leqslant p$. We must then show (note that we write $m_N \otimes \mathbf{1}$ because we have not yet shown that m_p descends, see Lemma 3.7)

$$\begin{aligned} &dm_{p+1}\big(\mathbf{1}^{\otimes l} \otimes m_N \cdot \mathbf{1} \otimes \mathbf{1}^{\otimes p-l}\big) \\ &\quad = (-1)^p\big(\mathbf{1}\cdot m_p\big(\mathbf{1}^{\otimes l-1} \otimes (m_N \otimes \mathbf{1}) \otimes \mathbf{1}^{\otimes p-l}\big) \\ &\qquad - m_p\big(\mathbf{1}^{\otimes l} \otimes (m_N \otimes \mathbf{1}) \otimes \mathbf{1}^{\otimes p-1-l}\big)\cdot \mathbf{1}\big). \end{aligned} \tag{21}$$

Considering the first term, we have by using Eq. (7), successively "shifting the u term all the way to the left," that $m_{p+1}(\mathbf{1}^{\otimes l} \otimes m_N \cdot \mathbf{1} \otimes \mathbf{1}^{\otimes p-l})$ is

$$m_N \cdot m_{p+1}\big(\mathbf{1}^{\otimes p+1}\big) + (-1)^{p+2}\sum_{i=0}^{l} m_{p+2}\big(\mathbf{1}^{\otimes i} \otimes dm_N \otimes \mathbf{1}^{\otimes p+1-i}\big). \tag{22}$$

Applying d to this, the first term in (21) is

$$dm_N \cdot m_{p+1}\big(\mathbf{1}^{\otimes p+1}\big) + m_N \cdot dm_{p+1}\big(\mathbf{1}^{\otimes p+1}\big) \tag{23}$$

$$-\sum_{i=1}^{l} \mathbf{1}\cdot m_{p+1}\big(\mathbf{1}^{\otimes i-1} \otimes dm_N \otimes \mathbf{1}^{\otimes p-i}\big) \tag{24}$$

$$- dm_N \cdot m_{p+1}\big(\mathbf{1}^{\otimes p+1}\big) \tag{25}$$

$$+\sum_{i=0}^{l} m_{p+1}\big(\mathbf{1}^{\otimes i} \otimes dm_N \otimes \mathbf{1}^{\otimes p-i}\big)\cdot \mathbf{1}. \tag{26}$$

The first term in (23) and the term in (25) cancel. By induction on the total degree of the argument, the second term in (23) is equal to

$$(-1)^p m_N \cdot \big(\mathbf{1}\cdot m_p\big(\mathbf{1}^{\otimes p}\big) - m_p\big(\mathbf{1}^{\otimes p}\big)\cdot \mathbf{1}\big). \tag{27}$$

But then again by using Eq. (7) and "shifting successively to the right," we see that the last two terms in (21) become equal to the sum of (27) and the terms in (26) and (24). □

Lemma 3.5.

$$\sum_{i=0}^{p} m_{p+1}\big(\mathbf{1}^{\otimes i} \otimes dm_N \otimes \mathbf{1}^{\otimes p-i}\big) = (-1)^p(m_N \cdot m_p - m_p \cdot m_N). \tag{28}$$

Proof. According to Lemma 3.4 we can assume that the equation

$$dm_{r+1} = (-1)^r(\mathbf{1}\cdot m_r - m_r\cdot\mathbf{1})$$

holds for $r \geqslant p$. Assuming $q \geqslant p+1$ and inserting this in the sum

$$\sum_{i=0}^{q-1} m_q\big(\mathbf{1}^{\otimes i}\otimes dm_{r+1}\otimes\mathbf{1}^{\otimes q-1-i}\big)$$

we get, observing that many terms will cancel because of Eq. (7), that this is equal to

$$(-1)^{r+q}\sum_{i=0}^{q} m_{q+1}\big(\mathbf{1}^{\otimes i}\otimes dm_r\otimes\mathbf{1}^{\otimes q-i}\big)+(-1)^r(m_q m_r - m_r m_q).$$

Note that the first term here is of the same type which we just expanded, so we may proceed inductively. We now start with the left-hand side of (28) and apply this inductively. In the end this becomes

$$\sum_{a\geqslant p+1,\, a+b=N+p}(-1)^{\eta(a,b)}(m_a m_b - m_b m_a) \tag{29}$$

where $\eta(a,b)\equiv\eta(b,a) \pmod 2$. Hence the terms in (29) cancel, except the term

$$(-1)^p(m_N\cdot m_p - m_p\cdot m_N).$$

This proves the lemma. □

Lemma 3.6. *Equation* (7) *holds when* $i = p+1$.

Proof. By the fact that Eq. (7) holds for $i \leqslant p$, and successively "shifting even factors all the way to the right outside of m_p," we are reduced to prove the $i = p+1$ case when all the a_i are linear. Also u may be written as $m_N(a_{p+1},\dots,a_{N+p})$ where these arguments are odd. The second term in (7), $m_p(\dots,a_p u)$ will then, by successively "shifting even terms all the way to the left and then outside of m_p," be equal to

$$a_1\cdots a_N m_p(a_{N+1},\dots,a_{N+p}) + (-1)^{p+1}\sum_{i=1}^{p} m_{p+1}\big(\dots,d(a_i\cdots a_{i+N-1}),\dots\big).$$

By the previous Lemma 3.5, this is equal to

$$m_p(a_1\cdots a_p)a_{p+1}\cdots a_{N+p} - (-1)^{p+1} m_{p+1}\big(a_1,\dots,d(a_{p+1}\cdots a_{N+p})\big). \qquad \square$$

Lemma 3.7. *The map*

$$\big(T(W)^{\mathrm{od}}\big)^{\otimes p}\xrightarrow{m_p} B^{\mathrm{ev}}$$

descends to a map $(B^{\mathrm{od}})^{\otimes p}\to B^{\mathrm{ev}}$.

Proof. We must show that m_p becomes zero if one of its p arguments have the form $W^{\otimes a} \otimes S \otimes W^{\otimes b}$. Using Eq. (7) and shifting even factors to the left or right outside of m_p, this is easily reduced to the case when all arguments are linear, except one which has the form $W^{\otimes a} \otimes S \otimes W^{\otimes b}$ where $a+b$ is $N+1$ or 1. Suppose $a=0$ (so $b=1$) and that the non-linear argument is in position p, the last one. We must show that m_p vanishes on $W^{\otimes p-1} \otimes (S \otimes W)$. By Lemma 3.1 this is contained in

$$W^{\otimes p} \otimes S + \bigcap_{i=1}^{p} \bigl(W^{\otimes p-i} \otimes S \otimes W^{\otimes i}\bigr)$$

and hence $W^{\otimes p-1} \otimes (S \otimes W)$ is contained in

$$\bigl(W^{\otimes p-1} \otimes S \otimes W\bigr) \cap \bigl(W^{\otimes p} \otimes S\bigr) + \bigcap_{i=1}^{p} \bigl(W^{\otimes p-i} \otimes S \otimes W^{\otimes i}\bigr). \tag{30}$$

Now if the argument of m_p is contained in the left term of (30), then m_p vanishes on it by using Eq. (7) and "shifting S to the right outside of m_p." If the argument of m_p is contained in the right term of (30) we may use Eq. (7) again and shift the argument of m_p successively to the left. Note that all terms with m_{p+1} will vanish since $d(S)$ is zero. In the second last step we are left with an argument in $(S \otimes W) \otimes W^{\otimes p-1}$ and in the last step we "shift the S-part to the left outside of m_p," and so m_p vanishes on this argument.

Assume now that it is the argument in position $p-1$ of m_p that is contained in $S \otimes W$. By Lemma 3.1, $W^{\otimes p-2} \otimes (S \otimes W) \otimes W$ is equal to

$$\begin{aligned} &\bigl(W^{\otimes p-2} \otimes (S \otimes W) \otimes W\bigr) \cap \bigl(W^{\otimes p-1} \otimes (S \otimes W)\bigr) \\ &\quad + \bigcap_{i=1}^{p-1} \bigl(W^{\otimes p-1-i} \otimes (S \otimes W) \otimes W^{\otimes i}\bigr). \end{aligned}$$

By the former case m_p vanishes on the first term and by the same kind of shift argument as above, it also vanishes on the second term. In this way we may continue, and this settles the $a=0$ case.

The case $a=1$ and $b=0$ may be reduced to the case just treated, by shifting the S-part one step to the right, using Eq. (7).

So suppose now $2 \leqslant a \leqslant N-1$ and $a+b=N+1$. Applying Lemma 3.1 with $a=m-1$ and tensoring Eq. (11) to the right with $W^{\otimes b-1}$, we derive

$$W^{\otimes a} \otimes S \otimes W^{\otimes b} \subseteq W^{\otimes a+1} \otimes S \otimes W^{\otimes b-1} + S \otimes W^{\otimes N+1}.$$

So by induction we get that m_p vanishes when this is one of the arguments. $\square$

4. PBW-deformations of $T(V)/(\bigwedge^N V)$

In [3], R. Berger showed that the algebra $T(V)/(\bigwedge^N V)$, where $\bigwedge^N V$ is naturally embedded in $V^{\otimes N}$ by

$$x_1 \wedge \cdots \wedge x_N \mapsto \sum_{\sigma \in S_N} (-1)^{\operatorname{sgn}(\sigma)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(N)}$$

is an N-Koszul algebra (char $k = 0$). Theorem 1.1 therefore applies and we shall use this to compute the possible α_i's for this algebra such that the associated algebra U is of PBW-type.

When $N = 2$ (and $\alpha_2 = 0$) the classical PBW-theorem says that U is of PBW-type iff $\alpha_1 : \bigwedge^2 V \to V$ is a Lie bracket. In the general case when N is even the result involves a Lie algebra together with symplectic forms of successive even degrees and the Lie algebra is related to each form by an equation which can be viewed as generalizing the Jacobi identity.

The case N odd is distinct and simply involves choosing freely a linear form and successive symplectic forms of even degree. We shall now state the main theorem, but first we explain some notation.

Notation. (1) If we have a map

$$\psi : \bigwedge^p V \to U \tag{31}$$

the domain may be considered as a quotient space of $V^{\otimes p}$. Thus we get a map which we denote by the same symbol

$$\psi : V^{\otimes p} \to U. \tag{32}$$

On the other hand, given map (32) then by restricting it to the subspace $\bigwedge^p V$ of the domain we get a map (31). We shall be free to switch back and forth like this. Thus if $\psi' : \bigwedge^q V \to U'$ is another map we may form $\psi \otimes \psi'$ which we may consider as defined on $\bigwedge^{p+q} V$. If we go from (31) to (32) and back to (31) again the new map will differ from the old by a constant, but only in a few cases will it be of any importance to keep track of this.

(2) There is also a natural map

$$\operatorname{Hom}_k\left(\bigwedge^p V, \bigwedge^r V\right) \xrightarrow{T_a} \operatorname{Hom}_k\left(\bigwedge^{a+p} V, \bigwedge^{a+r} V\right) \tag{33}$$

given by (arrange I, P and A in ascending order)

$$T_a(\phi)(v_I) = \sum_{I = A \cup P} (-1)^{\operatorname{sgn}(A,P)} v_A \wedge \phi(v_P),$$

where we sum over all partitions $A \cup P$ of I with P of cardinality p. The sign is the sign of the permutation we get by concatenating P with A. Given a map ϕ we shall often by abuse of notation denote $T_a(\phi)$ simply by ϕ, like in (35) below.

(3) Let $\mathbf{1}^{\otimes p}$ denote the identity on $V^{\otimes p}$. For a linear map $L : V \otimes V \to V$ we introduce the notation

$$\underline{\mathbf{1}^{2a} L^b} = \sum_{\sum i_l = a, \sum j_l = b} \mathbf{1}^{\otimes 2i_1} \otimes L^{\otimes j_1} \otimes \cdots \otimes \mathbf{1}^{\otimes 2i_s} \otimes L^{\otimes j_s}$$

which is a map from $V^{\otimes 2a+2b}$ to $V^{\otimes 2a+b}$, and

$$\underline{\pm \mathbf{1}^a L} = \mathbf{1}^{\otimes a} \otimes L - \mathbf{1}^{\otimes a-1} \otimes L \otimes \mathbf{1} + \cdots + (-1)^a L \otimes \mathbf{1}^{\otimes a}.$$

Note that

$$\left[\mathbf{1}, \underline{\mathbf{1}^{2c} L}\right] = \underline{\pm \mathbf{1}^{2c+1} L}.$$

On some occasions we will also allow L to be a linear map from $V^{\otimes s}$ to V (notably in Proposition 5.6).

Now we shall consider PBW-deformations U of the algebra $T(V)/(\bigwedge^N V)$ given by linear maps $\alpha_i : \bigwedge^N V \to V^{\otimes N-i}$.

Theorem 4.1. *Suppose N is odd and $\dim V \geqslant N+2$. Let $l : V \to k$ be a linear map and $\Phi_{2r} : \bigwedge^{2r} V \to k$ arbitrary symplectic forms for $2 \leqslant 2r < N$. Let*

$$\begin{aligned} \alpha_{2r} &= \mathbf{1}^{\otimes N-2r} \otimes \Phi_{2r}, \\ \alpha_{2r+1} &= \mathbf{1}^{\otimes N-2r-1} \otimes l \otimes \Phi_{2r}. \end{aligned} \tag{34}$$

Then these α_i's give an algebra U of PBW-type. Conversely if U is of PBW-type there are l and Φ_{2r} such that the α_i are of the form above.

Theorem 4.2. *Suppose $N = 2n$ is even and $\dim V \geqslant N+2$. Let $L : \bigwedge^2 V \to V$ be a Lie bracket and $\Phi_{2r} : \bigwedge^{2r} V \to k$ for $0 \leqslant 2r < N$ be symplectic forms (with $\Phi_0 = 1$) such that the compositions*

$$L \circ \Phi_{2r} \circ L : \bigwedge^{2r+3} V \to V \tag{35}$$

are zero. Let α_{2r} be

$$\sum_{i=0}^{r} \underline{\mathbf{1}^{2(n-2r+i)} L^{2r-2i}} \otimes \Phi_{2i} \tag{36}$$

and let α_{2r+1} be

$$\sum_{i=0}^{r} \underline{\mathbf{1}^{2(n-2r-1+i)} L^{2r+1-2i}} \otimes \Phi_{2i} + \mathbf{1}^{\otimes 2n-2r-1} \otimes r\Phi_{2r}\bigl(\mathbf{1}^{\otimes 2r-1} \otimes L\bigr) \tag{37}$$

where $2r, 2r+1$ assumes values from 1 to $N-1$ and $\alpha_N = 0$. Then these α_i give an algebra U of PBW-type. Conversely if U is of PBW-type and $\alpha_N = 0$, there exists L and Φ_{2r} such that the α_i are of the form above.

If $\alpha_N \neq 0$, the U of PBW-type are obtained by also choosing a $\Phi_{2n} : \bigwedge^{2n} V \to k$ such that the following composition is zero

$$\Phi_{2n}\bigl(\mathbf{1}^{\otimes 2n-1} \otimes L\bigr) : \bigwedge^{2n+1} V \to k.$$

Remark 4.3. If the dimension of V is $\leqslant N$ the α_i may be chosen arbitrarily.

Remark 4.4. The case when V has dimension $N+1$ seems quite different and would have needed a separate treatment. So we do not consider it here.

Remark 4.5. When $N = 4$ and $\alpha_4 = 0$ we have given a Lie bracket L and a quadratic symplectic form Φ_2 such that the composition $L \circ \Phi_2 \circ L$ is zero. Up to coordinate change a quadratic symplectic form is given by its rank, an even number. For each such rank we thus get a class of Lie algebras.

Having stated the above results for the algebra $T(V)/(\bigwedge^N V)$, one might wonder about what happens concerning PBW-deformations of the algebra $T(V)/(S^N(V))$, the relations being the symmetric tensors of degree N. We state the following without proof (the case $N = 2$ is well known for Clifford algebras).

Theorem 4.6. *Consider the algebra $T(V)/(S^N(V))$. To give linear maps $S^N(V) \xrightarrow{\alpha_i} V^{\otimes N-i}$ such that U is of PBW-type is equivalent to giving arbitrary symmetric forms $\Phi_r : S^r(V) \to k$ for $r = 1, \ldots, N$. (The α_i's are then given by suitable expressions in these forms.)*

Before embarking on the proofs of the theorems above we shall need auxiliary results. We shall gradually move towards the proof of these theorems by investigating the conditions (4) and (5) of Theorem 1.1. Concerning notation, a monomial $\bigwedge^p V$ will be denoted by $x_1 \wedge \cdots \wedge x_p$ while a monomial in $V^{\otimes p}$ will be denoted simply by $x_1 \cdots x_p$.

5. Results on the α-mappings

This section is mainly devoted to study linear maps

$$\bigwedge^N V \xrightarrow{\alpha} V^{\otimes N-s}.$$

But first we need a result on the map T_a given in (33).

5.1. The linear map T_a

Proposition 5.1. *If $v = \dim_k V \geqslant a + p + r$ then T_a is injective. In particular it is an isomorphism when* $\dim_k V = a + p + r$.

Proof. Fix a basis $e_1, \dots, e_v$ of V and let $V = V' \oplus \langle e\rangle$ where e is one of the e_i's. There is a decomposition $\bigwedge^p V = \bigwedge^p V' \oplus \bigwedge^{p-1} V' \wedge e$. For short denote $\mathrm{Hom}_k(\bigwedge^s V', \bigwedge^t V')$ by $H(s,t)$. There is then a decomposition of $\mathrm{Hom}_k(\bigwedge^p V, \bigwedge^r V)$

$$H(p,r) \oplus H(p,r-1) \oplus H(p-1,r) \oplus H(p-1,r-1)$$

and a corresponding decomposition for $\mathrm{Hom}_k(\bigwedge^{a+p} V, \bigwedge^{a+r} V)$.

Check that $H(p,r-1)$ only maps to $H(a+p,a+r-1)$. By induction this map is injective. Also $H(p-1,r)$ only maps to $H(a+p-1,a+r)$ and so by induction is injective. The kernel of T_a is therefore contained in $H(p,r) \oplus H(p-1,r-1)$. Take note that this maps to the subspace $H(a+p,a+r) \oplus H(a+p-1,a+r-1)$.

For a subset I of $\{1,\dots,v\}$ denote by e_I the product $\bigwedge_{i\in I} e_i$. Now the maps $\phi_{P,R}$ in $\mathrm{Hom}_k(\bigwedge^p V, \bigwedge^r V)$, sending e_P to e_R and all other $e_{P'}$ to zero, constitute a basis for this space. Suppose then $\sum \gamma_{P,R}\phi_{P,R}$ is in the kernel of T_a. Let e be in $P\backslash R$. Then by what was said just above $\gamma_{P,R} = 0$. Similarly if we choose an e in $R\backslash P$. Hence if $p \neq r$ we are done. We may therefore assume that $p = r$ and that the kernel consists of elements $\sum \gamma_{P,P}\phi_{P,P}$.

So let $I(p)$ be the subspace of $\mathrm{Hom}_k(\bigwedge^p V, \bigwedge^p V)$ generated by $\phi_{P,P}$. Then T_a restricts to a map $I(p) \to I(p+a)$ which may be identified as follows. Let A be the commutative ring $k[e_1,\dots,e_v]/(e_1^2,\dots,e_v^2)$. We can identify A_p with $I(p)$ via $e_P \mapsto \phi_{P,P}$. If σ_a is the ath elementary symmetric polynomial in the variables e_i, then T_a can be identified with the map

$$A_p \xrightarrow{\cdot\sigma_a} A_{a+p}.$$

Note that $a!\sigma_a = (\sigma_1)^a$ in A. Also $\sigma_b \cdot \sigma_a = \binom{a+b}{a}\sigma_{a+b}$. To show that σ_a is injective it is therefore enough to show that σ_{a+b} is injective where the dimension of V is $a+b+2p$. Replacing a with $a+b$ and σ_a with σ_{a+b} we may now assume the dimension of V to be $a+2p$.

Let $I'(p)$ be the $I(p)$ intersected with $\mathrm{Hom}_k(\bigwedge^p V', \bigwedge^p V')$ and let

$$A' = k[e_1,\dots,e_{v-1}]\big/\big(e_1^2,\dots,e_{v-1}^2\big)$$

and so $A_p = A'_p \oplus e_v \cdot A'_{p-1}$. We are now left with showing that the map

$$I'(p) \oplus I'(p-1) \to I'(a+p) \oplus I'(a+p-1)$$

which is the same as

$$A'_p \oplus e_v A'_{p-1} \xrightarrow{\binom{\alpha\ 0}{\gamma\ \beta}} A'_{a+p} \oplus e_v A'_{a+p-1}$$

is injective, where $\alpha = \sigma'_a$, $\beta = \sigma'_a$ and $\gamma = e_v\sigma'_{a-1}$.

Suppose $f \oplus e_v g$ is in the kernel. This means $\beta(e_v g) = -\gamma(f)$ and $\alpha(f) = 0$. By induction γ is an isomorphism, so this says

$$\alpha \circ \gamma^{-1} \circ \beta(e_v g) = 0.$$

But

$$\alpha = a!\big(\sigma_1'\big)^a, \qquad \beta = a!\big(\sigma_1'\big)^a, \qquad \gamma = e_v(a-1)!\big(\sigma_1'\big)^{a-1}.$$

Since β is injective by induction we get $\gamma^{-1} \circ \beta = a\sigma_1'/e_v$ and $\alpha \circ \gamma^{-1} \circ \beta$ becomes $aa!(\sigma_1')^{a+1}/e_v$ which is an isomorphism by induction on p. This demonstrates the proposition. □

5.2. Conditions for $\mathbf{1} \otimes \alpha$ *to map to* $\bigwedge^N V$

Let E be the exterior algebra on the vector space V of dimension v. Let $y_0, y_1, \ldots, y_j$ be linearly independent elements in V and $y_J = y_1 \wedge y_2 \wedge \cdots \wedge y_j$ (not including y_0).

Lemma 5.2. *Let M be a homogeneous element in E such that $M \wedge y_J$ has degree $\leqslant v-2$. If for every y in V, $M \wedge y_J \wedge y$ has y_0 as a factor* (*i.e., can be written $M' \wedge y_0$ where M' may depend on y*), *then $M \wedge y_J = N \wedge y_0 \wedge y_J$ for a homogeneous element N.*

Proof. Let $\{y_i\} \cup \{z_k\}$ be a basis for V. Write $M \wedge y_J$ as $N \wedge y_0 \wedge y_J + N' \wedge y_J$, where N' is an expression in the z_k's. We then see that N' must be zero. □

Now we study maps $\alpha : \bigwedge^N V \to V^{\otimes N-s}$. We shall also consider the maps from the dual perspective. Let $W = V^*$. Dualizing α we get linear maps

$$W^{\otimes N-s} \xrightarrow{\beta} \bigwedge^N W.$$

Proposition 5.3. *Let $v \geqslant N+2$. Then the image of*

$$\bigwedge^{N+1} V \xrightarrow{\mathbf{1}\otimes\alpha} V^{\otimes N+1-s}$$

is contained in $\bigwedge^{N+1-s} V$ if and only if α is $\mathbf{1}^{\otimes N-s} \otimes \Phi$ for some symplectic form $\Phi : \bigwedge^s V \to k$.

Proof. The image of $\mathbf{1} \otimes \alpha$ is in $\bigwedge^{N+1-s} V$ iff the expression

$$x_1 \wedge \beta(x_2, \ldots, x_{N+1-s})$$

is alternating under permutations of $x_1, \ldots, x_{N+1-s}$. Let $x_2, \ldots, x_{N+1-s}$ be linearly independent. We can then apply Lemma 5.2 as follows. First take $y = x_1$, $y_0 = x_2$ and $j = 0$.

It follows that $\beta(x_2, \dots, x_{N+1-s})$ contains x_2 as a factor. Next let $y = x_1$, $y_0 = x_3$, and $y_1 = x_2$. It follows that $x_2 \wedge x_3$ is a factor. Thus we may proceed and get

$$\beta(x_2, \dots, x_{N+1-s}) = x_2 \wedge \cdots \wedge x_{N+1-s} \wedge a$$

for some a (which a priori may depend on the x's). We also see that

$$\beta(x_{\sigma(2)}, \dots, x_{\sigma(N+1-s)}) = x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(N+1-s)} \wedge a \tag{38}$$

for an arbitrary permutation σ of $\{2, \dots, N+1-s\}$.

If, on the other hand, $x_2, \dots, x_{N+1-s}$ span a space of dimension $N-s-1$, adjoin an x_1 linearly independent of these, and suppose if we omit x_{N+1-s} the others span a space of dimension $N-s$. Then switching the first two elements

$$\begin{aligned} x_1 \wedge \beta(x_2, \dots, x_{N+1-s}) &= -x_2 \wedge \beta(x_1, \dots, x_{N+1-s}) \\ &= -x_2 \wedge x_1 \wedge x_3 \wedge \cdots \wedge x_{N+1-s} \wedge a' \\ &= 0. \end{aligned}$$

Since this is true for all x_1, we must have $\beta(x_2, \dots, x_{N+1-s})$ equal to zero. By descending induction on the dimension of the space spanned by the arguments, we see that $\beta(x_2, \dots, x_{N+1-s})$ is zero when the elements are dependent.

Now let $x_1, \dots, x_v$ be a basis for W. Let I and J be disjoint subsets of $\{1, \dots, v\}$ of cardinalities $N-s$ and s. Let I be $\{i_1, \dots, i_{N-s}\}$ and I' be $\{i_2, \dots, i_{N-s+1}\}$ where also I' and J are disjoint. We now let $\beta(x_{i_1}, \dots, x_{i_{N-s}})$, which we write $\beta(x_I)$, contain the term $c_J \cdot x_I \wedge x_J$. Since

$$x_{i_1} \wedge \beta(x_{i_2}, \dots, x_{i_{N+1-s}}) = (-1)^{N-s} \cdot x_{i_{N+1-s}} \wedge \beta(x_{i_1}, \dots, x_{i_{N-s}})$$

we get that $\beta(x_{I'})$ will contain the term $c_J \cdot x_{I'} \wedge x_J$. By using (38) together with this observation, we easily obtain

$$\beta(x_I) = c_J \cdot x_I \wedge x_J$$

for arbitrary I which is disjoint from J. Letting $F = \sum_J c_J \cdot x_J$, we have $\beta(x_I) = x_I \wedge F$ for all I (where now repetitions in I is allowed). Letting Φ be the dual of F we get α equal to $\mathbf{1}^{\otimes N-s} \otimes \Phi$. $\square$

5.3. *Conditions for* $[\mathbf{1}, \alpha]$ *to be zero*

Now note that when R is $\bigwedge^N V$ then $(V \otimes R) \cap (R \otimes V)$ is $\bigwedge^{N+1} V$ by [3, Lemma 3.12]. Also note that if $\alpha_1, \alpha_2, \dots, \alpha_i$ have been found, we can find a particular solution α_{i+1} to the equation

$$\alpha_i \circ [\mathbf{1}, \alpha_1] = [\mathbf{1}, \alpha_{i+1}]$$

of (5), and then add solutions of the homogeneous equation

$$[\mathbf{1}, \alpha_{i+1}] = 0.$$

So we want to find conditions on linear maps

$$\bigwedge^N V \xrightarrow{\alpha} V^{\otimes N-s},$$

such that the induced map

$$\bigwedge^{N+1} V \xrightarrow{[\mathbf{1},\alpha]} V^{\otimes N-s+1}$$

is zero. Considering this from the dual perspective, we have the dual of α

$$W^{\otimes N-s} \xrightarrow{\beta} \bigwedge^N W,$$

and the dual of $[\mathbf{1}, \alpha]$ is given by the map sending $x_1 \cdots x_{N-s+1}$ to

$$x_1 \wedge \beta(x_2, \dots, x_{N-s+1}) - \beta(x_1, \dots, x_{N-s}) \wedge x_{N-s+1}.$$

Proposition 5.4. *Consider the following maps from* $\mathrm{Hom}(\bigwedge^N V, V^{\otimes N-s})$ *to* $\mathrm{Hom}(\bigwedge^{N+1} V, V^{\otimes N+1-s})$:

(a) $\alpha \mapsto \mathbf{1} \otimes \alpha + (-1)^s \cdot \alpha \otimes \mathbf{1}$.
If either $v \geqslant N+2$, *or* $v = N+1$ *with* $N-s$ *even, this map is injective.*
(b) $\alpha \mapsto \mathbf{1} \otimes \alpha + (-1)^{s-1} \cdot \alpha \otimes \mathbf{1}$.
If $v \geqslant N+2$ *then* α *is in the kernel if and only if* α *is* $\mathbf{1}^{\otimes N-s} \otimes \Phi$ *for a form* Φ *in* $\mathrm{Hom}(\bigwedge^s V, k)$.

Proof. Let δ be 0 or 1. The expression $\mathbf{1} \otimes \alpha + (-1)^{s-\delta} \cdot \alpha \otimes \mathbf{1}$ is zero iff

$$\begin{aligned} &x_1 \wedge \beta(x_2, \dots, x_{N+1-s}) \\ &\quad = (-1)^{s+1-\delta} \cdot \beta(x_1, \dots, x_{N-s}) \wedge x_{N+1-s} \\ &\quad = (-1)^{N+s+1-\delta} \cdot x_{N+1-s} \wedge \beta(x_1, \dots, x_{N+1-s}). \end{aligned} \tag{39}$$

Shifting $x_1, \dots, x_{N-s}, x_{N+1-s}$ cyclically $N+1-s$ times gives

$$x_1 \wedge \beta(x_2, \dots, x_{N+1-s}) = (-1)^{(N+1-s)(N+1-s-\delta)} \cdot x_1 \wedge \beta(x_2, \dots, x_{N+1-s}).$$

Hence if $\delta = 0$ and $N-s$ is even we get that β must be zero.

Now assume $v \geqslant N+2$. The same argument as in Proposition 5.3 gives (assuming α is in the kernel)

$$\beta(x_1, \dots, x_{N-s}) = x_1 \wedge \cdots \wedge x_{N-s} \wedge a$$

for some a (which a priori may depend on the x's). In particular if x_i and x_j are equal for some $i < j$ then $\beta(x_1, \dots, x_{N-s})$ is zero. But then β factors through $\bigwedge^{N-s} V$ so

$$\beta(x_{\sigma(1)}, \dots, x_{\sigma(N-s)}) = x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(N-s)} \wedge a$$

for any permutation σ.

Again exactly the same argument as in Proposition 5.3 gives that there exists a form F of degree s such that $\beta(x_I)$ is $x_I \wedge F$. Putting this into (39) we see that it holds if δ is one. If δ is zero one gets

$$x_1 \wedge \cdots \wedge x_{N+1-s} \wedge F = -x_1 \wedge \cdots \wedge x_{N+1-s} \wedge F$$

for all $x_1, \dots, x_{N+1-s}$, and so in this case F is zero. □

5.4. *Conditions for* $[\mathbf{1}, \alpha]$ *to map to* $\bigwedge^N V$

Now we consider condition (4) of Theorem 1.1. This will separate into two cases. For naturality of proof we formulate slightly more general results than what is needed later on. For

$$\bigwedge^N V \xrightarrow{\alpha} V^{\otimes N-s}$$

we have the bracket

$$\{\mathbf{1}, \alpha\} = \mathbf{1} \otimes \alpha + (-1)^s \cdot \alpha \otimes \mathbf{1}$$

which is a linear map from $\bigwedge^{N+1} V$ to $V^{\otimes N+1-s}$.

Proposition 5.5. *Suppose $N - s$ is even and $v \geqslant N + 1$. In the map*

$$\mathrm{Hom}\big(\bigwedge^N V, V^{\otimes N-s}\big) \to \mathrm{Hom}\big(\bigwedge^{N+1} V, V^{\otimes N+1-s}\big)$$

given by sending α to $\{\mathbf{1}, \alpha\}$, the latter is contained in $\mathrm{Hom}(\bigwedge^{N+1} V, \bigwedge^{N+1-s} V)$ *if and only if α is $\mathbf{1}^{\otimes N-s} \otimes \Phi$ for a form Φ in* $\mathrm{Hom}(\bigwedge^s V, k)$.

Proposition 5.6. *Suppose $N - s$ is odd, say N is $2p + s + 1$, and $v \geqslant N + 2$. In the map*

$$\mathrm{Hom}\big(\bigwedge^N V, V^{\otimes N-s}\big) \to \mathrm{Hom}\big(\bigwedge^{N+1} V, V^{\otimes N+1-s}\big)$$

given by sending α to $\{\mathbf{1}, \alpha\}$, the latter is contained in

$$\mathrm{Hom}\big(\bigwedge^{N+1} V, \bigwedge^{N+1-s} V\big) \tag{40}$$

if and only if α is $\underline{\mathbf{1}^{2p} L}$ for a linear map L in $\mathrm{Hom}(\bigwedge^{s+1} V, V)$.

Proof. We shall only prove the latter proposition since the former is similar but easier. It is easy to see that if α is $\underline{\mathbf{1}^{2p}L}$, then $\{\mathbf{1}, \alpha\}$ is in (40). We therefore assume that $\{\mathbf{1}, \alpha\}$ is in (40) and will show that it is of the form $\underline{\mathbf{1}^{2p}L}$.

Step 1. Suppose first $v = N + 2$. Then (40) is naturally isomorphic to $\mathrm{Hom}(\bigwedge^{s+1} V, V)$. By Proposition 5.1 an isomorphism is given by

$$\mathrm{Hom}\Big(\bigwedge^{s+1} V, V\Big) \xrightarrow{T_{N-s}} \mathrm{Hom}\Big(\bigwedge^{N+1} V, \bigwedge^{N+1-s} V\Big).$$

We consider $\bigwedge^{s+1} V$ as a quotient space of $V^{\otimes s+1}$ and therefore this as the domain of L. Also consider $\bigwedge^{N+1} V$ and $\bigwedge^{N+1-s} V$ as subspaces of $V^{\otimes N+1}$ and $V^{\otimes N+1-s}$, respectively. The map T_{N-s} is then given by

$$L \mapsto 1/(s+1)! \cdot \big\{\mathbf{1}, \underline{\mathbf{1}^{2p}L}\big\}.$$

Thus if $\{\mathbf{1}, \alpha\}$ is in (40) we get a form L in $\mathrm{Hom}(\bigwedge^{s+1} V, V)$ such that

$$\big\{\mathbf{1}, \alpha - 1/(s+1)! \cdot \underline{\mathbf{1}^{2p}L}\big\}$$

becomes zero. Proposition 5.4(a) then gives α equal to $1/(s+1)! \cdot \underline{\mathbf{1}^{2p}L}$.

Step 2. Suppose now that $v \geqslant N + 3$ and write $V = V' \oplus \langle e\rangle$. Our approach to show that α is of the form $\underline{\mathbf{1}^{2p}L}$ will be to use the decomposition of V to write $\mathrm{Hom}(\bigwedge^{s+1} V, V)$ as

$$\begin{aligned} &\mathrm{Hom}\Big(\bigwedge^{s+1} V', V'\Big) \oplus \mathrm{Hom}\Big(\bigwedge^{s} V' \wedge e, V'\Big) \\ &\quad \oplus \mathrm{Hom}\Big(\bigwedge^{s+1} V', \langle e\rangle\Big) \oplus \mathrm{Hom}\Big(\bigwedge^{s} V' \wedge e, \langle e\rangle\Big) \end{aligned} \tag{41}$$

and find the components of L in each of these summands by induction.

The map α now lives in

$$\begin{aligned} &\mathrm{Hom}\Big(\bigwedge^{N} V', V'^{\otimes N-s}\Big) \oplus \mathrm{Hom}\Big(\bigwedge^{N} V', \bigoplus_{a+b=N-s-1} V'^{\otimes a} \otimes e \otimes V'^{\otimes b}\Big) \\ &\quad \oplus \mathrm{Hom}\Big(\bigwedge^{N-1} V' \wedge e, V'^{\otimes N-s}\Big) \\ &\quad \oplus \mathrm{Hom}\Big(\bigwedge^{N-1} V' \wedge e, \bigoplus_{a+b=N-s-1} V'^{\otimes a} \otimes e \otimes V'^{\otimes b}\Big) \end{aligned} \tag{42}$$

together with Hom-terms where the codomain involves two or more e's, and is sent to $\{\mathbf{1}, \alpha\}$ which lives in

$$\begin{aligned} &\mathrm{Hom}\Big(\bigwedge^{N+1} V', V'^{\otimes N+1-s}\Big) \oplus \mathrm{Hom}\Big(\bigwedge^{N+1} V', \bigoplus_{a+b=N-s} V'^{\otimes a} \otimes e \otimes V'^{\otimes b}\Big) \\ &\quad \oplus \mathrm{Hom}\Big(\bigwedge^{N} V' \wedge e, V'^{\otimes N+1-s}\Big) \end{aligned}$$

$$\oplus \operatorname{Hom}\Bigl(\bigwedge^{N} V' \wedge e, \bigoplus_{a+b=N-s} V'^{\otimes a} \otimes e \otimes V'^{\otimes b}\Bigr) \tag{43}$$

together with Hom-terms whose codomain also involves two or more e's.

We are then assuming $\{\mathbf{1}, \alpha\}$ is in the subspace

$$\operatorname{Hom}\bigl(\bigwedge^{N+1}\bigl(V' \oplus \langle e\rangle\bigr), \bigwedge^{N+1-s}\bigl(V' \oplus \langle e\rangle\bigr)\bigr)$$

of (43).

Step 3. The first summand of (42) is the only one mapping non-zero to the first summand in (43). By induction the component of α in the first summand of (42) is given by $\underline{\mathbf{1}^{2p} L'}$ for some L' in $\operatorname{Hom}(\bigwedge^{s+1} V', V')$. By extending this by zero in (41) we can consider it as a map in $\operatorname{Hom}(\bigwedge^{s+1} V, V)$ and subtract $\underline{\mathbf{1}^{2p} L'}$ from α. This new α will map $\bigwedge^{N} V'$ to zero in $V'^{\otimes N-s}$.

Step 4. Consider the mapping to the third summand in (43). The image of the element α comes from the third summand in (42). The map between these two identifies as

$$\operatorname{Hom}\bigl(\bigwedge^{N-1} V', V'^{\otimes N-s}\bigr) \to \operatorname{Hom}\bigl(\bigwedge^{N} V', V'^{\otimes N+1-s}\bigr)$$

given by

$$\gamma \mapsto \mathbf{1} \otimes \gamma + (-1)^{s-1} \cdot \gamma \otimes \mathbf{1}.$$

By induction this is in $\operatorname{Hom}(\bigwedge^{N} V', \bigwedge^{N+1-s} V')$ iff γ is $\underline{\mathbf{1}^{2p} L''}$ for some L'' in $\operatorname{Hom}(\bigwedge^{s} V', V')$. Thus letting γ be the restriction of α (to the third summand in (42)) and extending L'' by zero in (41) we get a map in $\operatorname{Hom}(\bigwedge^{s+1} V, V)$. We subtract $\underline{\mathbf{1}^{2p} L''}$ from α. Now the components of α in the first and third summand in (42) become zero.

Step 5. We now consider the mapping to the second summand in (43). The image of the element α in this summand comes from the second summand in (42). Let α have component α^{i} in $\operatorname{Hom}(\bigwedge^{N} V', V'^{\otimes i} \otimes e \otimes V'^{\otimes j})$ where i ranges from 0 to $N-s-1$. The component of $\{\mathbf{1}, \alpha\}$ in $\operatorname{Hom}(\bigwedge^{N+1} V', V'^{\otimes a} \otimes e \otimes V'^{\otimes b})$ is then

$$\mathbf{1} \otimes \alpha^{a-1} + (-1)^{s} \alpha^{a} \otimes \mathbf{1}. \tag{44}$$

Identifying these spaces with $\operatorname{Hom}(\bigwedge^{N+1} V', V'^{\otimes N-s})$ we wish that all the (44) be in the subspace $\operatorname{Hom}(\bigwedge^{N+1} V', \bigwedge^{N-s} V')$ and that they are alternately equal in the sense that for $0 \leqslant a \leqslant N-s$

$$\mathbf{1} \otimes \alpha^{a} + (-1)^{s} \cdot \alpha^{a+1} \otimes \mathbf{1} = -\bigl(\mathbf{1} \otimes \alpha^{a-1} + (-1)^{s} \cdot \alpha^{a} \otimes \mathbf{1}\bigr).$$

In particular $\mathbf{1} \otimes \alpha^{N-s-1}$ is in this subspace and so by Proposition 5.3 we have α^{N-s-1} equal to $\mathbf{1}^{\otimes N-s-1} \otimes \Phi'$ for a form Φ' in $\operatorname{Hom}(\bigwedge^{s+1} V', k)$. But then we see by Proposition 5.4 that α^{N-s-j} is $\mathbf{1}^{\otimes N-s-1} \otimes \Phi'$ for j odd and zero when j is even.

Now extend Φ' by zero in (41) to a map in $\mathrm{Hom}(\bigwedge^{s+1} V, V)$. Note that $\underline{\mathbf{1}^{2p}\Phi'}$ is equal to $p+1$ times $\mathbf{1}^{\otimes 2p} \otimes \Phi'$. Replace Φ' by a multiple and subtract $\underline{\mathbf{1}^{2p}\Phi'}$ from α, giving us that its three first summands in (42) are zero.

Step 6. We now consider the mapping to the fourth summand in (43). Note that the image of the element α now exclusively comes from the fourth summand in (42). Again let α have component α^i in

$$\mathrm{Hom}\big(\textstyle\bigwedge^{N-1} V' \wedge e, V'^{\otimes i} \otimes e \otimes V'^{\otimes j}\big) \cong \mathrm{Hom}\big(\textstyle\bigwedge^{N-1} V', V'^{\otimes i} \otimes e \otimes V'^{\otimes j}\big).$$

We see that the component of $\{\mathbf{1}, \alpha\}$ in

$$\mathrm{Hom}\big(\textstyle\bigwedge^{N} V' \wedge e, V'^{\otimes a} \otimes e \otimes V'^{\otimes b}\big) \cong \mathrm{Hom}\big(\textstyle\bigwedge^{N} V', V'^{\otimes a} \otimes e \otimes V'^{\otimes b}\big) \tag{45}$$

becomes

$$\mathbf{1} \otimes \alpha^{a-1} - (-1)^s \cdot \alpha^a \otimes \mathbf{1}. \tag{46}$$

Identifying again the right-hand side of (45) with $\mathrm{Hom}(\bigwedge^N V', V'^{\otimes N-s})$ we wish that the (46) shall be in $\mathrm{Hom}(\bigwedge^N V', \bigwedge^{N-s} V')$ and be alternately equal. In particular $\mathbf{1} \otimes \alpha^{N-s-1}$ shall be in $\mathrm{Hom}(\bigwedge^N V', \bigwedge^{N-s} V')$. Proposition 5.3 then gives α^{N-s-1} equal to $\mathbf{1}^{\otimes N-s-1} \otimes \Phi''$ for a form Φ'' in $\mathrm{Hom}(\bigwedge^s V', k)$. But then we see that α^{N-s-j} is $\mathbf{1}^{\otimes N-s-1} \otimes \Phi''$ for j odd and zero when j is even.

Extend Φ'' by zero in (41) to a map in $\mathrm{Hom}(\bigwedge^{s+1} V, V)$, replace it by a suitable multiple, and subtract $\underline{\mathbf{1}^{2p}\Phi''}$ from α. The result is that all components of α in (42) are zero.

Step 7. If some summand of α lives in a Hom-term whose domain is $\bigwedge^N V'$ and whose codomain involves two or more e's, then our hypothesis on the image $\{\mathbf{1}, \alpha\}$ says that this summand must map to zero. Using Proposition 5.3 we easily deduce that this summand of α must be zero.

Similarly we can argue for summands of α living in a Hom-term whose domain is $\bigwedge^{N-1} V' \wedge e$ and whose codomain involves two or more e's.

The upshot is now that the original α is $\underline{\mathbf{1}^{2p}L}$ where L is $L' + L'' + \Phi' + \Phi''$. $\square$

6. Auxiliary results

Now let A be the algebra $T(V)/(xyt - txy)_{x,y,t\in V}$. Note that $x_I = x_{\sigma I}$ in A for an arbitrary even permutation σ. The following will be useful in the proof of Theorem 4.2.

Lemma 6.1. *For $p \geqslant 2$ there is an exact sequence*

$$0 \to \textstyle\bigwedge^p V \xrightarrow{i} A_p \xrightarrow{j} S_p(V) \to 0$$

where j is the natural quotient map and i is given by

$$x_{i_1} \wedge \cdots \wedge x_{i_p} \mapsto x_I - x_{\sigma I} \tag{47}$$

for an arbitrary odd permutation σ.

Proof. Fix a basis $x_1, \ldots, x_v$ for V. Let x_I be a monomial in A_p. As noted $x_I = x_{\sigma I}$ for any even permutation σ. If two indices i_a and i_b in I are equal, $\sigma I = \sigma(a, b)I$ and so $x_I = x_{\sigma I}$ holds for all permutations.

In any multihomogeneous component of A of degree $(b_1, \ldots, b_p)$, where some $b_i \geqslant 2$, j therefore becomes an isomorphism. Clearly the elements on the right in (47) are non-zero and linearly independent elements in the kernel when I varies over all increasing sequences of indices, and they also generate the kernel. □

This lemma will be applied as follows. Suppose N is even and $\rho : \bigwedge^N V \to V^{\otimes N-s}$ is a map where s is even. Then $[\mathbf{1}, \rho]$ applied to $x_1 \wedge \cdots \wedge x_{N+1}$ is (up to a non-zero constant)

$$\sum_i x_i \rho(x_{i+1}, \ldots, x_{N+1}, x_1, \ldots, x_{i-1}) - \rho(x_{i+1}, \ldots, x_{N+1}, x_1, \ldots, x_{i-1}) x_i.$$

The crucial thing to note is that this is zero in A_{N+1-s}. Hence a necessary condition for the equation of Theorem 1.1

$$\alpha_{s-1} \circ [\mathbf{1}, \alpha_1] = [\mathbf{1}, \alpha_s]$$

to be valid is that the image of the left-hand side is zero in A_{N+1-s}.

Before taking on the proof of Theorems 4.1 and 4.2 we need some extra lemmas to make the argument go smoother.

In the following let

$$L : \bigwedge^2 V \to V, \qquad \Phi_{2r} : \bigwedge^{2r} V \to k$$

be linear maps. We may consider them as maps from $V^{\otimes 2}$ and $V^{\otimes 2r}$, respectively, by composition with the natural quotient maps. By combining these maps and the identity maps in various ways we get maps defined on various $V^{\otimes q}$ and we shall again restrict these maps to the subspace $\bigwedge^q V$ of $V^{\otimes q}$.

Lemma 6.2.

(1) $L \circ (\mathbf{1} \otimes L) = -L \circ (L \otimes \mathbf{1})$ *on the subspace* $\bigwedge^3 V$ *of* $V^{\otimes 3}$.
(2) $\Phi_{2r}(\mathbf{1}^{\otimes a} \otimes L \otimes \mathbf{1}^{\otimes b}) = (-1)^b \Phi_{2r}(\mathbf{1}^{\otimes a+b} \otimes L)$ *on the subspace* $\bigwedge^{2r+1} V$ *of* $V^{\otimes 2r+1}$.
(3) *Let M be a tensor monomial in L and $\mathbf{1}$ of degree $2r$ with at least two L's. If $\Phi_{2r} \circ M$ is defined on $V^{\otimes q}$, it is zero on the subspace $\bigwedge^q V$.*

Proof. (1) It is immediate to see that this holds for an element $a \wedge b \wedge c$ in $\bigwedge^3 V$ which is equal to $abc - acb + \cdots$ in $V^{\otimes 3}$.

(2) When $r = 1$ we easily check $\Phi_2(\mathbf{1} \otimes L) = -\Phi_2(L \otimes \mathbf{1})$ on $\bigwedge^3 V$. The general argument is analogous.

(3) Note that $\Phi_2(L \otimes L) = 0$ on $\bigwedge^4 V$ because the image of $L \otimes L$ is in the symmetric tensors $S^2(V)$ in $V^{\otimes 2}$. The argument in the general case is a straightforward extension. □

Corollary 6.3. *If*

$$L \otimes \Phi_{2r}\big(\underline{\pm \mathbf{1}^{2r-1} L}\big) : \bigwedge^{2r+3} V \to V$$

is zero, then the following is also the zero map when $b \geqslant 1$

$$\underline{\mathbf{1}^{2a} L^b} \otimes \Phi_{2r}\big(\underline{\pm \mathbf{1}^{2r-1} L}\big) : \bigwedge^{2(r+a+b)+1} V \to V^{\otimes 2a+b}.$$

Proof. Note first that

$$\mathbf{1}^{\otimes 2} \otimes \Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big) = \Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big) \otimes \mathbf{1}^{\otimes 2}$$

on $\bigwedge^{2r+3} V$. Let M be a tensor monomial in L and $\mathbf{1}^{\otimes 2}$. By Lemma 6.2(2) it will be sufficient to show that $M \otimes \Phi_{2r}(\mathbf{1}^{\otimes 2r-1} \otimes L)$ becomes zero. But if $M = M' \otimes L \otimes \mathbf{1}^{\otimes 2b}$ then

$$M \otimes \Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big) = M' \otimes L \otimes \Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big) \otimes \mathbf{1}^{\otimes 2b}. \tag{48}$$

For an element $x_1 \wedge \cdots \wedge x_m$ (where $m = 2(b + a + r) + 1$) which we consider as

$$\sum (-1)^{\operatorname{sgn}(\sigma)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}$$

fix the $\sigma(i)$ indices which are in positions corresponding to M' and to $\mathbf{1}^{\otimes 2b}$ in the right-hand side of (48). By varying the middle terms, the right-hand side of (48) applied to this becomes zero. Varying also the end terms, we get the corollary. □

Lemma 6.4. *Suppose L is a Lie bracket. The maps*

$$L \otimes \Phi_{2r}\big(\underline{\pm \mathbf{1}^{2r-1} L}\big) : \bigwedge^{2r+3} V \to V \tag{49}$$

and

$$L \circ T_2(\Phi_{2r}) \circ T_{2r+1}(L) : \bigwedge^{2r+3} V \to \bigwedge^{2r+2} V \to \bigwedge^2 V \to V \tag{50}$$

are equal up to multiplication by a non-zero scalar.

Proof. The first map in (50) identifies up to non-zero scalar as $\underline{\pm\mathbf{1}^{2r+1}L}$ restricted to the subspace $\bigwedge^{2r+3} V$ of $V^{\otimes 2r+3}$. The middle map in (50) identifies as $\mathbf{1}^{\otimes 2} \otimes \Phi_{2r}$ up to a scalar. That the composition in (50) now is equal to the one in (49) up to a scalar follows since $L(\mathbf{1} \otimes L)$ and $L(L \otimes \mathbf{1})$ are zero by the Jacobi identity. □

Lemma 6.5. *Suppose L is a Lie bracket. Then*

$$\underline{\mathbf{1}^{2a}L^b} \circ \bigl[\mathbf{1}, \underline{\mathbf{1}^{2(a+b-1)}L}\bigr] = \bigl[\mathbf{1}, \underline{\mathbf{1}^{2(a-1)}L^{b+1}}\bigr]. \tag{51}$$

Proof. We use induction on $a+b$. The first of the terms in (51) can be written

$$\underline{\mathbf{1}^{2a}L^b} = \mathbf{1}^{\otimes 2} \otimes \underline{\mathbf{1}^{2(a-1)}L^b} + L \otimes \underline{\mathbf{1}^{2a}L^{b-1}}. \tag{52}$$

The second of the terms in (51) can be written

$$\mathbf{1}^{\otimes 2} \otimes \bigl[\mathbf{1}, \underline{\mathbf{1}^{2(a+b-2)}L}\bigr] + [\mathbf{1}, L] \otimes \mathbf{1}^{\otimes 2(a+b-1)}. \tag{53}$$

The terms in (53) composed with the first term in (52) become by induction

$$\mathbf{1}^{\otimes 2} \otimes \bigl[\mathbf{1}, \underline{\mathbf{1}^{2(a-2)}L^{b+1}}\bigr] + [\mathbf{1}, L] \otimes \underline{\mathbf{1}^{2(a-1)}L^b}, \tag{54}$$

and the terms in (53) composed with the second term in (52) become (using that $L \circ [\mathbf{1}, L]$ is zero since L is a Lie bracket)

$$L \otimes \bigl[\mathbf{1}, \underline{\mathbf{1}^{2(a-1)}L^b}\bigr]. \tag{55}$$

Thus the left-hand side of Eq. (51) is the sum of (54) and (55) which is

$$\begin{aligned}&\mathbf{1}^{\otimes 3} \otimes \underline{\mathbf{1}^{2(a-2)}L^{b+1}} - \mathbf{1}^{\otimes 2} \otimes \underline{\mathbf{1}^{2(a-2)}L^{b+1}} \otimes \mathbf{1}\\ &\quad + \mathbf{1} \otimes L \otimes \underline{\mathbf{1}^{2(a-1)}L^b} - L \otimes \underline{\mathbf{1}^{2(a-1)}L^b} \otimes \mathbf{1}.\end{aligned}$$

By using (52) this is seen to be equal to the right-hand side of (51). □

7. Proof of Theorems 4.1 and 4.2

We are now ready to give the proofs of the main theorems about PBW-deformations of $T(V)/(\bigwedge^N V)$. First we do the case when N is even.

Proof of Theorem 4.2.

Step 1. By Proposition 5.6, $\alpha_1 = \underline{\mathbf{1}^{2(n-1)}L}$. When $N = 2$ we know L is a Lie bracket. So assume $N \geqslant 4$. The composition $\alpha_1 \circ [\mathbf{1}, \alpha_1]$ is then

$$\underline{\mathbf{1}^{2(n-1)}L} \circ \underline{\pm\mathbf{1}^{2n-1}L}.$$

We do not yet know that L is a Lie bracket, but the same argument as in Lemma 6.5 shows (or simply check it directly) that this is

$$\big[\mathbf{1}, \underline{\mathbf{1}^{2(n-2)}L^2}\big] + \sum_{a+b=n-1} \mathbf{1}^{\otimes 2a} \otimes L \circ (\mathbf{1} \otimes L - L \otimes \mathbf{1}) \otimes \mathbf{1}^{\otimes 2b}. \tag{56}$$

Now this is to be equal to $[\mathbf{1}, \alpha_2]$ and so shall be zero in A_{N-1} of Lemma 6.1. The first term is zero in A_{N-1}. The other term is 0 in $S(V)_{N-1}$, seen as follows. Recall that we consider $[\mathbf{1}, \alpha_1]$ on the subspace $\bigwedge^{N+1} V$ of $V^{\otimes N+1}$, when $a \geqslant 1$ just keep $x_{\sigma(3)} \cdots x_{\sigma(N+1)}$ fixed and switch $x_{\sigma(1)}$ and $x_{\sigma(2)}$. Similarly we may argue when $b \geqslant 1$. Hence the image in A_{N-1} of the second term lies naturally in $\bigwedge^{N-1} V$ by the sequence of Lemma 6.1. In fact it is equal to a multiple of

$$T_{N-2}\big(L \circ (\mathbf{1} \otimes L - L \otimes \mathbf{1})\big)$$

which by Lemma 6.2 is a multiple of $T_{N-2}(L \circ (\mathbf{1} \otimes L))$. Applying Proposition 5.1 and noting that $v \geqslant (N-2)+3+1$ we must have $L \circ (\mathbf{1} \otimes L)$ equal to zero if (56) is zero in A_{N-1}. This is the Jacobi identity and so L is a Lie bracket.

Step 2. We shall now proceed by induction on $2r, 2r+1$. Suppose α_{2i} has the form (36) for $1 \leqslant i \leqslant r$ and that (35) holds for $i < r$. By the equivalence of Lemma 6.4 we have

$$L \otimes \Phi_{2i}\big(\underline{\pm \mathbf{1}^{2i-1}L}\big) = 0 \tag{57}$$

for $i < r$. We shall show that α_{2r+1} must have the form given. Since $[\mathbf{1}, \alpha_1]$ is equal to $\underline{\pm \mathbf{1}^{2n-1}L}$ we get by Lemma 6.5 that the composition of $[\mathbf{1}, \alpha_1]$ and α_{2r} is the sum of terms

$$\sum_{i=0}^{r} \big[\mathbf{1}, \underline{\mathbf{1}^{2(n-1-2r+i)}L^{2r+1-2i}}\big] \otimes \Phi_{2i} \tag{58}$$

and terms

$$\sum_{i=1}^{r} \underline{\mathbf{1}^{2(n-2r+i)}L^{2r-2i}} \otimes \Phi_{2i}\big(\underline{\pm \mathbf{1}^{2i-1}L}\big). \tag{59}$$

By the induction hypothesis and Corollary 6.3 we get that all terms in (59) vanish except when $i = r$ where we get

$$\mathbf{1}^{\otimes 2n-2r} \otimes 2r\Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big). \tag{60}$$

Now let α'_{2r+1} be given by (37) in the statement of Theorem 4.2. Since $\mathbf{1} \otimes \Phi_{2i}$ is equal to $\Phi_{2i} \otimes \mathbf{1}$ on $\bigwedge^{2i+1} V$ and

$$\mathbf{1} \otimes \Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big) = -\Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big) \otimes \mathbf{1}$$

on $\bigwedge^{2r+2} V$ we see that $[\mathbf{1}, \alpha'_{2r+1}]$ is equal to the composition of $[\mathbf{1}, \alpha_1]$ and α_{2r}, the sum of (58) and (60). By Proposition 5.4(a) we get $\alpha_{2r+1} = \alpha'_{2r+1}$.

Step 3. Now we assume that $0 \leqslant r \leqslant n-1$, that α_{2r+1} is given by (37), and (35) holds for $i < r$, which is equivalent to (57). Now $[\mathbf{1}, \alpha_1]$ is equal to $\underline{\pm\mathbf{1}^{2n-1}L}$. Then the composition of $[\mathbf{1}, \alpha_1]$ and α_{2r+1} is equal to the sum of terms (using the vanishing (57), Corollary 6.3 and Lemma 6.5)

$$\sum_{i=0}^{r} \big[\mathbf{1}, \underline{\mathbf{1}^{2(n-2r-2+i)} L^{2r+2-2i}}\,\big] \otimes \Phi_{2i} \tag{61}$$

and terms (using Lemma 6.2(3) and the Jacobi identity $L(\mathbf{1} \otimes L) = 0$)

$$\underline{\mathbf{1}^{2(n-r-1)} L} \otimes \Phi_{2r}\big(\underline{\pm\mathbf{1}^{2r-1} L}\big) - \underline{\pm\mathbf{1}^{2n-2r-2} L} \otimes r\Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big). \tag{62}$$

Again by Lemma 6.2(2) the sum of the terms in (62) is

$$\left(\sum_{a+b=2n-2r-2} \mathbf{1}^{\otimes a} \otimes L \otimes \mathbf{1}^{\otimes b}\right) \otimes r\Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big). \tag{63}$$

We now want to show that

$$L \otimes \Phi_{2r}\big(\mathbf{1}^{\otimes 2r-1} \otimes L\big) = 0, \tag{64}$$

which by Lemma 6.4 is the same as (35). If the sum of (61) and (62) is to be equal to $[\mathbf{1}, \alpha_{2r+2}]$ then it has to be zero in $A_{N+1-2r-2}$ of Lemma 6.1. Since (61) vanishes there and (63) becomes zero in $S(V)_{N+1-2r-2}$ we must have (63) zero in $\bigwedge^{N+1-2r-2} V$. By Proposition 5.1, since $v \geqslant (N-2r-2) + (2r+3) + 1$, we get (64). Then we see that (36) for α_{2r+2} makes $[\mathbf{1}, \alpha_{2r+2}]$ equal to the composition of $[\mathbf{1}, \alpha_1]$ and α_{2r+1} and by Proposition 5.4(b) every solution α_{2r+2} has this form. □

Now we give the much easier proof of the case when N is odd.

Proof of Theorem 4.1. Proposition 5.5 gives α_1 is $\mathbf{1}^{\otimes N-1} \otimes l$. Since $l \otimes \mathbf{1} = -\mathbf{1} \otimes l$ on $\bigwedge^2 V$ we get $[\mathbf{1}, \alpha_1]$ equal to $\mathbf{1}^{\otimes N} \otimes 2l$.

Now if α_{2r} is given as $\mathbf{1}^{\otimes N-2r} \otimes \Phi_{2r}$ then $\alpha_{2r} \circ [\mathbf{1}, \alpha_1]$ is

$$\mathbf{1}^{\otimes N-2r} \otimes 2l \otimes \Phi_{2r}. \tag{65}$$

On the other hand, if

$$\alpha'_{2r+1} = \mathbf{1}^{\otimes N-2r-1} \otimes l \otimes \Phi_{2r}$$

then $[\mathbf{1}, \alpha'_{2r+1}]$ is given by (65) also. Proposition 5.4 gives α_{2r+1} equal to α'_{2r+1}.

Lastly if α_{2r+1} is given as in (34), then $\alpha_{2r+1} \circ [\mathbf{1}, \alpha_1]$ is zero since $l \circ (\mathbf{1} \otimes l)$ vanishes on $\bigwedge^2 V$. Hence Proposition 5.4(b) gives the form of α_{2r+2}. □

8. PBW-deformations of cubic Artin–Schelter algebras

This section is an extended example of Poincaré–Birkhoff–Witt deformations of an interesting class of 3-Koszul algebras; the cubic *Artin–Schelter* algebras. So we will present equations for these deformations for the generic forms of these algebras, as presented in [1]. We give the full details of the argument in one case, a semi-detailed description in one case, and as the other cases are similar, we simply present the results.

Throughout this section, the vector space V of generators for our algebras will be two-dimensional, with basis x, y. The space of generators R will also be two-dimensional, with basis f, g. The generators are cubic, so $N = 3$. Finally, in [1] it is proved that in all the situations considered here, the vector space of syzygies is one-dimensional, and it has a generator w. We use the conditions from Theorem 1.1 throughout.

Type E. From [1] we have that this algebra is generated by the two cubic polynomials

$$f = y^3 + x^3, \qquad g = y^2x + \zeta yxy + \zeta^2 xy^2.$$

Here ζ is a primitive third root of unity. In our notation, $R = (f, g)$, and then $(V \otimes R) \cap (R \otimes V)$ is

$$(w) = \left(y^3x + \zeta y^2xy + \zeta^2 yxy^2 + xy^3 + x^4\right).$$

We define general maps $\alpha_1 : R \to V \otimes V$, $\alpha_2 : R \to V$, and $\alpha_3 : R \to k$ with parameters

$$\begin{aligned}
\alpha_1(f) &= a_{11} x \otimes x + a_{12} x \otimes y + a_{13} y \otimes x + a_{14} y \otimes y,\\
\alpha_2(f) &= a_{21} x + a_{22} y,\\
\alpha_3(f) &= a_3,
\end{aligned}$$

and $\alpha_i(g)$ is given similarly with coefficients b_{ij}. Condition (4) can be written

$$[\mathbf{1}, \alpha](w) = \beta f + \gamma g,$$

where β and γ are taken from the ground field. Now w can be written

$$w = fx + \zeta gy = xf + yg.$$

Thus

$$\beta f + \gamma g = x\alpha_1(f) + y\alpha_1(g) - \alpha_1(f)x - \zeta\alpha_1(g)y.$$

By comparing the coefficients of monomials in x and y, we get the following equations:

$$x^3:\quad \beta = a_{11} - a_{11}, \tag{66}$$

$$x^2y:\quad 0 = a_{12} - \zeta b_{11}, \tag{67}$$

$$xyx:\quad 0 = a_{13} - a_{12}, \tag{68}$$

$$xy^2:\quad \gamma\zeta^2 = a_{14} - \zeta b_{12}, \tag{69}$$

$$yx^2:\quad 0 = b_{11} - a_{13}, \tag{70}$$

$$yxy:\quad \gamma\zeta = b_{12} - \zeta b_{13}, \tag{71}$$

$$y^2x:\quad \gamma = b_{13} - a_{14}, \tag{72}$$

$$y^3:\quad \beta = b_{14} - \zeta b_{14}. \tag{73}$$

Now (66) gives $\beta = 0$, and therefore (73) gives $b_{14} = 0$. (67), (68) and (70) give $b_{11} = a_{12} = a_{13} = 0$, whereas (69), (71) and (72) give $b_{12} + b_{13} = 0$, and then the same three equations easily give $b_{12} = \gamma\zeta/(1+\zeta)$, $b_{13} = -\gamma\zeta/(1+\zeta)$ and $a_{14} = \gamma(1+2\zeta^2)/(\zeta+1)$.

Condition (5) gives $\alpha_1 \circ [\mathbf{1}, \alpha_1] = [\mathbf{1}, \alpha_2]$. We write out the coefficients as above, using the fact that β has to be zero:

$$\gamma\alpha_1(g) = x\alpha_2(f) + y\alpha_2(g) - \alpha_2(f)x - \zeta\alpha_2(g)y$$

giving the equations

$$x^2:\quad 0 = a_{21} - a_{21}, \tag{74}$$

$$xy:\quad \gamma b_{12} = a_{22} - \zeta b_{21}, \tag{75}$$

$$yx:\quad \gamma b_{13} = b_{21} - a_{22}, \tag{76}$$

$$y^2:\quad 0 = b_{22} - \zeta b_{22}. \tag{77}$$

From (77) we get that $b_{22} = 0$, from (75) and (76), using also $b_{12} + b_{13}$, we get $b_{21} = 0$. From (75) and the computation for b_{12} above, we get $a_{22} = \gamma^2\zeta/(1+\zeta)$.

Condition (5) also gives that $\alpha_2 \circ [\mathbf{1}, \alpha_1] = [\mathbf{1}, \alpha_3]$. Going about as above, we find equations

$$x:\quad 0 = a_3 - a_3, \tag{78}$$

$$y:\quad 0 = b_3 - \zeta b_3. \tag{79}$$

This gives $b_3 = 0$. Finally $\alpha_3 \circ [\mathbf{1}, \alpha_1] = 0$ gives $0 = \beta a_3 + \gamma b_3$, which we already knew.

There are no restrictions on a_{11}, a_{21} and a_3, and all the other coefficients are zero or expressed in terms of γ. To conclude, the deformations we are interested in have two generators x and y, and two relations

$$y^3 + x^3 + a_{11}x^2 + \frac{\gamma(1+2\zeta^2)}{(1+\zeta)}y^2 + a_{21}x + \frac{\gamma^2\zeta}{(1+\zeta)}y + a_3,$$
$$y^2x + \zeta yxy + \zeta^2 xy^2 + \frac{\gamma\zeta}{(1+\zeta)}xy - \frac{\gamma\zeta}{(1+\zeta)}yx.$$

For the other cases, we will give the coefficients explicitly; then the expressions for the deformations are given as above.

Type H. With notation as in the previous example, we get from [1] that

$$f = -\zeta^3y^3 + \zeta^2yx^2 + \zeta xyx + x^2y, \qquad g = y^2x - \zeta yxy + \zeta^2xy^2 + \zeta^3x^3.$$

Here ζ is a primitive eighth root of unity. Then

$$w = y^3x - \zeta y^2xy + \zeta^2yxy^2 - \zeta^3xy^3 + x^3y + \zeta x^2yx + \zeta^2xyx^2 + \zeta^3yx^3.$$

Define the coefficients of the α_i as before, and also β and γ. Note that

$$w = xf + yg = \zeta fx - \zeta gy.$$

From condition (4) we get

$$x\alpha_1(f) + y\alpha_1(g) - \zeta\alpha_1(f)x + \zeta\alpha_1(g) = \beta f + \gamma g.$$

Similarly, from (5) we get

$$\alpha_1(\beta f + \gamma g) = x\alpha_2(f) + y\alpha_2(g) - \zeta\alpha_2(f)x + \zeta\alpha_2(g)y,$$
$$\alpha_2(\beta f + \gamma g) = x\alpha_3(f) + y\alpha_3(g) - \zeta\alpha_3(f)x + \zeta\alpha_3(g)y,$$
$$\alpha_3(\beta f + \gamma g) = 0.$$

As before, by rewriting these equations monomial by monomial, we end up with expressing the a's and b's in terms of β and γ. These expressions are linear for the a_{1i}, quadratic for the a_{2i}, cubic for a_3 and similarly for the b's. In addition, there is one quartic relation between γ and β, which turns out to be trivially fulfilled.

We get the following expressions for the coefficients, expressed in terms of γ and β (calculated in Maple, valid in characteristic different from 2):

$$a_{11} = 1/2\gamma\left(\zeta^3 - \zeta^2 - \zeta - 1\right),$$
$$a_{12} = -1/2\beta\left(3\zeta^3 + 3\zeta^2 + 3\zeta + 1\right),$$
$$a_{13} = -1/2\beta\left(3\zeta^3 + 3\zeta^2 - \zeta - 3\right),$$
$$a_{14} = 3/2\gamma\left(\zeta^3 + \zeta^2 - \zeta + 1\right),$$
$$a_{21} = \gamma\beta\zeta\left(\zeta^2 + 2\zeta + 1\right),$$

$$\begin{aligned}
a_{22} &= 3/2\beta^2\zeta^3 - 3/2\zeta\beta^2 - 2\beta^2 + 3/2\gamma^2\zeta^3 - 3\gamma^2\zeta^2 + 3/2\gamma^2\zeta,\\
a_3 &= -1/2\gamma\left(2\beta^2\zeta^3 + 2\gamma^2\zeta^3 + \beta^2\zeta^2 - \gamma^2\zeta^2 - \zeta\beta^2 - \gamma^2\zeta - 2\beta^2 + 2\gamma^2\right),\\
b_{11} &= -3/2\beta\left(\zeta^3 - \zeta^2 - \zeta - 1\right),\\
b_{12} &= 1/2\gamma\left(3\zeta^3 - 3\zeta^2 - \zeta + 3\right),\\
b_{13} &= 1/2\gamma\left(3\zeta^3 - 3\zeta^2 + 3\zeta - 1\right),\\
b_{14} &= -1/2\beta\left(\zeta^3 + \zeta^2 - \zeta + 1\right),\\
b_{21} &= -3/2\beta^2\zeta^3 - 3\beta^2\zeta^2 + 3/2\gamma^2\zeta - 3/2\gamma^2\zeta^3 - 2\gamma^2 - 3/2\zeta\beta^2,\\
b_{22} &= -\gamma\beta\zeta\left(\zeta^2 - 2\zeta + 1\right),\\
b_3 &= 1/2\beta\left(2\beta^2\zeta^3 + 2\gamma^2\zeta^3 + \beta^2\zeta^2 - \gamma^2\zeta^2 - \zeta\beta^2 - \gamma^2\zeta - 2\beta^2 + 2\gamma^2\right).
\end{aligned}$$

We introduce the other cases by giving the generators f, g of the ideal, and w, a generator for $(V \otimes R) \cap (R \otimes V)$. These polynomials we take directly from [1]. Then we give the conditions on the coefficients of the a's and b's, in terms of γ and β. Note that in some cases, there are also conditions on γ and β. In a couple of cases, we have used simplifications like $2\beta = 0$ implies $\beta = 0$, so there are some variants in characteristic two that are left out. Many of the calculations have been carried out using Maple.

Type A. Here (a and b are general parameters)

$$f = ay^2x + byxy + axy^2 + x^3, \qquad g = y^3 + ayx^2 + bxyx + ax^2y$$

and thus

$$w = y^4 + a\left(x^2y^2 + xy^2x + y^2x^2 + yx^2y\right) + b(xyxy + yxyx) + x^4.$$

In this case $\beta = \gamma = 0$, and the only conditions on the a's and b's are

$$a_{14} = b_{13} = b_{12}, \qquad a_{13} = a_{12} = b_{11}, \qquad b_{21} = a_{22}.$$

So there are no conditions on the other coefficients.

Type S_1. In the case S_1, we have (a is a general parameter, α is invertible)

$$f = \alpha xy^2 + \alpha^2y^2x + a\alpha yxy, \qquad g = x^2y + \alpha yx^2 + axyx$$

and

$$w = x^2y^2 + \alpha xy^2x + \alpha^2y^2x^2 + \alpha yx^2y + axyxy + a\alpha yxyx.$$

We will consider first the case of generic α, meaning different from 1, and then the special case afterwards.

Generic α: In this case, there is no condition on γ or β, and all the coefficients can be expressed using these parameters, and in addition b_{21} (we could also have used a_{22}). The equations are:

$$\begin{aligned}
a_{11} &= 0,\\
a_{12} &= -\gamma(2+a)\alpha/(\alpha-1),\\
a_{13} &= -\gamma(2\alpha+a)\alpha/(\alpha-1),\\
a_{14} &= \beta\alpha^2(1+a+\alpha)/(\alpha-1),\\
a_{21} &= \gamma^2(1+a+\alpha)\alpha/(\alpha-1)^2,\\
a_{22} &= \alpha b_{21},\\
a_3 &= -\gamma\big(\beta\gamma\alpha+\beta\gamma\alpha a+\alpha^2\beta\gamma+b_{21}-2b_{21}\alpha+b_{21}\alpha^2\big)\alpha/(\alpha-1)^3,\\
b_{11} &= -\gamma(1+a+\alpha)/(\alpha-1),\\
b_{12} &= \alpha\beta(2+a)/(\alpha-1),\\
b_{13} &= \alpha\beta(2\alpha+a)/(\alpha-1),\\
b_{14} &= 0,\\
b_{22} &= \beta^2\alpha^2(1+a+\alpha)/(\alpha-1)^2,\\
b_3 &= \beta\big(\alpha b_{21}-2\alpha b_{21}\alpha+\alpha b_{21}\alpha^2+\alpha^2\beta\gamma+\alpha^2\beta\gamma a+\alpha^3\beta\gamma\big)/(\alpha-1)^3.
\end{aligned}$$

The case $\alpha=1$: We get $\gamma=\beta=0$ (unless $a=-2$). Then

$$b_{11}=a_{12}=a_{13},\qquad b_{12}=b_{13}=a_{14},\qquad a_{22}=b_{21}.$$

There are no other relations, and the other coefficients are free.

When $a=-2$ (and still $\alpha=1$), we get a number of conditions as follows:

$$\begin{gathered}
a_{12}=b_{11}+\gamma,\qquad a_{13}=b_{11}-\gamma,\qquad b_{12}=a_{14}-\beta,\qquad b_{13}=a_{14}+\beta,\\
\beta a_{14}+\gamma b_{14}=0=\beta a_{14}+\gamma b_{14},\qquad \beta a_{12}+\gamma b_{12}=0=\beta a_{13}+\gamma b_{13},\\
\beta b_{11}+\gamma a_{14}=0,\qquad \beta a_{21}+\gamma b_{21}=0=\beta a_{22}+\gamma b_{22},\\
b_{21}=a_{22},\qquad \beta^2 a_{21}=\gamma^2 b_{22},\qquad \beta a_3+\gamma b_3=0.
\end{gathered}$$

Type S_2. We have

$$f=xy^2+\alpha y^2x,\qquad g=-\alpha x^2y+\alpha^2yx^2$$

and

$$w=x^2y^2+\alpha xy^2x+\alpha^2y^2x^2-\alpha yx^2y.$$

As in the previous case, there are special values of α that require separate treatment: $\alpha = \pm 1$.

First consider the generic case ($\alpha \neq \pm 1$). Here all the coefficients can be expressed using γ and β, which are free parameters:

$$a_{11} = 0, \qquad a_{12} = -\frac{2\alpha\gamma}{1+\alpha}, \qquad a_{13} = -\frac{2\alpha^2\gamma}{1+\alpha}, \qquad a_{14} = \frac{(1+\alpha)\beta}{\alpha - 1},$$
$$a_{21} = \frac{\alpha^2\gamma^2}{1+\alpha}, \qquad a_{22} = \frac{2\alpha\beta\gamma}{1-\alpha}, \qquad a_3 = \frac{\alpha^2\beta\gamma^2}{\alpha^2 - 1},$$
$$b_{11} = \frac{\alpha^2\gamma(1-\alpha)}{1+\alpha}, \qquad b_{12} = \frac{2\beta\alpha}{1-\alpha}, \qquad b_{13} = \frac{2\beta\alpha^2}{\alpha - 1}, \qquad b_{14} = 0,$$
$$b_{21} = -\frac{2\alpha^2\beta\gamma}{1+\alpha}, \qquad b_{22} = \frac{\alpha\beta^2}{\alpha - 1}, \qquad b_3 = \frac{\alpha^2\beta^2\gamma}{1-\alpha^2}.$$

The case $\alpha = 1$: The relations are as follows:

$$\beta = b_{11} = b_{14} = b_{21} = b_{22} = b_3 = 0,$$
$$a_{12} = a_{13} = -\gamma, \qquad a_{14} = b_{13} = -b_{12}, \qquad a_{22} = \gamma b_{12}.$$

The case $\alpha = -1$: We get

$$\gamma = a_{11} = a_{14} = a_{21} = a_{22} = a_3 = 0,$$
$$b_{12} = b_{13} = -\beta, \qquad b_{11} = a_{12} = -a_{13}, \qquad b_{21} = \beta a_{13}.$$

Type S_2'. Now

$$f = y^2x + xy^2 + x^3, \qquad g = yx^2 - x^2y,$$
$$w = x^2y^2 + xy^2x + y^2x^2 - yx^2y + x^4.$$

We get

$$\gamma = b_{11} = b_{14} = b_{21} = b_{22} = b_3 = 0,$$
$$a_{12} = a_{13} = -\gamma, \qquad a_{14} = b_{13} = -b_{12}, \qquad a_{22} = \gamma b_{12}.$$

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