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A homotopy operation spectral sequence for the computation of homotopy groups

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Let X be an $(r-1)$-connected space with $r \geqslant 2$. Whitehead embedded the Hurewicz homomorphism h in the long exact sequence

$$\cdots \to H_{n+1}X \to \Gamma_n X \to \pi_n X \xrightarrow{h} H_n X \to \Gamma_{n-1}X \to \cdots. \tag{1}$$

The classical Hurewicz theorem is equivalent to the fact that $\Gamma_j X = 0$ for $j \leqslant r$. Whitehead [32] computed the group $\Gamma_{r+1}X$ and the group $\Gamma_{r+2}X$ was computed by Baues in [5]. In this paper we show that there is a "homotopy operation spectral sequence" which converges to the groups $\Gamma_n X$, $n > r$. The E^2-term of the spectral sequence is given by certain homotopy operation functors Γ_r^k and their derived functors $L_i\Gamma_r^k$. The functors Γ_r^k are algebraically determined by properties of homotopy groups of spheres and can be described explicitly for low k. The spectral sequence is the canonical generalization of the computation of $\Gamma_{r+1}X$ and hence it establishes the algebraic structure of homotopy groups Whitehead's program asked for 50 years ago. A different but related spectral sequence converging to the homology of X is due to Blanc [10].

The classical homotopy operations on homotopy groups $\pi_*(X)$ of the space X are given by the *Whitehead product* $[\alpha, \beta] \in \pi_{n+m-1}(X)$ for $\alpha \in \pi_n(X)$ and $\beta \in \pi_m X$ and by *composites* $\alpha \circ \eta \in \pi_j(X)$. Here $\eta \in \pi_j(S^n)$ is an element in a homotopy group of a sphere S^n, $j > n$. The Hilton–Milnor theorem shows that the classical operations, in fact, generate all homotopy operations on $\pi_*(X)$. Hence if we know the homotopy groups $\pi_r(X), \pi_{r+1}(X), \ldots, \pi_{r+k-1}(X)$ we can construct elements in $\pi_{r+k}(X)$ by applying appropriate homotopy operations to elements in $\pi_r(X), \ldots, \pi_{r+k-1}(X)$. This way we obtain the "decomposable" part in the group $\pi_{r+k}(X)$ which lies in the kernel of the Hurewicz homomorphism. We show that the "decomposable" part is the image of a natural homomorphism

$$\eta^k : \Gamma_r^k(\eta^1, \ldots, \eta^{k-1}) \xrightarrow{\gamma} \Gamma_{r+k}(X) \longrightarrow \pi_{r+k}(X). \tag{2}$$

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Here Γ_r^k denotes the homotopy operation functor defined algebraically on tuples $(\eta^1, \dots, \eta^{k-1})$ of previous homomorphisms.

The homotopy operation spectral sequence converges to the groups $\Gamma_{r+*}(X)$ and contains γ in (2) as the edge homomorphism. The spectral sequence describes exactly those properties of the unknown group $\Gamma_{r+k}(X)$ which are determined by the known groups $\pi_r(X), \dots, \pi_{r+k-1}(X)$ and the known homomorphisms $\eta^1, \dots, \eta^{k-1}$. Together with the exact sequence (1) this yields information on the unknown group $\pi_{r+k}(X)$. The procedure is a fundamental new tool to compute the homotopy groups $\pi_{r+k}(X)$ of a space inductively. In particular, since the spectral sequence admits a vanishing line, we obtain for small k the information about $\Gamma_{r+k}(X)$ already by exact sequences. For small k we describe the algebraic functors Γ_r^k and the corresponding exact sequences for $\Gamma_{r+k}(X)$ explicitly. This extends the computation of $\Gamma_{r+1}(X)$ and $\Gamma_{r+2}(X)$ of Whitehead [32] and Baues [5].

The key homological ingredient for the construction of the homotopy operation spectral sequence is the E_2-model category of simplicial spaces introduced and studied by Dwyer et al. [19, 2]; see Section 6.

We discuss a few applications of the spectral sequence. For example, we obtain a new homotopy invariant of a simply connected closed six-dimensional manifold; see (5.11). Moreover, we obtain the explicit primary obstruction for the realizability of a π-algebra; see (3.3). We also compare the homology of the classical groups $\mathrm{SL}(\mathbb{Z})$, $\mathrm{St}(\mathbb{Z})$ with the K-theory of $\mathbb{Z}$; see (3.6) and (5.16).

1. Homotopy operations and the associated homotopy operation functors

We consider the category of homotopy operations given by the homotopy category consisting of one-point unions of spheres. Such homotopy operations act on the homotopy groups $\pi_*(X)$ of a space X. Moreover, homotopy operations determine canonically a sequence of functors Γ_r^k, $r \geqslant 2$, $k \geqslant 1$ which we call homotopy operation functors. The derived functors $L_i \Gamma_r^k$ yield the E_2-term of the spectral sequence in the next section.

Let $\mathbf{Top}^*/\simeq$ be the homotopy category of pointed topological spaces.

(1.1) Definition. The *category Π of homotopy operations* is the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of finite one point unions of spheres of dimensions $\geqslant 1$. We also consider the full subcategories ($r \geqslant 2$, $k \geqslant 0$)

$$\Pi_r^k \subset \Pi_r \subset \Pi.$$

Here Π_r consists of all finite one point unions of spheres of dimension $\geqslant r$ and Π_r^k consists of all finite one point unions of spheres S^n with $r \leqslant n \leqslant r+k$. The one point union of objects is the categorical coproduct in Π.

Recall that a *theory* $\mathbf{T}$ is a small category in which finite coproducts denoted by $A \vee B$ exist. The empty coproduct is the initial object $*$ of $\mathbf{T}$. Let $\mathbf{Set}$ be the category of sets. A *model* of the theory $\mathbf{T}$ is a contravariant functor $M : \mathbf{T} \to \mathbf{Set}$ which carries finite coproducts to products. A morphism between such models is a natural transformation of functors. The resulting category of models is denoted by model($\mathbf{T}$). (cf. [7, 8]).

For example, the categories of homotopy operations Π, Π_r, Π_r^k above are theories. A model of Π is also termed a Π-*algebra*; see Stover [26, 21, 10]. A model of Π_r^k is denoted by $\pi : \Pi_r^k \to \mathbf{Set}$ with

$$\pi_n = \pi(S^n) \quad \text{for} \quad r \leqslant n \leqslant r+k. \tag{1.2}$$

Since S^n is a cogroup in Π we see that π_n is a group which is abelian for $n \geqslant 2$. Moreover, one has the canonical functor

$$G_n : \mathrm{model}(\Pi_r^k) \to \mathbf{Ab}. \tag{1}$$

Here $\mathbf{Ab}$ is the category of abelian groups and G_n carries the model π to the abelian group $G_n(\pi) = \pi_n$ in (1.2). In fact, for $k = 0$ and $r \geqslant 2$ the functor

$$G_r : \mathrm{model}(\Pi_r^0) \cong \mathbf{Ab} \tag{2}$$

is an isomorphism of categories which we shall use as an identification. Since π carries finite coproducts to products we see that the values of π on objects in Π_r^k are completely determined by the sequence of groups $(\pi_r, \pi_{r+1}, \dots, \pi_{r+k})$.

(1.3) Remark. In view of Hilton's analysis of the homotopy groups of a one point union of spheres ([34] XI) one can consider a Π-algebra π as a graded group $(\pi_1, \pi_2, \dots)$ with π_n abelian for $n \geqslant 2$, together with Whitehead product homomorphisms

$$[-,-] : \pi_n \otimes \pi_m \to \pi_{n+m-1}$$

$(n, m \geqslant 2)$ and composition functions $\alpha^* : \pi_m \to \pi_n$ for $\alpha \in \pi_n(S^m)$ and a left action of π_1 on the π_n $(n \geqslant 2)$ which commutes with these operations. Moreover, these operations satisfy "all the identities that hold for the homotopy groups of pointed topological spaces". This statement (as used in [11, 21]) is rather vague. A complete and deep algebraic analysis of the relations in terms of James–Hopf invariants was achieved in the thesis of Dreckmann [17], cf. [18].

One has the canonical functor

$$\mathbf{Top}^*/\simeq \ \to \mathrm{model}(\Pi_r^k) \tag{1.4}$$

which carries a topological space X to the model

$$\pi_*(X) = [-, X] : \Pi_r^k \to \mathbf{Set}$$

defined on $S \in \Pi_r^k$ by the set of homotopy classes $[S, X]$. If S is the finite one point union $S = S^{n_1} \vee \dots \vee S^{n_t}$ of spheres S^{n_i} with $r \leqslant n_i \leqslant r+k$ then clearly

$$[S, K] = \pi_{n_1}(X) \times \dots \times \pi_{n_t}(X)$$

is a product of homotopy groups of X. A map $S \to S'$ in Π_r^k induces a "homotopy operation" between such products of homotopy groups.

(1.5) Definition. The homotopy operation functor Γ_r^{k+1} with $r \geqslant 2$, $k \geqslant 0$ is defined by the composite

$$\Gamma_r^{k+1} : \mathrm{model}(\Pi_r^k) \xrightarrow{\Delta} \mathrm{model}(\Pi_r^{k+1}) \xrightarrow{G_{r+k+1}} \mathbf{Ab}$$

where Δ is the left adjoint of the functor

$$\lambda^* : \mathrm{model}(\Pi_r^{k+1}) \longrightarrow \mathrm{model}(\Pi_r^k)$$

induced by the inclusion $\lambda : \Pi_r^k \subset \Pi_r^{k+1}$. In 7.4 [8] an explicit description of the left adjoint Δ is given.

In the next result we use the following well-known *comma category* $\Gamma\mathbf{A}$ of a functor $\Gamma : \mathbf{C} \to \mathbf{A}$. The objects of the category $\Gamma\mathbf{A}$ are triples (X, A, η) where X is an object in $\mathbf{C}$ and $\eta : \Gamma(X) \to A$ is a homomorphism in $\mathbf{A}$. Morphisms $(X, A, \eta) \to (Y, B, \lambda)$ in $\Gamma\mathbf{A}$ are pairs (f, g) where $f : X \to Y$ is a morphism in $\mathbf{C}$ such that the diagram

$$\begin{tikzcd} \Gamma(X) \arrow[r, "\Gamma(f)"] \arrow[d, "\eta"] & \Gamma(Y) \arrow[d, "\lambda"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

commutes in $\mathbf{A}$. An object (X, A, η) is also denoted by η.

(1.6) Proposition. *Let Γ_r^{k+1} $\mathbf{Ab}$ be the comma category of the homotopy operation functor Γ_r^{k+1}. Then there is a canonical isomorphism of categories*

$$\mathrm{model}(\Pi_r^{k+1}) = \Gamma_r^{k+1}\,\mathbf{Ab}.$$

Proof. Since Π_r^{k+1} is a graded theory in the sense of (8.1″) [8] we obtain the result by (8.4) and (7.7) in [8]. Q.E.D.

Using isomorphism (1.6) as an identification we obtain by (1.2)(2) inductively the sequence of functors $\Gamma_r^1, \Gamma_2^2, \ldots$ (also termed homotopy operation functors) with

$$\Gamma_r^1 : \mathbf{Ab} \to \mathbf{Ab}$$
$$\Gamma_r^{k+1} : \Gamma_r^k\,\mathbf{Ab} \to \mathbf{Ab}, \quad k \geqslant 1 \tag{1.7}$$

This is an iterated comma category satisfying $\mathrm{model}(\Pi_r^k) = \Gamma_r^k\,\mathbf{Ab}$. Hence a model π of $\mathbf{\Pi}_r^k$ can be identified with a graded abelian group $\pi_* = (\pi_r, \pi_{r+1}, \ldots, \pi_{r+k})$ together with a sequence of homomorphisms

$$\begin{aligned} &\eta^1 : \Gamma_r^1(\pi_r) \to \pi_{r+1} \\ &\eta^2 : \Gamma_r^2(\eta^1) \to \pi_{r+2} \\ &\eta^3 : \Gamma_r^3(\eta^1, \eta^2) \to \pi_{r+3} \\ &\quad\vdots \\ &\eta^k : \Gamma_r^k(\eta^1, \ldots, \eta^{k-1}) \to \pi_{r+k}. \end{aligned} \tag{1}$$

Such sequences describe objects in the iterated comma category. For any model π of Π_r^k the graded abelian group π_* has a graded subgroup $P(\pi) \subset \pi_*$, generated by all elements which are in the image of a nontrivial primary homotopy operation (i.e. any homotopy operation which vanishes in homology). In fact, we have in degree t with $r \leqslant t \leqslant r+k$

$$P_t(\pi) = \begin{cases} 0 & \text{for} \quad t = r, \\ \text{image}\,(\eta^\tau : \Gamma_r^\tau(\eta^1, \dots, \eta^{\tau-1}) \to \pi_t) & \text{for} \quad t > r, \ \tau = t - r. \end{cases} \tag{2}$$

The quotient group

$$Q_t(\pi) = \pi_t / P_t(\pi) = \begin{cases} \pi_r & \text{for} \quad t = r, \\ \text{cok}\,(\eta^\tau) & \text{for} \quad t > r, \ \tau = t - r, \end{cases} \tag{3}$$

will be called the *indecomposables* in degree t of the model π. Hence Q_t yields a functor $Q_t : \text{model}\,(\Pi_r^k) \to \mathbf{Ab}$.

(1.8) Remark. A Π-algebra π is the same as a π_1-object in the category model(Π_2) and hence the Π-algebra π can be equivalently described by a group π_1 and a sequence $(\pi_2, \pi_3, \dots)$ of left π_1-modules together with a sequence of π_1-equivariant homomorphisms

$$\eta^1 : \Gamma_2^1(\pi_2) \to \pi_3$$
$$\eta^{k+1} : \Gamma_2^{k+1}(\eta^1, \dots, \eta^k) \to \pi_{2+k+1}, \quad k \geqslant 1.$$

Hence only the homotopy operation functors Γ_2^k, $k \geqslant 1$, are needed to define a Π-algebra π.

Whitehead [32] computed the homotopy operation functor Γ_r^1 by

$$\Gamma_r^1(\pi_r) = \begin{cases} \Gamma(\pi_2) & \text{for} \quad r = 2, \\ \pi_r \otimes \mathbb{Z}/2 & \text{for} \quad r \geqslant 3. \end{cases} \tag{1.9}$$

Here Γ is the universal quadratic functor defined as follows. A function $f : A \to B$ between abelian groups is termed quadratic if $f(-a) = f(a)$ and if the function $A \times A \to B$ with $(a,b) \mapsto f(a+b) - f(a) - f(b)$ is bilinear. For each abelian group A there is a universal quadratic function $\gamma : A \to \Gamma(A)$ which defines the functor Γ. If $\pi = \pi_* X$ is given by (1.4) then the composite

$$\pi_2 \xrightarrow{\gamma} \Gamma(\pi_2) \xrightarrow{\eta^1} \pi_3$$

coincides with the map $\pi_2 X \to \pi_3 X$ induced by the Hopf map $\eta_2 \in \pi_3(S^2)$. cf. [5].

As a further example the homotopy operation functor Γ_r^2 is given as follows where $\eta^1 : \Gamma_r^1(\pi_r) \to \pi_{r+1}$. See 11.3.3, 9.3.3 and 8.3.5 in [5].

$$\Gamma_r^2(\eta^1) = \begin{cases} \Gamma_2^2(\eta^1) & \text{for} \quad r = 2, \\ \pi_4 \otimes \mathbb{Z}/2 \oplus \Lambda^2(\pi_3) & \text{for} \quad r = 3, \\ \pi_{r+1} \otimes \mathbb{Z}/2 & \text{for} \quad r \geqslant 4. \end{cases} \tag{1.10}$$

Here $\Lambda^2(A) = A \otimes A/(a \otimes a \sim 0)$ is the exterior square of the abelian group A and $\Gamma_2^2(\eta^1)$ is defined by the pushout diagram in $\mathbf{Ab}$

$$\begin{tikzcd} \pi_3 \otimes (\mathbb{Z}/2 \oplus \pi_2) \arrow[r, "q_*"] & \Gamma_2^2(\eta^1) \\ \Gamma(\pi_2) \otimes (\mathbb{Z}/2 \oplus \pi_2) \arrow[u, "\eta^1 \otimes 1"] \arrow[r, "q"] & L(\pi_2, 1)_3 \oplus \Gamma(\pi_2) \otimes \mathbb{Z}/2 \arrow[u, "\eta^1_*"'] \end{tikzcd}$$

The group $L(A, 1)_3$ is the image of the homomorphism $A^{\otimes 3} \to A^{\otimes 3}$ which carries $a \otimes b \otimes c$ to

$$[[a, b], c] = (a \otimes b + b \otimes a) \otimes c - c \otimes (a \otimes b + b \otimes a) \tag{2}$$

with $a, b, c \in A$. Moreover, the quotient map q is the inclusion on $\Gamma(\pi_2) \otimes \mathbb{Z}/2$ and satisfies

$$q(\gamma(a) \otimes b) = -[[b, a], a] + (\gamma(a + b) - \gamma(a) - \gamma(b)) \otimes 1 \tag{3}$$

with $1 \in \mathbb{Z}/2$ and $a, b \in \pi_2$.

The next somewhat technical remark is of importance only in Section 5 below so that it can be skipped. On the other hand, a reader interested in the paper Baues [8] finds in this remark the simplest examples of "reduction functors" which play a crucial role in [8].

(1.11) Remark. For $s \geqslant 0$ let $\mathbf{Ab}^s = \mathbf{Ab} \times \dots \times \mathbf{Ab}$ be the s-fold product of the category $\mathbf{Ab}$. We define in Section 4 [8] a system of *reduction functors*

$$\Gamma^{n+1}_{r,\varepsilon} : (\Gamma^n_r\, \mathbf{Ab}) \times \mathbf{Ab}^{|\varepsilon|} \longrightarrow \mathbf{Ab}$$

associated to the iterated comma category $(\Gamma^1_r, \Gamma^2_r, \dots)$. Here $\varepsilon = (\varepsilon^1, \varepsilon^2, \dots, \varepsilon^t)$ is a finite sequence of integers $\varepsilon^i \geqslant 0$ with $t \geqslant 0$ and $|\varepsilon|$ is the number of integers $\varepsilon^i > 0$ in ε. Clearly, for the empty sequence $t = 0$ the functor coincides with Γ^{n+1}_r. The reduction functors are useful to obtain information on the vanishing line for the derived functors $L_i\Gamma^k_r$ in (1.12) below with $k = n + 1 + t$. As an example we consider the case $k = r = 2$. Using (1.10)(1) above one gets the exact sequence in the top row of the following diagram where $Q(\eta^1) = \operatorname{cok}(\eta^1)$ and where we use the reduction functors $\Gamma^1_{2,(0)}$ and $\Gamma^1_{2,(1)}$ with $\Gamma^0_r\, \mathbf{Ab} = \mathbf{Ab}$.

$$\begin{tikzcd}
\Gamma^1_{2,(0)}(\pi_2) \arrow[d, equal] & & \Gamma^1_{2,(1)}(\pi_2, Q(\eta^1)) \arrow[d, equal] & \\
L(\pi_2, 1)_3 \oplus \Gamma(\pi_2) \otimes \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \Gamma^2_2(\eta^1) \arrow[r] \arrow[d, equal] & Q(\eta^1) \otimes (\mathbb{Z}/2 \oplus \pi_2) \arrow[r] \arrow[d, equal] & 0 \\
\Lambda(I_1 \pi_2) \arrow[r] & \Lambda(\eta^1) \arrow[r] & \Lambda_{(1)}(\eta^1) \arrow[r] & 0
\end{tikzcd}$$

By (2.6) [5] we obtain for the functor $\Lambda = \Gamma^2_2 : \Gamma^1_2\, \mathbf{Ab} \to \mathbf{Ab}$ the exact sequence in the bottom row of the diagram which shows that the reduction functors $\Gamma^1_{2,(0)}$ and $\Gamma^1_{2,(1)}$ are given by the equations in the diagram. The exact sequence is short exact if η^1 is split injective. In Section 5 we shall use the exact sequence to study the derived functors $L_i\Gamma^2_2$. We point out that the exact sequence in the diagram is also studied in Appendix 11.3A [5] where a relation with the "nilization functor" is described.

As usual we obtain for the category $\mathbf{C}$ the category $s\mathbf{C}$ of simplicial objects in $\mathbf{C}$.

(1.12) Definition. For each object

$$X \in \operatorname{model}(\Pi^k_r) = \Gamma^k_r\, \mathbf{Ab}$$

there is a *resolution*

$$X_\bullet \in s\operatorname{model}(\Pi^k_r) = s\,\Gamma^k_r\, \mathbf{Ab}$$

which either can be chosen to be a free simplicial resolution defined by Blanc (3.2.2) [10] or a resolution as in Baues 3.3 [8]. Then the *derived functors*

$$L_i\Gamma_r^{k+1} : \Gamma_r^k\,\mathbf{Ab} \to \mathbf{Ab}$$

are defined by the homotopy groups

$$L_i\Gamma_r^{k+1}(X) = \pi_i(\Gamma_r^{k+1}(X_\bullet))$$

of the simplicial abelian group $\Gamma_r^{k+1}(X_\bullet)$. One has $L_0\Gamma_r^{k+1} = \Gamma_r^{k+1}$.

For example the derived functors of Γ_r^1 are the derived functors in the sense of Dold and Puppe [16] satisfying

$$L_1\Gamma_r^1(A) = \begin{cases} \Gamma\,T(A) & \text{for} \quad r = 2, \\ A * \mathbb{Z}/2 & \text{for} \quad r \geqslant 3. \end{cases} \tag{1.13}$$

Here $\Gamma\,T(A)$ is the Γ-torsion in (11.12) [5] (see also (5.5)(3) below) and $A * B = \mathrm{Tor}(A, B)$ is the torsion product of abelian group A and B. The cross effect of the Γ-torsion is $\Gamma\,T(A\,|\,B) = A * B$ and for cyclic groups we have $\Gamma\,T(\mathbb{Z}) = 0$ and $\Gamma T(\mathbb{Z}/n) = \mathbb{Z}/n * \mathbb{Z}/2$. Moreover $L_i\Gamma_r^1 = 0$ for $i \geqslant 2$; see 6.14.9 [5]. The next result is a consequence of the vanishing line of Blanc [10]; an additional proof of this result is contained in [8].

(1.14) Proposition. *The derived functors $L_i\Gamma_r^k$ are trivial*, that is $L_i\Gamma_r^k = 0$, *for* $r \geq 3$ and $i \geqslant 2k$. *Moreover for $r = 2$ one has $L_i\Gamma_2^k = 0$ if $i > 2k$.*

We conjecture that also $L_i\,\Gamma_2^k = 0$ for $i = 2k$; for example, we have $L_2\,\Gamma_2^1 = 0$ by (1.13) and in (5.12)(8) we shall show $L_4\Gamma_2^2 = 0$. For the proof of (1.14) we observe that we have by 3.6 [8] the following connection between the derived functors $L_i\Gamma_r^k$ and the derived functors $L_j\,Q_t$ of the functor Q_t of indecomposables in (1.7)(3):

$$L_{i+1}\,Q_{r+k} = L_i\Gamma_r^k \quad \text{for} \quad i \geqslant 1. \tag{1.15}$$

Moreover, for a model π of Π_r^k the sequence

$$0 \to (L_1\,Q_{r+k})(\pi) \to \Gamma_r^k(\eta^1, \dots, \eta^{k-1}) \xrightarrow{\eta^k} \pi_{r+k} \to Q_{r+k}(\pi) \to 0 \tag{1.16}$$

is exact with $Q_{r+k} = L_0\,Q_{r+k}$. The derived functors $L_j\,Q_t$ were studied by Blanc [10]. Therefore (1.14) is a consequence of 4.1 [10]. The exact sequence (1.16) shows that $L_1\,Q_t$ can be computed by use of the homotopy operation functor. In addition, 5.1.1 [10] shows

$$(L_{2k-1}\Gamma_r^k)(\pi) = \pi_r * \mathbb{Z}/2 \quad \text{for} \quad r \geqslant 1,\ k \geqslant 1. \tag{1.17}$$

For $k = 1$ this is compatible with (1.13).

For small k ($k \leqslant 19$) the categories Π_r^k are completely known by the results in Toda's book [27]. Therefore, in principle, it is possible to compute the corresponding homotopy operation functors Γ_r^k and the derived functors $L_i\Gamma_r^k$. The rich structure of homotopy groups of spheres, however, implies that such a computation is of high complexity.

2. The spectral sequence based on homotopy operations

Let X be an $(r-1)$-connected CW-complex with $r \geqslant 2$. The Γ-groups $\Gamma_n X$ in the "certain exact sequence" of Whitehead [32]

$$\cdots \longrightarrow H_{n+1}X \xrightarrow{b} \Gamma_n X \xrightarrow{i} \pi_n X \xrightarrow{h} H_n X \xrightarrow{b} \Gamma_{n-1} X \to \cdots$$

are defined by

$$\Gamma_n X = \text{image}\,\{\pi_n X^{n-1} \xrightarrow{j} \pi_n X^n\}. \tag{2.1}$$

Here X^n denotes the n-skeleton of X and the map j is induced by $X^{n-1} \subset X^n$. Moreover, the operator i in the exact sequence is induced by $X^n \subset X$. The operator h is the Hurewicz homomorphism and the secondary boundary b is induced by attaching maps of n-cells, cf. for example [5], where many further properties of Whitehead's exact sequence are studied. Since X is $(r-1)$-connected we may assume that $X^{r-1} = *$ is a point and therefore $\Gamma_j X = 0$ for $j \leqslant r$. Hence the sequence terminates as follows where we omit X in the notation so that $\pi_t = \pi_t X$.

$$\begin{tikzcd}
 & & \Gamma_r^2(\eta^1) \arrow[d, "\gamma"'] \arrow[dr, "\eta^2"] & & & \Gamma_r^1(\pi_r) \arrow[d, "\gamma"'] \arrow[dr, "\eta^1"] & & & \\
\cdots \arrow[r] & H_{r+3} \arrow[r] & \Gamma_{r+2} \arrow[r] & \pi_{r+2} \arrow[r] & H_{r+2} \arrow[r] & \Gamma_{r+1} \arrow[r] & \pi_{r+1} \arrow[r] & H_{r+1} \arrow[r] & 0
\end{tikzcd}$$

Here $\eta^1, \eta^2, \ldots$ are given by the model $\pi_*(X)$ of Π_r defined as in (1.4). We claim that for $k \geqslant 1$ there exists a natural commutative diagram

$$\begin{tikzcd}
\Gamma_r^k(\eta^1,\ldots,\eta^{k-1}) \arrow[d, "\gamma"'] \arrow[dr, "\eta^k"] & \\
\Gamma_{r+k}(X) \arrow[r, "i"] & \pi_{r+k}(X)
\end{tikzcd} \tag{2.2}$$

where η^k is defined as in (1.7)(1); that is, there is a transformation γ which is natural in X and which satisfies $\eta^k = i\gamma$. In fact, using (2.1) we see that γ exists since all homotopy operations involved in the definition of Γ_r^k are defined on elements in degree $< r+k$ and such elements live in the skeleton X^{r+k-1}. Whitehead [32] showed that

$$\gamma : \Gamma_r^1(\pi_r(X)) \xrightarrow{\cong} \Gamma_{r+1}X \tag{2.3}$$

is an isomorphism for all $(r-1)$-connected spaces X. This result now is generalized by the following spectral sequence which is constructed in Section 6 below.

(2.4) Theorem. *Let X be an $(r-1)$-connected space with $r \geqslant 2$. Then there is a natural spectral sequence $\{E^r_{p,q}\}$ with differential $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ satisfying*

$$E^0_{0,q} = \Gamma_r^q(\eta^1, \ldots, \eta^{q-1})$$

$$E^2_{p,q} = (L_p\,\Gamma_r^q)(\eta^1, \ldots, \eta^{q-1})$$

with p, $q \geqslant 1$. The sequence converges to $\Gamma_{r+*}X$ such that the edge homomorphism

$$E^2_{0,q} = \Gamma^q_r(\eta^1, \dots, \eta^{q-1}) \twoheadrightarrow E^\infty_{0,q} \subset \Gamma_{r+q}(X)$$

coincides with γ in (2.2).

Using the vanishing line (1.14) we see that the spectral sequence looks like

$$\begin{array}{lllllllll}
 & \vdots & & & & & & & \\
\Gamma_r + 4 \longleftarrow & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\Gamma_r + 3 \longleftarrow & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & \\
\Gamma_r + 2 \longleftarrow & \bullet & \bullet & \bullet & \bullet & & & & \\
\Gamma_r + 1 \longleftarrow & \bullet & \bullet & & & & & &
\end{array}$$

Only for $p < 2q$ with $r \geqslant 3$ (resp. $p \leqslant 2q$ with $r = 2$) the E^2-term is non trivial. The terms $E^2_{p,q}$ with $p + q = k$ contribute to $\Gamma_{r+k}(X)$. This, in fact, proves that for $k = 1$ the map γ in (2.3) is an isomorphism. Moreover, we derive from the spectral sequence the results in the following sections.

(2.5) Addendum. If X is a connected pointed CW-complex with universal covering $\tilde{X}$ then we can apply the spectral sequence (2.4) to $\tilde{X}$. Hence, there exists a natural spectral sequence $\{E^r_{p,q}\}$ consisting of $\pi_1(X)$-modules with $\pi_1(X)$-equivariant differentials satisfying

$$E^2_{0,q} = \Gamma^q_2(\eta^1, \dots, \eta^{q-1})$$

$$E^2_{p,q} = (L_p\Gamma^q_2)(\eta^1, \dots, \eta^{q-1})$$

with p, $q \geqslant 1$. This sequence converges to the $\pi_1(X)$-module $\Gamma_{2+*}(X) = \Gamma_{2+*}(\tilde{X})$ such that the edge homomorphism coincides with γ in (2.2) (cf.) (1.8)).

3. The group $\Gamma_{r+2}X$ of an $(r-1)$-connected space

The next theorem is an immediate consequence of the spectral sequence (2.4) since $L_2\Gamma^1_r = 0$ by (1.12). In this section we describe various specifications and applications of this result.

(3.1) Theorem. *Let X be an* $(r-1)$*-connected space with $r \geqslant 2$. Then there is a natural short exact sequence*

$$0 \longrightarrow \Gamma^2_r(\eta^1) \to \Gamma_{r+2}X \longrightarrow (L_1\Gamma^1_r)(\pi_r) \longrightarrow 0$$

where $\eta^1 : \Gamma^1_r(\pi_r) \to \pi_{r+1}$ with $\pi_* = \pi_*(X)$.

Using (1.10) and (1.13) we see that there are the following three cases of (3.1).

(1) Corollary. *Let X be a* 1*-connected space and let*

$$\eta^1 : \Gamma(\pi_2(X)) \longrightarrow \pi_3(X)$$

be induced by the Hopf map $\eta_2 \in \pi_3(S^2)$, *that is*, $\eta^1\gamma(\alpha) = \alpha\eta_2$ *for* $\alpha \in \pi_2(X)$; *see* (1.9). *Then there is a natural short exact sequence*

$$0 \to \Gamma_2^2(\eta^1) \xrightarrow{\gamma} \Gamma_4(X) \xrightarrow{\mu} \Gamma T(\pi_2(X)) \longrightarrow 0$$

where $\Gamma_2^2(\eta^1)$ *is defined in* (1.10)(1) *and where* ΓT *is the* Γ*-torsion in* (1.13).

(2) Corollary. *Let* X *be a* 2*-connected space and let*

$$\eta^1 : \pi_3(X) \otimes \mathbb{Z}/2 \longrightarrow \pi_4(X)$$

be induced by the Hopf map $\eta_3 \in \pi_4(S^3)$, *that is* $\eta^1(\alpha \otimes 1) = \alpha\eta_3$ *for* $\alpha \in \pi_3(X)$. *Then there is a natural short exact sequence*

$$0 \longrightarrow \pi_4(X) \otimes \mathbb{Z}/2 \oplus \Lambda^2(\pi_3(X)) \xrightarrow{\gamma} \Gamma_5(X) \xrightarrow{\mu} \pi_3(X) * \mathbb{Z}/2 \to 0.$$

where Λ^2 *is the exterior square in* (1.10).

(3) Corollary. *Let* X *be an* $(r-1)$*-connected space with* $r \geqslant 4$ *and let*

$$\eta^1 : \pi_r(X) \otimes \mathbb{Z}/2 \to \pi_{r+1}(X)$$

be induced by the Hopf map $\eta_r \in \pi_{r+1}(S^r)$, *that is* $\eta^1(\alpha \otimes 1) = \alpha\eta_r$ *for* $\alpha \in \pi_r(X)$. *Then one has a natural short exact sequence*

$$0 \to \pi_{r+1}(X) \otimes \mathbb{Z}/2 \xrightarrow{\gamma} \Gamma_{r+2}(X) \xrightarrow{\mu} \pi_r(X) * \mathbb{Z}/2 \longrightarrow 0.$$

The exact sequence (3) was already achieved by Whitehead [33], who also determined the extension in (3). Moreover, the exact sequences (1)–(3) were obtained using different methods by Baues [5] 11.3.3, 9.3.3 and 8.3.5, respectively. Also, the extension problems for (1)–(3) are solved in [5] as follows.

(3.2) Remark on the extension problem. In (3.1)(3) the extension is determined by the homomorphism

$$E : \pi_r * \mathbb{Z}/2 \longrightarrow \pi_{r+1} \otimes \mathbb{Z}/2 \tag{1}$$

defined by $E(x) = \gamma^{-1} 2 \mu^{-1}(x)$ for $x \in \pi_r * \mathbb{Z}/2$. Here $E = E_r(\eta^1)$ coincides with the composite.

$$E_r(\eta^1) : \pi_r * \mathbb{Z}/2 \xrightarrow{i} \pi_r \xrightarrow{q} \pi_r \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_{r+1} \xrightarrow{q} \pi_{r+1} \otimes \mathbb{Z}/2 = \Gamma_r^2(\eta^1) \tag{2}$$

where i and q are the canonical maps, $r \geqslant 4$. For extension (3.1)(2) we know that $\Lambda^2(\pi_3)$ is a direct summand of $\Gamma_5 X$ and the remaining summand is an extension given as in (2) by $q\eta^1 qi$. Let $E_3(\eta^1)$ be the composite

$$\begin{aligned} E_3(\eta^1) : \pi_3 * \mathbb{Z}/2 \xrightarrow{i} \pi_3 \xrightarrow{q} \pi_3 \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_4 \xrightarrow{q} \\ \xrightarrow{q} \pi_4 \otimes \mathbb{Z}/2 \xrightarrow{j} \pi_4 \otimes \mathbb{Z}/2 \oplus \Lambda^2(\pi_3) = \Gamma_3^2(\eta^1) \end{aligned} \tag{3}$$

where j is the inclusion.

The situation for extension (3.2)(1) is more complicated. Let $A = \pi_2$ and let K be a basis of the $\mathbb{Z}/2$-vector space $A \otimes \mathbb{Z}/2$. Then the cross effects of the functors Γ and ΓT show that the choice of such a basis yields canonically an isomorphism

$$\psi_K : \Gamma T\,(A \otimes \mathbb{Z}/2) \cong \Gamma\,(A \otimes \mathbb{Z}/2) \otimes \mathbb{Z}/2.$$

Hence we get an extension

$$0 \longrightarrow \Gamma(A) \otimes \mathbb{Z}/2 \xrightarrow{\Delta} \pi_4''(A) \xrightarrow{\mu} \Gamma\,T\,(A) \longrightarrow 0$$

which is determined by $E = \Delta^{-1}(\cdot 2)\,\mu^{-1}$ where E is the composite

$$E_K : \Gamma\,T(\,A) \xrightarrow{q^*} \Gamma\,T\,(A \otimes \mathbb{Z}/2) \xrightarrow{\psi_K} \Gamma\,(A \otimes \mathbb{Z}/2) \otimes \mathbb{Z}/2 = \Gamma\,(A) \otimes \mathbb{Z}/2. \tag{4}$$

Now consider the following push out of abelian groups:

$$\begin{tikzcd} 0 \arrow[r] & L(A,1)_3 \oplus \Gamma(A) \otimes \mathbb{Z}/2 \arrow[r, "1\oplus\Delta"] \arrow[d, "\eta^1_*"] \arrow[dr, phantom, "\text{push}"] & L(A,1)_3 \oplus \pi_4''(A) \arrow[r, "(0,\mu)"] \arrow[d] & \Gamma T(A) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Gamma_2^2(\eta^1) \arrow[r] & \Gamma_4 X \arrow[r] & \Gamma T(\pi_2) \arrow[r] & 0 \end{tikzcd} \tag{5}$$

Here η^1_* is the homomorphism in (1.10)(1). Then the bottom row of (5) determines the extension in (3.1)(1). We define the homomorphism $E_2(\eta^1)$ by the following composite with $A = \pi_2$:

$$E_2(\eta^1) : \Gamma\,T\,(A) \xrightarrow{E_K} \Gamma\,(A) \otimes \mathbb{Z}/2 \xrightarrow{j} L(A,\,1)_3 \oplus \Gamma\,(A) \otimes \mathbb{Z}/2 \xrightarrow{\eta^1_*} \Gamma_2^2\,(\eta^1)\,. \tag{6}$$

Here j denotes the inclusion and E_K and η^1_* are given by (4) and (5). The homomorphisms $E_r(\eta^1)$ above with $r \geq 2$ are used in the following theorem. We point out that all extensions (3.1)(1)–(3) are completely determined by η^1.

Blanc [12] considers the problem of realizing an abstract Π-algebra π by a space X so that $\pi \cong \pi_* X$ in model(Π). Concerning this problem we get as a consequence of (3.1) and (3.2) the following new result which describes the *first obstruction for realizability*.

(3.3) Theorem. *Let $r \geq 2$.*

(A) *Let π be a model of Π_r^1 given by $\eta^1 : \Gamma_r^1(\pi_r) \to \pi_{r+1}$. Then π is realizable by an* $(r-1)$*-connected space X. The first k-invariant of X is given by η^1.*

(B) *Let π be a model of Π_r^2 given by η^1 and $\eta^2 : \Gamma_r^2(\eta^1) \to \pi_{r+2}$. Then π is realizable by an* $(r-1)$*-connected space if and only if the obstruction $O(\pi) = \eta^2 \circ E_r(\eta^1)$ vanishes. Here $O(\pi)$ is the composite of η^2 and the homomorphism $E_r(\eta^1)$ in* (3.2).

Proof. Proposition (A) is an easy consequence of Whitehead's classification of $(r-1)$-connected $(r+2)$-dimensional homotopy types; cf. for example 3.5.6 [5]. Next, we prove (B). If π is realizable by a space X then the existence of the commutative diagram (2.2) with $k = 2$ implies that $O(\pi) = 0$; here we use (3.2). On the other hand, if $O(\pi) = 0$ then there exists a commutative diagram with exact row and column as follows:

$$\begin{tikzcd}
 & \Gamma_r^2(\eta^1) \arrow[d] \arrow[dr, "\eta^2"] & & & & \\
H_{r+3} \arrow[r] & \Gamma_{r+2} \arrow[r] \arrow[d] & \pi_{r+2} \arrow[r] & H_{r+2} \arrow[r] & \Gamma_r^1(\pi_r) \arrow[r, "\eta^1"] & \pi_{r+1} \\
 & L_1\Gamma_r^1(\pi_r) & & & &
\end{tikzcd}$$

Here the column is given by the extension in (3.2) and H_{r+3} is free abelian. Now the "theorem on the realizability of the Hurewicz homomorphism" 3.4.7 [5] implies that the exact row of the diagram is realizable by an $(r-1)$-connected $(r+3)$-dimensional space. Q.E.D.

(3.4) Example. Cochran and Habegger [15] compute the homotopy group $\pi_4(M)$ of a closed simply connected four-dimensional manifold M. This result is a consequence of (3.1). We have the homology groups $H_4M = \mathbb{Z}$, $H_3M = 0$ and H_2M is free abelian. Hence we have the exact sequence

$$0 \longrightarrow \Gamma_4 M \longrightarrow \pi_4 M \longrightarrow H_4 M \xrightarrow{b} \Gamma(H_2 M) \xrightarrow{\eta_1} \pi_3 M \longrightarrow 0.$$

Here b carries the generator to the intersection form of M; see (8.10) [6]. Hence b is injective and η^1 is the cokernel map of b. Moreover, by (3.1) we see that

$$\pi_4 M = \Gamma_4 M = \Gamma_2^2(\eta^1)$$

is algebraically determined by η^1 since $\Gamma T(H_2 M) = 0$.

(3.5) Example. Let M be a closed simply connected five-dimensional manifold or Poincaré complex. Then we get the natural diagram

$$\begin{tikzcd}
 & & {[M] \in H_5M = \mathbb{Z}} \arrow[d, "b"] & & \\
0 \arrow[r] & \Gamma_2^2(\eta^1) \arrow[r] & \Gamma_4 M \arrow[r, "\mu"] & \Gamma T(H_2 M) \arrow[r] & 0
\end{tikzcd}$$

by (3.1)(1). The fundamental class $[M]$ is the generator of $H_5 M$ and the element

$$\mu b[M] \in \Gamma T(H_2 M)$$

is a homotopy invariant of the manifold M. This is the *torsion invariant* in 9.8 [4]; compare also 1.16 [9]. We generalize this torsion invariant for simply connected six-dimensional manifolds in (5.4) below.

(3.6) Example. Let $GL(\mathbb{Z})$ (resp. $SL(\mathbb{Z})$) be the infinite general (resp. special) linear group. Let $\tilde{K}(\mathbb{Z})$ be the simply connected cover of the algebraic K-theory space $K(\mathbb{Z}) = B(GL(\mathbb{Z}))^+$. Then we have

$$\begin{aligned} K_n(\mathbb{Z}) &= \pi_n \tilde{K}(\mathbb{Z}) \\ H_n\mathrm{SL}(\mathbb{Z}) &= H_n\tilde{K}(\mathbb{Z}) \end{aligned} \tag{1}$$

for $n \geqslant 2$. Hence Whitehead's exact sequence for the space $\tilde{K}(\mathbb{Z})$ has the form (see [31])

$$\begin{tikzcd}
\Gamma_5 \arrow[r] & K_5\mathbb{Z} \arrow[r] \arrow[d, equal] & H_5\mathrm{SL}(\mathbb{Z}) \arrow[r] & \Gamma_4 \arrow[r] \arrow[d, equal] & K_4\mathbb{Z} \arrow[r] \arrow[d, equal] & H_4\mathrm{SL}(\mathbb{Z}) \arrow[r] \arrow[d, equal] & \Gamma(\mathbb{Z}/2) \arrow[r, "\eta^1"] \arrow[d, equal] & K_3(\mathbb{Z}) \arrow[d, equal] \\
 & \mathbb{Z} \oplus T_3 & & (\mathbb{Z}/2)^3 & 0 & \mathbb{Z}/2 & \mathbb{Z}/4 & \mathbb{Z}/48
\end{tikzcd}$$

Here T_3 is 3-torsion and η^1 is the map $\mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2 \subset \mathbb{Z}/48$. Therefore we can compute Γ_4 by (3.1)(1) and (3.2) and this yields $\Gamma_4 = (\mathbb{Z}/2)^3$. Let $K(\eta^1, 2)$ be the fiber of the map $\eta^1 : K(\mathbb{Z}/2, 2) \to K(\mathbb{Z}/48, 4)$ given by the homomorphism η^1. Since $K_4\mathbb{Z} = 0$ we see that $\Gamma_5 = H_6 K(\eta^1, 2)$ is 2-torsion. In fact $\Gamma_5 = \Gamma_5 \tilde{K}(\mathbb{Z})$ is computed in (5.16) below. Therefore we get the short exact sequence

$$0 \to \mathbb{Z} \oplus T_3 \longrightarrow H_5\,\mathrm{SL}(\mathbb{Z}) \longrightarrow (\mathbb{Z}/2)^3 \longrightarrow 0. \tag{2}$$

This sequence is actually nonsplit so that $H_5\mathrm{SL}(\mathbb{Z}) = \mathbb{Z} \oplus T_3 \oplus (\mathbb{Z}/2)^2$. To see this we observe that the 2-connected cover $\tilde{\tilde{K}}(\mathbb{Z})$ of $K(\mathbb{Z})$ satisfies

$$H_n\,\mathrm{St}(\mathbb{Z}) = H_n\tilde{\tilde{K}}(\mathbb{Z}) \tag{3}$$

for $n \geqslant 3$ where $\mathrm{St}(\mathbb{Z})$ is the infinite Steinberg group. Then the map $g : \tilde{\tilde{K}}(\mathbb{Z}) \to \tilde{K}(\mathbb{Z})$ induces the following commutative diagram for the Whitehead sequences:

$$\begin{tikzcd}
0 \arrow[r] & K_5\mathbb{Z} \arrow[r] \arrow[d, equal] & H_5\mathrm{St}(\mathbb{Z}) \arrow[r] \arrow[d, "g_*"] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_5\mathbb{Z} \arrow[r] & H_5\mathrm{SL}(\mathbb{Z}) \arrow[r] & (\mathbb{Z}/2)^3 \arrow[r] & 0
\end{tikzcd} \tag{4}$$

in which the top row is nonsplit so that g_* is split injective with cokernel $(\mathbb{Z}/2)^2$ (cf. [1, 2]). In (5.16) we give a further argument that the bottom sequence of (4) is non split. The extensions of the short exact sequences in (4) characterize nontrivial k-invariants k_5 of the spaces $\tilde{\tilde{K}}(\mathbb{Z})$ and $\tilde{K}(\mathbb{Z})$, respectively. This follows from the "Theorem on Postnikov invariants" 2.5.10 [5]. The top row of (4) shows that the k-invariant k_5 of $\tilde{\tilde{K}}(\mathbb{Z})$ is the nontrivial element in

$$0 \neq k_5 \in H^6(K(\mathbb{Z}/48, 3), \mathbb{Z} \oplus T_3) = \mathbb{Z}/2. \tag{5}$$

One can apply the Whitehead sequence also to the space $\tilde{\tilde{K}}(\mathbb{Z})$ to obtain the exact sequence

$$K_6\mathbb{Z} \longrightarrow H_6\,\mathrm{St}(\mathbb{Z}) \longrightarrow \Gamma_5 \longrightarrow 0 \tag{6}$$

where $\Gamma_5 = \mathbb{Z}/2$ as is easily seen by (3.1)(2). Since $K_6\mathbb{Z}$ is odd torsion we thus have $H_6\mathrm{St}(\mathbb{Z}) = \mathbb{Z}/2 \oplus$ odd torsion.

4. On the functors Γ_r^k for $k < r-1$

In the stable range $k < r-1$ the suspension functor $\Sigma: \Pi_r^k \to \Pi_{r+1}^k$ is an isomorphism of additive categories. Therefore, the theory Π_r^k is completely determined by the k-skeleton $\pi_{*\leqslant k}^S(S^0)$ of the stable ring of homotopy groups of spheres $\pi_*^S(S^0)$. A model of Π_r^k is the same as a $\pi_{*\leqslant k}^S(S^0)$-module so that for $k < r-1$ we have the functors $\Gamma^k = \Gamma_r^k$ which do not depend on r. These functors form the iterated Grothendieck construction

$$\Gamma^1: \mathbf{Ab} \to \mathbf{Ab}$$

$$\Gamma^{k+1}: \Gamma^k\,\mathbf{Ab} \to \mathbf{Ab}, \quad k \geqslant 1. \tag{4.1}$$

Here all functors Γ^k and categories $\Gamma^k\,\mathbf{Ab}$ are additive. In low degrees $\leqslant 6$ we have in $\pi_*^S(S^0)$ only the algebra generators η and ν (in dimension 1 and 3, respectively) and α in dimension 3 with the relations

$$2\eta = 0, \quad 4\nu = \eta^3, \quad \eta\nu = \nu\eta = 0, \quad 2\nu^2 = 0 \quad \text{and} \quad 3\alpha = 0, \quad \alpha^2 = 0.$$

This implies that the functors Γ^k with $k \leqslant 6$ are given as follows. For a homomorphism $f: A \to B \in \mathbf{Ab}$ we use the notation

$$f_\otimes = f \otimes \mathbb{Z}/n : A \otimes \mathbb{Z}/n \longrightarrow B \otimes \mathbb{Z}/n$$

and $q: B \to B/fA$ is the quotient map for the cokernel of f. Let $\pi_0, \ldots, \pi_6$ be abelian groups. Then the following system of homomorphisms $\eta^1, \ldots, \eta^6$ defines an object in $\Gamma^6\,\mathbf{Ab}$.

$$\Gamma^1(\pi_0) = \pi_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_1 \quad \text{with} \quad \eta^1 = \eta^* \tag{1}$$

$$\Gamma^2(\eta^1) = \pi_1 \otimes \mathbb{Z}/2 \xrightarrow{\eta^2} \pi_2 \quad \text{with} \quad \eta^2 = \eta^* \tag{2}$$

$$\Gamma^3(\eta^1, \eta^2) = (\pi_0 \otimes \mathbb{Z}/3) \oplus P \xrightarrow{\eta^3} \pi_3. \tag{3}$$

Here P is given by the push out in $\mathbf{Ab}$

$$\begin{tikzcd} \pi_0 \otimes \mathbb{Z}/8 \arrow[r] & P \\ \pi_0 \otimes \mathbb{Z}/2 \arrow[u, "\pi_0 \otimes 4"] \arrow[r, "(\eta\eta)^*"] & \pi_2 \otimes \mathbb{Z}/2 \arrow[u] \end{tikzcd}$$

The left-hand column is induced by the inclusion $4: \mathbb{Z}/2 \subset \mathbb{Z}/8$ and the bottom row is the composite

$$(\eta\eta)^* = \eta_\otimes^2\,\eta_\otimes^1 : \pi_0 \otimes \mathbb{Z}/2 \longrightarrow \pi_1 \otimes \mathbb{Z}/2 \longrightarrow \pi_2 \otimes \mathbb{Z}/2.$$

Using η^3 we obtain the action of $\alpha, \eta, \nu \in \pi_*^S(S^0)$ by the composites

$$\alpha^*: \pi_0 \otimes \mathbb{Z}/3 \xrightarrow{\eta^3} \pi_3$$

$$\eta^*: \pi_2 \otimes \mathbb{Z}/2 \longrightarrow P \xrightarrow{\eta^3} \pi_3$$

$$\nu^*: \pi_0 \otimes \mathbb{Z}/8 \longrightarrow P \xrightarrow{\eta^3} \pi_3.$$

Next, we get

$$\Gamma^4(\eta^1,\eta^2,\eta^3) = \pi_1 \otimes \mathbb{Z}/3 \oplus Q \xrightarrow{\eta^4} \pi_4. \tag{4}$$

Here Q is the push out

$$\begin{tikzcd}
(\pi_1/\eta^*\pi_0)\otimes\mathbb{Z}/8 \arrow[r] & Q \\
\pi_1\otimes\mathbb{Z}/2 \arrow[u, "q\otimes 4"] \arrow[r, "q(\eta\eta)^*"] & (\pi_3/\nu^*\pi_0)\otimes\mathbb{Z}/2 \arrow[u]
\end{tikzcd}$$

where $(\eta\eta)^* = (\eta^*)_\otimes\, \eta^2_\otimes$ with $(\eta^*)_\otimes$ given by (3). Using η^4 we obtain again the action of α, η, ν by the composites

$$\alpha^*: \pi_1 \otimes \mathbb{Z}/3 \xrightarrow{\eta^4} \pi_3$$

$$\eta^*: \pi_3 \otimes \mathbb{Z}/2 \longrightarrow (\pi_3/\nu^*\pi_0)\otimes\mathbb{Z}/2 \longrightarrow Q \xrightarrow{\eta^4} \pi_4$$

$$\nu^*: \pi_1 \otimes \mathbb{Z}/8 \longrightarrow (\pi_1/\eta^*\pi_0)\otimes\mathbb{Z}/8 \longrightarrow Q \xrightarrow{\eta^4} \pi_4.$$

Next, we describe

$$\Gamma^5(\eta^1,\eta^2,\eta^3,\eta^4) = \pi_2 \otimes \mathbb{Z}/3 \oplus R \xrightarrow{\eta^5} \pi_5. \tag{5}$$

Here R is the push out

$$\begin{tikzcd}
(\pi_2/\eta^*\pi_1)\otimes\mathbb{Z}/8 \arrow[r] & R \\
\pi_2\otimes\mathbb{Z}/2 \arrow[u, "q\otimes 4"] \arrow[r, "q(\eta\eta)^*"] & (\pi_4/\nu^*\pi_1)\otimes\mathbb{Z}/2 \arrow[u]
\end{tikzcd}$$

where $(\eta\eta)^* = (\eta^*)_\otimes(\eta^*)_\otimes$ is given by η^* in (3) and (4), respectively. Again we get by η^5 the action

$$\alpha^*: \pi_2 \otimes \mathbb{Z}/3 \xrightarrow{\eta^5} \pi_5$$

$$\eta^*: \pi_4 \otimes \mathbb{Z}/2 \longrightarrow (\pi_4/\nu^*\pi_1)\otimes\mathbb{Z}/2 \longrightarrow R \xrightarrow{\eta^5} \pi_5$$

$$\nu^*: \pi_2 \otimes \mathbb{Z}/8 \longrightarrow (\pi_2/\eta^*\pi_1)\otimes\mathbb{Z}/8 \longrightarrow R \xrightarrow{\eta^5} \pi_5.$$

Finally, we describe

$$\Gamma^6(\eta^1,\eta^2,\eta^3,\eta^4,\eta^5) = (\pi_3/\alpha^*\pi_0)\otimes\mathbb{Z}/3 \oplus S \xrightarrow{\eta^6} \pi_6 \tag{6}$$

where S is given by the following double push out diagram:

$$\begin{tikzcd}
\pi_0\otimes\mathbb{Z}/2 \arrow[rr] & & S \\
\pi_0\otimes\mathbb{Z}/8 \arrow[u] \arrow[r, "q\nu^*"] & (\pi_3/\eta^*\pi_2)\otimes\mathbb{Z}/8 \arrow[r] & S' \arrow[u] \\
& \pi_3\otimes\mathbb{Z}/2 \arrow[u, "q\otimes 4"] \arrow[r, "q(\eta\eta)^*"] & (\pi_5/\nu^*\pi_2)\otimes\mathbb{Z}/2 \arrow[u]
\end{tikzcd}$$

Here $\pi_0 \otimes \mathbb{Z}/8 \to \pi_0 \otimes \mathbb{Z}/2$ is the quotient map and v^* is defined by (3) and $(\eta\eta)^*$ is defined by η^* in (4) and η^* in (5).

In the paper [8] one finds a general method for the computation of the derived functors $L_i\Gamma_r^k$. Since in the stable range all functors $\Gamma_r^k = \Gamma^k$ $(k < r-1)$ are additive the computation of $L_i\Gamma_r^k$ is possible only in terms of chain complexes of abelian groups. We obtain the computation of the derived functors

$$L_i\Gamma^{k+1} : \Gamma^k\,\mathbf{Ab} \longrightarrow \mathbf{Ab} \tag{4.2}$$

for small k as follows.

A *weak resolution* C_* of an abelian group A is a chain complex $C_* = (\cdots \to C_1 \to C_0)$ of abelian groups satisfying

$$H_iC_* = \begin{cases} A & \text{for} \quad i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is a *resolution* if all C_i are free abelian. Moreover C_* has *length* $\leqslant n$ if $C_i = 0$ for $i > n$.

Given an object $(\pi_0, \pi_1, \dots, \eta^1, \eta^2, \dots)$ in the iterated comma category (4.1) we choose a length 1 resolution of π_0

$$C_1^0 \longrightarrow C_0^0 \longrightarrow \pi_0 \tag{1}$$

and we choose inductively weak resolutions C_*^i of π_i of length $2i+1$ together with the following commutative diagrams in which all homomorphisms η_i^k, $k \geqslant 1$, $i \geqslant 0$, are split injective and have a free abelian group as cokernel.

$$\begin{array}{ccccccccc}
 & & & & \Gamma^1 C_1^0 & \to & \Gamma^1 C_0^0 & \to & \Gamma^1\pi_0 \\
 & & & & \downarrow{\scriptstyle \eta_1^1} & & \downarrow{\scriptstyle \eta_0^1} & & \downarrow{\scriptstyle \eta^1} \\
C_3^1 & \to & C_2^1 & \to & C_1^1 & \to & C_0^1 & \to & \pi_1
\end{array} \tag{2}$$

$$\begin{array}{ccccccccccccc}
 & & & & \Gamma^1 C_3^1 & \to & \Gamma^1 C_2^1 & \to & \Gamma^2\eta_1^1 & \to & \Gamma^2\eta_0^1 & \to & \Gamma^2\eta^1 \\
 & & & & \downarrow{\scriptstyle \eta_3^2} & & \downarrow{\scriptstyle \eta_2^2} & & \downarrow{\scriptstyle \eta_1^2} & & \downarrow{\scriptstyle \eta_0^2} & & \downarrow{\scriptstyle \eta^2} \\
C_5^2 & \to & C_4^2 & \to & C_3^2 & \to & C_2^2 & \to & C_1^2 & \to & C_0^2 & \to & \pi_2
\end{array} \tag{3}$$

$$\begin{array}{ccccccccccccccc}
 & & & & \Gamma^1(C_5^2) & \to & \Gamma^1(C_4^2) & \to & \Gamma^2(\eta_3^2) & \to & \Gamma^2(\eta_2^2) & \to & \Gamma^3(\eta_1^1, \eta_1^2) & \to & \Gamma^3(\eta_0^1, \eta_0^2) & \to & \Gamma^3(\eta^1, \eta^2) \\
 & & & & \downarrow{\scriptstyle \eta_5^3} & & \downarrow{\scriptstyle \eta_4^3} & & \downarrow{\scriptstyle \eta_3^3} & & \downarrow{\scriptstyle \eta_2^3} & & \downarrow{\scriptstyle \eta_1^3} & & \downarrow{\scriptstyle \eta_0^3} & & \downarrow{\scriptstyle \eta^3} \\
 & & & \to & C_5^3 & \to & C_4^3 & \to & C_3^3 & \to & C_2^3 & \to & C_1^3 & \to & C_0^3 & \to & \pi_3 \\
C_7^3 & \to & C_6^3 & \to & & & & & & & & & & & & &
\end{array} \tag{4}$$

The homology of the top row in (2) is $L_*\Gamma^1$ so that by (4.1)(1) we have

$$L_1\Gamma^1(\pi_0) = \pi_0 * \mathbb{Z}/2 \tag{5}$$

and $L_i\Gamma^1 = 0$ for $i > 1$. Moreover, the homology of the top row in (3) is $L_*\Gamma^2$ so that we get by use of (4.1)(2) and the argument below the following result.

(4.3) Proposition.

$$L_1\Gamma^2(\eta^1) = (\pi_1 * \mathbb{Z}/2)/\eta^1(\pi_0 \otimes \mathbb{Z}/2)$$

$$L_2\Gamma^2(\eta^1) = \ker(\eta^1)$$

$$L_3\Gamma^2(\eta^1) = \pi_0 * \mathbb{Z}/2$$

$$L_i\Gamma^2 = 0 \quad \text{for} \quad i > 3.$$

Next, the homology of the chain complex (4) is $L_*\Gamma^3$ and inductively one gets this way the derived functors $L_i\Gamma^k$ with $L_i\Gamma^k = 0$ for $i > 2k-1$; cf. (1.14). Now it is easy to prove (1.17) in the stable case.

Proof of Proposition 4.3. We obtain by (2) the short exact sequence of chain complexes

$$0 \longrightarrow \Gamma^1 C^0_* \longrightarrow C^1_* \longrightarrow B^1_* \longrightarrow 0 \tag{6}$$

with $H_0 B^1_* = \mathrm{cok}(\eta^1)$, $H_1 B^1_* = \ker(\eta^1)$ and $H_2 B^1_* = \pi_0 * \mathbb{Z}/2$. We can apply the functor $\otimes\,\mathbb{Z}/2$ to (6) and we get the short exact sequence of chain complexes

$$0 \to \Gamma^1 C^0_* \longrightarrow \Gamma^2(\eta^1_*) \to B^1_* \otimes \mathbb{Z}/2 \longrightarrow 0 \tag{7}$$

where the chain complex $\Gamma^2(\eta^1_*)$ coincides with (3). Now (7) yields a long exact sequence of homology groups. This shows

$$L_3\Gamma^2(\eta^1) = H_3\Gamma^2(\eta^1_*) = H_3(B^1_* \otimes \mathbb{Z}/2) = \pi_0 * \mathbb{Z}/2.$$

Moreover, we get this way the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & L_2\Gamma^2(\eta^1) \arrow[r] & H_2(B^1_* \otimes \mathbb{Z}/2) \arrow[r] \arrow[d, equal] & \pi_0 * \mathbb{Z}/2 \arrow[r] & 0 \\ & & \ker(\eta^1) \oplus \pi_0 * \mathbb{Z}/2 & & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & L_1\Gamma^2(\eta^1) \arrow[r] & H_1(B^1_* \otimes \mathbb{Z}/2) \arrow[r, "\partial"] \arrow[d, equal] & \pi_0 \otimes \mathbb{Z}/2 \arrow[r, "\eta^1_\otimes"] & \pi_1 \otimes \mathbb{Z}/2 \\ & & \ker(\eta^1) \oplus \mathrm{cok}(\eta^1) * \mathbb{Z}/2 & & \end{tikzcd}$$

Here ∂ is the canonical inclusion on $\ker(\eta^1)$ so that we get the exact sequence

$$0 \longrightarrow L_1\Gamma^2(\eta^1) \longrightarrow \mathrm{cok}(\eta^1) * \mathbb{Z}/2 \xrightarrow{\partial} \mathrm{im}(\eta^1) \longrightarrow \pi_1 \otimes \mathbb{Z}/2.$$

But here ∂ is the boundary associated to the 6-term exact sequence given by the short exact sequence

$$0 \longrightarrow \mathrm{im}(\eta^1) \longrightarrow \pi_1 \longrightarrow \mathrm{cok}(\eta^1) \longrightarrow 0.$$

This yields the formulas in (4.3). Q.E.D.

5. On the group $\Gamma_{r+3}X$ of an $(r-1)$-connected space

Again the spectral sequence (2.4) yields immediately the next result which is a crucial tool for the computation of the homotopy group $\pi_{r+3}(X)$. We describe various special cases and applications of this result.

(5.1) Theorem. *Let X be an $(r-1)$-connected space with $r \geqslant 2$. Then there is a natural exact sequence*

$$(L_2\Gamma_r^2)(\eta^1) \xrightarrow{d^2} \Gamma_r^3(\eta^1,\eta^2) \longrightarrow \Gamma_{r+3}X \longrightarrow (L_1\Gamma_r^2)(\eta^1) \to 0$$

where $\eta^1 : \Gamma_r^1(\pi_r X) \longrightarrow \pi_{r+1}(X)$ and $\eta^2 : \Gamma_r^2(\eta^1) \longrightarrow \pi_{r+2}(X)$.

Recall that the functors Γ_r^1 and Γ_r^2 are completely understood by (1.9) and (1.10). In the theorem the derived functors $L_i\Gamma_r^2$ and the functor Γ_r^3 are needed.

For example, we get by (4.1)(3) and (4.3) the following stable case of Theorem (5.1).

(5.2) Corollary. *Let X be an $(r-1)$-connected space with $r \geqslant 5$ and let*

$$\eta^1 = \eta^* : \pi_r(X) \otimes \mathbb{Z}/2 \longrightarrow \pi_{r+1}(X)$$

$$\eta^2 = \eta^* : \pi_{r+1}(X) \otimes \mathbb{Z}/2 \longrightarrow \pi_{r+2}(X)$$

be induced by the Hopf maps $\eta = \eta_r \in \pi_{r+1}S^r$ and $\eta = \eta_{r+1} \in \pi_{r+2}S^{r+1}$ respectively. Then there is a natural exact sequence

$$\ker(\eta^1) \xrightarrow{d^2} \pi_r(X) \otimes \mathbb{Z}/3 \oplus P \longrightarrow \Gamma_{r+3}(X) \longrightarrow \frac{\pi_{r+1}(X) * \mathbb{Z}/2}{\eta^1(\pi_r(X) \otimes \mathbb{Z}/2)} \longrightarrow 0.$$

Here P is the push out of

$$\pi_r(X) \otimes \mathbb{Z}/8 \xleftarrow{1 \otimes 4} \pi_r(X) \otimes \mathbb{Z}/2 \xrightarrow{(\eta\eta)^*} \pi_{r+2}(X) \otimes \mathbb{Z}/2.$$

(5.3) Remark. The exact sequence in (5.2) depends only on the $(r+2)$-type of X which is classified by an algebraic invariant (termed an "injective A^3-system") in 8.1.6 [5]. Therefore, it should be possible to compute d^2 and the extension problem in the exact sequence (5.2) only in terms of the A^3-system associated to X similarly as this is done in Remark (3.2).

A similar explicit form of Theorem (5.1) for the unstable cases $r = 2, 3, 4$ is needed though the relevant computations are quite involved. For CW-complexes with torsion free homology such computation can be found in Unsöld [28]. We need notation on quadratic functors as follows; cf. 6.13 [5].

(5.4) Definition. A quadratic $\mathbb{Z}$-module M is a diagram

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

in **Ab** satisfying $PHP = 2P$ and $HPH = 2H$. For an abelian group A we define the *quadratic tensor product* $A \otimes M$ generated by the symbols $a \otimes m$, $[a, b] \otimes n$ with $a, b \in A$, $m \in M_e$, $n \in M_{ee}$. The relations are

$$(a+b) \otimes m = a \otimes m + b \otimes m + [a, b] \otimes H(m)$$

$$[a, a] \otimes n = a \otimes P(n)$$

$$a \otimes m \quad \text{is linear in } m$$

$$[a, b] \otimes n \quad \text{is linear in } a, b \text{ and } n\text{, respectively.}$$

Each M yields the quadratic functor $\otimes M : \mathbf{Ab} \longrightarrow \mathbf{Ab}$ which carries A to $A \otimes M$. For example Whitehead's functor Γ and the exterior square Λ^2 yield the equations

$$\Gamma(A) \otimes C = A \otimes (C \xrightarrow{1} C \xrightarrow{2} C)$$

$$\Lambda^2(A) \otimes C = A \otimes (0 \to C \to 0)$$

where C is an abelian group. We identify $C = (C \to 0 \to C)$ since

$$A \otimes C = A \otimes (C \to 0 \to C)$$

is the usual tensor product of abelian groups. A map $\alpha = (\alpha_e, \alpha_{ee}) : M \to N$ between quadratic $\mathbb{Z}$-modules is given by maps $\alpha_e : M_e \longrightarrow N_e$, $\alpha_{ee} : M_{ee} \longrightarrow N_{ee}$ in **Ab** satisfying $H\alpha_e = \alpha_{ee} H$, $\alpha_e P = P\alpha_{ee}$. Such a map induces a homomorphism $1 \otimes \alpha : A \otimes M \to A \otimes N$ which is natural in A.

Homotopy groups of spheres in the metastable range yield the quadratic $\mathbb{Z}$-modules $(m < 3n - 2)$

$$\pi_m\{S^n\} = (\pi_m S^n \xrightarrow{H} \pi_m S^{2n-1} \xrightarrow{P} \pi_m S^n)$$

where H is the Hopf invariant and P is induced by the Whitehead product square $P(\alpha) = [i_n, i_n]\alpha$. For example we have by 6.15.4 [5]

$$\pi_6\{S^3\} = (\mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/4) \oplus \mathbb{Z}/3$$

$$\pi_7\{S^4\} = (\mathbb{Z} \oplus \mathbb{Z}/4 \xrightarrow{(1,0)} \mathbb{Z} \xrightarrow{(2,-1)} \mathbb{Z} \oplus \mathbb{Z}/4) \oplus \mathbb{Z}/3. \tag{5.5}$$

These quadratic $\mathbb{Z}$-modules are used in the next two corollaries of (5.1).

(5.6) Corollary. *Let X be a 3-connected space and let*

$$\eta^1 = \eta^* : \pi_4(X) \otimes \mathbb{Z}/2 \to \pi_5(X)$$

$$\eta^2 = \eta^* : \pi_5(X) \otimes \mathbb{Z}/2 \to \pi_6(X)$$

be induced by the Hopf maps $\eta = \eta_4 \in \pi_5 S^4$ and $\eta = \eta_5 \in \pi_6(S^5)$, respectively. Then there is a natural exact sequence

$$\ker(\eta^1) \xrightarrow{d} \pi_4(X) \otimes \mathbb{Z}/3 \oplus P_7 \longrightarrow \Gamma_7(X) \longrightarrow \frac{\pi_5(X) * \mathbb{Z}/2}{\eta^1 \ (\pi_4(X) \otimes \mathbb{Z}/2)} \longrightarrow 0.$$

Here P_7 is the push out in **Ab** *of*

$$\pi_4(X)\otimes M \xleftarrow{\mathrm{id}\otimes\alpha} \pi_4(X)\otimes\mathbb{Z}/2 \xrightarrow{(\eta\eta)^*} \pi_6(X)\otimes\mathbb{Z}/2$$

where $(\eta\eta)^* = \eta^2_\otimes\,\eta^1_\otimes$ and where α is the map between quadratic $\mathbb{Z}$-modules

$$\alpha:\mathbb{Z}/2 = (\mathbb{Z}/2 \longrightarrow 0 \longrightarrow \mathbb{Z}/2) \to M = (\mathbb{Z}\oplus\mathbb{Z}/4 \xrightarrow{(1,0)} \mathbb{Z} \xrightarrow{(2,-1)} \mathbb{Z}\oplus\mathbb{Z}/4)$$

given by $\alpha_e = (0,2)$ and $\alpha_{ee} = 0$.

For the proof of (5.6) we only observe that by (5.5) and relations in Toda [27]

$$\Gamma^3_4(\eta^1,\eta^2) = \pi_4\otimes\mathbb{Z}/3\oplus P_7 \tag{5.7}$$

with P_7 defined as in (5.6). In the next corollary we use the *exterior square torsion*

$$\Lambda^2\,T(A) = \Omega(A) = (L_1\Lambda^2)(A) \tag{5.8}$$

which is the first left derived functor of Λ^2. In the notation of Eilenberg and Mac Lane [22] this is the functor Ω; cf. 6.2.10 [5]. The cross effect of $\Lambda^2\,T$ is the torsion product $\Lambda^2\,T(A\,|\,B) = A * B$ of abelian groups and one has $\Lambda^2\,T(\mathbb{Z}) = 0$ and $\Lambda^2\,T(\mathbb{Z}/n) = \mathbb{Z}/n$. Using (1.10) it is easy to compute the derived functors $L_i\Gamma^2_3(\eta^1)$ by use of (4.3). This leads to the following result.

(5.9) Corollary. *Let X be a 2-connected space and let*

$$\eta^1 = \eta^*:\pi_3(X)\otimes\mathbb{Z}/2 \longrightarrow \pi_4(X)$$

$$\eta^2 = (\eta^*, [\,-\,,\,-\,]):\pi_4(X)\otimes\mathbb{Z}/2\oplus\Lambda^2\pi_3(X) \longrightarrow \pi_5(X)$$

be induced by the Hopf maps $\eta = \eta_3\in\pi_4 S^3$ and *$\eta = \eta_4\in\pi_5 S^4$, respectively, and the Whitehead product $[\,-\,,\,-\,]:\pi_3\otimes\pi_3\to\pi_5$. Then there is a natural exact sequence*

$$\ker(\eta^1) \xrightarrow{d^2} \pi_3(X)\otimes\mathbb{Z}/3\oplus P_6 \longrightarrow \Gamma_6(X) \longrightarrow \frac{\pi_4(X)*\mathbb{Z}/2}{\eta^1(\pi_3(X)\otimes\mathbb{Z}/2)}\oplus\Lambda^2\,T(\pi_3(X)) \longrightarrow 0.$$

Here P_6 is defined by the following double push out diagram in **Ab** *with $\pi_* = \pi_*(X)$:*

$$\begin{tikzcd}
\pi_3\otimes N \arrow[rrr] & & & P_6 \\
\pi_3\otimes\mathbb{Z}/2\oplus(\Lambda^2\pi_3)\otimes\mathbb{Z}/2 \arrow[u, "{(1\otimes\alpha,\,1\otimes\beta)}"] \arrow[r, "\eta^1_\otimes\oplus 1"] & \pi_4\otimes\mathbb{Z}/2\oplus(\Lambda^2\pi_3)\otimes\mathbb{Z}/2 \arrow[r, "\eta^2_\otimes"] & \pi_5\otimes\mathbb{Z}/2 \arrow[r] & P_6' \arrow[u] \\
 & \pi_3\otimes\pi_3\otimes\mathbb{Z}/2 \arrow[u, "q"] \arrow[rr, "1\otimes\eta^1"] & & \pi_3\otimes\pi_4 \arrow[u]
\end{tikzcd}$$

The map q is given by the quotient map $\pi_3\otimes\pi_3\longrightarrow\Lambda^2\pi_3$ and α and β are maps between quadratic $\mathbb{Z}$-modules

$$\alpha:\mathbb{Z}/2 = (\mathbb{Z}/2\longrightarrow 0\longrightarrow\mathbb{Z}/2)\longrightarrow N = (\mathbb{Z}/4\xrightarrow{1}\mathbb{Z}/2\xrightarrow{0}\mathbb{Z}/4)$$

$$\beta:(0\longrightarrow\mathbb{Z}/2\longrightarrow 0)\longrightarrow N = (\mathbb{Z}/4\xrightarrow{1}\mathbb{Z}/2\xrightarrow{0}\mathbb{Z}/4)$$

with $\alpha_e = 2$, $\alpha_{ee} = 0$ and $\beta_e = 0$, $\beta_{ee} = 1$.

For the proof of (5.9) we observe that by (5.5) and relations in Toda [27] we have

$$\Gamma_3^3(\eta^1,\eta^2) = \pi_3 \otimes \mathbb{Z}/3 \oplus P_6 \tag{5.10}$$

with P_6 defined as in (5.9). The remaining case $r = 2$ in Theorem 5.1 has the following important application.

(5.11) Example. Let M be a closed simply connected six-dimensional manifold or Poincaré complex. Then we get the following natural diagram which is the analogue of (3.4) and (3.5), respectively:

$$\begin{tikzcd}
 & & {[M] \in H_6(M) = \mathbb{Z}} \arrow[d, "b"] & & \\
(L_2\Gamma_2^2)(\eta^1) \arrow[r] & \Gamma_2^3(\eta^1,\eta^2) \arrow[r] & \Gamma_5 M \arrow[r, "\mu"] & (L_1\Gamma_2^2)(\eta^1) \arrow[r] & 0
\end{tikzcd}$$

The bottom row is exact by (5.1). The fundamental class $[M]$ thus yields the *torsion invariant*

$$\mu b[M] \in (L_1\Gamma_2^2)(\eta^1)$$

which is a homotopy invariant of M. The classification of simply connected six-dimensional manifolds and Poincaré-complexes is a deep problem which is not completely solved because of the lack of good invariants. The new torsion invariant above should ease the problem considerably; cf. [35, 30, 23, 14, 24]. For example, for $M = \mathbb{C}P_3$ we obtain the generator

$$\mu b[\mathbb{C}P_3] \in L_1\Gamma_2^2(\eta^1) \cong \Gamma_5\,\mathbb{C}P_3 \cong \mathbb{Z}$$

where $\eta^1 : \Gamma(\mathbb{Z}) \to 0$. Here the isomorphisms are compatible with the exact sequence in (5.12)(9) below.

We now study along the lines of Baues [8] the derived functors $L_i\Gamma_2^2$ of Γ_2^2 in (1.10)(1) which are needed for example in (5.11) above. Let N be the normalization functor from simplicial abelian groups to chain complexes and let K be the inverse of N; cf. [14].

For $\eta^1 : \Gamma(\pi_2) \longrightarrow \pi_3$ we choose a length 1 resolution

$$C_1^2 \xrightarrow{d} C_0^2 \longrightarrow \pi_2 \tag{5.12}$$

and we choose a weak resolution C_*^3 of π_3 of length 3 together with the following commutative diagram in which η_i^1 ($i = 0, 1, 2$) is split injective with torsion free cokernel B_i^3; cf. (4.2):

$$\begin{tikzcd}
 & C_1^2 \otimes C_1^2 \arrow[r, "\partial_2"] \arrow[d, "\eta_2^1"] & \Gamma(C_1^2) \oplus C_1^2 \otimes C_0^2 \arrow[r, "\partial_1"] \arrow[d, "\eta_1^1"] & \Gamma C_0^2 \arrow[r] \arrow[d, "\eta_0^1"] & \Gamma\pi_2 \arrow[d, "\eta^1"] \\
C_3^3 \arrow[r] \arrow[d, equal] & C_2^3 \arrow[r] \arrow[d] & C_1^3 \arrow[r] \arrow[d] & C_0^3 \arrow[r] \arrow[d] & \pi_3 \arrow[d] \\
B_3^3 \arrow[r] & B_2^3 \arrow[r] & B_1^3 \arrow[r] & B_0^3 \arrow[r] & \mathrm{cok}(\eta^1)
\end{tikzcd} \tag{1}$$

The columns are split short exact sequences of abelian groups. The top row in diagram (1) is given by d in (5.12) and by the Whitehead product $P: A \otimes A \longrightarrow \Gamma(A)$ for the functor Γ defined by $P(a \otimes b) = \gamma(a+b) - \gamma(a) - \gamma(b)$, that is

$$\partial_1 = (\Gamma(d), P(d \otimes 1))$$
$$\partial_2 = (P, -1 \otimes d). \tag{2}$$

We point out that ∂_1 is injective and that $C_1^2 \otimes C_1^2$ is free abelian. Therefore, it is possible to choose a weak resolution C_*^3 of π_3 of length 3 as in diagram (1). The homology of the top row in degree 1 is the *Γ-torsion*

$$(L_1\Gamma)(\pi_2) = \Gamma T(\pi_2) = \ker(\partial_1)/\operatorname{im}(\partial_2). \tag{3}$$

One can check that the top row of (1) coincides with $N\Gamma(KC_*^2)$. Hence (1) yields a short exact sequence of simplicial abelian groups

$$0 \longrightarrow \Gamma(K(C_*^2)) \xrightarrow{\eta^1_\bullet} K(C_*^3) \longrightarrow K(B_*^3) \longrightarrow 0. \tag{4}$$

We can apply the functor Γ_2^2 to the simplicial object $\eta^1_\bullet \in s\Gamma\,\mathbf{Ab}$ in (4) and we obtain the derived functors of Γ_2^2 by

$$(L_i\Gamma_2^2)(\eta^1) = \pi_i \Gamma_2^2(\eta^1_\bullet). \tag{5}$$

We now use the exact sequence (1.11) to study the derived functors $L^i\Gamma_2^2$. Since $\eta^1_\bullet$ in (4) is split injective in each degree we see that (1.11) induces the short exact sequences of simplicial abelian groups

$$0 \longrightarrow \Gamma^1_{2,(0)}(K(C_*^2)) \longrightarrow \Gamma_2^2(\eta^1_\bullet) \longrightarrow \Gamma^1_{2,(1)}(K(C_*^2), K(B_*^3)) \longrightarrow 0 \tag{6}$$

which in turn yields a long exact sequence of homotopy groups. The functors $\Gamma^1_{2,(0)}$ and $\Gamma^1_{2,(1)}$ are explicitly described in (1.11). Using the Eilenberg–Zilber theorem [16] we see that

$$\begin{aligned}\pi_i\Gamma^1_{2,(1)}(K(C_*^2), K(B_*^2)) &= H_i(B_*^3 \otimes (\mathbb{Z}/2 \oplus \pi_2))\\ &\cong H_i(B_*^3) \otimes (\mathbb{Z}/2 \oplus \pi_2) \oplus H_{i-1}(B_*^3) * (\mathbb{Z}/2 \oplus \pi_2).\end{aligned} \tag{7}$$

Here the direct sum decomposition is natural in the sense of the universal coefficient theorem, that is, the inclusion of the $\otimes$-summand is natural but the retraction of this summand is not natural. We deduce from the definition of B_*^3 the homology groups

$$H_i B_*^3 = \begin{cases} \operatorname{cok}(\eta^1) & \text{for} \quad i = 0 \\ \ker(\eta^1) & \text{for} \quad i = 1 \\ \Gamma T(\pi_2) & \text{for} \quad i = 2 \\ 0 & \text{for} \quad i \geqslant 3. \end{cases}$$

By a result of Buth [14] we have $L_i\Gamma^1_{2,(0)} = 0$ for $i \geqslant 3$. This shows by (6)

$$L_i\Gamma_2^2(\eta^1) = 0 \quad \text{for} \quad i \geqslant 4. \tag{8}$$

Moreover, the following sequence is exact

$$0 \longrightarrow L_3\Gamma_2^2(\eta^1) \longrightarrow \Gamma\, T(\pi_2) * (\mathbb{Z}/2 \oplus \pi_2) \longrightarrow \Gamma\, T(\pi_2) * \mathbb{Z}/2 \oplus L_2 L(\pi, 1)_3$$
$$\longrightarrow L_2\Gamma_2^2(\eta^1) \longrightarrow \left\{\begin{matrix} \Gamma\, T(\pi_2) \otimes (\mathbb{Z}/2 \oplus \pi_2) \\ \oplus \\ \ker(\eta^1) * (\mathbb{Z}/2 \oplus \pi_2) \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \Gamma(\pi_2) * \mathbb{Z}/2 \oplus \Gamma\, T(\pi_2) \otimes \mathbb{Z}/2 \\ \oplus \\ L_1 L(\pi_2, 1)_3 \end{matrix}\right\}$$
$$\longrightarrow L_1\Gamma_2^2(\eta^1) \longrightarrow \left\{\begin{matrix} \ker(\eta^1) \otimes (\mathbb{Z}/2 \oplus \pi_2) \\ \oplus \\ \operatorname{cok}(\eta^1) * (\mathbb{Z}/2 \oplus \pi_2) \end{matrix}\right\} \longrightarrow \Gamma(\pi_2) \otimes \mathbb{Z}/2 \oplus L(\pi_2, 1)_3$$
$$\longrightarrow \Gamma_2^2(\eta^1) \longrightarrow \operatorname{cok}(\eta^1) \otimes (\mathbb{Z}/2 \oplus \pi_2) \longrightarrow 0. \tag{9}$$

Hence this sequence is a continuation of the exact sequence in (1.11). The sequence is natural for maps $\alpha : \eta^1 \longrightarrow \eta_0^1$ in Γ**Ab** though the direct sum decompositions are only natural in the sense of the universal coefficient theorem for homology; see (7). Sequence (9) is an initial case of the general theory of reduction functors in Baues [8]. We point out that Buth showed that the derived functors $L_i L(-,1)_3$ satisfy $L_i L(\mathbb{Z}, 1)_3 = 0$ for $i \geqslant 0$, and for $k > 0$

$$L_i L(\mathbb{Z}/k, 1)_3 = \begin{cases} \mathbb{Z}/\gcd(3k, k^2) & \text{for} \quad i = 1 \\ \mathbb{Z}/\gcd(3, k) & \text{for} \quad i = 2 \\ 0 & \text{for} \quad i \geqslant 3 \quad \text{and} \quad i = 0. \end{cases} \tag{10}$$

This and the cross-effect formulas

$$L(A\,|\,B, 1)_3 = (A \otimes B) \otimes B \oplus (A \otimes B) \otimes A$$
$$L(A\,|\,B\,|\,C, 1)_3 = (A \otimes C) \otimes B \oplus (B \otimes C) \otimes A$$
$$L_1 L(A\,|\,B, 1)_3 = (A \otimes B) * B \oplus (A * B) \otimes B \oplus (A \otimes B) * A \oplus (A * B) \otimes A$$
$$L_1 L(A\,|\,B\,|\,C, 1)_3 = (A \otimes C) * B \oplus (A * C) \otimes B \oplus (B \otimes C) * A \oplus (B * C) \otimes A$$
$$L_2 L(A\,|\,B, 1)_3 = (A * B) * B \oplus (A * B) * A$$
$$L_2 L(A\,|\,B\,|\,C, 1)_3 = (A * C) * B \oplus (B * C) * A$$

now yield a complete computation of $L_i L(A,1)_3$ for all finitely generated abelian groups A.

(5.13) Definition. Let π_2, π_3, π_4 be abelian groups and let

$$\eta^1 : \Gamma(\pi_2) \longrightarrow \pi_3$$
$$\eta^2 : \Gamma_2^2(\eta^1) \longrightarrow \pi_4 \tag{1}$$

be homomorphisms in **Ab**. Then we define by $\gamma : \pi_2 \to \Gamma(\pi_2)$ the following elements $(x, y \in \pi_2)$

$$x\eta = \eta^1(\gamma(x)) \in \pi_3$$
$$[x, y] = \eta^1(\gamma(x + y) - \gamma(x) - \gamma(y)) \in \pi_3. \tag{2}$$

Moreover, we define by the composite

$$\pi_3 \otimes (\mathbb{Z}/2 \oplus \pi_2) \xrightarrow{q^*} \Gamma_2^2(\eta^1) \xrightarrow{\eta^2} \pi_4$$

the elements ($z \in \pi_3$)

$$z\eta = \eta^2 q^*(z \otimes 1) \in \pi_4$$

$$[z, y] = \eta^2 q^*(z \otimes y) \in \pi_4. \tag{3}$$

Using the notation in (2) and (3) we define an equivalence relation $\sim$ on the direct sum

$$U = \pi_4 \otimes (\mathbb{Z}/2 \oplus \pi_2) \oplus \Lambda^2(\pi_3) \tag{4}$$

which is generated by the following relations (5)–(7), with $1 \in \mathbb{Z}/2$, $z \in \pi_3$, $x, y \in \pi_2$.

$$(z\eta) \otimes y \sim [z, y] \otimes 1 \tag{5}$$

$$z \wedge (x\eta) \sim [z, x] \otimes 1 + [z, x] \otimes x \tag{6}$$

$$[z, x] \otimes y + [z, y] \otimes x \sim z \wedge [x, y]. \tag{7}$$

Here $z \wedge z' = q(z \otimes z')$ is defined by the quotient map $q: \pi_3 \otimes \pi_3 \to \Lambda^2(\pi_3)$ with $z, z' \in \pi_3$.

(5.14) Theorem. *There is a natural isomorphism*

$$\Gamma_2^3(\eta^1, \eta^2) = (\pi_4 \otimes (\mathbb{Z}/2 \oplus \pi_2) \oplus \Lambda^2(\pi_3))/\sim$$

where $\sim$ *is the equivalence relation in* (5.13).

In fact (5.13)(5), (6) are obtained by the Barcus–Barratt formula and (5.13)(7) is deduced from the Jacobi identity for Whitehead products. The tedious proof of (5.14) is achieved along the lines of the computations of Unsöld for the proof of III.1.5 p. 168 [28]. Using (5.14) we obtain as a final case $r = 2$ of (5.1) the following result.

(5.15) Corollary. *Let X be a simply connected space and let*

$$\eta^1 : \Gamma(\pi_2(X)) \longrightarrow \pi_3(X)$$

$$\eta^2 : \Gamma_2^2(\eta^1) \longrightarrow \eta_4(X)$$

be induced by the Hopf maps $\eta_2 \in \pi_3(S^2)$, $\eta_3 \in \pi_4(S^3)$ and the Whitehead product $[-,-]: \pi_3(X) \otimes \pi_2(X) \to \pi_4(X)$, that is $\eta^1\gamma(\alpha) = \alpha\eta_2$ with $\alpha \in \pi_2(X)$ as in (3.1)(1) *and $\eta^2 q_*(\beta \otimes 1) = \beta\eta_3$ and $\eta^2 q_*(\beta \otimes \alpha) = [\beta, \alpha]$ with $\beta \in \pi_3(X)$ and q_* as in* (1.10)(1). *Then there is a natural exact sequence*

$$L_2\Gamma_2^2(\eta^1) \xrightarrow{d^2} \Gamma_2^3(\eta^1, \eta^2) \longrightarrow \Gamma_5(X) \to L_1\Gamma_2^2(\eta^1) \longrightarrow 0.$$

Here the derived functors $L_i\Gamma_2^2$ are described in (5.12) *and $\Gamma_2^3(\eta^1, \eta^3)$ is computed in* (5.14).

(5.16) Example. We can continue the exact sequence in (3.6) and we obtain for $X = \tilde{K}(\mathbb{Z})$ the exact sequence

$$K_6\mathbb{Z} \longrightarrow H_6\,\mathrm{SL}(\mathbb{Z}) \longrightarrow \Gamma_5\tilde{K}(\mathbb{Z}) \longrightarrow 0 \tag{1}$$

where η^1 and η^2 for $\tilde{K}(\mathbb{Z})$ are given by $\eta^1 : \Gamma(\mathbb{Z}/2) = \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2 \subset \mathbb{Z}/48$ as in (3.6) and $\eta^2 : \Gamma^2_2(\eta^1) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$. Hence we see by (5.14) that $\Gamma^3_2(\eta^1, \eta^2) = 0$ so that by (5.15)

$$\Gamma_5 \tilde{K}(\mathbb{Z}) \cong L_1 \Gamma^2_2(\eta^1). \tag{2}$$

This group can be computed by (5.12)(9) as follows. Consider $\eta^1_0 : \Gamma(\mathbb{Z}/2) \to 0$ and the canonical map $\alpha : \eta^1 \to \eta^1_0$ in $\Gamma\mathbf{Ab}$ which is the identity on $\pi_2 = \mathbb{Z}/2$. Then α induces the commutative diagram:

$$\begin{array}{ccccccccccccccc}
\cdots \to & (\mathbb{Z}/2)^4 & \to & (\mathbb{Z}/2)^3 & \to & L_1\Gamma^2_2(\eta^1) & \to & (\mathbb{Z}/2)^4 & \to & \mathbb{Z}/2 & \to & (\mathbb{Z}/2)^2 & \to & (\mathbb{Z}/2)^2 & \to 0 \\
 & \| & & \| & & \downarrow{\scriptstyle\alpha_*} & & \downarrow & & \| & & \downarrow & & \downarrow & \\
\cdots \to & (\mathbb{Z}/2)^4 & \to & (\mathbb{Z}/2)^3 & \to & L_1\Gamma^2_2(\eta^1_0) & \to & (\mathbb{Z}/2)^2 & \to & \mathbb{Z}/2 & \to & 0 & \to & 0 & \to 0
\end{array} \tag{3}$$

Here the exact rows are given by (5.12)(9) and (1.12). Using (5.15) for the space $K(\mathbb{Z}/2,2)$ we see that

$$L_1 \Gamma^2_2(\eta_0) = \Gamma_5 K(\mathbb{Z}/2, 2) = H_6 K(\mathbb{Z}/2, 2) = \mathbb{Z}/2 \tag{4}$$

where we use the computations of $H_6 K(\mathbb{Z}/2, 2) = \mathbb{Z}/2$ of Eilenberg and Mac Lane [22]. Now (4) and (3) shows by (2)

$$\Gamma_5 \tilde{K}\mathbb{Z} = L_1 \Gamma^2_2(\eta^1) = (\mathbb{Z}/2)^3. \tag{5}$$

Moreover since $K_6\mathbb{Z}$ is odd torsion we get by (1)

$$H_6 \mathrm{SL}(\mathbb{Z}) = (\mathbb{Z}/2)^3 \oplus \text{odd torsion} \tag{6}$$

Using corollary 14 [3] we see that $H^6(\mathrm{SL}(\mathbb{Z}), \mathbb{Z}/2) = (\mathbb{Z}/2)^5$. This group maps surjectively to the group $\mathrm{Hom}(H_6(\mathrm{SL}(\mathbb{Z})), \mathbb{Z}/2) = (\mathbb{Z}/2)^3$. Therefore we have $\mathrm{Ext}(H_5(\mathrm{SL}(\mathbb{Z})), \mathbb{Z}/2) = (\mathbb{Z}/2)^2$. This implies that the extension in (3.6)(2) is nonsplit. According to an argument of H.W. Henn eq. (6) can also be proved by use of corollary 14 [3] and the Bockstein spectral sequence.

(5.17) Example. We also can continue the exact sequence in (3.6) for $X = \tilde{\tilde{K}}(\mathbb{Z})$. In this case we get $\pi_3 X = \mathbb{Z}/48, \pi_4 X = 0, \pi_5 X = \mathbb{Z} \oplus T_3$, so that $\eta^1 : \mathbb{Z}/2 \to 0$ and $\eta^2 : 0 \to \mathbb{Z} \oplus T_3 = K_5\mathbb{Z}$. Now (5.9) shows that there is a commutative diagram in which the row and the column are exact:

$$\begin{array}{ccccccccccccc}
 & & & & \mathbb{Z}/2 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & \mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
K_7\mathbb{Z} & \longrightarrow & H_7\mathrm{St}(\mathbb{Z}) & \longrightarrow & \Gamma_6\tilde{\tilde{K}}(\mathbb{Z}) & \longrightarrow & K_6\mathbb{Z} & \longrightarrow & H_6\mathrm{St}(\mathbb{Z}) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & & & \downarrow & & & & & & & & \\
 & & & & 0 & & & & & & & &
\end{array}$$

Since $K_6\mathbb{Z}$ is odd torsion this shows that the 2-torsion of $\mathrm{cok}(K_7\mathbb{Z} \to H_7\mathrm{St}(\mathbb{Z}))$ is $\mathbb{Z}/2$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. The diagram is also of interest with respect to the 3-torsion since the 3-torsion of $\Gamma_6 \tilde{\tilde{K}}(\mathbb{Z})$ is $\mathbb{Z}/3$.

6. The construction of the homotopy operation spectral sequence

In this section we prove Theorem 2.4. The key homological ingredient is a minor modification of the "E_2-model category" structure on simplicial spaces (as in [19, 2]). We begin by spelling out what we need from this work. Fix $r \geqslant 1$.

The category of simplicial pointed spaces $s\mathbf{Top}^*$ has a closed model category structure where $f: X \to Y$ is a weak equivalence if

$$\pi_p \pi_q f: \pi_p \pi_q X \longrightarrow \pi_p \pi_q Y \tag{6.1}$$

is an isomorphism for all $p \geqslant 0$ and $q \geqslant r$. The cofibrations (and, hence, cofibrant objects) can be described as follows.

If $X \in s\mathbf{Top}^*$, let

$$L_n X = \operatorname*{colim}_{\phi:[n]\to[k]} X_k$$

where ϕ runs over the morphisms in the ordinal number category which are surjections and $k < n$, that is, $\phi^*: X_k \to X_n$ is a composition of degeneracies. There is an obvious map $s: L_n X \to X_n$ and $f: X \to Y$ in $s\mathbf{Top}^*$ is a Reedy cofibration if the induced maps

$$X_n \cup_{L_n X} L_n Y \longrightarrow Y_n$$

are cofibrations for all $n \geqslant 0$. Then $f: X \to Y$ is an E_2-cofibration if it is a Reedy cofibration and for each $n \geqslant 0$ there is a space

$$Z_n = \bigvee_{\alpha \in I_n} S^m_\alpha$$

where α in some index set I_n and $m \geqslant r$ and a map

$$Z_n \longrightarrow Y_n$$

so that

$$[X_n \cup_{L_n X} L_n Y] \vee Z_n \longrightarrow Y_n \tag{6.2}$$

is a homotopy equivalence. In particular, an inductive argument shows that if X is cofibrant in the E_2-model category, then X_n is homotopy equivalent to a wedge of spheres S^m, $m \geqslant r$.

This E_2-model category structure is actually a simplicial model category in the standard simplicial structure on $s\mathbf{Top}^*$ (see [25, II. Section 2]). In particular, given a simplicial set K and $X \in s\mathbf{Top}^*$, we can form an object $X \otimes K \in s\mathbf{Top}^*$ with

$$(X \otimes K)_n = \bigvee_{K_n} X_n .$$

If K is pointed, we can form an object $X \wedge K \in s\mathbf{Top}^*$ defined by the push-out diagram

$$\begin{array}{ccc} X \cong X \otimes * & \longrightarrow & X \otimes K \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge K. \end{array} \tag{6.3}$$

Of particular interest is the object $S^j \wedge \Delta^i/\partial\Delta^i$, where $S^j \in s\mathbf{Top}^*$ is the sphere regarded as a constant simplicial object, Δ^i is the standard simplicial i-simplex and $\partial\Delta^i \subseteq \Delta^i$ is the boundary. The simplicial set $\Delta^i/\partial\Delta^i$ is a simplicial model for the i-sphere. If $X \in s\mathbf{Top}^*$ is E_2-fibrant (or Reedy fibrant), define

$$\pi_{i,j} X = [S^j \wedge \Delta^i/\partial\Delta^i, X] \tag{6.4}$$

where the homotopy classes are computed in the homotopy category of the E_2-model category of $s\mathbf{Top}^*$. This can be calculated as

$$\begin{aligned}\pi_{i,j} X &= \pi_0\, \mathbf{map}\,(S^j \wedge \Delta^i/\partial\Delta^i, X)\\ &= \pi_i\, \mathbf{map}\,(S^j, X)\end{aligned}$$

where $\mathbf{map}(\,\cdot\,,\,\cdot\,)$ is the simplicial mapping space of $s\mathbf{Top}^*$. The "spiral exact sequence" of [20] implies that

$$\pi_0\pi_j X \cong \pi_{0,j} X \tag{6.5}$$

and that there is a long exact sequence

$$\begin{aligned}\cdots &\longrightarrow \pi_{i-1,j+1} X \longrightarrow \pi_{i,j} X \longrightarrow \pi_i\pi_j X \longrightarrow \pi_{i-2,j+1} X\\ &\longrightarrow \pi_{i-1,j} X \longrightarrow \cdots \longrightarrow \pi_{1,j} X \longrightarrow \pi_1\pi_j X \longrightarrow 0.\end{aligned} \tag{6.6}$$

We assume $j \geqslant r$.

Splicing the exact triangles

$$\begin{tikzcd} \pi_{*+1,j+1} X \arrow[rr] & & \pi_{*,j} X \arrow[dl] \\ & \pi_*\pi_j X \arrow[ul] & \end{tikzcd}$$

together yields a spectral sequence

$$\pi_p\pi_q X \Rightarrow \pi_{p+q-r,r} X = [S^r \wedge \Delta^{p+q-r}/\partial\Delta^{p+q-r}, X]. \tag{6.7}$$

Taking geometric realizations using the fact that $|\Delta^i/\partial\Delta^i| \cong S^i$ (regarding a simplicial set as a simplicial disrete space) yields a homomorphism

$$[S^n \wedge \Delta^m/\partial\Delta^m, X] \longrightarrow [S^{n+m}, X].$$

or, in symbols, a map

$$\pi_{p+q-r,r} X \longrightarrow \pi_{p+q}|X|.$$

The following is a minor variation on the main theorem of [20].

(6.8) Proposition. *Suppose $X \in s\mathbf{Top}^*$ is Reedy cofibrant and X_m is $(r-1)$-connected for all $m \geqslant 0$. Then the homomorphism*

$$\pi_{m-r,r} X \longrightarrow \pi_m|X|$$

is an isomorphism.

Proof. In [20] it is shown that

$$\pi_{m-1,1} X \longrightarrow \pi_m |X|$$

is an isomorphism. This is the case $r = 1$. The spiral exact sequence for the case $r = 1$ shows

$$\pi_{m-j-1,j+1} X \cong \pi_{m-j,j} X$$

for $j < r$.

Combining (6.7) and (6.8) yields that for X satisfying the hypotheses of (6.8) there is a spectral sequence

$$\pi_p \pi_q X \Rightarrow \pi_{p+q} |X|. \tag{6.9}$$

One source of input for this spectral sequence arises from bisimplicial sets. If $X = \{X_{m,n}\}$ is a bisimplicial set we call the first (or m) index to be the horizontal index and the second (or n) index to be the vertical direction. If we write $|X|_v = \{|X_{m,*}|\}$ to be the level-wise geometric realization, then $|X|_v$ is automatically Reedy cofibrant. This is because the map

$$L_m X = \operatorname*{colim}_{\phi:[m]\to[k]} X_{k,*} \longrightarrow X_{m,*}$$

is automatically an inclusion of simplicial sets, so

$$|L_m X|_v \cong L_m |X| \cong \operatorname*{colim}_{\phi:[m]\to[k]} |X_{k,*}| \longrightarrow |X_{m,*}|$$

is a cofibration. If $X_{m,*}$ is connected as a simplicial set and X is pointed then (6.8) yields a spectral sequence

$$\pi_p \pi_q |X|_v \Rightarrow \pi_{p,q} \| X|_v|.$$

Since $\| X_v \| = |\operatorname{diag} X|$ and one can define $\pi_q X_{m,*} = \pi_q |X_{m,*}|$ we recover the Bousfield–Friedlander spectral sequence

$$\pi_p \pi_q X \Rightarrow \pi_{p+q} (\operatorname{diag} X). \tag{6.10}$$

See [13] Section B.

With this technology in hand, we can begin to construct our spectral sequence. Fix a Reedy cofibrant simplicial space Y, which will eventually be an E_2-cofibrant model for some fixed pointed space regarded as a constant simplicial space. Let Sing(Y) be the bisimplicial space obtained by applying the singular functor level-wise to Y. Then the natural map

$$|\operatorname{Sing}(Y)|_v \longrightarrow Y \tag{6.11}$$

is a level-wise weak equivalence between Reedy cofibrant simplicial spaces, so

$$|\operatorname{diag} \operatorname{Sing}(Y)| = \| \operatorname{Sing}(Y)|_v| \longrightarrow |Y| \tag{6.12}$$

is a weak equivalence. Note that if Y is pointed and level-wise connected, so is Sing(Y).

Now let $\mathbb{Z}\operatorname{Sing}(Y)$ be the free bisimplicial abelian group of Sing(Y). Assume Y is pointed and let

$$\bar{\mathbb{Z}}\operatorname{Sing}(Y) = \mathbb{Z}\operatorname{Sing}(Y)/\mathbb{Z}\{*\}.$$

Then

$$\pi_q \bar{\mathbb{Z}}\ \mathrm{Sing}(Y) \cong \tilde{H}_q \,\mathrm{Sing}\,(Y) \cong \tilde{H}_q\, Y \cong \{\tilde{H}_q\, Y_m\}$$

and

$$\pi_q\, Y \xrightarrow{\cong} \pi_q \mathrm{Sing}(Y) \longrightarrow \pi_q \bar{\mathbb{Z}}\, \mathrm{Sing}(Y) \cong \tilde{H}_q\, Y$$

is the Hurewicz homomorphism. Factor $\mathrm{Sing}(Y) \longrightarrow \bar{\mathbb{Z}}\,\mathrm{Sing}(Y)$ as

$$\mathrm{Sing}(Y) \xrightarrow{i} Z \xrightarrow{f} \bar{\mathbb{Z}}\, \mathrm{Sing}(Y)$$

where i is a level-wise weak equivalence and f is a fibration in the closed model category structure of Bousfield–Friedlander [13], Theorem B.6. Then, among other things, f is a level-wise fibration and the realization $\mathrm{diag}(f)$ is a fibration. Let

$$F \longrightarrow Z \xrightarrow{f} \bar{\mathbb{Z}}\,\mathrm{Sing}(Y)$$

be the resulting fiber sequence. The next task is to identify the homotopy type of $|\mathrm{diag}\, F|$.

If X is a pointed, connected cofibrant space, let $S_\infty X$ be the infinite symmetric product on X and let ΓX be the homotopy fiber of the natural map $X \to S_\infty X$.

(6.14) Lemma. *There is a weak equivalence*

$$|\mathrm{diag}\, F| \longrightarrow \Gamma |Y|.$$

Proof. We have morphisms of simplicial spaces

$$Y \xleftarrow{p} |\mathrm{Sing}(Y)|_v \xrightarrow{|i|} |Z|_v \xrightarrow{|f|} |\bar{\mathbb{Z}}\,\mathrm{Sing}(Y)|_v.$$

Both p and $|i|$ are level-wise weak equivalences between Reedy cofibrant objects. Since $\| \cdot |_v| \cong |\mathrm{diag}(\,\cdot\,)|$ we have

$$|Y| \xleftarrow{|p|} |\mathrm{diag}\,\mathrm{Sing}(Y)| \xrightarrow{\|i\|} |\mathrm{diag}\, Z| \xrightarrow{\|f\|} |\mathrm{diag}\,\bar{\mathbb{Z}}\ \mathrm{Sing}\,(Y)|$$

and $|p|$ and $\|i\|$ are weak equivalences. In particular $|p|$ has a homotopy inverse, since the source and target are cofibrant. Also

$$\mathrm{diag}\,\bar{\mathbb{Z}}\ \mathrm{Sing}(Y) = \bar{\mathbb{Z}}\ \mathrm{diag}\ \mathrm{Sing}\,(Y),$$

whence

$$\pi_* |\mathrm{diag}\,\bar{\mathbb{Z}}\ \mathrm{Sing}(Y)| = H_* |\mathrm{diag}\ \mathrm{Sing}\,(Y)| = H_* |Y|.$$

Since $|\bar{\mathbb{Z}}\ \mathrm{diag}\,\mathrm{Sing}(Y)|$ is an abelian topological monoid, there is a diagram

$$\begin{array}{ccc} |Y| & \longrightarrow & |\mathrm{diag}\, Z| \\ \downarrow & & \downarrow \\ S_\infty |Y| & \longrightarrow & |\bar{\mathbb{Z}}\ \mathrm{diag}\ \mathrm{Sing}\,(Y)| \end{array}$$

and the horizontal maps are weak equivalences. Now $f: Z \to \bar{\mathbb{Z}}\,\mathrm{Sing}(Y)$ was chosen so that

$$\mathrm{diag}\,(f): \mathrm{diag}\,Z \longrightarrow \mathrm{diag}\,\bar{\mathbb{Z}}\,\mathrm{Sing}(Y)$$

is a fibration of simplicial sets. Furthermore, the diagonal functor preserves pullbacks, so the fiber of $\mathrm{diag}\,(f)$ is $\mathrm{diag}(F)$. Then

$$|\,\mathrm{diag}\,F\,| \longrightarrow |\,\mathrm{diag}\,Z\,| \longrightarrow |\bar{\mathbb{Z}}\,\mathrm{diag}\,\mathrm{Sing}(Y)|$$

is a homotopy fiber sequence. Q.E.D.

We now can construct our spectral sequence. Let X be an $(r-1)$-connected space where $r \geqslant 2$, regard X as a constant simplicial space. Let $Y \to X$ be a cofibrant model for X in the E_2-model category structure based on the spheres S^m, $m \geqslant 1$. Then Y is Reedy cofibrant level-wise connected and

$$\pi_p \pi_q Y \cong \begin{cases} \pi_q X, & p = 0, \quad q \geqslant r \\ 0, & p \neq 0 \quad \text{or} \quad q < r. \end{cases}$$

Since X is $(r-1)$-connected the spectral sequence (6.9) implies that the evident map

$$|\,Y\,| \longrightarrow X$$

is a weak equivalence. To finish the construction apply the process above to produce a bisimplicial set F so that

$$|\,\mathrm{diag}\,F\,| \simeq \Gamma\,|\,Y\,| \simeq \Gamma X.$$

Then the spectral sequence (6.9) becomes

$$\pi_p \pi_q F \Rightarrow \pi_{p+q} \Gamma X = \Gamma_{p+q} X. \tag{6.15}$$

It remains to identify the E_2-term. For fixed p, there is a fiber sequence

$$F_{p,*} \longrightarrow Z_{p,*} \longrightarrow \mathbb{Z}\,\mathrm{Sing}(Y_p)$$

and a weak equivalence $Z_{p,*} \longrightarrow \mathrm{Sing}(Y_p)$. Since Y_p is a wedge of spheres, the Hurewicz homomorphism for Y_p is onto, so there is a short exact sequence

$$0 \longrightarrow \pi_q F_{p,*} \longrightarrow \pi_q Z_{p,*} \longrightarrow \pi_q \mathbb{Z}\,\mathrm{Sing}(Y_p) \longrightarrow 0. \tag{6.16}$$

More is true. Since $\pi_* Z_{p,*}$, is a free model in Π_r,

$$\pi_q \mathbb{Z}\,\mathrm{Sing}(Y_p) = \tilde{H}_q Y_p = Q_q(\pi_* Z_{p,*})$$

and $(L_1 Q_p)(\pi_* Z_{p,*}) = 0$. Hence (1.16) and (6.16) imply

$$\pi_{r+q} F_{p,*} \cong \Gamma_r^q(\eta^1, \ldots, \eta^{q-1}).$$

Combining this with (6.15) and reindexing the q variable gives

$$L_p \Gamma_r^q(\eta^1, \ldots, \eta^{q-1}) = \pi_p \Gamma_r^q(\eta_\bullet^1, \ldots, \eta_\bullet^{q-1}) \Rightarrow \Gamma_{r+p+q} X. \tag{6.17}$$

This is the desired spectral sequence. The statement about the edge homomorphism is clear.

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