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Spreads Admitting Regular Elliptic Covers

G. L. Ebert

We say that a spread **S** of $PG(3, q)$ admits a regular elliptic cover if and only if **S** contains $q-1$ pairwise disjoint reguli (necessarily covering all but two fixed lines of **S**). Clearly, any André spread satisfies this condition. In this paper, we construct an infinite family of non-André spreads admitting regular elliptic covers by replacing $(q+1)$-nests of reguli in a regular spread. These are the only known non-André spreads to admit such a cover. The collineation groups of these spreads are also discussed in detail.

1. Introduction

In [7], $(q+1)$-nests of reguli in a regular spread Ω were constructed for all odd prime powers q. Moreover, if U denotes the set of lines contained in the reguli of such a $(q+1)$-nest, it was shown that U can, in fact, be realized as the line set of two different $(q+1)$-nests as well as the line set of $(q+1)/2$ (disjoint) reguli in a partial linear set. Clearly, this line set U can be replaced in Ω by simultaneously reversing the $(q+1)/2$ reguli in the above linear set, thereby yielding an André spread. However, it was also remarked (without proof) in [7] that the $(q+1)$-nests themselves are replaceable. That is, U can be replaced by judiciously choosing $(q+1)/2$ lines from the opposite regulus of each regulus in the $(q+1)$-nest in question. Hence, for a given prime power q, the same line set U in general can be replaced in three projectively inequivalent ways, only one of which yields an André spread. Examples for $q=7$ were given in [7]. In this paper, we prove the above assertions on replacement and describe in detail the resulting spreads as well as their collineation groups. The two-dimensional translation planes corresponding to the non-André spreads so constructed (see [1] or [5]) appear to be new.

Finally and most interestingly, we prove here that the above non-André spreads admit two different regular elliptic covers which share $(q-3)/2$ reguli in a partial linear set. These are the only non-André spreads known to the author which admit a regular elliptic cover.

2. Preliminary Results

Let $\Sigma = PG(3, q)$ denote projective 3-space over the finite field $GF(q)$. A *spread* of Σ is any collection of q^2+1 skew lines, necessarily partitioning the points of Σ. By the well known correspondence of André [1] or Bose [5], every such spread determines a two-dimensional translation plane, and conversely every two-dimensional translation plane arises from such a spread. A regulus of Σ is any set R of $q+1$ skew lines such that any line transversal to three lines of R is transversal to all lines of R. The $q+1$ lines transversal to R are pairwise skew, forming another regulus R^{opp}, called the *opposite regulus* to R. Any three skew lines of Σ uniquely determine a regulus, and a spread Ω of Σ is called *regular* iff the regulus determined by any three of its lines is contained in Ω. The translation plane corresponding to a regular spread is desarguesian, and hence to obtain non-desarguesian planes we must construct non-regular spreads.

Given any two lines of a regular spread Ω, the remaining q^2-1 lines of Ω can be partitioned into $q-1$ pairwise disjoint reguli, called a *complete linear set* of reguli. Any subset of a complete linear set of reguli in Ω is called a *linear set*. Of course, there are many non-linear sets of pairwise disjoint reguli in Ω as well.

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One method for constructing new spreads is to start with a regular spread and then replace some subset of lines by another partial spread covering the same set of points. The simplest example of this is reversing a regulus (i.e. replacing a regulus by its opposite regulus) in a regular spread, thereby obtaining a spread corresponding to a Hall plane (see [4]). Reversing each regulus of a linear set of (pairwise disjoint) reguli generates a spread corresponding to a two-dimensional André plane, while simultaneously reversing the reguli in any set of disjoint reguli yields what is called a *subregular spread* (see [4]).

In what follows, $q \geqslant 5$ will always denote an odd prime power. Following the terminology established in [8], if **S** is any spread of Σ, we say that **S** admits a *regular elliptic cover* provided that **S** contains $q-1$ pairwise disjoint reguli (partitioning all but two fixed lines of **S**). Clearly, the regular spread as well as any André spread admits a regular elliptic cover. One motivating force behind this paper is the search for other spreads that admit such covers.

Let Ω be a regular spread of Σ. A *t-nest* of Ω is defined to be any collection N of t reguli in Ω such that each line of Ω is contained in exactly 0 or 2 reguli of N. Let U denote the set of lines contained in the reguli of the t-nest N. Then U (or N) is called *replaceable* iff there exists a partial spread V of Σ covering the same points as U and having no line in common with U. In [2] and [3], replaceable t-nests were constructed for $t=q$ and $t=q-1$. We shall soon see that replaceable $(q+1)$-nests exist for all odd prime powers q, and the resulting spreads all admit regular elliptic covers.

Let β denote a primitive element of $GF(q^2)$, $w=\beta^{q+1}$ a primitive element of $GF(q)$, $\varepsilon=\beta^{(q+1)/2}$, and $\alpha=\beta^{q-1}$. We now define some lines of Σ by using co-ordinates of the underlying four-dimensional vector space. In particular, let $l_\infty=\langle(1,0,0,0),\ (0,1,0,0)\rangle$ and $l_{(x,y)}=\langle(x,wy,1,0),\ (y,x,0,1)\rangle$ for any $x,y\in GF(q)$. Then, as shown in [4], $\Omega=\{l_\infty\}\cup\{l_{(x,y)}: x,y\in GF(q)\}$ is a regular spread of Σ. Now let b denote any element of $GF(q)$ such that $b+1$ is a non-square of $GF(q)$, and as in [7], let $\mathbf{R}_0$ denote the regulus of Ω corresponding to the circle

$$D_0=\begin{pmatrix}1 & 1\\ b & -1\end{pmatrix}$$

in the miquelian inversive plane $M(q)$. Using the Bruck [4] correspondence between the points and circles of $M(q)$ and the lines and reguli of Ω, it is easy to compute that $\mathbf{R}_0=\{l_{(x,y)}: bx^2-2x-1=wby^2\}$ (see [3] or [7]).

Now let $a=(\alpha+\alpha^{-1})/2$ and $c=(\alpha-\alpha^{-1})/2\varepsilon$. Since $\alpha^q=\alpha^{-1}$ and $\varepsilon^q=-\varepsilon$, we see that a and c are (non-zero) elements of $GF(q)$. Let T_1 denote the collineation of Σ induced by the matrix

$$\begin{bmatrix} a & c & 0 & 0\\ wc & a & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

acting on column vectors. Of course, matrices which are scalar multiples of one another induce the same collineation of Σ. A straightforward computation shows that T_1: $\langle(x,wy,1,0)\rangle\to\langle(\bar{x},w\bar{y},1,0)\rangle$ and T_1: $\langle(y,x,0,1)\rangle\to\langle(\bar{y},\bar{x},0,1)\rangle$, where $\bar{x}=ax+wcy$ and $\bar{y}=cx+ay$. If Θ denotes the collineation of $M(q)$ given by Θ: $z\to\alpha z$ for z in $GK(q^2)\cup\{\infty\}$ (see [7]), then expressing each element z of $GF(q^2)$ uniquely as $x+y\varepsilon$ for $x,y\in GF(q)$, we see that Θ: $x+y\varepsilon\to\bar{x}+\bar{y}\varepsilon$. Since T_1: $l_{(x,y)}\to l_{(\bar{x},\bar{y})}$ and T_1: $l_\infty\to l_\infty$, we see that T_1 is a collineation of Ω that is a pre-image of Θ under the natural map $\mathrm{Aut}(\Omega)\to\mathrm{Aut}(M(q))$. Here we are explicitly using the Bruck correspondence $x + y\varepsilon \leftrightarrow l_{(x,y)}$. As shown in [7], the orbit $N = \{\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_q\}$ of $\mathbf{R}_0$ under T_1 (or equivalently the orbit of D_0 under Θ) is a $(q+1)$-nest. Moreover, it is easily computed that $\mathbf{R}_i = \{l_{(x,y)}: bx^2 - a_i x - 1 = wby^2 - wc_i y\}$, where $a_i = \alpha^i + \alpha^{-i}$ and $c_i = (\alpha^i - \alpha^{-i})/\varepsilon$. In particular, $a_0 = 2$, $c_0 = 0$, $a_1 = 2a$, and $c_1 = 2c$.

As a final piece of notation in this preliminary section, we let T_2 denote the collineation of Σ induced by the matrix

$$\begin{bmatrix} s & w^{-1}t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & w^{-1}t \\ 0 & 0 & t & s \end{bmatrix}$$

acting on column vectors, where s and t are elements of $GF(q)$ chosen so that the minimal polynomial of β over $GF(q)$ has the form $f(x) = (x-s)^2 - w^{-1}t^2$ (see [2]). Then, as shown in [2], T_2 is a collineation of order $q+1$ leaving invariant every line of the regular spread Ω by permuting its points in a cycle of length $q+1$. It is also trivial to check that T_2 commutes with T_1. Finally, observe that

$$\det\left(\begin{bmatrix} s & w^{-1}t \\ t & s \end{bmatrix}\right) = s^2 - w^{-1}t^2 = \beta\beta^q = w$$

is a non-square of $GF(q)$.

3. The Replacement Question

Let U denote the set of lines in the reguli of a $(q+1)$-nest N constructed as in the previous section. Clearly, $|U| = (q+1)^2/2$ as each line of U is contained in exactly two reguli of N. As shown in [7], U can also be realized as the line set of $(q+1)/2$ (necessarily disjoint) reguli in a partial linear set of reguli in Ω with carriers $l_{(0,0)}$ and l_∞. Hence, U can certainly be replaced by simultaneously reversing each regulus in this partial linear set, thereby obtaining an André spread. In this section, we show that U can also be replaced with 'opposite half-reguli', one for each regulus of the original nest. We will always let $\mathbf{R}^{\text{opp}}$ denote the opposite regulus to the regulus $\mathbf{R}$.

Lemma 1. *Let $q \geq 5$ denote an odd prime power. Using the notation established in the previous section, let L denote any line of $\mathbf{R}_0^{\text{opp}}$. Then the orbit of L under T_1 is a collection of $q+1$ skew lines, one from $\mathbf{R}_i^{\text{opp}}$ for $i = 0, 1, 2, \ldots, q$.*

Proof. Since T_1 is a collineation of Σ of order $q+1$, it suffices to show that $L \cap T_1^k(L) = \varnothing$ for $k = 1, 2, 3, \ldots, q$. By way of contradiction, suppose that P is a point of Σ incident with both L and $T_1^k(L)$ for some integer k, where $1 \leq k \leq q$. Then P lies on some line $l_{(x,y)}$ of $\mathbf{R}_0$, where $x, y \in GF(q)$ with $bx^2 - 2x - 1 = wby^2$. Since $b+1$ is a non-square of $GF(q)$, a discriminant argument shows that $y \neq 0$. Using an appropriate power of the collineation T_2, we may assume without loss of generality that $P = \langle (x, wy, 1, 0) \rangle$. Then L is the unique line of $\mathbf{R}_0^{\text{opp}}$ passing through P, and a straightforward computation shows that $L = \langle (x, wy, 1, 0), (x+1, wy, bx-1, wby) \rangle$. Moreover, computations similar to those given in Section 2 show that a matrix representation for T_1^k looks like

$$\begin{bmatrix} \tilde{a} & \tilde{c} & 0 & 0 \\ w\tilde{c} & \tilde{a} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\tilde{a} = (\alpha^k + \alpha^{-k})/2$ and $\tilde{c} = (\alpha^k - \alpha^{-k})/2\varepsilon$.

Since P lies on $T_1^k(L)$, $P = T_1^k(Q)$ for some point Q on L. Either $Q = P$ or $Q = \langle \lambda(x, wy, 1, 0) + (x+1, wy, bx-1, wby)\rangle$ for some $\lambda \in GF(q)$. In the first case, we obtain

$$\begin{bmatrix} \tilde{a} & \tilde{c} & 0 & 0 \\ w\tilde{c} & \tilde{a} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ wy \\ 1 \\ 0 \end{bmatrix} \sim \begin{bmatrix} x \\ wy \\ 1 \\ 0 \end{bmatrix},$$

where $\sim$ denotes that the two column vectors are $GF(q)$-scalar multiples of one another. This implies that

$$\begin{Bmatrix} (\tilde{a}-1)x + w\tilde{c}y = 0 \\ \tilde{c}x + (\tilde{a}-1)y = 0 \end{Bmatrix}.$$

Solving simultaneously and using the fact that $y \neq 0$ as above, we obtain $w\tilde{c}^2 = (\tilde{a}-1)^2$. Since w is a non-square in $GF(q)$, this forces $\tilde{c} = 0$ and hence $\tilde{a} = 1$. Hence T_1^k is the identity collineation, contradicting $1 \leq k \leq q$. In the second case, we similarly obtain

$$\begin{bmatrix} \lambda + bx - 1 \\ wby \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

contradicting the facts that $w \neq 0$, $y \neq 0$, and $b \neq 0$. □

THEOREM 2. *Let $q \geq 5$ denote an odd prime power, and let U denote the line set of a $(q+1)$-nest $N = \{\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_q\}$ constructed as above. Then a replacement partial spread V for U in Ω exists which consists of $(q+1)/2$ lines from $\mathbf{R}_i^{\text{opp}}$ for $i = 0, 1, 2, \ldots, q$. Such a V can be viewed as the union of $(q+1)/2$ T_1-orbits of size $q+1$ or as $q+1$ T_2^2-orbits of size $(q+1)/2$.*

PROOF. If L denotes any line of $\mathbf{R}_0^{\text{opp}}$, Lemma 1 implies that the orbit of L under T_1 consists of $q+1$ skew lines, one from each R_i^{opp}. Letting T_2^2 act on such an orbit, we obtain $(q+1)/2$ T_1-orbits containing $(q+1)^2/2$ lines of $\bigcup_{i=0}^q \mathbf{R}_i^{\text{opp}}$. Since $|U| = (q+1)^2/2$ and $U = \bigcup_{i=0}^q \mathbf{R}_i$, it suffices to show that no two of the lines generated as above meet. Since T_1 and T_2 commute, the generated line set V may be viewed either as $(q+1)/2$ T_1-orbits or $q+1$ T_2^2-orbits.

By way of contradiction, suppose that $T_2^k T_1^i(L) \cap L \neq \varnothing$ for some integers i and k, where $1 \leq i$, $k \leq q$ and k is even. (Note that $k \neq 0$ by Lemma 1 and $i \neq 0$ by definition of T_2.) That is, suppose there is a point P on L such that $P = T_2^k(Q)$ for some point Q on $T_1^i(L)$. Now P lies on some line $l_{(x,y)}$ of $\mathbf{R}_0$. As in the proof of Lemma 1, $y \neq 0$ and we may assume that $P = \langle (x, wy, 1, 0)\rangle$ and hence $L = \langle (x, wy, 1, 0), (x+1, wy, bx-1, wby)\rangle$. Now $Q = T_1^i(R)$ for some R on L. But R also lies on $l_{(\bar{x},\bar{y})}$ for some line $l_{(\bar{x},\bar{y})}$ of $\mathbf{R}_0$. Since $P = T_2^k(Q)$ and T_2 leaves invariant every line of Ω, Q lies on $l_{(x,y)}$. Thus $T_1^i\colon l_{(\bar{x},\bar{y})} \to l_{(x,y)}$. Previous computations now imply that $x = \tilde{a}\bar{x} + w\tilde{c}\bar{y}$ and $y = \tilde{c}\bar{x} + \tilde{a}\bar{y}$, where $\tilde{a} = (\alpha^i + \alpha^{-i})/2$ and $\tilde{c} = (\alpha^i - \alpha^{-i})/2\varepsilon$. Solving simultaneously, we obtain $x + y\varepsilon = \alpha^i(\bar{x} + \bar{y}\varepsilon)$ and $x - y\varepsilon = \alpha^{-i}(\bar{x} - \bar{y}\varepsilon)$. Hence, $x^2 - wy^2 = \bar{x}^2 - w\bar{y}^2$. Since $l_{(x,y)}$ and $l_{(\bar{x},\bar{y})}$ are both lines of $\mathbf{R}_0$, this implies that $(2x+1)/b = (2\bar{x}+1)/b$ and hence $\bar{x} = x$. Therefore, $\bar{y}^2 = y^2$, implying $\bar{y} = -y$ as $i \neq 0$. Thus, $R = L \cap l_{(\bar{x},\bar{y})} = L \cap l_{(x,-y)} = \langle (x+1, wy, bx-1, wby)\rangle$, the last equality holding by a direct computation.

As $Q \neq P$ and Q lies on $l_{(x,y)}$, we may write $Q = \langle (y, x, 0, 1) + \lambda(x, wy, 1, 0)\rangle$ for some $\lambda \in GF(q)$. A straightforward induction shows that the matrix representation for

T_2^k has the form

$$\begin{bmatrix} e & w^{-1}f & 0 & 0 \\ f & e & 0 & 0 \\ 0 & 0 & e & w^{-1}f \\ 0 & 0 & f & e \end{bmatrix}$$

for some $e, f \in GF(q)$. Since $k \neq 0$, we necessarily have $f \neq 0$. Now, $T_2^k(Q) = P$ implies that $f\lambda + e = 0$. Moreover, since k is even, a simple determinant argument shows that $e^2 - w^{-1}f^2$ must be a non-zero square in $GF(q)$. Substituting $e = -f\lambda$ and using the fact that w is a non-square, we see that $w\lambda^2 - 1$ must be a non-square in $GF(q)$. But $Q = T_1^i(R)$ implies that $\langle(y + \lambda x, x + \lambda wy, \lambda, 1)\rangle = \langle(\bar{a}(x+1) + \bar{c}wy, \bar{c}w(x+1) + \bar{a}wy, bx - 1, wby)\rangle$ and hence $\lambda = (bx-1)/wby$. Thus $w\lambda^2 - 1 = w[(bx-1)^2 - wb^2y^2]/w^2b^2y^2 = (b+1)/wb^2y^2$ since $l_{(x,y)} \in \mathbf{R}_0$ implies that $wby^2 = bx^2 - 2x - 1$. Hence, $w\lambda^2 - 1$ is a square in $GF(q)$ as $b+1$ and w are both non-squares, contradicting the previous assertion on its quadratic character. This proves the result. □

Using the notation of Theorem 2, it was pointed out in [7] that there is a companion $(q+1)$-nest $\bar{N} = \{\bar{\mathbf{R}}_0, \bar{\mathbf{R}}_1, \bar{\mathbf{R}}_2, \ldots, \bar{\mathbf{R}}_q\}$ to N, the line set of which is also U. An argument analogous to that given in Theorem 2 shows that $\bar{N}$ is also replaceable. That is, there exists a replacement set $\bar{V}$ for U consisting of $(q+1)/2$ lines from $\bar{\mathbf{R}}_i^{\text{opp}}$ for $i = 0, 1, 2, \ldots, q$. Once again, $\bar{V}$ can be viewed as the union of $(q+1)/2$ T_1-orbits or $q+1$ T_2^2-orbits (see [7]) for examples). Thus, we have the following result.

COROLLARY 3. *Let $q \geq 5$ denote an odd prime power, and let $b \in GF(q)$ be chosen so that $b+1$ is a non-square in $GF(q)$. Let U denote the line set of a $(q+1)$-nest generated as in Section 2. The U is replaceable in three ways. The resulting spreads are projectively inequivalent, in general, one being an André spread.*

It should be noted that choosing another $b \in GF(q)$ will in general yield different spreads. Once again, see [7] for examples.

4. REGULI AND COVERS

In this section, we discuss the reguli contained in a spread $\mathbf{S} = (\Omega \backslash U) \cup V$ obtained by replacing the line set U of a $(q+1)$-nest as indicated in Theorem 2. As pointed out in [7], the lines of $\Omega \backslash U$ are partitioned into two fixed lines, namely $l_{(0,0)}$ and l_∞, and $(q-3)/2$ orbits of size $q+1$ under T_1. These orbits are, in fact, reguli in a (partial) linear set of Ω with carriers $l_{(0,0)}$ and l_∞. Thus, $\mathbf{S}$ contains at least $(q-3)/2$ reguli. The following results are concerned with reguli contained in V. First, we need a technical lemma.

LEMMA 4. *Let β be a primitive element of $GF(q^2)$, $\alpha = \beta^{q-1}$, and $w = \beta^{q+1}$. Let $s, t \in GF(q)$ be chosen so that $f(x) = (x-s)^2 - w^{-1}t^2$ is the minimal polynomial of β over $GF(q)$. Let $a = (\alpha + \alpha^{-1})/2$ and $c = (\alpha - \alpha^{-1})/2\varepsilon$, and let*

$$M_1 = \begin{bmatrix} a & c \\ wc & a \end{bmatrix} \quad \textit{and} \quad M_2 = \begin{bmatrix} s & w^{-1}t \\ t & s \end{bmatrix}.$$

Then either $M_1M_2^2$ or $M_1M_2^{-2}$ is a diagonal 2×2 matrix.

PROOF. M_2 is invertible since w^{-1} is a non-square of $GF(q)$. By definition of the polynomial f, we have $\beta^2 = 2s\beta - (s^2 - w^{-1}t^2)$. However, since β and β^q are the two roots of f, we have $s^2 - w^{-1}t^2 = \beta\beta^q = w$ and $2s = \beta + \beta^q$. Thus, $\beta^2 = 2s\beta - w$ and, using the Frobenious automorphism of $GF(q^2)$, we have $\beta^{2q} = 2s\beta^q - w$. Therefore, $\beta^{2q} + \beta^2 = 2s(\beta + \beta^q) - 2w = 4s^2 - 2w$. Hence, letting $\varepsilon = \beta^{(q+1)/2}$ as before, $[(\beta^q - \beta)\varepsilon]^2 = (\beta^{2q} - 2w + \beta^2)w = 4w(s^2 - w) = 4t^2$. Since $\beta + \beta^q = 2s$, we have $1 + \alpha = 1 + \beta^{q-1} = 2s/\beta$ and therefore $1 + \alpha^{-1} = 1 + \alpha^q = 2s/\beta^q$. Subtracting these equations, we obtain $2c\varepsilon = \alpha - \alpha^{-1} = 2s(\beta^q - \beta)/w$ and hence $c^2w^4 = s(\beta^q - \beta)\varepsilon = 4s^2t^2$ from our previous calculation.

Using the fact that $\alpha = a + c\varepsilon$ and $\alpha^{-1} = a - c\varepsilon$, we have $\alpha^2 = wc^2 + 1$. Also, $s^2 - w^{-1}t^2 = w$ as above, and hence $ws^2 - t^2 = w^2$. A simple computation now shows that $[cw(s^2 + w^{-1}t^2) - 2ast][cw(s^2 + w^{-1}t^2) + 2ast] = 0$ iff $c^2w^4 = 4s^2t^2$, which is always true by our work in the previous paragraph. Since

$$M_1M_2^2 = \begin{bmatrix} a(s^2 + w^{-1}t^2) + 2cst & 2aw^{-1}st + c(s^2 + w^{-1}t^2) \\ cs(s^2 + w^{-1}t^2) + 2ast & 2cst + a(s^2 + w^{-1}t^2) \end{bmatrix}$$

and

$$M_1M_2^{-2} = w^{-2}\begin{bmatrix} a(s^2 + w^{-1}t^2) - 2cst & c(s^2 + w^{-1}t^2) - 2aw^{-1}st \\ cs(s^2 + w^{-1}t^2) - 2ast & a(s^2 + w^{-1}t^2) - 2cst \end{bmatrix},$$

it follows immediately that one of $M_1M_2^2$ and $M_1M_2^{-2}$ is necessarily a diagonal matrix. □

With a little more work, it can be shown that $M_1M_2^2$ is diagonal when $t = -(\beta^q - \beta)\varepsilon/2$, while $M_1M_2^{-2}$ is diagonal when $t = (\beta^q - \beta)\varepsilon/2$. However, we will not need this stronger result here.

THEOREM 5. *Let $q \geq 5$ denote an odd prime power, and let $\mathbf{S} = (\Omega \backslash U) \cup V$ denote a spread of Σ obtained by replacing the line set of a $(q+1)$-nest as indicated in Theorem 2. Then the $(q+1)^2/2$ lines of V can be partitioned in two different ways as $(q+1)/2$ (disjoint) reguli.*

PROOF. As shown in the proof of Theorem 2, V can be partitioned as $(q+1)/2$ orbits of size $q+1$ under T_1. We claim that each of these T_1-orbits is a regulus. Let $x, y \in GF(q)$ with $wby^2 = bx^2 - 2x - 1$, and let $L = \langle (x, wy, 1, 0), (x+1, wy, bx-1, wby) \rangle$. Then $L \in \mathbf{R}_0^{\text{opp}}$ as before. In order to show that the T_1-orbit of L is a regulus, we first consider a collineation T_0 of Σ induced by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ w & 1 & 0 & 0 \\ 0 & 0 & 1 & e \\ 0 & 0 & 1 & f \end{bmatrix},$$

acting on column vectors, where $e, f \in GF(q)$ with $e \neq f$. A straightforward computation shows that T_0: $L \to \langle (x + wy, wx + wy, 1, 1), (x + 1 + wy, wx + w + wy, bx - 1 + weby, bx - 1 + wfby) \rangle$. Recalling that $\Omega = \{l_\infty\} \cup \{l_{(\bar{x},\bar{y})}: \bar{x}, \bar{y} \in GF(q)\}$, where $l_{(\bar{x},\bar{y})} = \langle (\bar{x}, w\bar{y}, 1, 0), (\bar{y}, \bar{x}, 0, 1) \rangle$, another straightforward (but lengthy) computation shows that if we choose $e = -(2wy + 1)/w(2x + 1)$ and $f = -(2y + 1)/(2x + 1)$, then $T_0(L) \in \Omega$. It should be noted that $2x + 1 \neq 0$ since $wby^2 = bx^2 - 2x - 1$ and w is a non-square of $GF(q)$. It should also be remarked that $T_0(L) \neq l_{(0,0)}$ and $T_0(L) \neq l_\infty$.

Hence, $T_0(L) = l_{(\bar{x},\bar{y})}$ for some $\bar{x}, \bar{y} \in GF(q)$ (not both zero), and thus $T_1^i(T_0(L)) = l_{(\tilde{x},\tilde{y})}$, where $\tilde{x} + \tilde{y}\varepsilon = \alpha^i(\bar{x} + \bar{y}\varepsilon)$. Therefore, $\{T_1^i(T_0(L)): i = 0, 1, 2, \ldots, q\}$ is a regulus of Σ, namely the regulus corresponding to the circle $\{(\bar{x}+\bar{y}\varepsilon)\alpha^i: i=0, 1, 2, \ldots, q\}$ in the miquelian plane $M(q)$. But T_1 and T_0 commute, and hence the T_1-orbit of L is also a regulus. Since the other T_1-orbits comprising V in the proof of Theorem 2 are obtained by letting T_2^2 act on the T_1-orbit just discussed, we see that V is partitioned in this fashion into $(q+1)/2$ reguli.

We now claim that V can be partitioned in yet another way into $(q+1)/2$ reguli. Using the notation of Lemma 4, we let T denote $T_1T_2^2$ or $T_1T_2^{-2}$ accordingly as $M_1M_2^2$ or $M_1M_2^{-2}$ is diagonal. Thus, T is represented by a matrix of the form

$$\begin{bmatrix} \lambda I & 0 \\ 0 & A \end{bmatrix}$$

for some $0 \neq \lambda \in GF(q)$, where $A = M_2^2$ or $A = M_2^{-2}$. It is easy to see that any collineation $\bar{T}$ represented by a matrix of the form

$$\begin{bmatrix} 1 & e & 0 & 0 \\ 1 & f & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & w & 1 \end{bmatrix},$$

where $e, f \in GF(q)$ with $e \neq f$, commutes with T. Choosing $L \in \mathbf{R}_0^{\text{opp}}$ as above and proceeding as in the previous case, it is not hard to show that distinct elements e and f of $GF(q)$ may be chosen so that $\bar{T}(L) \in \Omega\backslash\{l_{(0,0)}, l_\infty\}$. Since T_2 leaves invariant each line of Ω, T is a collineation of Ω, the action of which on the lines of Ω is the same as that of T_1. Hence, an argument analogous to that given above shows that the orbit of L under T is some regulus $\mathbf{R}$, and this regulus is contained in V. Once again, $\mathbf{R}$ contains one line from $\mathbf{R}_i^{\text{opp}}$ for $i=0, 1, 2, \ldots, q$. Finally, it is easy to see that the images of $\mathbf{R}$ under powers of T_2^2 are also reguli contained in V, and, in fact, V is partitioned by these $(q+1)/2$ images. Clearly, the reguli in these two partitions of V are all distinct. □

COROLLARY 6. *Using the notation of Theorem* 5, $\mathbf{S} = (\Omega\backslash U) \cup V$ *contains at least* $(3q-1)/2$ *reguli.*

COROLLARY 7. *Using the notation of Theorem* 5, $\mathbf{S} = (\Omega\backslash U) \cup V$ *admits two different regular elliptic covers. These covers share* $(q-3)/2$ *reguli in a (partial) linear set of* Ω. *Given either of these two covers, there is some collineation of* $\mathbf{S}$ *(either* T_1 *or* T*) such that each regulus in the cover is a line orbit under this collineation.*

It should be noted that for all examples checked with $q \geq 7$, $\mathbf{S}$ contained exactly $(3q-1)/2$ reguli, and we conjecture this always to be true. See [7] for some interesting examples when $q=5$. We also believe that the only reguli contained in U are the $(q+1)/2$ reguli from the (partial) linear set with carriers $l_{(0,0)}$ and l_∞ as well as the $2(q+1)$ reguli from the $(q+1)$-nest N and its companion nest $\bar{N}$ (see the discussion following Theorem 2).

5. COLLINEATION GROUPS AND ORBIT STRUCTURE

Using the notation of Theorem 5, our previous work shows that T_1 and T_2^2 are collineations of $\mathbf{S} = (\Omega\backslash U) \cup V$ inherited from the collineation group of the regular spread Ω. Using these two collineations, $\mathbf{S}$ is partitioned into two fixed lines (namely, $l_{(0,0)}$ and l_∞), $(q-3)/2$ orbits of size $q+1$ (namely, the reguli of the partial linear set contained in $\Omega\backslash U$), and one orbit of size $(q+1)^2/2$ (namely, V). We now look for other collineations of $\mathbf{S}$.

LEMMA 8. *Let U denote the line set of a $(q+1)$-nest as described in Section 2. Then $U = \{l_{(x,y)}: [1-(x^2-wy^2)b]^2 - 4(x^2-wy^2)$ is a non-square in $GF(q)\}$.*

PROOF. If $N = \{\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_q\}$ denotes the $(q+1)$-nest, it was shown in Section 2 that $\mathbf{R}_i = \{l_{(x,y)}: bx^2 - a_ix - 1 = wby^2 - wc_iy\}$, where $a_i = \alpha^i + \alpha^{-i}$ and $c_i = (\alpha^i - \alpha^{-i})/\varepsilon$. So, if $l_{(x,y)} \in U$, then $l_{(x,y)} \in \mathbf{R}_i$ for some $i = 0, 1, 2, \ldots, q$, and hence $[1-(x^2-wy^2)b]^2 = w^2c_i^2y^2 - 2wa_ic_ixy + a_i^2x^2$ as above. Using $a_i^2 - wc_i^2 = 4$, one easily computes that $[1-(x^2-wy^2)b]^2 - 4(x^2-wy^2) = w(c_ix - a_iy)^2$ and hence is a non-square in $GF(q)$. It should be noted that $c_ix - a_iy \neq 0$ as otherwise $x^2 - wy^2 = (2 + b \pm 2\sqrt{1+b})/b^2$, contradicting the fact that $b+1$ is a non-square of $GF(q)$.

Conversely, suppose that $[1-(x^2-wy^2)b]^2 - 4(x^2-wy^2)$ is some non-square d in $GF(q)$. Let $\delta = \frac{1}{2}[1-(x^2-wy^2)b + \sqrt{d}]$. Since d is a non-square of $GF(q)$, $\delta \neq 0$ and $\delta^q = \frac{1}{2}[1-(x^2-wy^2)b - \sqrt{d}]$. Hence $\delta^{q+1} = x^2 - wy^2$ and $\delta + \delta^q = 1-(x^2-wy^2)b$. Next, let $\gamma = (x+y\varepsilon)/\delta$. Then $\gamma^q = (x-y\varepsilon)/\delta^q$ and $\gamma^{q+1} = 1$. Therefore, $\gamma = \alpha^i$ for some $i = 0, 1, 2, \ldots, q$, by definition of α. If we write $2\gamma = 2\alpha^i$ uniquely as $a_i + c_i\varepsilon$ for some $a_i, c_i \in GF(q)$, then $2\gamma^q = 2\alpha^{-i} = a_i - c_i\varepsilon$. Hence

$$\begin{aligned}1-(x^2-wy^2)b &= \delta + \delta^q = (x+y\varepsilon)\alpha^{-i} + (x-y\varepsilon)\alpha^i \\ &= \tfrac{1}{2}(x+y\varepsilon)(a_i - c_i\varepsilon) + \tfrac{1}{2}(x-y\varepsilon)(a_i + c_i\varepsilon) = a_ix - 2c_iy\end{aligned}$$

and $l_{(x,y)} \in \mathbf{R}_i \subseteq U$ as described above. □

THEOREM 9. *Let $q \geq 5$ denote an odd prime power, and let $\mathbf{S} = (\Omega\backslash U) \cup V$ be a spread of Σ obtained by replacing a $(q+1)$-nest, as described in Theorem 2. Let T_3 denote the collineation of Σ induced by the matrix*

$$\begin{bmatrix} 0 & -w^{-1} & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -w^{-1} \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

acting on column vectors. Then either T_3 or T_2T_3 is an involutionary collineation of $\mathbf{S}$ *inherited from* $\mathrm{Aut}(\Omega)$ *accordingly as $w-1$ is a square or a non-square of $GF(q)$.*

PROOF. Straightforward computations show that $T_3: l_\infty \to l_\infty$ and $T_3: l_{(x,y)} \to l_{(x,-y)}$ for any $x, y \in GF(q)$. Thus $T_3 \in \mathrm{Aut}(\Omega)$. Moreover, $T_3: U \to U$ by Lemma 8, and hence $T_3: \Omega\backslash U \to \Omega\backslash U$. It may easily be checked that $T_3^2 = I$, $T_3T_2 = T_2^{-1}T_3$, and $T_3T_1 = T_1^{-1}T_3$. In particular, both T_3 and T_2T_3 are involutions in $\mathrm{Aut}(\Omega)$. Of course, $T_2: \Omega\backslash U \to \Omega\backslash U$ as T_2 leaves invariant each line of Ω. Thus, it only remains to check the action of T_3 and T_2T_3 on the lines of V.

Let $x, y \in GF(q)$ with $bx^2 - 2x - 1 = wby^2$. Then $l_{(x,y)} \in \mathbf{R}_0$ and $L = \langle (x, wy, 1, 0), (x+1, wy, bx-1, wby)\rangle \in \mathbf{R}_0^{\mathrm{opp}}$, as before. As in the proof of Theorem 2, we may take $V = \{T_1^iT_2^{2j}(L): i = 0, 1, 2, \ldots, q \text{ and } j = 0, 1, 2, \ldots, (q-1)/2\}$, where T_1 and T_2 commute. Since $T_3(T_1^iT_2^{2j}(L)) = T_1^{-i}T_2^{-2j}T_3(L)$, $T_1: V \to V$, and $T_2^2: V \to V$, we see that $T_3: V \to V$ (and hence $T_3 \in \mathrm{Aut}(\mathbf{S})$) iff $T_3(L) \in V$. We now determine when $T_3(L) \in V$. As $T_3: l_{(x,y)} \to l_{(x,-y)}$ and $\mathbf{R}_0 = \{l_{(x,y)}: bx^2 - 2x - 1 = wby^2\}$, $T_3: \mathbf{R}_0 \to \mathbf{R}_0$ and hence $T_3: \mathbf{R}_0^{\mathrm{opp}} \to \mathbf{R}_0^{\mathrm{opp}}$. Therefore, $T_3(L) = T_2^k(L)$ for some $k = 0, 1, 2, \ldots, q$ by definition of

T_2. Since T_2^2: $V \to V$, $T_3(L) \in V$, provided that k is even. Representing T_2^k by the matrix

$$\begin{bmatrix} e & w^{-1}f & 0 & 0 \\ f & e & 0 & 0 \\ 0 & 0 & e & w^{-1}f \\ 0 & 0 & f & e \end{bmatrix},$$

as in the proof of Theorem 2, the usual determinant argument shows that k is even iff $e^2 - w^{-1}f^2$ is a square in $GF(q)$.

In order to simplify the notation, let P and Q denote the projective points $\langle(x, wy, 1, 0)\rangle$ and $\langle(x+1, wy, bx-1, wby)\rangle$, respectively. Then $P = L \cap l_{(x,y)}$ and $Q = L \cap l_{(x,-y)}$. Since T_3: $l_{(x,y)} \to l_{(x,-y)}$ and T_2^k: $l_{(x,-y)} \to l_{(x,-y)}$, $T_3(L) = T_2^k(L)$ implies that $T_3(P) = T_2^k(Q)$. That is, $\langle(x-y, x-wy, 1, 1)\rangle = \langle(ex+e+y, x+1+wey, ebx - e + by, bx - 1 + weby)\rangle$. In particular, this forces $e = f(bx - by - 1)/(bx - wby - 1)$. Note that if $bx - wby - 1 = 0$, then $wby = by$ and $b = 0$ or $y = 0$, both of which are contradictions as before. A straightforward computation now shows that $e^2 - w^{-1}f^2 = f^2[b^2(1-w^{-1})x^2 - 2b(1-w^{-1})x + (1-w^{-1}) + b^2(1-w)y^2]/(bx - wby - 1)^2$. Using the fact that $wby^2 = bx^2 - 2x - 1$ to rewrite the last term of the numerator, one easily obtains that $e^2 - w^{-1}f^2$ and $(1-w^{-1})(b+1)$ have the same quadratic character. That is, since $b+1$ and w are both non-squares of $GF(q)$, $e^2 - w^{-1}f^2$ is a (non-zero) square of $GF(q)$ iff $w-1$ is a (non-zero) square of $GF(q)$. Thus, by our previous work, we have that $T_3 \in \mathrm{Aut}(\mathbf{S})$ provided that $w-1$ is a square in $GF(q)$.

Finally, suppose that $w-1$ is a non-square in $GF(q)$. Then the above paragraph shows that $T_3(L) = T_2^{2n+1}(L)$ for some integer n. Now consider the collineation $T_2T_3 = T_3T_2^{-1}$. Letting $T_1^iT_2^{2j}(L)$ again denote an arbitrary line in V, we have $T_2T_3(T_1^iT_2^{2j}(L)) = T_1^{-i}T_2^{-2j}T_2T_3(L) = T_1^{-i}T_2^{-2j}T_2^{2n+2}(L) \in V$. Arguing as above, we see that $T_2T_3 \in \mathrm{Aut}(\mathbf{S})$ iff $w-1$ is a non-square in $GF(q)$. The result now follows. □

THEOREM 10. *Using the same notation as in Theorem 9, let T_4 denote the collineation of Σ induced by the matrix*

$$\begin{bmatrix} 0 & 0 & b^{-1} & w^{-1}b^{-1} \\ 0 & 0 & b^{-1} & b^{-1} \\ 1 & w^{-1} & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

acting on column vectors. Then either T_4 or T_2T_4 is a collineation of **S** *inherited from* Aut(Ω) *accordingly as $1 - w$ is a square or a non-square in $GF(q)$.*

PROOF. Straightforward computations show that T_4: $l_{(0,0)} \leftrightarrow l_\infty$ and T_4: $l_{(x,y)} \to l_{(\bar{x},\bar{y})}$, where $\bar{x} = x/b(x^2 - wy^2)$ and $\bar{y} = -y/b(x^2 - wy^2)$. Hence, $T_4 \in \mathrm{Aut}(\Omega)$. Since $\bar{x}^2 - w\bar{y}^2 = 1/b^2(x^2 - wy^2)$, an application of Lemma 8 shows that T_4: $U \to U$ and hence T_4: $\Omega \backslash U \to \Omega \backslash U$. Thus $T_4 \in \mathrm{Aut}(\mathbf{S})$ iff T_4: $V \to V$. Once again, if L denotes any line of $\mathbf{R}_0^{\mathrm{opp}}$, we may take $V = \{T_1^iT_2^{2j}(L)$: $i = 0, 1, 2, \ldots, q$ and $j = 0, 1, 2, \ldots, (q-1)/2\}$. We proceed in a fashion somewhat analogous to the proof of Theorem 9, but with a few added wrinkles.

We first define $\bar{T}_1$ to be the collineation of Σ induced by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & wc & a \end{bmatrix},$$

where $a = (\alpha + \alpha^{-1})/2$ and $c = (\alpha - \alpha^{-1})/2\varepsilon$ as previously defined. Then, it is easy to compute that $T_4 T_1 T_4^{-1} = \bar{T}_1$, $T_1\bar{T}_1 = \bar{T}_1 T_1$, $T_2\bar{T}_1 = \bar{T}_1 T_2$, and also $T_2 T_4 = T_4 T_2$. Hence $T_4(T_1^i T_2^{2j}(L)) = \bar{T}_1^i T_2^{2j} T_4(L)$. That is, T_4: $V \to V$ (and hence $T_4 \in \mathrm{Aut}(\mathbf{S})$) provided that $T_4(L) \in V$ and $\bar{T}_1$: $V \to V$. Again, letting $T_1^i T_2^{2j}(L)$ denote an arbitrary line in V, we have $\bar{T}_1(T_1^i T_2^{2j}(L)) = T_1^i T_2^{2j} \bar{T}_1(L)$. Hence, to show $\bar{T}_1$: $V \to V$, it suffices to show that $\bar{T}_1(L) \in V$. Summarizing the above work, we have that $T_4 \in \mathrm{Au}(\mathbf{S})$ provided that both $T_4(L)$ and $\bar{T}_1(L)$ are lines of V for some fixed line L of $\mathbf{R}_0^{\mathrm{opp}}$. It should also be observed that $\bar{T}_1$: $l_{(x,y)} \to l_{(x',y')}$, where $x' = ax - wcy$ and $y' = ay - cx$, and hence $\bar{T}_1$ and T_1^{-1} have the same action on lines of Ω (see previous computations concerning T_1 in Section 2).

Let $x, y \in GF(q)$ with $bx^2 - 2x - 1 = wby^2$, let $P = \langle (x, wy, 1, 0) \rangle$, and let $Q = \langle (x+1, wy, bx-1, wby) \rangle$. Then $l_{(x,y)} \in \mathbf{R}_0$ and $L = \langle P, Q \rangle \in \mathbf{R}_0^{\mathrm{opp}}$ as before. We now show that $\bar{T}_1(L) \in V$. As T_1^{-1}: $\mathbf{R}_0 \to \mathbf{R}_q$ and $\bar{T}_1$ acts like T_1^{-1} on lines of Ω, $\bar{T}_1$: $\mathbf{R}_0 \to \mathbf{R}_q$ and hence $\bar{T}_1(L)$ and $T_1^{-1}(L)$ are both lines of $\mathbf{R}_q^{\mathrm{opp}}$. Thus, T_2^k: $\bar{T}_1(L) \to T_1^{-1}(L)$ for some integer k. In particular, T_2^k: $\bar{T}_1(P) \to T_1^{-1}(P)$ as $T_1^{-1}(l_{(x,y)}) = \bar{T}_1(l_{(x,y)})$. Since T_1^{-1} may be represented by the matrix

$$\begin{bmatrix} a & -c & 0 & 0 \\ -wc & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

this implies that T_2^k: $\langle (x, wy, a, wc) \rangle \to \langle (ax - wcy, way - wcx, 1, 0) \rangle$ and hence $f = -wcea^{-1}$, where we represent T_2^k as in the proof of Theorem 9. This, in turn, implies that $e^2 - w^{-1}f^2 = e^2(a^2 - wc^2)/a^2 = e^2/a^2$. That is, $e^2 - w^{-1}f^2$ is always a square in $GF(q)$, and therefore the usual determinant argument shows that k is necessarily even. Hence, since $T_1^{-1}(L) \in V$ and T_2^2: $V \to V$, T_2^k: $\bar{T}_1(L) \to T_1^{-1}(L)$ implies that $\bar{T}_1(L) \in V$.

We now address the question of whether $T_4(L)$ is a line of V. A tedious computation shows that T_4: $\mathbf{R}_0 \to \mathbf{R}_{(q+1)/2}$. Thus $T_4(L)$ and $T_1^{(q+1)/2}(L)$ are lines of $\mathbf{R}_{(q+1)/2}^{\mathrm{opp}}$, and therefore T_2^k: $T_1^{(q+1)/2}(L) \to T_4(L)$ for some integer k. Once again, $T_4(L)$ will be a line of V provided that k is even. It is easy to check that $T_1^{(q+1)/2}$ and T_4^{-1} may be represented by the matrices

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & b^{-1} & -w^{-1}b^{-1} \\ 0 & 0 & -b^{-1} & b^{-1} \\ 1 & -w^{-1} & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix},$$

respectively. Now $T_1^{(q+1)/2}$: $l_{(x,y)} \to l_{(-x,-y)}$ and T_4^{-1}: $l_{(-x,-y)} \to l_{(-x/(2x+1),y/(2x+1))} \in \mathbf{R}_0$, using the fact that $wby^2 = bx^2 - 2x - 1$ in the latter computation. One then shows directly that $R = L \cap l_{(-x/(2x+1),y/(2x+1))} = \langle (0, -b^{-1}, y, x) \rangle$, and thus $T_4(R) = \langle (wy + x, wy + wx, -1, -w) \rangle$. Since $T_4(R) = T_4(L) \cap l_{(-x,-y)}$ and T_2^k: $T_1^{(q+1)/2}(L) \to T_4(L)$, we have T_2^k: $T_1^{(q+1)/2}(P) \to T_4(R)$. Computing $T_1^{(q+1)/2}(P) = \langle (-x, -wy, 1, 0) \rangle$ and representing T_2^k as in the above paragraph, one easily obtains that $we = f$ and hence $e^2 - w^{-1}f^2 = (1 - w)e^2$. The usual determinant argument now shows that k is even iff $1 - w$ is a square in $GF(q)$. Therefore, by our work above, we have that $T_4(L) \in V$ and thus $T_4 \in \mathrm{Aut}(\mathbf{S})$ provided that $1 - w$ is a square in $GF(q)$.

Finally, if $1 - w$ is a non-square in $GF(q)$, then $T_4(L) = T_2^{2n+1} T_1^{(q+1)/2}(L)$ for some integer n as in the above paragraph. Hence $T_2 T_4(L) = T_2^{2n+2} T_1^{(q+1)/2}(L) \in V$, and an analogous argument shows that $T_2 T_4 \in \mathrm{Aut}(\mathbf{S})$ in this case. □

COROLLARY 11. *Assume the same notation as in Theorems* 9 *and* 10:
(i) *If* $q \equiv 1 \pmod 4$, *let* $G = \langle T_1, T_2^2, T_3, T_4\rangle$ *or* $G = \langle T_1, T_2^2, T_2T_3, T_2T_4\rangle$ *accordingly as* $w-1$ *is a square or a non-square of* $GF(q)$. *Then* G *is a collineation group of* **S** *inherited from* Aut(Ω) *that partitions* **S** *into one orbit of size* 2, *one orbit of size* $(q+1)^2/2$, *one orbit of size* $q+1$, *and* $(q-5)/4$ *orbits of size* $2(q+1)$.
(ii) *If* $q \equiv 3 \pmod 4$, *let* $G = \langle T_1, T_2^2, T_3, T_2T_4\rangle$ *or* $G = \langle T_1, T_2^2, T_2T_3, T_4\rangle$ *accordingly as* $w-1$ *is a square or a non-square of* $GF(q)$. *Then* G *is a collineation group of* **S** *inherited from* Aut(Ω) *that partitions* **S** *into one orbit of size* 2, *one orbit of size* $(q+1)^2/2$, *and* $(q-3)/4$ *orbits of size* $2(q+1)$ *if* b *is a square of* $GF(q)$, *while* G *partitions* **S** *into one orbit of size* 2, *one orbit of size* $(q+1)^2/2$, *two orbits of size* $q+1$, *and* $(q-7)/4$ *orbits of size* $2(q+1)$ *if* b *is a non-square of* $GF(q)$.

PROOF. The fact that G is a collineation group of **S** inherited from Aut(Ω) in all cases follows from Theorems 9 and 10, the discussion immediately preceding Theorem 9, and the fact that -1 is a square in $GF(q)$ iff $q \equiv 1 \pmod 4$. Similarly, previous work implies that **S** will have one G-orbit of size $(q+1)^2/2$ (namely, V) and one G-orbit of size 2 (namely, $\{l_{(0,0)}, l_\infty\}$). The only remaining question is how G acts on the $(q-3)/2$ reguli of the partial linear set with carriers $l_{(0,0)}$ and l_∞ which comprise the lines of $(\Omega\backslash U)\backslash\{l_{(0,0)}, l_\infty\}$.

A simple calculation (see [7], for instance) shows that the reguli of Ω in the linear set with carriers $l_{(0,0)}$ and l_∞ have the form $\mathbf{R}_a = \{l_{(x,y)}: x^2 - wy^2 = a\}$, one for each non-zero element a of $GF(q)$. From Lemma 8, we know that $\mathbf{R}_a \subseteq \Omega\backslash U$ iff $(1-ab)^2 - 4a$ is a (non-zero) square of $GF(q)$. Note that $(1-ab)^2 - 4a \neq 0$ from a standard discriminant argument and the fact that $b+1$ is a non-square. Also, from our earlier discussion, T_1: $\mathbf{R}_a \to \mathbf{R}_a$ for each $0 \neq a \in GF(q)$ by permuting the lines of $\mathbf{R}_a$ in a cycle of length $q+1$, and T_2 leaves invariant each line of Ω. Thus each of the $(q-3)/2$ $\mathbf{R}_a$'s contained in $\Omega\backslash U$ is a $\langle T_1, T_2^2\rangle$-orbit of **S**, and we only need decide if T_3 and/or T_4 combine any of these orbits. In fact, since T_3: $l_{(x,y)} \to l_{(x,-y)}$, T_3: $\mathbf{R}_a \to \mathbf{R}_a$ for all $0 \neq a \in GF(q)$, and we may concentrate our efforts on the action of T_4.

The proof of Theorem 10 shows that T_4: $\mathbf{R}_a \to \mathbf{R}_{a^{-1}b^{-2}}$. In particular, T_4 leaves invariant the reguli $\mathbf{R}_{b^{-1}}$ and $\mathbf{R}_{-b^{-1}}$ while pairing off the remaining reguli in the (complete) linear set. Thus, it only remains to be seen when $\mathbf{R}_{b^{-1}}$ and $\mathbf{R}_{-b^{-1}}$ are contained in $\Omega\backslash U$. In this regard, note that $(1-ab)^2 - 4a = -4b^{-1}$ if $a = -b^{-1}$.

Suppose first that $q \equiv 1 \pmod 4$, and thus -1 is a square in $GF(q)$. Recalling that $b+1$ is a non-square in $GF(q)$, our work in the above paragraph now implies that precisely one of $\{\mathbf{R}_{b^{-1}}, \mathbf{R}_{-b^{-1}}\}$ will be contained in $\Omega\backslash U$. Hence G will partition **S** as indicated in part (i) of the theorem. On the other hand, if $q \equiv 3 \pmod 4$, then -1 is a non-square in $GF(q)$. In this case, both $\mathbf{R}_{b^{-1}}$ and $\mathbf{R}_{-b^{-1}}$ will be contained in $\Omega\backslash U$ if b is a non-square, while neither regulus will be contained in $\Omega\backslash U$ if b is a square. The result now follows. □

It is not known if G is the full collineation group of **S** or even if the full collineation group of **S** is inherited from Aut(Ω). The computational group theory package CAYLEY has been used to examine several examples for $q = 5$ and $q = 7$, and a fairly sophisticated pruning algorithm was developed to show that $G = \text{Aut}(\mathbf{S}) \cap \text{Aut}(\Omega)$ in all these examples. The author believes this always to be true. Finally, it should be noted that $|G| = 2(q+1)^2$ in all cases.

6. CONCLUDING REMARKS

In this paper, an infinite family of (new) two-dimensional translation planes has been constructed by replacing $(q+1)$-nests of reguli in a regular spread of $PG(3, q)$. As indicated in Section 3, several non-isomorphic planes can be constructed for a given q using this method, and completely answering the isomorphism question remains an open problem. Other remaining problems include determining if the group G from Corollary 11 is the full collineation group of the corresponding spread **S**, and deciding if the reguli described in Section 4 are indeed the only reguli of **S**.

However, a more important question may be the following. The spreads corresponding to the above planes are the only non-André spreads known to the author which admit regular elliptic covers. As spreads admitting regular conical and regular hyperbolic covers have recently been characterized (see [8] and [9]), we close with this question: Is every spread admitting a regular elliptic cover either an André spread or a spread obtained from a $(q+1)$-nest?

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G. L. Ebert
Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, U.S.A.