Boundary-layer effects for the 2-D Boussinesq equations with vanishing diffusivity limit in the half plane

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1. Introduction

This paper is mainly concerned with the vanishing diffusivity limit of the 2-D Boussinesq system for incompressible flows, the equations of which have the following form:
\[ u_t^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = \nu \Delta u^\varepsilon + \theta^\varepsilon e_2, \]  
\[ \theta_t^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon = \varepsilon \Delta \theta^\varepsilon, \]  
\[ \text{div} u^\varepsilon = 0 \]  
\[ \text{(1.1)} \]
\[ \text{(1.2)} \]
\[ \text{(1.3)} \]
in the half plane \( \mathbb{R}^2_+ = \{(x, y): x > 0, -\infty < y < \infty\} \), with initial and boundary conditions
\[ u^\varepsilon(x, y, 0) = u_0(x, y), \quad \theta^\varepsilon(x, y, 0) = \theta_0(x, y), \quad (x, y) \in \mathbb{R}^2_+, \]  
\[ u^\varepsilon(0, y, t) = 0, \quad \theta^\varepsilon(0, y, t) = \alpha(y, t), \quad (y, t) \in \mathbb{R} \times (0, T). \]  
\[ \text{(1.4)} \]
\[ \text{(1.5)} \]
Here, the unknowns are \( u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon), \theta^\varepsilon, \) and \( p^\varepsilon \) which respectively denote the vector velocity field, the scalar temperature, and the scalar pressure; \( \nu > 0 \) and \( \varepsilon \geq 0 \) are the viscosity and the diffusivity coefficients, respectively. As usual, \( e_2 := (0, 1), \nabla := (\partial_x, \partial_y), \) and \( \Delta := \partial_x^2 + \partial_y^2. \)

The Boussinesq system plays an important role in atmospheric and oceanographic sciences (see [30,31,35]), and has received significant attention in the mathematical fluid dynamics community because of its close connection to the 3-D incompressible flows, see, for example, [4–8,10,11,19–23,25,27,30–32,35,38,45] and the references therein for both the analytical and the numerical studies.}

In particular, Chae [5] (see also [22]) obtained the global regularity and proved the global existence of smooth solution to the Cauchy problem of the 2-D Boussinesq equations with "partial viscosity," that is, with either the zero diffusivity (\( \varepsilon = 0, \nu > 0 \)) or the zero viscosity (\( \nu = 0, \varepsilon > 0 \)). The vanishing diffusivity/viscosity limit was also justified in [5]. Recently, Lai, Pan and Zhao [27] considered an initial-boundary value problem for the 2-D Boussinesq equations with zero diffusivity in smooth bounded domains, and established the global well-posedness theory of classical solutions with \( H^2 \)-initial data and non-slip boundary condition. Zhao [45] proved the global existence of classical solutions to the 2-D inviscid-diffusive Boussinesq equations with slip boundary conditions. On the other hand, the global regularity/singularity question for the 2-D Boussinesq equations with both zero diffusivity and zero viscosity, which possess many similarities to the 3-D axi-symmetric Euler equations with swirl away from the symmetric axis \( r = 0 \), is still an outstanding open problem in mathematical fluid mechanics, see, for example, [6–8,11,23] for studies in this direction. It is worth pointing out that if \( \theta^\varepsilon = 0 \), then the system (1.1)–(1.3) reduces to the Navier–Stokes equations for incompressible fluids, which have been extensively studied by a great number of mathematician in a large variety of contexts, see, for example, [9,14,29,39], etc.

In the present paper, we are interested in the questions related to the vanishing diffusivity limit of the initial and boundary value problem (1.1)–(1.5) in the half plane. As the diffusivity vanishes (i.e. \( \varepsilon \to 0 \)), it is clear that Eq. (1.2) becomes hyperbolic (instead of parabolic), and the boundary \( x = 0 \) is characteristic for the temperature due to the non-slip boundary condition \( u|_{x=0} = 0 \). Thus, by the classical theory in [33], the boundary condition of temperature should be dropped when the Boussinesq equations are of zero diffusivity coefficient. Formally, when \( \varepsilon = 0 \), the 2-D Boussinesq system (1.1)–(1.3) turns into:

\[ u_t^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = \nu \Delta u^0 + \theta^0 e_2, \]  
\[ \theta_t^0 + u^0 \cdot \nabla \theta^0 = 0, \]  
\[ \text{div} u^0 = 0 \]  
\[ \text{(1.6)} \]
\[ \text{(1.7)} \]
\[ \text{(1.8)} \]
in \( \mathbb{R}^2_+ \times (0, T) \), with the following initial and boundary conditions
\[ u^0(x, y, 0) = u_0(x, y), \quad \theta^0(x, y, 0) = \theta_0(x, y), \quad (x, y) \in \mathbb{R}^2_+, \]  
\[ u^0(0, y, t) = 0, \quad (y, t) \in \mathbb{R} \times (0, T). \]  
\[ \text{(1.9)} \]
\[ \text{(1.10)} \]
Note that the boundary value of $\theta^0|_{x=0}$ can be uniquely determined from (1.7) by using $u^0$, $\theta_0$ and the method of characteristics.

Our first purpose is to justify the vanishing diffusivity limit from the problem (1.1)–(1.5) to the problem (1.6)–(1.10), and to prove the convergence rates. Precisely, we shall prove

**Theorem 1.1.** Assume that the initial and boundary data $(u_0, \theta_0, a)$ satisfies

\[
\begin{align*}
& u_0 \in H^2(\mathbb{R}^2_+), \quad \theta_0 \in H^1(\mathbb{R}^2_+) \cap W^{1,\infty}(\mathbb{R}^2_+), \\
& a \in L^\infty(0, T; H^3(\mathbb{R})), \quad a_t \in L^\infty(0, T; H^1(\mathbb{R})), \\
& \nabla \cdot u_0 = 0, \quad u_0(0, y) = 0, \quad \theta_0(0, y) = a(y, 0).
\end{align*}
\]

(1.11)

Then for each fixed $\varepsilon > 0$, there is a unique global strong solution $(u^\varepsilon, \theta^\varepsilon)$ to the initial-boundary value problem (1.1)–(1.5), such that for any given $T > 0$,

\[
\begin{align*}
& u^\varepsilon \in L^\infty(0, T; H^1_0(\mathbb{R}^2_+) \cap H^2(\mathbb{R}^2_+)), \quad \theta^\varepsilon \in L^\infty(0, T; H^1(\mathbb{R}^2_+) \cap L^2(0, T; H^2(\mathbb{R}^2_+))).
\end{align*}
\]

Moreover, there exists a positive constant $C$, independent of $\varepsilon$, such that for any $p \geq 2$,

\[
\begin{align*}
& \sup_{0 \leq t \leq T} \left\{ \|u^\varepsilon(t)\|_{L^2(\mathbb{R}^2_+)}^2 + \|\theta^\varepsilon(t)\|_{L^2(\mathbb{R}^2_+)}^2 + \|\nabla u^\varepsilon(t)\|_{L^2(\mathbb{R}^2_+)}^2 \right\} \\
& \quad + \int_0^T \left\{ \|\nabla u^\varepsilon(t)\|_{L^2(\mathbb{R}^2_+)}^2 + \|\nabla \theta^\varepsilon(t)\|_{L^2(\mathbb{R}^2_+)}^2 + \|\nabla^2 u^\varepsilon(t)\|_{L^p(\mathbb{R}^2_+)}^2 \right\} dt \leq C,
\end{align*}
\]

(1.12)

\[
\begin{align*}
& \sup_{0 \leq t \leq T} \left\{ \|\theta^\varepsilon(t)\|_{L^2(\mathbb{R}^2_+)}^2 + \|\theta^\varepsilon(t)\|_{L^\infty(\mathbb{R}^2_+)}^2 \right\} + \sup_{0 \leq t \leq T} \int_{\mathbb{R}^2_+} e^{-(\alpha + |y|)} |\nabla \theta^\varepsilon| dx \, dy \leq C,
\end{align*}
\]

(1.13)

and

\[
\begin{align*}
& \varepsilon^{1/2} \sup_{0 \leq t \leq T} \|\nabla \theta^\varepsilon(t)\|_{L^2(\mathbb{R}^2_+)}^2 + \varepsilon^{3/2} \int_0^T \|\nabla^2 \theta^\varepsilon(t)\|_{L^2(\mathbb{R}^2_+)}^2 dt \leq C.
\end{align*}
\]

(1.14)

As a result, there exists a subsequence of $\varepsilon$, still denoted by $\varepsilon$, such that as $\varepsilon \to 0$,

\[
\begin{align*}
& u^\varepsilon \to u^0 \quad \text{strongly in } L^\infty(0, T; H^1(\mathbb{R}^2_+)), \quad \theta^\varepsilon \to \theta^0 \quad \text{strongly in } L^\infty(0, T; L^2(\mathbb{R}^2_+)),
\end{align*}
\]

where the pair of limit functions $(u^0, \theta^0)$ is a global weak solution of the problem (1.6)–(1.10) in the sense of distributions. In particular, it holds that

\[
\begin{align*}
& \sup_{0 \leq t \leq T} \left\{ \|u^\varepsilon(t) - u^0(t)\|_{H^1(\mathbb{R}^2_+)}^2 + \|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\mathbb{R}^2_+)}^2 \right\} \\
& \quad + \int_0^T \left\{ \|\nabla^2 (u^\varepsilon(t) - u^0(t))\|_{L^2(\mathbb{R}^2_+)}^2 + \|\nabla (p^\varepsilon - p^0)\|_{L^2(\mathbb{R}^2_+)}^2 \right\} dt \leq C \varepsilon^{1/2}.
\end{align*}
\]

(1.15)
Let $S(\mathbb{R}^{2+1})$ be the space of rapidly decreasing functions, and denote by $S^+_T$ the space of all functions $\Phi = (\phi_1, \phi_2) \in S(\mathbb{R}^{2+1})^2$ such that $\Phi(0, y, t) = 0$ in $\mathbb{R} \times (0, T)$, $\Phi(x, y, t) = 0$ for $t \geq T$, and $\text{div} \Phi = 0$ in $\mathbb{R}^2_x \times (0, T)$. Then, the global weak solution to (1.6)-(1.10) mentioned in Theorem 1.1 is defined in a way similar to that for the Navier–Stokes equations.

**Definition 1.1.** A pair of functions $(u^0, \theta^0)$ is said to be a global weak solution to the initial-boundary value problem (1.6)-(1.10), if for every $T > 0$, it holds that

$$u^0 \in L^\infty(0, T; L^2(\mathbb{R}^2_x)) \cap L^2(0, T; H^1_0(\mathbb{R}^2_x)), \quad \theta^0 \in L^\infty(0, T; L^2(\mathbb{R}^2_x)).$$

Moreover, it holds in the sense of distributions that

$$\int_0^T \int_{\mathbb{R}^2_x} \left\{ u^0 \cdot (\phi_t + u^0 \cdot \nabla \Phi + v \Delta \Phi) + \theta^0 \phi_2 \right\} \, dx \, dy \, dt = - \int_0^T \int_{\mathbb{R}^2_x} u_0(x, y) \cdot \Phi(x, y, 0) \, dx \, dy,$$

for any $\Phi = (\phi_1, \phi_2) \in S^+_T$, and that

$$\int_0^T \int_{\mathbb{R}^2_x} \theta^0 (\psi_t + u^0 \cdot \nabla \psi) \, dx \, dy \, dt = - \int_{\mathbb{R}^2_x} \theta_0(x, y) \psi(x, y, 0) \, dx \, dy$$

for any $\psi \in S(\mathbb{R}^{2+1})$ satisfying $\psi(0, y, t) = 0$ and $\psi(x, y, t) = 0$ for $t \geq T$.

The global weak solution of problem (1.1)-(1.5) is defined in the same way, but with additional diffusive and boundary terms in the integral formulation. For a strong solution, it is a weak solution with the derivatives in the equations at least belonging to $L^2(0, T; L^2(\mathbb{R}^2_x))$.

The proof of Theorem 1.1 will be completed in Section 3, using the estimates proved in Lemmas 2.1–2.5 and 3.1. The existence of global solutions to the problem (1.1)-(1.5) with fixed $v, \varepsilon > 0$ can be proved in the same manner as that used for the incompressible Navier–Stokes equations (see, e.g., [14,29]). In order to take the limit as $\varepsilon \to 0$ and to prove the convergence rates, we need to obtain some uniform-in-$\varepsilon$ estimates of $(u^\varepsilon, \theta^\varepsilon)$, which are somewhat complicated due to the strong nonlinearity, the coupling between the velocity and the temperature, and the boundary effects on the temperature. The difficulties induced by the nonlinearity and the coupling will be circumvented by using the estimates of the Stokes system and the Sobolev inequalities in a subtle way, so that, the uniform estimates of the first and second order derivatives of $u^\varepsilon$ can be obtained step by step (see Lemmas 2.2 and 2.3).

On the other hand, it is very difficult to deal with the boundary effects on $\theta^\varepsilon$. As a result, the method used in [5,22,27] to show the global regularity of $\theta^0$ for the problem (1.6)-(1.10) seems not working for our problem. For example, if we apply the operator $\nabla^\perp := (-\partial_y, \partial_x)$ to both sides of (1.7) and take the $L^2$-inner product with $|\nabla^\perp \theta^\varepsilon|^{p-2} \nabla \theta^\varepsilon$, we easily see that $\|\nabla \theta^0\|_{L^p(\mathbb{R}^2_x)}$ is bounded for any $2 \leq p \leq \infty$ (cf. Lemma 3.1). However, the same procedure cannot be used any more to prove the uniform estimates of $\nabla \theta^\varepsilon$, since some of the boundary terms, for instance, $\varepsilon \theta_x \theta_{xx} \theta_0$, cannot be uniformly bounded with respect to $\varepsilon$. Indeed, the uniform-in-$\varepsilon$ estimates of $\theta^\varepsilon$ are much fewer than those of $\theta^0$, and we have only (1.14) and the weighted $L^1$-estimate (1.13) for $\nabla \theta^\varepsilon$, instead of the uniform $L^2-L^\infty$ estimates (see Lemmas 2.4 and 2.5).

As mentioned above, the presence of boundary will significantly affect the behavior of $\theta^\varepsilon$ when the diffusivity coefficient $\varepsilon$ goes to zero. So, our second and main purpose in this paper is to study the boundary-layer effects for the 2-D Boussinesq system (1.1)-(1.5) with vanishing diffusivity limit. In fact, due to the disparity of boundary conditions in (1.5) and (1.10), we cannot expect that as
Equipped with the norm: a positive function \( A \) near the boundary. Inspired by this, we introduce the concept of boundary-layer thickness to the one of the problem (1.6)–(1.10) away from the boundary, while there is a sharp change of that the Navier–Stokes solution tends asymptotically (as \( \varepsilon \to 0 \)) to an Euler solution outside a boundary layer and to a solution of Prandtl’s equations within the boundary layer. Grenier [16] showed that there is a sequence of solutions to the Navier–Stokes equations, such that as \( \varepsilon \to 0 \), the convergence from the Navier–Stokes solution to the Prandtl solution does not hold in \( L^\infty(0, T; H^1_{loc}) \) for arbitrary small \( T > 0 \). The methods used in [3] and [16] are both the so-called “asymptotic expansion” method which is based on the inner scaled variable \( Y = y/\varepsilon^q \) where \( y > 0 \) is the flow domain and \( q > 0 \) is the scaling exponent. As it was pointed out in [13] that the completely different results obtained in [3] and [16] imply that, to apply the asymptotic expansion method, one should first determine a suitable choice of the inner scaled variable \( Y = y/\varepsilon^q \) or the value of boundary-layer thickness \( \delta(\varepsilon) = \varepsilon^q \). Thus, in order to shed some light on the further mathematical analysis of boundary layers for the Boussinesq equations, in this paper we try to estimate the boundary-layer thickness from the mathematical aspect.

Similar to the relations among the Euler, Navier–Stokes and Prandtl equations (see, for example, [28,36,34]), it is expected that as \( \varepsilon \to 0 \), the solution of the problem (1.1)–(1.5) converges uniformly to the one of the problem (1.6)–(1.10) away from the boundary, while there is a sharp change of gradient near the boundary. Inspired by this, we introduce the concept of boundary-layer thickness (BL-thickness) as follows.

**Definition 1.2.** A positive function \( \delta(\varepsilon) \) is said to be a BL-thickness for the problem (1.1)–(1.5) with vanishing diffusivity coefficient \( \varepsilon \), if \( \delta(\varepsilon) \downarrow 0 \) as \( \varepsilon \downarrow 0 \) and

\[
\begin{align*}
\lim_{\varepsilon \to 0} \theta^\varepsilon - \theta^0 & \in L^\infty(0, T; W^{1,1}_g(\Omega_{\delta(\varepsilon)})) = 0, \\
\liminf_{\varepsilon \to 0} \theta^\varepsilon - \theta^0 & \in L^\infty(0, T; W^{1,1}_g(\mathbb{R}^2_+)) > 0,
\end{align*}
\]

where \( \Omega_{\delta} := \{(x, y) \in \mathbb{R}^2_+ : \delta < x < \infty, \ -\infty < y < \infty\} \) and \( W^{1,1}_g(\Omega) \) is a weighted Sobolev space equipped with the norm:

\[
\|v\|_{W^{1,1}_g(\Omega)} = \|v\|_{L^1(\Omega)} + \|\nabla v\|_{L^2(\Omega)}, \quad \|v\|_{L^2(\Omega)} = \int_\Omega e^{-(x+y)|v|} dx dy.
\]

**Remark 1.1.** The definition of BL-thickness is only given for \( \theta^\varepsilon \) and \( \theta^0 \), since there is no boundary-layer effect between \( u^\varepsilon \) and \( u^0 \). This is mainly due to the smooth mechanism of \( \Delta u^\varepsilon \), and it holds that \( u^\varepsilon \to u^0 \) strongly in \( C(0, T; H^1_0(\mathbb{R}^2_+)) \). Note that \( H^1_0(\mathbb{R}^2_+) \hookrightarrow H^{1/2}(\partial \mathbb{R}^2_+) \).

**Remark 1.2.** The above definition does not determine the BL-thickness uniquely, since any function \( \delta(\varepsilon) \), satisfying \( \delta(\varepsilon) \geq \delta(\varepsilon) \) and \( \delta(\varepsilon) \downarrow 0 \) as \( \varepsilon \downarrow 0 \), is also a BL-thickness. Thus, there should exist a minimal BL-thickness \( \delta_\varepsilon(\varepsilon) \) which may be considered as the true BL-thickness.

**Remark 1.3.** The BL-thickness for scalar conservation laws was also defined by Frid and Shelukhin [13] in a similar manner. The BL-thickness for the one-dimensional cylindrical compressible Navier–Stokes equations, in this paper we try to estimate the boundary-layer thickness from the mathematical aspect.
equations with vanishing shear viscosity limit was studied in [12,24], where the weighted Sobolev space $W^{1,1}_g(\Omega)$ in (1.16) and (1.17) is replaced by $L^\infty(\Omega)$. Here, the choice of the weighted space is due to the unboundedness of the domain $\mathbb{R}^2_+$. In view of Definition 1.2, the violation of (1.17) would imply that there is no boundary-layer effect on the temperature.

We shall prove that a function $\delta_0(\varepsilon) = \varepsilon^{1/2-1/n}$ with any $n > 2$ is a BL-thickness in the sense of Definition 1.2. This is in agreement with the Stokes–Blasius law for laminar boundary-layer thickness (see, e.g. [36]), since it holds that $\liminf_{n \to \infty} \delta_0(\varepsilon) = \sqrt{\varepsilon}$. To make the analysis of BL-thickness simpler, as that in [12,13,24], we restrict ourselves to the special case of vanishing initial data:

$$u_0(x, y) = 0 \quad \text{and} \quad \theta_0(x, y) = 0, \quad (x, y) \in \mathbb{R}^2_+. \quad (1.18)$$

Then, our second and main result in this paper can be stated in the following theorem.

**Theorem 1.2.** Let the conditions of Theorem 1.1 and (1.18) be satisfied. Then there exists only trivial solution $(0, 0)$ to the problem (1.6)–(1.10). Furthermore, any positive function $\delta(\varepsilon) \in C((0, 1])$ is a BL-thickness in the sense of Definition 1.2, such that

$$\lim_{\varepsilon \to 0} \|\theta^\varepsilon\|_{L^\infty(\Omega \times (0, \varepsilon^2(\Omega^{(1)}))} = 0, \quad (1.19)$$

provided $\delta(\varepsilon) \to 0$ and $\delta(\varepsilon)/\varepsilon^{1/2} \to \infty$ as $\varepsilon \to 0$, and that

$$\liminf_{\varepsilon \to 0} \|\theta^\varepsilon\|_{L^\infty(\Omega^{(1)}; W^{1,1}_g(\mathbb{R}^2_+))} > 0, \quad (1.20)$$

provided the boundary data $a(y, t)$ is not identically zero.

**Remark 1.4.** The analogous results in Theorems 1.1 and 1.2 also hold for the domains of the form $\Omega = \{(x, y) \in \mathbb{R}^2_+: x \in (-l, l), \ y \in (-\infty, \infty)\}$, where $l > 0$ is a fixed positive constant, see Remark 4.2.

**Remark 1.5.** As that in [12,13,24], the special vanishing initial data is used to simplify the proof of BL-thickness, but it is sufficient to show the boundary-layer effect and the boundary-layer thickness. In fact, the same method can be applied to prove a weaker result (i.e. $\delta(\varepsilon) \sim \varepsilon^{1/8}$) for general initial data, provided the solutions of the limit system (1.6)–(1.10) are sufficiently smooth, see Remark 4.3.

The proof of Theorem 1.2 is based on some weighted estimates established in Section 4, as well as the uniform estimates proved in Sections 2 and 3. The inequality (1.20) is an immediate consequence of non-zero boundary data, since the limit problem (1.6)–(1.10) has only trivial solution $(0, 0)$ under the conditions of Theorem 1.2. So, to prove Theorem 1.2, it suffices to prove (1.19), which will be shown in a much simpler manner than that in [12,13]. Roughly speaking, the method due to Frid and Shelukhin strongly depends on a uniform pointwise bound of viscous solutions (cf. [12, Lemma 3.1] and [13, Lemma 5.1]) and a boundary cut-off function which vanishes on the boundary. The pointwise bounds in [12,13] were obtained by using the maximal principle which is not valid for our problem. Moreover, the construction of a desired boundary cut-off function generally depends on the geometry of the domain considered. In the present paper, unlike that in [12,13], the proof of (1.19) will be circumvented by showing the following weighted estimate

$$\|x^2e^{-(x^2+|y|^2)}\nabla \theta^\varepsilon\|_{L^1(\mathbb{R}^2_+)} \leq C\varepsilon,$$
since it is clear that as $\varepsilon \to 0$,
\[
\|e^{-(x+|y|)}\nabla \theta^\varepsilon\|_{L^1(\Omega_2)} \leq \frac{1}{\delta^2} \|x^2 e^{-(x+|y|)} \nabla \theta^\varepsilon\|_{L^1(\Omega_2)} \leq \frac{C\varepsilon}{\delta^2} \to 0,
\]
provided $\delta = \delta(\varepsilon)$ satisfies the conditions in Theorem 1.2. This is the most important estimate in the analysis of BL-thickness. We note here that the proof of this weighted estimate does not depend on the pointwise bounds of $u^\varepsilon$, nor the special choice of boundary cut-off function. Indeed, as observed in [24], it relies only on the $t$-integrability of $\|\nabla u^\varepsilon\|_{L^\infty(\mathbb{R}^2_+)}$.

The rest of this paper is outlined as follows. By virtue of the estimates of the Stokes system and the Sobolev inequalities, we first derive some uniform-in-$\varepsilon$ estimates of the solution $(u^\varepsilon, \theta^\varepsilon)$ to the problem (1.1)-(1.5) in Section 2. With the help of these uniform estimates we are able to pass the limit as $\varepsilon \to 0$ and prove the convergence rates (i.e. Theorem 1.1) in Section 3. Finally, in Section 4 we estimate the boundary-layer thickness and complete the proof of Theorem 1.2.

**Notations.** Let $W^{k,p}(\mathbb{R}^2_+)$ ($W^{k,2}(\mathbb{R}^2_+)$ = $H^k(\mathbb{R}^2_+)$), $W^{0,p}(\mathbb{R}^2_+) = L^p(\mathbb{R}^2_+)$) with $k \in \mathbb{Z}_+$ and $p \geq 1$ be the usual Sobolev space defined over $\mathbb{R}^2_+$ with the norm $\|\cdot\|_{W^{k,p}(\mathbb{R}^2_+)}$. For simplicity, throughout this paper we use the following abbreviations:

\[
\|\cdot\|_{W^{m,p}} := \|\cdot\|_{W^{m,p}(\mathbb{R}^2_+)}, \quad \|\cdot\|_{H^k} := \|\cdot\|_{H^k(\mathbb{R}^2_+)} , \quad \|\cdot\|_{L^p} := \|\cdot\|_{L^p(\mathbb{R}^2_+)}. 
\]

The norms of $L^p(\mathbb{R}_+)$ and $W^{k,p}(\mathbb{R}_+)$ are respectively denoted by $\|\cdot\|_{L^p_{\varepsilon}}$ and $\|\cdot\|_{W^{k,p}_{\varepsilon}}$, i.e.,

\[
\|v\|_{L^p_{\varepsilon}} := \left(\int_0^{\infty} |v(x, y, t)|^p \, dx\right)^{1/p}, \quad \|v\|_{W^{k,p}_{\varepsilon}} := \left(\sum_{0 \leq \alpha \leq k} \int_0^{\infty} |\partial_\alpha^\varepsilon v(x, y, t)|^p \, dx\right)^{1/p}.
\]

Note that $W^{k,2}(\mathbb{R}_+) = H^k(\mathbb{R}_+)$. The spaces of $L^p_{\varepsilon}(\mathbb{R}_+)$, $W^{k,p}_{\varepsilon}(\mathbb{R}_+)$ ($W^{k,2}_{\varepsilon}(\mathbb{R}_+) = H^k_{\varepsilon}(\mathbb{R}_+)$) and the norms are defined in a similar manner. $L^p(I, B)$ (resp. $\|\cdot\|_{L^p(I, B)}$) is used to denote the space of all strongly measurable $p$th-power integrable functions (essentially bounded functions if $p = \infty$) from $I$ to $B$ (resp. the norm), where $I \subset \mathbb{R}$ and $B$ is a Banach space.

### 2. Uniform estimates

This section is devoted to the uniform estimates of the solution $(u^\varepsilon, \theta^\varepsilon)$ to the initial-boundary value problem (1.1)-(1.5). For simplicity, in this section we omit the index $\varepsilon$ and denote the solution by $(u, \theta)$. Moreover, throughout the rest of this paper, we shall use $C$ to denote the generic positive constant, which may change from line to line, but is independent of $\varepsilon$.

We start with the following uniform $L^2$-estimate of $(u, \theta)$ and $L^\infty$-estimate of $\theta$.

**Lemma 2.1.** Under the conditions of Theorem 1.1, it holds that

\[
\sup_{0 \leq t \leq T} \left\{ \|\theta(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \right\} + \int_0^T (\varepsilon \|\nabla \theta(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2) \, dt \leq C.
\]

Moreover, if we set $M := \max(\|\theta_0\|_{L^\infty}, \|u\|_{L^\infty(0, T; L^\infty)})$, then we have

\[
\|\theta\|_{L^\infty(Q_T)} \leq M \quad \text{with} \quad Q_T := \mathbb{R}^2_+ \times (0, T).
\]
Proof. To prove the first part, we set
\[ \tilde{\theta}(x, y, t) = \theta(x, y, t) - a(y, t)e^{-X}. \]
Then, it is clear that \( \tilde{\theta} = \tilde{\theta}(x, y, t) \) satisfies \( \tilde{\theta}|_{x=0} = 0 \) and
\[ \tilde{\theta}_t + u \cdot \nabla \tilde{\theta} - \varepsilon \Delta \tilde{\theta} = \varepsilon \Delta (a e^{-X}) - u \cdot \nabla (a e^{-X}) - a_t e^{-X}. \quad (2.1) \]
Taking the \( L^2 \)-inner product of (2.1) with \( \tilde{\theta} \) and integrating it by parts over \( \mathbb{R}^2_+ \), we have from the Cauchy–Schwarz inequality that
\[ \frac{1}{2} \frac{d}{dt} \| \tilde{\theta} \|^2_{L^2} + \varepsilon \| \nabla \tilde{\theta} \|^2_{L^2} = \int \int_{\mathbb{R}^2_+} \tilde{\theta} \{ \varepsilon \Delta (a e^{-X}) - u \cdot \nabla (a e^{-X}) - a_t e^{-X} \} \, dx \, dy \]
\[ \leq \frac{\varepsilon}{2} \| \nabla \tilde{\theta} \|^2_{L^2} + C \| \tilde{\theta} \|^2_{L^2} + C \int \int_{\mathbb{R}^2_+} e^{-2x} \{(1 + |u|^2)(a^2 + a_t^2) + a_t^2 \} \, dx \, dy. \quad (2.2) \]
where we have used the divergence-free condition (1.3) and the fact that \( \tilde{\theta}|_{x=0} = 0 \). By virtue of the Sobolev type inequality:
\[ \int_0^\infty \sup_{y \in \mathbb{R}} |u|^2 \, dx \leq C \int_0^{\infty} \left( \| u \|^2_{L^2_+} + \| u_y \|^2_{L^2_+} \right) \, dx \leq C \left( \| u \|^2_{L^2_+} + \| \nabla u \|^2_{L^2_+} \right), \]
we find that
\[ \int \int_{\mathbb{R}^2_+} e^{-2x} |u|^2 (a^2 + a_t^2) \, dx \, dy \leq C \| a \|^2_{H^1_+} \int_0^\infty \sup_{y \in \mathbb{R}} |u|^2 \, dx \leq C \left( \| u \|^2_{L^2_+} + \| \nabla u \|^2_{L^2_+} \right). \]
On the other hand, it is easy to see that
\[ \int \int_{\mathbb{R}^2_+} e^{-2x} (a^2 + a_t^2 + a_t^2) \, dx \, dy \leq \int_0^\infty e^{-2x} \int_{-\infty}^\infty (a^2 + a_t^2 + a_t^2) \, dy \leq C. \]
Hence, we infer from (2.2) that
\[ \frac{d}{dt} \| \tilde{\theta} \|^2_{L^2} + \varepsilon \| \nabla \tilde{\theta} \|^2_{L^2} \leq C \left( 1 + \| \tilde{\theta} \|^2_{L^2} + \| u \|^2_{L^2} + \| \nabla u \|^2_{L^2} \right). \quad (2.3) \]
Similarly, multiplying (1.1) by \( u \) in \( L^2 \), we obtain after integrating by parts that
\[ \frac{d}{dt} \| u \|^2_{L^2} + \| \nabla u \|^2_{L^2} \leq C \| \theta \|_{L^2} \| u \|_{L^2} \leq C \left( 1 + \| \tilde{\theta} \|^2_{L^2} + \| u \|^2_{L^2} \right). \quad (2.4) \]
Thus, combining (2.3) and (2.4), by virtue of Gronwall’s lemma we easily obtain the \( L^2 \)-estimate of \( (u, \tilde{\theta}) \), which in turn completes the proof of the first part of Lemma 2.1 due to the fact that \( \| a e^{-X} \|_{L^\infty(0,T; H^1)} \) is uniformly bounded under the conditions of Theorem 1.1.
To prove the second part, we first observe that $\|a\|_{L^\infty_t L^\infty_y} \leq \|a\|_{H_y^1}$ is bounded. Thus, if we denote by $v_+ := \max(v, 0)$ and multiply (1.2) by $(\theta - M)_+$ in $L^2$, we deduce after integrating by parts and using the divergence-free condition (1.3) that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2_+} (\theta - M)_+^2 \, dx \, dy + \varepsilon \int_{\mathbb{R}^2_+} \|\nabla (\theta - M)_+\|^2 \, dx \, dy = 0,
\]
which, integrated in $t$, yields $\|(\theta - M)_+(t)\|_{L^2} \leq 0$. This implies $\theta \leq M$ for a.e. $(x, y, t) \in Q_T$. Similarly, if we take the $L^2$-inner product of (1.2) with $(\theta + M)_- := \max(-(\theta + M), 0)$, we then arrive at $\|(\theta + M)_-(t)\|_{L^2} \leq 0$, which immediately gives $\theta \geq -M$ for a.e. $(x, y, t) \in Q_T$. The proof of Lemma 2.1 is therefore complete. □

The next lemma is concerned with the uniform $L^2$-estimates of the first and second order derivatives of $u$, which will be achieved by using the estimates of the Stokes system and the Sobolev inequalities.

**Lemma 2.2.** Under the conditions of Theorem 1.1, one has
\[
\sup_{0 \leq t \leq T} \left\| \nabla u(t) \right\|_{L^2}^2 + \int_0^T \left( \left\| u_t(t) \right\|_{L^2}^2 + \left\| \nabla^2 u(t) \right\|_{L^2}^2 \right) \, dt \leq C. \tag{2.5}
\]
Furthermore, it also holds that
\[
\sup_{0 \leq t \leq T} \left\{ \left\| u_t(t) \right\|_{L^2}^2 + \left\| \nabla^2 u(t) \right\|_{L^2}^2 \right\} + \int_0^T \left\| \nabla u_t(t) \right\|_{L^2}^2 \, dt \leq C. \tag{2.6}
\]
**Proof.** Keeping in mind that $\text{div} \, u = 0$ and $\text{div} \, u_t = 0$, by Cauchy–Schwarz inequality we obtain after taking the $L^2$-inner product of (1.1) with $u_t$ and integrating by parts that
\[
\frac{\nu}{2} \frac{d}{dt} \left\| \nabla u \right\|_{L^2}^2 + \left\| u_t \right\|_{L^2}^2 \leq \frac{1}{2} \left\| u_t \right\|_{L^2}^2 + C \int_{\mathbb{R}^2_+} \left( |u|^2 |\nabla u|^2 + \theta^2 \right) \, dx \, dy,
\]
which, together with Lemma 2.1 and the Hölder inequality, implies
\[
\frac{d}{dt} \left\| \nabla u \right\|_{L^2}^2 + \left\| u_t \right\|_{L^2}^2 \leq C + C \left\| u \right\|_{L^4}^2 \left\| \nabla u \right\|_{L^4}^2. \tag{2.7}
\]
Thanks to the Gagliardo–Nirenberg inequality (cf. [1]):
\[
\left\| u \right\|_{L^4}^2 \leq C \left\| u \right\|_{L^2} \left\| \nabla u \right\|_{L^2}, \quad \left\| \nabla u \right\|_{L^4}^2 \leq C \left\| \nabla u \right\|_{L^2}^2 + C \left\| \nabla u \right\|_{L^2} \left\| \nabla^2 u \right\|_{L^2},
\]
we have from (2.7) and Lemma 2.1 that
\[
\frac{d}{dt} \left\| \nabla u \right\|_{L^2}^2 + \left\| u_t \right\|_{L^2}^2 \leq C + C \left\| u \right\|_{L^2} \left\| \nabla u \right\|_{L^2}^3 + C \left\| u \right\|_{L^2} \left\| \nabla u \right\|_{L^2}^2 \left\| \nabla^2 u \right\|_{L^2}^2 \leq C + C \left\| u \right\|_{L^2}^3 + C \left\| \nabla u \right\|_{L^2}^2 \left\| \nabla^2 u \right\|_{L^2}. \tag{2.8}
\]
To deal with the term $\|\nabla^2 u\|_{L^2}$, we write Eq. (1.1) of $u$ in the form

$$-\nu \Delta u + \nabla p = \theta e_2 - u_t - u \cdot \nabla u, \quad \text{div} \ u = 0 \quad \text{on} \ \mathbb{R}^2_+$$

with non-slip boundary condition $u|_{x=0} = 0$. Thus, owing to the estimates for the Stokes system (see, for example, [26,39]), we find

$$\|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 \leq C \left( \|\theta\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u \cdot \nabla u\|_{L^2}^2 \right)$$

$$\leq C \left( 1 + \|u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right)$$

$$\leq C \left( 1 + \|u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \right), \quad (2.9)$$

where we have also used the Gagliardo–Nirenberg inequality and Lemma 2.1. Thus, using the Cauchy–Schwarz inequality, we immediately get from (2.9) that

$$\|\nabla^2 u\|_{L^2}^2 \leq C \left( 1 + \|u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right),$$

that is,

$$\|\nabla^2 u\|_{L^2} \leq C \left( 1 + \|u_t\|_{L^2} + \|\nabla u\|_{L^2}^2 \right). \quad (2.10)$$

Thus, plugging (2.10) into (2.8), we obtain by the Cauchy–Schwarz inequality that

$$\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 \leq C + C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

This, combined with Gronwall’s lemma, leads to the desired estimates in (2.5), since it follows from Lemma 2.1 that $\|\nabla u(t)\|_{L^2}^2 \in L^1(0,T)$. Note that, the estimate of $\|\nabla^2 u\|_{L^2(Q_T)}$ in (2.5) is an immediate consequence of (2.10) and Lemma 2.1.

We now turn to the proof of (2.6). To do this, we differentiate (1.1) with respect to $t$, multiply the resulting equation by $u_t$ in $L^2$, and integrate by parts over $\mathbb{R}^2_+$ to get

$$\frac{1}{2} \frac{d}{dt} \|u_t\|_{L^2}^2 + \nu \|\nabla u_t\|_{L^2}^2 = -\iint_{\mathbb{R}^2_+} u_t \cdot \nabla u \cdot u_t \, dx \, dy + \iint_{\mathbb{R}^2_+} \theta_t e_2 \cdot u_t \, dx \, dy$$

$$:= I_1 + I_2. \quad (2.11)$$

With the help of (2.5), the Hölder and Gagliardo–Nirenberg inequalities, we have

$$I_1 \leq \|\nabla u\|_{L^2} \|u_t\|_{L^4} \|\nabla u_t\|_{L^2} \leq C \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \leq \nu \|\nabla u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2,$$

while the second term $I_2$ can be estimated as follows, using (1.2), (1.3) and Lemma 2.1.

$$I_2 = \iint_{\mathbb{R}^2_+} (\epsilon \Delta \theta - u \cdot \nabla \theta) u_{2t} \, dx \, dy = \iint_{\mathbb{R}^2_+} (-\epsilon \nabla \theta \cdot \nabla u_{2t} + \theta u \cdot \nabla u_{2t}) \, dx \, dy$$

$$\leq \epsilon \|\nabla \theta\|_{L^2} \|\nabla u_t\|_{L^2} + \|\theta\|_{L^\infty} \|u\|_{L^2} \|\nabla u_t\|_{L^2} \leq \frac{\nu}{4} \|\nabla u_t\|_{L^2}^2 + C \epsilon \|\nabla \theta\|_{L^2}^2 + C.$$
Thus, we deduce from (2.11) that
\[
\frac{d}{dt} \| u_t \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2 \leq C \| u_t \|_{L^2}^2 + C \varepsilon \| \nabla \theta \|_{L^2}^2 + C,
\]
which, together with Lemma 2.1 and (2.5), yields
\[
\| u_t (\tau) \|_{L^2}^2 + \int_T^\tau \| \nabla u_t (s) \|_{L^2}^2 ds \leq C + C \| u_t (\tau) \|_{L^2}^2, \quad 0 < \tau < t \leq T. \tag{2.12}
\]
On the other hand, taking the \(L^2\)-inner product of (1.1) with \(u_t\) and integrating by parts, we find
\[
\iint_{\mathbb{R}^2_+} |u_t|^2 dxdy = \iint_{\mathbb{R}^2_+} (\theta e_2 - u \cdot \nabla u + \nu \Delta u) \cdot u_t dxdy
\leq \frac{1}{2} \iint_{\mathbb{R}^2_+} |u_t|^2 dxdy + C \iint_{\mathbb{R}^2_+} (\theta^2 + |u|^2 |\nabla u|^2 + |\Delta u|^2) dxdy,
\]
since \(\iint_{\mathbb{R}^2_+} \nabla p \cdot u_t dxdy = -\iint_{\mathbb{R}^2_+} p \text{div} u_t dxdy = 0\) due to (1.3). So, similar to the proof of (2.8), it follows from Lemma 2.1 and (2.5) that
\[
\limsup_{\tau \to 0^+} \| u_t (\tau) \|_{L^2}^2 \leq C (1 + \| \nabla^2 u_0 \|_{L^2}^2) \leq C.
\]
Therefore, the proof of (2.6) is complete by letting \(\tau \to 0^+\) in (2.12). Note that, it follows readily from (2.10) and (2.5) that \(\| \nabla^2 u \|_{L^2(0,T;L^2)}\) is uniformly bounded in \(\varepsilon\).

Using Lemmas 2.1 and 2.2, we can prove that \(\| \nabla u(t) \|_{L^\infty}\) is integrable in \(t\) over \((0,T)\), which is crucial for the analysis of BL-thickness. Indeed, by Lemmas 2.1 and 2.2 we have

**Lemma 2.3.** Under the conditions of Theorem 1.1, it holds that
\[
\| u \|_{L^\infty(0,T;L^r)} + \| \nabla u \|_{L^\infty(0,T;L^p)} + \| u_t \|_{L^2(0,T;L^r)} \leq C, \quad \forall r \geq 2, \tag{2.13}
\]
\[
\| \nabla^2 u \|_{L^2(0,T;L^p)} + \| \nabla u \|_{L^2(0,T;L^\infty)} \leq C, \quad \forall p \geq 2. \tag{2.14}
\]

**Proof.** As a result of Lemmas 2.1 and 2.2, the estimate (2.13) follows from the Sobolev embedding inequality immediately. On the other hand, using (2.13) and the estimates of the Stokes system (see, e.g. [26,39]), one gets that
\begin{align*}
\|\nabla^2 u\|_{L^p} + \|\nabla p\|_{L^p} &\leq \|\theta e_2 - u_t - u \cdot \nabla u\|_{L^p} \\
&\leq C\left(\|\theta\|_{L^p} + \|u_t\|_{L^p} + \|u\|_{L^\infty} \|\nabla u\|_{L^p}\right)
\leq C\left(1 + \|\theta\|_{L^p} + \|u_t\|_{L^p}\right),
\end{align*}

where the second term on the right-hand side can be easily bounded as follows, using the interpolation inequality and Lemma 2.1.

\[\|\theta\|_{L^p} \leq C\|\theta\|_{L^2}^{2/p}\|\theta\|_{L^\infty}^{(p-2)/p} \leq C, \quad \forall p \in [2, \infty),\]

so that,

\[\int_0^T \|\nabla^2 u(t)\|_{L^p}^2 \, dt \leq C + C \int_0^T \|u(t)\|_{L^p}^2 \, dt \leq C, \quad \forall p \in [2, \infty).\]

In view of this and the Sobolev embedding inequality, one can take \(p\) suitably large (e.g. \(p > 2\)) to get that \(\|\nabla u\|_{L^2(0,T;L^\infty)}\) is uniformly bounded in \(\epsilon\). This finishes the proof of Lemma 2.3. \(\square\)

In the next, we shall derive some estimates on the \(L^2\)-norm of the derivatives of \(\theta\), which will be used in the proof of convergence rates and the analysis of BL-thickness.

**Lemma 2.4.** Under the conditions of Theorem 1.1, there exists a positive constant \(C\), independent of \(\epsilon > 0\), such that for sufficiently small \(\epsilon\),

\[\epsilon^{1/2} \sup_{0 \leq t \leq T} \|\nabla \theta(t)\|_{L^2}^2 + \epsilon^{3/2} \int_0^T \|\nabla^2 \theta(t)\|_{L^2}^2 \, dt \leq C.\]

**Proof.** Differentiation of (2.1) with respect to \(x\) gives

\[\tilde{\theta}_x + u \cdot \nabla \tilde{\theta}_x - \epsilon \Delta \tilde{\theta}_x = \{\epsilon \Delta (ae^{-x}) - u \cdot \nabla (ae^{-x}) - a_t e^{-x}\}_x - u_x \cdot \nabla \tilde{\theta}.\]

Multiplying this by \(\epsilon \tilde{\theta}_x\) in \(L^2\) and integrating the resulting equation by parts over \(\mathbb{R}^2_+\) yield

\[\frac{\epsilon}{2} \frac{d}{dt} \int_{\mathbb{R}^2_+} |\tilde{\theta}_x|^2 \, dx \, dy + \epsilon^2 \int_{\mathbb{R}^2_+} |\nabla \tilde{\theta}_x|^2 \, dx \, dy = \epsilon \int_{\mathbb{R}^2_+} \tilde{\theta}_x \{\epsilon \Delta (ae^{-x}) - u \cdot \nabla (ae^{-x}) - a_t e^{-x}\}_x \, dx \, dy\]

\[-\epsilon^{1/2} \int_{\mathbb{R}^2_+} \tilde{\theta}_x u_x \cdot \nabla \tilde{\theta} \, dx \, dy - \epsilon \int_{-\infty}^{\infty} \epsilon \tilde{\theta}_x \tilde{\theta}_x \big|_{x=0} \, dy,\]

in which the right-hand side can be handled term by term as follows.
First, similar to the treatment of (2.2), we have from a direct computation and (2.13) that

\[ \varepsilon \iint_{\mathbb{R}^2_+} \tilde{\theta}_x \{ \varepsilon \Delta (ae^{-x}) - u \cdot \nabla (ae^{-x}) - a_t e^{-x} \} \, dx \, dy \]

\[ = \varepsilon \iint_{\mathbb{R}^2_+} \tilde{\theta}_x \{ [\varepsilon (a + a_{yy}) + (u_1 a - u_2 a_y) - a_t] e^{-x} \} \, dx \, dy \]

\[ \leq \varepsilon \| \tilde{\theta}_x \|_{L^2_x}^2 + C \varepsilon \| a \|_{H^2_y}^2 + \| a_t \|_{L^2_y}^2 \]

\[ \leq \varepsilon \| \tilde{\theta}_x \|_{L^2_x}^2 + C \varepsilon (1 + \| \nabla u \|_{L^\infty}^2). \]

The second term on the right-hand side of (2.15) can be easily bounded by

\[ \left| \varepsilon \iint_{\mathbb{R}^2_+} \tilde{\theta}_x u_x \cdot \nabla \tilde{\theta} \, dx \, dy \right| \leq C \varepsilon \| \nabla u \|_{L^\infty} \| \nabla \tilde{\theta} \|_{L^2_x}^2. \]

Finally, since \( \tilde{\theta} = u = 0 \) on \( x = 0 \), it follows from (2.1) that

\[ -\varepsilon \tilde{\theta}_{xx} \big|_{x=0} = \{ \varepsilon \Delta (ae^{-x}) - a_t e^{-x} \} \big|_{x=0} = \varepsilon (a + a_{yy}) - a_t, \]

which, combined with the Sobolev type inequality:

\[ \| \tilde{\theta}_x \|_{L^\infty}^2 \leq \| \tilde{\theta}_x \|_{L^2_x}^2 + \| \tilde{\theta}_x \|_{L^2_x} \| \tilde{\theta}_{xx} \|_{L^2_x}, \]

implies that

\[ \left| \varepsilon \int_{-\infty}^{\infty} \varepsilon \tilde{\theta}_{xx} \tilde{\theta}_x \big|_{x=0} \, dy \right| \leq \varepsilon \int_{-\infty}^{\infty} \left( |a| + |a_{yy}| + |a_t| \right) \left( \| \tilde{\theta}_x \|_{L^2_x} + \| \tilde{\theta}_x \|_{L^2_x}^{1/2} \| \tilde{\theta}_{xx} \|_{L^2_x}^{1/2} \right) \, dy \]

\[ \leq \frac{\varepsilon^2}{2} \| \tilde{\theta}_{xx} \|_{L^2_x}^2 + C \varepsilon \| \tilde{\theta}_x \|_{L^2_x}^2 + C \varepsilon^{1/2} \left( \| a \|_{H^2_y}^2 + \| a_t \|_{L^2_y}^2 \right) \]

\[ \leq \frac{\varepsilon^2}{2} \| \tilde{\theta}_{xx} \|_{L^2_x}^2 + C \varepsilon \| \tilde{\theta}_x \|_{L^2_x}^2 + C \varepsilon^{1/2} \]

for enough small \( \varepsilon > 0 \). Here, we have used the following Young inequality:

\[ \varepsilon a^{1/2} b^{1/2} c = \left( \varepsilon^{1/4} a \right) \left( \varepsilon^{1/4} b \right) \left( \varepsilon^{1/2} c \right) \leq C \left( \varepsilon^{1/2} a^2 + \varepsilon b^2 + \varepsilon^{1/2} c^2 \right), \quad \forall a, b, c \in \mathbb{R}. \]

Thus, putting the above estimates into (2.15), we obtain for enough small \( \varepsilon > 0 \) that

\[ \varepsilon \frac{d}{dt} \| \tilde{\theta}_x \|_{L^2_x}^2 + \varepsilon^2 \| \nabla \tilde{\theta}_x \|_{L^2_x}^2 \leq C \varepsilon \left( 1 + \| \nabla u \|_{L^\infty} \right) \| \nabla \tilde{\theta} \|_{L^2_x}^2 + C \varepsilon^{1/2} \left( 1 + \| \nabla u \|_{L^\infty}^2 \right). \]

(2.16)

Similar to the proof of (2.16), differentiating (2.1) with respect to \( y \) and taking the \( L^2 \)-inner product of the resulting equation with \( \tilde{\theta}_y \), we get that (keeping in mind that \( \tilde{\theta}_y = 0 \) on \( x = 0 \))
\[ \varepsilon \frac{d}{dt} \| \tilde{\theta}_y \|^2_{L^2} + \varepsilon^2 \| \nabla \tilde{\theta}_y \|^2_{L^2} \leq C \varepsilon (1 + \| \nabla u \|_{L^\infty}) \| \nabla \tilde{\theta} \|^2_{L^2} + C \varepsilon (\| a \|^2_{H^1_y} + \| a_t \|^2_{H^1_y} + \| a \|^2_{H_y^1} \| \nabla u \|^2_{L^\infty}) \]

\[ \leq C \varepsilon (1 + \| \nabla u \|_{L^\infty}) \| \nabla \tilde{\theta} \|^2_{L^2} + C (1 + \| \nabla u \|^2_{L^\infty}). \]  

(2.17)

Combining (2.16) and (2.17), we know that

\[ \varepsilon^{1/2} \frac{d}{dt} \| \nabla \tilde{\theta} \|^2_{L^2} + \varepsilon^{3/2} \| \nabla^2 \tilde{\theta} \|^2_{L^2} \leq C \varepsilon^{1/2} (1 + \| \nabla u \|_{L^\infty}) \| \nabla \tilde{\theta} \|^2_{L^2} + C (1 + \| \nabla u \|^2_{L^\infty}), \]

provided \( \varepsilon > 0 \) is small enough. This, together with (2.14) and Gronwall's lemma, gives

\[ \varepsilon^{1/2} \sup_{0 \leq t \leq T} \| \nabla \tilde{\theta}(t) \|^2_{L^2} + \varepsilon^{3/2} \int_0^T \| \nabla^2 \tilde{\theta}(t) \|^2_{L^2} dt \leq C, \]

from which and the facts that \( a \in L^\infty(0, T; H^2_y) \) and \( a e^{-x} \in L^\infty(0, T; H^2) \), we obtain the desired estimate of Lemma 2.4. \( \square \)

To be continued, let \( \varphi_\eta(r) = (r^2 + \eta^2)^{1/2} \). Then,

\[ |\varphi_\eta'(r)| \leq 1, \quad \varphi_\eta''(r) \geq 0, \quad \text{and} \quad \lim_{\eta \to 0} \varphi_\eta(r) = \lim_{\eta \to 0} r \varphi_\eta'(r) = |r|. \]

Also let \( \rho \in C^2(\mathbb{R}_+) \) be a non-increasing function such that \( \rho(y) = 1 \) for \( y \in [0, 1/2] \), \( \rho(y) \) is a non-negative polynomial for \( y \in [1/2, 1] \), and \( \rho(y) = e^{-y} \) for \( y \geq 1 \). Define the even function \( g(y) = \rho(|y|) \). Then, it is clear that \( g \in L^2_+(\mathbb{R}) \) and there are positive constants \( k_1, k_2 \) such that

\[ k_1 e^{-|y|} \leq g(y) \leq k_2 e^{-|y|} \quad \text{and} \quad |g_y|, |g_{yy}| \leq k_2 g(y). \]

**Lemma 2.5.** Under the conditions of Theorem 1.1, we have

\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^2_+} e^{-(x+|y|)} |\nabla \theta| \, dx \, dy \leq C. \]

**Proof.** Differentiating (2.1) with respect to \( x \), multiplying the resulting equation by \( e^{-x} g(y) \varphi_\eta''(\tilde{\theta}_x) \) in \( L^2 \), and integrating by parts over \( \mathbb{R}^2_+ \), we get

\[ \frac{d}{dt} \int_{\mathbb{R}^2_+} e^{-x} g(y) \varphi_\eta(\tilde{\theta}_x) \, dx \, dy + \varepsilon \int_{\mathbb{R}^2_+} e^{-x} g(y) \varphi_\eta''(\tilde{\theta}_x) \| \nabla \tilde{\theta}_x \|^2 \, dx \, dy \]

\[ = \int_{\mathbb{R}^2_+} e^{-x} g(y) \varphi_\eta''(\tilde{\theta}_x) \{ e \Delta (ae^{-x}) - u \cdot \nabla (ae^{-x}) - a_t e^{-x} \} \, dx \, dy \]

\[ - \int_{\mathbb{R}^2_+} e^{-x} g(y) \varphi_\eta'(\tilde{\theta}_x) u_x \cdot \nabla \tilde{\theta} \, dx \, dy + \int_{\mathbb{R}^2_+} \varphi_\eta(\tilde{\theta}_x) u \cdot \nabla (e^{-x} g(y)) \, dx \, dy \]

\[ - \varepsilon \int_{\mathbb{R}^2_+} \varphi_\eta'(\tilde{\theta}_x) \nabla \tilde{\theta}_x \cdot \nabla (e^{-x} g(y)) \, dx \, dy - \varepsilon \int_{-\infty}^\infty g(y) \{ \tilde{\theta}_{xx} \varphi_\eta(\tilde{\theta}_x) \} \big|_{x=0} \, dy. \]  

(2.18)
We are now in a position of dealing with the terms on the right-hand side of (2.18). First, noting that \( |\varphi'(r)| \leq 1 \) and \( e^{-x}g(y) \in L^2 \), we have by (2.13) that

\[
\int\int_{\mathbb{R}^2_+} e^{-x}g(y)\varphi'_\eta(\tilde{\theta}_x)\left[ \varepsilon \Delta (ae^{-x}) - u \cdot \nabla (ae^{-x}) - a_t e^{-x} \right] \, dx \, dy
\]

\[
\leq \int\int_{\mathbb{R}^2_+} e^{-x}g(y) \left\{ \left[ \varepsilon (a + a_{yy}) + (u_1 a - u_2 a_y) - a_t e^{-x} \right] \right\} \, dx \, dy
\]

\[
\leq C \left( 1 + \|a\|_{H^2_y}^2 + \|a_t\|_{L^2_y}^2 + \|a\|_{H^1_y} \|\nabla u\|_{L^\infty}^2 \right) \leq C \left( 1 + \|\nabla u\|_{L^\infty}^2 \right).
\]

It is easy to see that the second and the third terms are respectively bounded by

\[
\left| \int\int_{\mathbb{R}^2_+} e^{-x}g(y)\varphi'_\eta(\tilde{\theta}_x)u_x \cdot \nabla \tilde{\theta} \, dx \, dy \right| \leq \|\nabla u\|_{L^\infty} \int\int_{\mathbb{R}^2_+} e^{-x}g(y)|\nabla \tilde{\theta}| \, dx \, dy
\]

and

\[
\left| \int\int_{\mathbb{R}^2_+} \varphi'_\eta(\tilde{\theta}_x)u \cdot \nabla (e^{-x}g(y)) \, dx \, dy \right| \leq C \|u\|_{L^\infty} \int\int_{\mathbb{R}^2_+} e^{-x}g(y)\varphi'_\eta(\tilde{\theta}_x) \, dx \, dy
\]

\[
\leq C \int\int_{\mathbb{R}^2_+} e^{-x}g(y)\varphi'_\eta(\tilde{\theta}_x) \, dx \, dy,
\]

since there holds \( |\nabla (e^{-x}g(y))| \leq Ce^{-x}g(y) \in L^2 \). As for the fourth term, thanks to the Hölder and Young inequalities, we find that

\[
\left| \varepsilon \int\int_{\mathbb{R}^2_+} \varphi'_\eta(\tilde{\theta}_x) \nabla \tilde{\theta}_x \cdot \nabla (e^{-x}g(y)) \, dx \, dy \right| \leq \varepsilon \|\nabla (e^{-x}g(y))\|_{L^2} \|\nabla \tilde{\theta}_x\|_{L^2}
\]

\[
\leq C \varepsilon^{1/2} + C \varepsilon^{3/2} \|\nabla \tilde{\theta}_x\|_{L^2}^2.
\]

Finally, Eq. (2.1) implies that \( \varepsilon \tilde{\theta}_{xx} |_{x=0} = a_t - \varepsilon (a + a_{yy}) \in L^2_y \), and thus, the last term on the right-hand side of (2.18) is uniformly bounded in \( \varepsilon \) due to the fact that \( g \in L^2_y \).

Thus, keeping in mind that \( \varphi''_\eta(r) \geq 0 \) and the second term on the left-hand side of (2.18) is non-negative, we obtain, after putting the above estimates into (2.18) and letting \( \eta \to 0 \), that

\[
\frac{d}{dt} \int\int_{\mathbb{R}^2_+} e^{-x}g(y)|\tilde{\theta}_x| \, dx \, dy \leq \left( 1 + \|\nabla u\|_{L^\infty} \right) \int\int_{\mathbb{R}^2_+} e^{-x}g(y)|\nabla \tilde{\theta}| \, dx \, dy
\]

\[
+ C \left( 1 + \|\nabla u\|_{L^\infty}^2 + \varepsilon^{3/2} \|\nabla \tilde{\theta}_x\|_{L^2}^2 \right).
\]
\[
\frac{d}{dt} \int \int_{\mathbb{R}^2_+} e^{-xg(y)}|\tilde{\theta}_y| \, dx \, dy \leq \left(1 + \|\nabla u\|_{L^\infty}\right) \int \int_{\mathbb{R}^2_+} e^{-xg(y)}|\nabla \tilde{\theta}| \, dx \, dy \\
+ C\left(1 + \|\nabla u\|_{L^\infty}^2 + \varepsilon^{3/2} \|\nabla \tilde{\theta}_y\|_{L^2}^2\right).
\]

Due to Lemmas 2.3 and 2.4, it holds that
\[\|\nabla u\|_{L^\infty}^2 + \varepsilon^{3/2} \|\nabla^2 \tilde{\theta}\|_{L^2}^2 \in L^1(0, T),\]
and hence, it follows from (2.19), (2.20) and Gronwall’s lemma that
\[\int \int_{\mathbb{R}^2_+} e^{-xg(y)}|\nabla \tilde{\theta}| \, dx \, dy \leq C,\]
which, together with the fact that \(g(y) \geq k_1e^{-|y|}\), completes the proof of Lemma 2.5. Note that the Hölder inequality and the conditions (1.11) imply that \(\|e^{-(x+|y|)}\nabla(\varepsilon e^{-x})\|_{L^1}\) is uniformly bounded in \(\varepsilon\). □

3. Vanishing diffusivity limit

By mollifying the initial-boundary data, the existence and uniqueness of global smooth solution to the initial-boundary value problem (1.1)–(1.5) with positive coefficients \(\nu\) and \(\varepsilon\) can be proved in a manner similar to that in [27] by the method used for the Navier–Stokes equations (see, for example, [14,29]). Thus, by virtue of the estimates given in Lemmas 2.1–2.4 which can be considered as the global a priori estimates, one can take the limit to obtain the global existence of strong solutions to (1.1)–(1.5) under the conditions of Theorem 1.1.

In light of the uniform estimates stated in the previous section, it is not difficult to see that there exists a subsequence of \(\varepsilon\), still denoted by \(\varepsilon\), such that as \(\varepsilon \to 0\),
\[
\begin{align*}
\varepsilon \Delta \theta^\varepsilon & \to u^0 \quad \text{strongly in } C(0, T; H^1) , \\
\theta^\varepsilon & \to \theta^0 \quad \text{weakly-* in } L^\infty(0, T; L^2) , \\
\varepsilon \varepsilon \nabla \theta^\varepsilon & \to 0 \quad \text{strongly in } L^2(0, T; L^2) , \\
\varepsilon \nabla \theta^\varepsilon & \to 0 \quad \text{strongly in } L^\infty(0, T; L^2) ,
\end{align*}
\]

where the limit functions \(u^0, \theta^0\) solve Eqs. (1.6)–(1.8) in the sense of distributions and satisfy the initial condition \(u^0|_{t=0} = u_0\) and the non-slip boundary condition \(u^0|_{x=0} = 0\). Moreover, for any \(\phi \in C_0^\infty(\mathbb{R}^2_+)\), one has
\[
\begin{align*}
\int \int_{\mathbb{R}^2_+} (\theta^0 - \theta_0) \phi \, dx \, dy & = \lim_{\varepsilon \to 0} \int \int_{\mathbb{R}^2_+} (\theta^\varepsilon - \theta_0) \phi \, dx \, dy = \lim_{\varepsilon \to 0} \int_0^t \int \int_{\mathbb{R}^2_+} \theta^\varepsilon \phi \, dx \, dy \, ds \\
& = \lim_{\varepsilon \to 0} \int_0^t \int \int_{\mathbb{R}^2_+} (\varepsilon \Delta \theta^\varepsilon - u^\varepsilon \cdot \nabla \theta^\varepsilon) \phi \, dx \, dy \, ds \\
& = \lim_{\varepsilon \to 0} \int_0^t \int \int_{\mathbb{R}^2_+} (-\varepsilon \nabla \theta^\varepsilon \cdot \nabla \phi + \theta^\varepsilon u^\varepsilon \cdot \nabla \phi) \, dx \, dy \, ds
\end{align*}
\]
\[
\int_0^t \int_{\mathbb{R}^2} \phi \cdot \nabla \theta \, dx \, dy \, ds \leq t \left\| \nabla \phi \right\|_{L^\infty} \left\| \theta^0 \right\|_{L^2} \left\| u^0 \right\|_{L^2},
\]

which tends to zero as \( t \to 0 \). This shows that \( \theta^0 \) satisfies the initial data in the sense of trace.

In order to show the convergence rates, we still need the estimate of \( \left\| \nabla \theta^0 \right\|_{L^\infty} \).

**Remark 3.1.** It is worth pointing out that all the estimates derived in Lemmas 2.1–2.5 except Lemma 2.4 also hold for the Boussinesq system (1.6)–(1.10) with zero diffusivity.

**Lemma 3.1.** Under the conditions of Theorem 1.1, one has

\[
\sup_{0 \leq t \leq T} \left\{ \left\| \nabla \theta^0(t) \right\|_{L^2} + \left\| \nabla \theta^0(t) \right\|_{L^\infty} \right\} \leq C.
\]

**Proof.** An application of the operator \( \nabla^\bot = (-\partial_y, \partial_x) \) to Eq. (1.7) results in

\[
\nabla^\bot \theta^0_t + (u^0 \cdot \nabla) \nabla^\bot \theta^0 = \nabla^\bot \theta^0 \cdot \nabla u^0.
\]

Taking the \( L^2 \)-inner product of the above equation with \( |\nabla^\bot \theta^0|^{p-2} \nabla^\bot \theta^0 \) \((p \geq 2)\), we see that

\[
\frac{1}{p} \frac{d}{dt} \left\| \nabla \theta^0 \right\|_{L^p} = \iint_{\mathbb{R}^2} \nabla^\bot \theta^0 \cdot \nabla u^0 \cdot \nabla \theta^0 \left| \nabla^\bot \theta^0 \right|^{p-2} \, dx \, dy \leq \left\| \nabla u^0 \right\|_{L^\infty} \left\| \nabla \theta^0 \right\|_{L^p}^p,
\]

which in particular implies that

\[
\frac{d}{dt} \left\| \nabla \theta^0 \right\|_{L^p} \leq \left\| \nabla u^0 \right\|_{L^\infty} \left\| \nabla \theta^0 \right\|_{L^p}.
\]

(3.1)

Thanks to Lemma 2.3 and Remark 3.1, we have \( \left\| \nabla u^0(t) \right\|_{L^\infty} \in L^1(0, T) \). So, a direct calculation from (3.1) yields

\[
\left\| \nabla \theta^0 \right\|_{L^p} \leq \left\| \nabla \theta^0 \right\|_{L^p} \exp \left\{ \int_0^t \left\| \nabla u^0(s) \right\|_{L^\infty} \, ds \right\} \leq C \left\| \nabla \theta^0 \right\|_{L^p}.
\]

Thus, choosing \( p = 2 \) in the above inequality gives the estimate of \( \left\| \nabla \theta^0 \right\|_{L^2} \). On the other hand, letting \( p \to \infty \) yields the desired estimate of \( \left\| \nabla \theta^0 \right\|_{L^\infty} \) immediately. The proof of Lemma 3.1 is therefore complete. \( \square \)

**Remark 3.2.** From Lemma 3.1, we see that there indeed exists a strong solution \((u^0, \theta^0)\) to the problem (1.6)–(1.10) under the conditions of Theorem 1.1. Moreover, in addition to the estimates of \((u^0, \theta^0)\) in Lemmas 2.1–2.3 and 3.1, one can prove more global regularity for \((u^0, \theta^0)\), and thus, show the global existence of a classical solution to the problem (1.6)–(1.10) in the same manner as that in [27], provided the given initial data \((u_0, \theta_0)\) are in \( H^3 \).

The proof of Theorem 1.1 will be completed by the following proposition.
Proposition 3.1. Let the conditions of Theorem 1.1 be satisfied. Assume that the two pairs of functions \((u^\varepsilon, \theta^\varepsilon, p^\varepsilon)\) and \((u^0, \theta^0, p^0)\) are respectively the solutions of the problems (1.1)-(1.5) and (1.6)-(1.10) with the same initial data and the same non-slip boundary condition. Then,

\[
\sup_{0 \leq t \leq T} \left\{ \left\| (u^\varepsilon - u^0)(t) \right\|_{H^1}^2 + \left\| (\theta^\varepsilon - \theta^0)(t) \right\|_{L^2}^2 \right\} + \int_0^T \left\{ \left\| \nabla^2 (u^\varepsilon - u^0)(t) \right\|_{L^2}^2 + \left\| \nabla (p^\varepsilon - p^0) \right\|_{L^2}^2 \right\} dt \leq C \varepsilon^{1/2}.
\]

Proof. The proof is based on the standard \(L^2\)-energy method. To this end, we set

\[v = u^\varepsilon - u^0, \quad \omega = \theta^\varepsilon - \theta^0, \quad q = p^\varepsilon - p^0.\]

Then, it follows from the problems (1.1)-(1.5) and (1.6)-(1.10) that \((v, \omega, q)\) satisfies

\[\begin{align*}
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 & = -\int \int v \cdot \nabla u^0 \cdot v \, dx \, dy + \int \int \omega e_2 \cdot v \, dx \, dy \\
& \leq \|\nabla u^0\|_{L^\infty} \|v\|_{L^2}^2 + \|v\|_{L^2} \|\omega\|_{L^2} \\
& \leq (1 + \|\nabla u^0\|_{L^\infty}) \|v\|_{L^2}^2 + \|\omega\|_{L^2}^2,
\end{align*}\]

where we have used the condition \(\text{div} u^\varepsilon = 0\) in (1.3).

Similarly, if we take the \(L^2\)-inner product of (3.3) with \(\omega\) and utilize Lemma 3.1, we obtain after integrating by part that

\[\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 = \varepsilon \int \int \Delta \theta^\varepsilon \omega \, dx \, dy - \int \int v \cdot \nabla \theta^0 \omega \, dx \, dy \leq \varepsilon \|\Delta \theta^\varepsilon\|_{L^2} \|\omega\|_{L^2} + \|\nabla \theta^0\|_{L^\infty} \|v\|_{L^2} \|\omega\|_{L^2} \leq \varepsilon^2 \|\Delta \theta^\varepsilon\|_{L^2}^2 + C \left( \|v\|_{L^2}^2 + \|\omega\|_{L^2}^2 \right).\]

Summing up (3.5) and (3.4), we find

\[\frac{d}{dt} \left( \|v\|_{L^2}^2 + \|\omega\|_{L^2}^2 \right) + \|\nabla v\|_{L^2}^2 \leq \varepsilon^2 \|\Delta \theta^\varepsilon\|_{L^2}^2 + C \left( \|v\|_{L^2}^2 + \|\omega\|_{L^2}^2 \right).\]
By virtue of the estimates (2.14) for \(\|\nabla u^0\|_{L^\infty}\) (see also Remark 3.1) and Lemma 2.4, we deduce from Gronwall’s lemma that

\[
\sup_{0 \leq t \leq T} \left\{ \|v(t)\|^2_{L^2} + \|\omega(t)\|^2_{L^2} \right\} + \int_0^T \|\nabla v(t)\|^2_{L^2} \, dt \leq C \varepsilon^{1/2}. \tag{3.6}
\]

Next, taking the \(L^2\)-inner product of (3.2) with \(v_t\) and integrating by parts, one gets

\[
\frac{v}{2} \frac{d}{dt} \|\nabla\|^2_{L^2} + \|v_t\|^2_{L^2} \leq \int \int (-u^\varepsilon \cdot \nabla v - v \cdot \nabla u^0 + \omega e_2) \cdot v_t \, dx \, dy
\leq C(\|u^\varepsilon\|_{L^\infty} \|\nabla v\|_{L^2} + \|\nabla u^0\|_{L^\infty} \|v\|_{L^2} + \|\omega\|_{L^2} \|v_t\|_{L^2})
\leq \frac{1}{2} \|v_t\|^2_{L^2} + C \|\nabla v\|^2_{L^2} + C \varepsilon^{1/2}(1 + \|v^0\|^2_{L^\infty}),
\]
where we have used (2.13) and (3.6). Consequently,

\[
\frac{d}{dt} \|\nabla\|^2_{L^2} + \|v_t\|^2_{L^2} \leq C \|\nabla v\|^2_{L^2} + C \varepsilon^{1/2}(1 + \|v^0\|^2_{L^\infty}),
\]
and it follows from (2.14) (for \(\|\nabla u^0\|_{L^\infty}\), (3.6) and Gronwall’s lemma that

\[
\sup_{0 \leq t \leq T} \|\nabla v(t)\|^2_{L^2} + \int_0^T \|v(t)\|^2_{L^2} \, dt \leq C \varepsilon^{1/2}. \tag{3.7}
\]

Finally, similar to the derivation of (2.9), using (2.13), (3.6) and (3.7), we deduce from (3.2) and the estimates of Stokes system that

\[
\|\nabla^2 v\|^2_{L^2} + \|\nabla q\|^2_{L^2} \leq C(\|\omega\|^2_{L^2} + \|v_t\|^2_{L^2} + \|u^\varepsilon \cdot \nabla v\|^2_{L^2} + \|v \cdot \nabla u^0\|^2_{L^2})
\leq C(\|\omega\|^2_{L^2} + \|v_t\|^2_{L^2} + \|u^\varepsilon\|^2_{L^\infty} \|\nabla v\|^2_{L^2} + \|\nabla u^0\|^2_{L^\infty} \|v\|^2_{L^2})
\leq C \varepsilon^{1/2}(1 + \|v^0\|^2_{L^\infty}) + C \|v_t\|^2_{L^2},
\]
from which and (3.7) we conclude that

\[
\int_0^T (\|\nabla^2 v(t)\|^2_{L^2} + \|\nabla q(t)\|^2_{L^2}) \, dt \leq C \varepsilon^{1/2}.
\]

This, together with (3.6) and (3.7), finishes the proof of Proposition 3.1. \(\square\)

4. Boundary-layer thickness

In this section, we study the boundary layer effects for the 2-D Boussinesq system with vanishing diffusivity limit. As mentioned in the introduction, to simplify the analysis, we focus on the special
case of vanishing initial data, that is, we suppose that \( u_0 = 0 \) and \( \theta_0 = 0 \). Under this assumption, we can show that the limit problem (1.6)–(1.10) has only the trivial solution \((0,0)\) (see Proposition 4.1), which proves the first part of Theorem 1.2.

**Proposition 4.1.** If the initial data \( u_0 \) and \( \theta_0 \) are both zero, then the initial-boundary value problem (1.6)–(1.10) has only trivial solution \((0,0)\).

**Proof.** For the proof, it suffices to show the uniqueness of solutions to the problem (1.6)–(1.10). Indeed, similar to the proof of (3.6), let

\[
v = u^{0.1} - u^{0.2}, \quad \omega = \vartheta^{0.1} - \vartheta^{0.2}, \quad q = p^{0.1} - p^{0.2},
\]

where the pairs \((u^{0.1}, \vartheta^{0.1}, p^{0.1})\) and \((u^{0.2}, \vartheta^{0.2}, p^{0.2})\) are the solutions of (1.6)–(1.10) with the initial data \((u_{0,1}, \theta_{0,1})\) and \((u_{0,2}, \theta_{0,2})\), respectively. Then,

\[
v_t + u^{0.1} \cdot \nabla v + \nabla q = \nu \Delta v - v \cdot \nabla u^{0.2} + \omega e_2, \tag{4.1}
\]

\[
\omega_t + u^{0.1} \cdot \nabla \omega = -v \cdot \nabla \vartheta^{0.2}, \tag{4.2}
\]

and \( \text{div} v = 0 \). Multiplying (4.1) and (4.2) by \( \nu \) and \( \omega \) in \( L^2 \) respectively, integrating the resulting equations by parts over \( \mathbb{R}^2_+ \), and summing them up, by Lemma 3.1 we obtain

\[
\frac{1}{2} \frac{d}{dt} \left\{ \| v \|_{L^2}^2 + \| \omega \|_{L^2}^2 \right\} + \nu \| \nabla v \|_{L^2}^2
\]

\[
= \iint_{\mathbb{R}^2_+} \left( -v \cdot \nabla u^{0.2} \cdot v + \nu \omega e_2 \cdot v - v \cdot \nabla \vartheta^{0.2} \omega \right) dxdy
\]

\[
\leq C \left( \| \nabla u^{0.2} \|_{L^\infty} \| v \|_{L^2}^2 + \| \omega \|_{L^2} \| v \|_{L^2} + \| \nabla \vartheta^{0.2} \|_{L^\infty} \| v \|_{L^2} \| \omega \|_{L^2} \right)
\]

\[
\leq C \left( 1 + \| \nabla u^{0.2} \|_{L^\infty} \right) \left( \| v \|_{L^2}^2 + \| \omega \|_{L^2}^2 \right),
\]

which, together with Lemma 2.3, Remark 3.1 and Gronwall’s lemma, yields

\[
\| v(t) \|_{L^2}^2 + \| \omega(t) \|_{L^2}^2 \leq C(T) \left( \| v(0) \|_{L^2}^2 + \| \omega(0) \|_{L^2}^2 \right).
\]

This is equivalent to the following stability result:

\[
\left\| (u^{0.1} - u^{0.2})(t) \right\|_{L^2}^2 + \left\| (\vartheta^{0.1} - \vartheta^{0.2})(t) \right\|_{L^2}^2 \leq C(T) \left( \| u_{0,1} - u_{0,2} \|_{L^2}^2 + \| \theta_{0,1} - \theta_{0,2} \|_{L^2}^2 \right).
\]

As a result, we also obtain the uniqueness of solutions to (1.6)–(1.10), and hence, the proof of Proposition 4.1 is complete. \( \square \)

**Remark 4.1.** In view of the uniqueness theorem in Proposition 4.1, we see that the convergence results established in the last section are actually valid for any sequence \( \varepsilon_j \to 0 \).

Now we turn to the proof of BL-thickness. As in Section 2, the index \( \varepsilon \) is omitted and the solution \((u^\varepsilon, \vartheta^\varepsilon)\) of the problem (1.1)–(1.5) with zero initial data (1.18) is still denoted by \((u, \vartheta)\) in the rest of this section. The analysis of the BL-thickness is based on the following two lemmas. First, we aim to prove
Lemma 4.1. Under the conditions of Theorem 1.2, it holds that

$$\sup_{0 \leq t \leq T} \iint_{\mathbb{R}^2_+} e^{-(x+|y|)}|\theta| \, dx \, dy \leq C \varepsilon^{1/2}.$$  

Proof. Let $\varphi_\eta(\theta) = (\theta^2 + \eta^2)^{1/2}$ and $g(y)$ be the same functions defined in Section 2. Taking the $L^2$-inner product of (1.2) with $e^{-x}g(y)\varphi_\eta'(\theta)$ and integrating by parts, we deduce that

$$\frac{d}{dt} \iint_{\mathbb{R}^2_+} e^{-x}g(y)\varphi_\eta(\theta) \, dx \, dy + \iint_{\mathbb{R}^2_+} e^{-x}g(y)\varphi_\eta''(\theta)|\nabla \theta|^2 \, dx \, dy$$

$$= -\varepsilon \iint_{\mathbb{R}^2_+} \varphi_\eta'(\theta) \nabla \theta \cdot \nabla (e^{-x}g(y)) \, dx \, dy - \varepsilon \int_{-\infty}^{\infty} g(y)\{\theta_x\varphi_\eta'(\theta)\} |x=0 \, dy$$

$$+ \iint_{\mathbb{R}^2_+} \varphi_\eta(\theta)u \cdot \nabla (e^{-x}g(y)) \, dx \, dy$$

$$:= I_1 + I_2 + I_3. \tag{4.3}$$

We now estimate each term on the right-hand side of (4.3). First, using Lemma 2.4 and the estimate $|\nabla (e^{-x}g(y))| \leq Ce^{-(x+|y|)} \in L^2$, we have

$$|I_1| \leq \varepsilon \|\nabla \theta\|_{L^2} \|\nabla (e^{-x}g(y))\|_{L^2} \leq C \varepsilon \|\nabla \theta\|_{L^2} \leq C \varepsilon^{3/4}.$$  

Secondly, due to the Sobolev type inequality:

$$\|\theta_x\|_{L^\infty}^2 \leq C \left(\|\theta_x\|_{L^2}^2 + \|\theta_x\|_{L^2} \|\theta_{xx}\|_{L^2}\right),$$

it follows from Lemma 2.4 and the Hölder inequality that

$$|I_2| \leq C \varepsilon \int_{-\infty}^{\infty} e^{-|y|} \left(\|\theta_x\|_{L^2}^2 + \|\theta_x\|_{L^2} \|\theta_{xx}\|_{L^2} \right) \, dy$$

$$\leq C \varepsilon \|\theta_x\|_{L^2}^2 + C \varepsilon \|\theta_x\|_{L^2} \|\theta_{xx}\|_{L^2} \right) \leq C \varepsilon^{3/4} + C \varepsilon^{7/8} \|\theta_{xx}\|_{L^2} \right).$$

Finally, by employing (2.13) and keeping in mind that $|\nabla (e^{-x}g(y))| \leq Ce^{-x}g(y)$, we find

$$|I_3| \leq C \|u\|_{L^\infty} \iint_{\mathbb{R}^2_+} \varphi_\eta(\theta) |\nabla (e^{-x}g(y))| \, dx \, dy$$

$$\leq C \iint_{\mathbb{R}^2_+} e^{-x}g(y)\varphi_\eta(\theta) \, dx \, dy.$$
Since \( \phi''_\eta(r) \geq 0 \) and the second term on the left-hand side of (4.3) is non-negative, putting the above estimates of \( I_i \) into (4.3) and letting \( \eta \to 0 \), we infer for small \( \varepsilon \in (0, 1) \) that

\[
\frac{d}{dt} \iint_{\mathbb{R}^2_+} e^{-x} g(y) |\theta' dx dy \leq C \iint_{\mathbb{R}^2_+} e^{-x} g(y) |\theta| dx dy + C \varepsilon^{3/4} + C \varepsilon^{7/8} \|\theta_{xx}\|_{L^2}^{1/2}. \tag{4.4}
\]

Using Lemma 2.4 again, we get from the Hölder inequality that

\[
\varepsilon^{7/8} \int_0^t \left\| \theta_{xx}(s) \right\|_{L^2}^{1/2} ds \leq C \varepsilon^{1/2} \left\{ \varepsilon^{3/2} \int_0^t \left\| \theta_{xx}(s) \right\|_{L^2}^2 ds \right\}^{1/4} \leq C \varepsilon^{1/2},
\]

which, inserted into (4.4), immediately leads to the desired estimate of Lemma 4.1, using Gronwall’s lemma and the fact that \( g(y) \geq k_1 e^{-|y|} \). \( \square \)

**Lemma 4.2.** Under the conditions of Theorem 1.2, it holds that

\[
\sup_{0 \leq t \leq T} \int \int_{\mathbb{R}^2} x^2 e^{-x(|x|+|y|)} |\nabla \theta| dx dy \leq C \varepsilon.
\]

**Proof.** To prove this lemma, we introduce the function \( \zeta(x) = x^2 e^{-x} \), which clearly satisfies

\[
\zeta(0) = \zeta'(0) = 0 \quad \text{and} \quad \zeta(\infty) = \zeta'(\infty) = 0.
\]

Then, by setting \( \pi = \theta_x \) and differentiating (1.2) with respect to \( x \), we get

\[
\pi_t + u \cdot \nabla \pi = \varepsilon \Delta \pi - u_x \cdot \nabla \theta.
\]

Multiplying this by \( \zeta(x) g(y) \varphi'_\eta(\pi) \) in \( L^2 \) and integrating by parts over \( \mathbb{R}^2_+ \), we obtain

\[
\frac{d}{dt} \iint_{\mathbb{R}^2_+} \zeta(x) g(y) \varphi'_\eta(\pi) dx dy + \varepsilon \iint_{\mathbb{R}^2_+} \zeta(x) g(y) \varphi''_\eta(\pi) |\nabla \pi|^2 dx dy
\]

\[
= -\varepsilon \iint_{\mathbb{R}^2_+} \varphi'_\eta(\pi) \nabla \pi \cdot \nabla (\zeta(x) g(y)) dx dy - \iint_{\mathbb{R}^2_+} \zeta(x) g(y) \varphi'_\eta(\pi) u_x \cdot \nabla \theta dx dy
\]

\[
+ \iint_{\mathbb{R}^2_+} \varphi'_\eta(\pi) u \cdot \nabla (\zeta(x) g(y)) dx dy
\]

\[ := J_1 + J_2 + J_3. \tag{4.5} \]

By the definitions of \( \zeta(x) \) and \( g(y) \), one sees from the Cauchy–Schwarz inequality that

\[
|\Delta (\zeta(x) g(y))| \leq C(x^2 + 1) e^{-x} g(y) \in L^2,
\]

so that, a direct computation gives
\[ J_1 = \varepsilon \iint_{\mathbb{R}^2_+} \Delta (\zeta(x)g(y)) \varphi_\eta(\pi) \, dx \, dy \leq C \varepsilon \iint_{\mathbb{R}^2_+} (x^2 + 1) e^{-x} g(y) \varphi_\eta(\pi) \, dx \, dy \]

\[ \leq C \varepsilon \iint_{\mathbb{R}^2_+} \zeta(x) g(y) \varphi_\eta(\pi) \, dx \, dy \]

The second term \( J_2 \) on the right-hand side of (4.5) can be easily bounded by

\[ |J_2| \leq C \|u_x\|_{L^\infty} \iint_{\mathbb{R}^2_+} \zeta(x) g(y) |\nabla \theta| \, dx \, dy. \]

In order to deal with \( J_3 \), we write it in the following form (noting that \( u = (u_1, u_2) \)):

\[ J_3 = \iint_{\mathbb{R}^2_+} \zeta(x) \varphi_\eta(\pi) u_2 \partial_y g(y) \, dx \, dy + \iint_{\mathbb{R}^2_+} g(y) \varphi_\eta(\pi) u_1 \partial_x \zeta(x) \, dx \, dy \]

\[ = \iint_{\mathbb{R}^2_+} \zeta(x) \varphi_\eta(\pi) u_2 \partial_y g(y) \, dx \, dy - \iint_{\mathbb{R}^2_+} g(y) \varphi_\eta(\pi) u_1 (x^2 e^{-x}) \, dx \, dy \]

\[ + \iint_{\mathbb{R}^2_+} g(y) \varphi_\eta(\pi) u_1 (2xe^{-x}) \, dx \, dy \]

\[ \leq \int \int_{\mathbb{R}^2_+} \zeta(x) g(y) \varphi_\eta(\pi) \, dx \, dy + \int \int_{\mathbb{R}^2_+} x |u_1| e^{-x} g(y) \varphi_\eta(\pi) \, dx \, dy, \]

where we have used (2.13) and the fact that \( |g_y| \leq k_2 g \). To estimate the second term on the right-hand side of the last inequality, we observe that \( u|_{x=0} = 0 \) and

\[ \left| u(x, y, t) \right| \leq \int_{0}^{x} \left\| u_x(\cdot, y, t) \right\|_{L^\infty} \, dx \leq x \left\| u_x(t) \right\|_{L^\infty}, \quad \forall x \in [0, \infty), \]

from which and the definition of \( \zeta(x) \) we deduce

\[ \iint_{\mathbb{R}^2_+} x |u_1| e^{-x} g(y) \varphi_\eta(\pi) \, dx \, dy \leq \|u_x\|_{L^\infty} \int \int_{\mathbb{R}^2_+} x^2 e^{-x} g(y) \varphi_\eta(\pi) \, dx \, dy \]

\[ \leq \|\nabla u\|_{L^\infty} \int \int_{\mathbb{R}^2_+} \zeta(x) g(y) \varphi_\eta(\pi) \, dx \, dy, \]

and hence,

\[ J_3 \leq C (1 + \|\nabla u\|_{L^\infty}) \int \int_{\mathbb{R}^2_+} \zeta(x) g(y) \varphi_\eta(\pi) \, dx \, dy. \]
Thus, substituting the above estimates for $J_i$ ($i = 1, 2, 3$) into (4.5) and sending $\eta \to 0$, by Lemma 2.5 we obtain that
\[
\frac{d}{dt} \iint_{\mathbb{R}^2} \zeta(x) g(y) |\theta_x| \, dx \, dy \leq C \left(1 + \|\nabla u\|_{L^\infty}\right) \iint_{\mathbb{R}^2} \zeta(x) g(y) |\nabla \theta| \, dx \, dy + C \varepsilon \iint_{\mathbb{R}^2} e^{-x} g(y) |\theta_x| \, dx \, dy \\
\leq C \left(1 + \|\nabla u\|_{L^\infty}\right) \iint_{\mathbb{R}^2} \zeta(x) g(y) |\nabla \theta| \, dx \, dy + C \varepsilon.
\] (4.6)

In exactly the same way, we also have
\[
\frac{d}{dt} \iint_{\mathbb{R}^2} \zeta(x) g(y) |\theta_y| \, dx \, dy \leq C \left(1 + \|\nabla u\|_{L^\infty}\right) \iint_{\mathbb{R}^2} \zeta(x) g(y) |\nabla \theta| \, dx \, dy + C \varepsilon.
\] (4.7)

Combining (4.6) and (4.7), we conclude that
\[
\frac{d}{dt} \iint_{\mathbb{R}^2} \zeta(x) g(y) |\nabla \theta| \, dx \, dy \leq C \left(1 + \|\nabla u\|_{L^\infty}\right) \iint_{\mathbb{R}^2} \zeta(x) g(y) |\nabla \theta| \, dx \, dy + C \varepsilon.
\]

This, together with (2.14) and Gronwall’s lemma, finishes the proof of Lemma 4.2. Note that, in view of the definition of $g(y)$ one has $g(y) \geq k_1 e^{-|y|}$. □

By virtue of Lemmas 4.1 and 4.2, we can complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Indeed, it follows from Lemma 4.2 that
\[
\|\nabla \theta\|_{L^1_{\delta}(\Omega_\delta)} \leq \frac{1}{\delta^2} \left\| x^2 e^{-(x+|y|)} \nabla \theta \right\|_{L^1_{\delta}(\Omega_\delta)} \leq \frac{1}{\delta^2} \left\| x^2 e^{-(x+|y|)} \nabla \theta \right\|_{L^1} \leq \frac{C \varepsilon}{\delta^2},
\]

where $\Omega_\delta$ and $L^1_g$ are the same as in Definition 1.2. Thus,
\[
\|\nabla \theta\|_{L^1_{\delta}(\Omega_\delta)} \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]

provided $\delta$ is chosen to be $\delta(\varepsilon) = \varepsilon^a$ with $a \in (0, 1/2)$. This, together with Lemma 4.1, completes the proof of (1.19). On the other hand, due to the non-zero boundary data and the following inequality:
\[
e^{-x} \theta(x, y, t) \leq \int_{-\infty}^{\infty} \left| \left( e^{-x} \theta \right)_x \right| \, dx \leq \int_{0}^{\infty} \left| \left( e^{-x} \theta \right)_y \right| \, dx,
\]

we know that
\[
0 < \int_{-\infty}^{\infty} e^{-|y|} \left\| e^{-x} \theta \right\|_{L^\infty} \, dy \leq \int \int e^{-(x+|y|)} \left( |\theta| + |\theta_x| \right) \, dx \, dy = \|\theta\|_{W^{1,1}_g(\mathbb{R}^2)}.
\]

This proves (1.20). So, the proof of Theorem 1.2 is complete. □
Next we present two remarks on the BL-thickness for the 2-D Boussinesq equations with vanishing diffusivity limit, which prove the claims stated in Remarks 1.4 and 1.5.

**Remark 4.2.** Let \( l > 0 \) be a given positive constant. Then, the above analysis of BL-thickness also holds for the domains of the form: \( \Omega := \{(x, y): -l < x < l, -\infty < y < \infty\} \). To see this, we consider the 2-D Boussinesq equations (1.1)–(1.3) in the domain \( \Omega \times (0, T) \) with the following initial-boundary conditions:

\[
\begin{align*}
&u^\varepsilon (x, y, 0) = 0, \quad \theta^\varepsilon (x, y, 0) = 0, \\
u^\varepsilon (-l, y, t) = 0, \quad u^\varepsilon (l, y, t) = 0, \\
&\theta^\varepsilon (-l, y, t) = a(y, t), \quad \theta^\varepsilon (l, y, t) = b(y, t).
\end{align*}
\]

(4.8)

Following the analogous arguments in Section 2 line by line, we find that all the uniform estimates in Lemmas 2.1–2.5 still hold for the problem (1.1)–(1.3) and (4.8). Indeed, the minor difference in the proof is that, instead of the transformation \( \tilde{\theta} = \theta - ae^{-x} \) introduced for (2.1), we now should set \( \tilde{\theta} = \theta - [(x+l)b - (x-l)a]/(2l) \) which also vanishes on the boundaries \( x = -l, l \). Moreover, multiplying (1.2) by \( g(y)\varphi_\eta(\tilde{\theta}) \) in \( L^2 \) and letting \( \eta \to 0 \), we can prove in the manner similar to the proof of Lemma 4.1 that

\[
\int_{-\infty}^{\infty} \int_{-l}^{l} e^{-|y|} |\theta| \, dx \, dy \leq C \int_{-\infty}^{\infty} \int_{-l}^{l} g(y)|\theta| \, dx \, dy \leq C e^{1/2}. \tag{4.9}
\]

To obtain the estimate analogous to that in Lemma 4.2, we set \( \xi(x) = (x-l)^2(x+l)^2 \) which satisfies \( \xi(\pm l) = \xi'(\pm l) = 0 \). After multiplying (1.2), by \( \xi(x)g(y)\varphi_\eta(\tilde{\theta}_x) \) in \( L^2 \) and integrating by parts, we obtain the same identity as the one in (4.5) with \( \zeta(x) \) replaced by \( \xi(x) \). Due to the boundedness of the x-interval \((-l, l)\), one has \( |\Delta(\xi(x)g(y))| \leq C(\xi(x) + 1)g(y) \). Thus, the first and the second terms can be estimated in exactly the same way as in Lemma 4.2. Due to the non-slip boundary condition, we observe that

\[
|u(x, y, t)| \leq \int_{x}^{1} \|u_{x}(\cdot, y, t)\|_{L^\infty} \, dx \leq \|\nabla u\|_{L^\infty}(l-x), \quad \forall x \in (-l, l)
\]

and

\[
|u(x, y, t)| \leq \int_{-l}^{x} \|u_{x}(\cdot, y, t)\|_{L^\infty} \, dx \leq \|\nabla u\|_{L^\infty}(x+l), \quad \forall x \in (-l, l),
\]

and hence, the third term can be estimated as follows:

\[
J_3 = \int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x)\varphi_\eta(\pi)u_2\partial_y g(y) \, dx \, dy + \int_{-\infty}^{\infty} \int_{-l}^{l} g(y)\varphi_\eta(\pi)u_1\partial_x \xi(x) \, dx \, dy
\]

\[
= \int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x)\varphi_\eta(\pi)u_2\partial_y g(y) \, dx \, dy + 2 \int_{-\infty}^{\infty} \int_{-l}^{l} g(y)\varphi_\eta(\pi)u_1 \{l_0 - l_1^2\} \, dx \, dy
\]
+ 2 \int_{-\infty}^{\infty} \int_{l}^{1} g(y) \varphi_\eta(\pi) u_1 \left((x - l)^2 + (x + l)\right) \, dx \, dy \\
\leq C \|u\|_{L^\infty} \int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x) g(y) \varphi_\eta(\pi) \, dx \, dy + C \|\nabla u\|_{L^\infty} \int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x) g(y) \varphi_\eta(\pi) \, dx \, dy \\
\leq C (1 + \|\nabla u\|_{L^\infty}) \int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x) g(y) \varphi_\eta(\pi) \, dx \, dy,

where we have used (2.13) and the inequality \(|g_y| \leq k_2 g\). Thus, by letting \(\eta \to 0\) and using Lemma 2.5, we get that

\[
\frac{d}{dt} \int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x) g(y) \nabla \theta \, dx \, dy \leq C (1 + \|\nabla u\|_{L^\infty}) \int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x) g(y) \nabla \theta \, dx \, dy + C \varepsilon.
\]

The same estimate also holds for \(\theta_y\). So, it follows from (2.14) and Gronwall's lemma that

\[
\int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x) e^{-\|\nabla \theta\|} \, dx \, dy \leq C \int_{-\infty}^{\infty} \int_{-l}^{l} \xi(x) g(y) \nabla \theta \, dx \, dy \leq C \varepsilon. \tag{4.10}
\]

Since it holds for any \(\delta \in (0, l)\) that

\[
\delta^2 \int_{-\infty}^{\infty} \int_{-l+\delta}^{l-\delta} e^{-\|\nabla \theta\|} \, dx \, dy \\
= \delta^2 \int_{-\infty}^{\infty} \int_{0}^{l-\delta} e^{-\|\nabla \theta\|} \, dx \, dy + \delta^2 \int_{-\infty}^{\infty} \int_{0}^{l-\delta} e^{-\|\nabla \theta\|} \, dx \, dy \\
\leq \int_{-\infty}^{\infty} \int_{-l+\delta}^{l-\delta} (x + l)^2 e^{-\|\nabla \theta\|} \, dx \, dy + \int_{-\infty}^{\infty} \int_{0}^{l-\delta} (x - l)^2 e^{-\|\nabla \theta\|} \, dx \, dy \\
\leq \frac{1}{l^2} \int_{-\infty}^{\infty} \int_{-l+\delta}^{l-\delta} \left(\int_{-\infty}^{\infty} + \int_{0}^{l-\delta}\right) (x - l)^2 (x + l)^2 e^{-\|\nabla \theta\|} \, dx \, dy \\
= \frac{1}{l^2} \int_{-\infty}^{\infty} \int_{-l+\delta}^{l-\delta} \xi(x) e^{-\|\nabla \theta\|} \, dx \, dy, \quad \xi(x) = (x - l)^2 (x + l)^2,
\]

we thus have from (4.10) that

\[
\|e^{-\|\nabla \theta\|}\|_{L^1(\Omega_\delta)} \leq C \varepsilon/\delta^2 \to 0 \quad \text{with} \quad \Omega_\delta = (-l + \delta, l - \delta) \times (-\infty, \infty),
\]
provided $\delta = \delta(\epsilon)$ satisfies the conditions of Theorem 1.2. This, together with (4.9), proves the BL-thickness for the problem (1.1)–(1.3), (4.8). Note that, the BL-thickness for such domains is defined in the same way as that in Definition 1.2 with $\Omega_\delta = (-l + \delta, l - \delta) \times (-\infty, \infty)$.

**Remark 4.3.** As claimed in Remark 1.5, by the same method we can also prove a weaker result (i.e. $\delta(\epsilon) \sim \epsilon^{1/8}$) than that in Theorem 1.2 on the BL-thickness for general initial data, the proof of which requires more regularity of $\theta^0$. In the next, we sketch the proof. First, recalling that it has been shown in Lemma 3.1 that

$$\left\| \nabla \theta^0(t) \right\|_{L^2} \leq C, \quad \forall t \in [0, T],$$

we can thus use Lemmas 2.1–2.3 and Remark 3.1 to infer from (1.6) that

$$\int_0^T \left\| u^0(t) \right\|_{H^3}^2 dt \leq C \int_0^T \left\{ \left( \left\| \theta^0(t) \right\|_{H^1}^2 + \left\| \left( u^0 \cdot \nabla u^0 \right)(t) \right\|_{L^2}^2 + \left\| u^0(t) \right\|_{H^1}^2 \right) \right\} dt \leq C,$$

where Lemmas 2.2 and 2.3 were used to get

$$\left\| u^0 \cdot \nabla u^0 \right\|_{H^1} \leq C \left( \left\| u^0 \right\|_{L^\infty} \left\| \nabla u^0 \right\|_{L^2}^2 + \left\| \nabla u^0 \right\|_{L^\infty} \left\| \nabla^2 u^0 \right\|_{L^2} \right) \leq C \left( \left\| \theta^0 \right\|_{L^\infty} + \left\| \nabla \theta^0 \right\|_{L^\infty} \right) \in L^1(0, T).$$

In addition to the conditions of Theorem 1.1, we now assume further that $\theta_0 \in H^3(\mathbb{R}^2_+)$. Differentiating (1.7) twice with respect to $x$ yields

$$\theta_{xx}^0 + u^0 \cdot \nabla \theta_{xx}^0 + u_{xx}^0 \cdot \nabla \theta^0 + 2u_x^0 \cdot \nabla \theta_x^0 = 0,$$

which, multiplied by $\theta_{xx}^0$ in $L^2$, gives

$$\frac{1}{2} \frac{d}{dt} \left\| \theta_{xx}^0 \right\|_{L^2}^2 = -\iint_{\mathbb{R}^2_+} (u_{xx}^0 \cdot \nabla \theta^0 + 2u_x^0 \cdot \nabla \theta_x^0) \theta_{xx}^0 \, dx \, dy \leq C \left( \left\| \nabla \theta^0 \right\|_{L^\infty} \left\| \nabla^2 u^0 \right\|_{L^2} + \left\| \nabla u^0 \right\|_{L^\infty} \left\| \nabla \theta_{xx}^0 \right\|_{L^2} \right) \left\| \theta_{xx}^0 \right\|_{L^2} \leq C \left( 1 + \left\| \nabla^2 u^0 \right\|_{L^2}^2 \right) + C \left( 1 + \left\| \nabla u^0 \right\|_{L^\infty} \right) \left\| \nabla^2 \theta^0 \right\|_{L^2}^2,$$

where we have used Lemma 3.1. Similarly, we can also prove

$$\frac{1}{2} \frac{d}{dt} \left\{ \left\| \theta_{yy}^0 \right\|_{L^2}^2 + \left\| \theta_{xy}^0 \right\|_{L^2}^2 \right\} \leq C \left( 1 + \left\| \nabla^2 u^0 \right\|_{L^2}^2 \right) + C \left( 1 + \left\| \nabla u^0 \right\|_{L^\infty} \right) \left\| \nabla^2 \theta^0 \right\|_{L^2}^2.$$

Thus, summing up and applying Lemmas 2.2, 2.3 and Remark 3.1, we deduce from Gronwall's lemma that

$$\left\| \nabla^2 \theta^0(t) \right\|_{L^2}^2 \leq C \left( 1 + \left\| \nabla^2 \theta^0 \right\|_{L^2}^2 \right) \leq C, \quad \forall t \in [0, T].$$

Following a procedure similar to that in the derivation of (4.13) and using the Sobolev embedding inequality, we find that
Proposition 3.1.

we find (1.2) and (1.7) that mas 4.1 and 4.2 for general initial data. To do so, let

on the right-hand side, using (4.11), (4.13), (4.14) and Proposition 3.1, we have

The left-hand side of (4.17) can be estimated in exactly the same way as in Lemma 4.2. For the terms

which, multiplied by

Thus, after letting

Hence, it readily follows from (4.12) and Gronwall's lemma that

With the help of (4.11), (4.13) and (4.14), we can prove the estimates analogous to those in Lemmas 4.1 and 4.2 for general initial data. To do so, let \( v = u^\varepsilon - u^0 \) and \( \omega = \theta^\varepsilon - \theta^0 \). Then one has from (1.2) and (1.7) that

Taking the \( L^2 \)-inner product of (4.15) with \( e^{-x} g(y) \varphi'_{\eta}(\omega) \) gives

where the left-hand side can be handled in the same manner as in Lemma 4.1, while the right-hand side can be bounded as follows, using (4.11), (4.13) and keeping in mind that \( \| v \|_{L^2} \leq C \varepsilon^{1/4} \) due to Proposition 3.1.

Thus, after letting \( \eta \to 0 \) and integrating in \( t \), one infers from Lemma 2.4 that

Similar to the proof of Lemma 4.2, by letting \( \pi = \omega_x \) and differentiating (4.15) with respect to \( x \), we find

which, multiplied by \( \zeta(x) g(y) \varphi'_{\eta}(\pi) \) in \( L^2 \), yields

The left-hand side of (4.17) can be estimated in exactly the same way as in Lemma 4.2. For the terms on the right-hand side, using (4.11), (4.13), (4.14) and Proposition 3.1, we have
R.H.S. of (4.17) ≤ Cε ∥ Δθ₀ ∥ _L^2_ + C ∥ v ∥ _L^2_ ∥ ∇θ₀ ∥ _L^2_ + C ∥ vₓ ∥ _L^2_ ∥ ∇θ₀ ∥ _L^2_ \\
+ C ∥ u_x ∥ _L^∞_ ∫∫₆ (y) ∥ ∇ω ∥ dxdy \\
≤ Cε^{1/4} + C ∥ u_x ∥ _L^∞_ ∫∫₆ (y) ∥ ∇ω ∥ dxdy,

and hence, it follows from (4.17) that

\[
\frac{d}{dt} ∫∫₆ (y) ∥ ∇ω ∥ dxdy ≤ C(1 + ∥ u^f ∥ _L^∞_ ) ∫∫₆ (y) ∥ ∇ω ∥ dxdy \\
+ Cε ∫∫₆ e^{-x} (y) ∥ π ∥ dxdy + Cε^{1/4} \\
≤ C(1 + ∥ u^f ∥ _L^∞_ ) ∫∫₆ (y) ∥ ∇ω ∥ dxdy + Cε^{1/4},
\]

since ε ∥ π ∥ _L^2_ = ε ∥ θ^f_y - θ^f_x ∥ _L^2_ ≤ Cε^{3/4} due to Lemma 2.4 and (4.11). The same estimate also holds for ω_y. So, an application of Gronwall’s lemma leads to

\[
∫∫₆ x^2 e^{-(x+y)} | ∇ω | dxdy ≤ C ∫∫₆ (y) ∥ ∇ω ∥ dxdy ≤ Cε^{1/4}. \tag{4.18}
\]

Following a procedure similar to the proof of Theorem 1.2, we can use (4.16) and (4.18) to show that any positive function δ(ε) = ε^β with β ∈ (0, 1/8) is a BL-thickness in the sense of Definition 1.2 for the 2-D Boussinesq system with general initial data, provided the given boundary data a(y, t) is not equivalent to the determined value of θ^f|_x=0. Note that, the boundary value of θ^f|_x=0 is uniquely determined by u^0 and θ^0, using Eq. (1.7) and the method of characteristics.

References


