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Sparsity of orthogonal matrices with restrictions

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Abstract

The sparsity of orthogonal matrices which have $k \geqslant 1$ columns of nonzeros is studied. It is shown that the minimum number of nonzero entries in such an m by m matrix is

$$\left(\left\lfloor \lg\left(\frac{m}{k}\right)\right\rfloor + k + 2\right) m - k2^{\lfloor \lg(m/k)\rfloor + 1}.$$

As a consequence it is shown that if A is an m by n matrix with $m < n$ and the properties that its rows are pairwise orthogonal, and it has less than

$$\left(\left\lfloor \lg \frac{n}{n-m}\right\rfloor + 2\right) n - (n-m)2^{\lfloor \lg(n/(n-m))\rfloor + 1}$$

nonzero entries, then each vector orthogonal to the rows of A has at least one entry equal to 0. Also, for integers k and n with $k \leqslant n$, the minimum number of nonzero entries in an n by n, connected, orthogonal matrix having a column with at least k nonzero entries is determined.

Keywords: Sparse matrices; Orthogonal matrices

1. Introduction

In 1990, Miroslav Fiedler [4] catalyzed several investigations (see [1–3,5,7,8]) into the sparsity of certain types of orthogonal matrices by asking: how sparse can an n by n orthogonal matrix (whose rows and columns cannot be permuted to give a matrix which is a direct sum of matrices) be? The assumption excluding direct sums

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is necessary, since otherwise the answer is trivially n. Fiedler's question is answered in [1] (see also [7]), where it is shown that each n by n orthogonal matrix which is not direct summable has at least $4n - 4$ nonzero entries, and that for $n \geqslant 2$, there exist such orthogonal matrices with exactly $4n - 4$ nonzero entries. This result is extended in [2] to m by n matrices which are not direct summable, and whose rows are pairwise orthogonal.

Define a vector or a matrix to be *full*, provided each of its entries is nonzero. In [3], it is shown that an n by n orthogonal matrix with a full column has at least

$$(\lfloor \lg n \rfloor + 3)n - 2^{\lfloor \lg n \rfloor + 1} \tag{1}$$

nonzero entries, where lg denotes the base-2 logarithm function. This is perhaps a surprising result, as it implies that the presence of a full column in an n by n orthogonal matrix forces the number of nonzeros to be super-linear (at least of order $n \lg n$) in n. The n by n orthogonal matrices with a full column which achieve the sparsity in (1) are closely related to the discrete Haar wavelet (see [3]). It will be beneficial to describe these matrices here. Throughout we let $\#(A)$ denote the number of nonzero entries in A.

We first describe a way of constructing an $(m + 1)$ by $(m + 1)$ orthogonal matrix with a full column, from such an m by m matrix. The jth column of the matrix A is denoted by $A_{\cdot,j}$, and the ith row by $A_{i,\cdot}$. Let A be an m by m orthogonal matrix with a full column, and let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2 by 2 orthogonal matrix with no entry equal to 0. Then the matrix

$$\widehat{A} = \left[\begin{array}{c|c} a & bA_{i,\cdot} \\ \hline 0 & A_{1,\cdot} \\ \vdots & \vdots \\ 0 & A_{i-1,\cdot} \\ c & dA_{i,\cdot} \\ 0 & A_{i+1,\cdot} \\ \vdots & \vdots \\ 0 & A_{m,\cdot} \end{array}\right]$$

is an $(m + 1)$ by $(m + 1)$ orthogonal matrix with a full column.

We next use this construction to recursively define a family, $\mathscr{H}_m$, of m by m orthogonal matrices which have a full column. The family $\mathscr{H}_1$ consists of [1] and [−1]. Assuming that the family $\mathscr{H}_m$ is defined, then the family $\mathscr{H}_{m+1}$ consists of all matrices which after row and column permutations can be obtained by choosing a matrix A in $\mathscr{H}_m$, choosing an i so that $\#(A_{i,\cdot}) = \min_{j=1,\ldots,m} \#(A_{j,\cdot})$, and applying the above construction. We define $\mathscr{H}$ to be the union of the $\mathscr{H}_i$ $(i \geqslant 1)$. In [2] it is shown that the n by n matrices in $\mathscr{H}$ are precisely the n by n, orthogonal matrices with a full column and exactly (1) nonzero entries.

In this paper, we study the sparsity of orthogonal matrices that have a fixed number, $k > 0$, of full columns. In particular, Corollary 4.2 asserts that minimum number of nonzero entries in an m by m orthogonal matrix with k full columns is

$$\left(\left\lfloor \lg\left(\frac{m}{k}\right)\right\rfloor + k + 2\right) m - k2^{\lfloor \lg(m/k)\rfloor+1}. \tag{2}$$

In order to prove Corollary 4.2, we prove something stronger. A matrix is *row-orthogonal* provided each of its rows is nonzero and its rows are pairwise orthogonal. Corollary 4.1 asserts that the minimum number of nonzero entries in an m by n row-orthogonal matrix with k full columns and no column of zeros is

$$\left(\left\lfloor \lg\left(\frac{m}{k}\right)\right\rfloor + k + 2\right) m - k2^{\lfloor \lg(m/k)\rfloor+1} + (n-m).$$

A consequence of this result is that if A is an m by n row-orthogonal matrix with $n > m$, with no column of zeros and

$$\#(A) < \left(\left\lfloor \lg\left(\frac{n}{n-m}\right)\right\rfloor + 2\right) n - (n-m)2^{\lfloor \lg(n/(n-m))\rfloor+1},$$

then each vector orthogonal to the rows of A has an entry equal to 0.

2. Examples of sparse orthogonal matrices

Throughout we let k be a fixed positive integer. For each integer m with $m \geqslant k$ let $q_{m,k}$ and $r_{m,k}$ denote the quotient and remainder when m is divided by k, and let $\ell_{m,k} = \lfloor \lg q_{m,k} \rfloor$. We begin this section by constructing sparse m by m orthogonal matrices whose first k columns are full.

For each integer t, let H_t denote a matrix in $\mathscr{H}_t$ whose first column is full. Let $\widehat{H}_t$ denote the t by $(t-1)$ matrix obtained from H_t by deleting its first column, and let $\hat{h}_t$ denote the first column of H_t. Since $H_t \in \mathscr{H}_t$,

$$\#(\widehat{H}_t) = (\lfloor \lg t \rfloor + 2)t - 2^{\lfloor \lg t \rfloor + 1}. \tag{3}$$

Let U be a k by k orthogonal matrix which has no entry equal to 0 and let $u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \ldots, u_k^{\mathrm{T}}$ denote its rows.[1] Finally, set

$$F_{m,k} = \begin{bmatrix} \hat{h}_{q_{m,k}} u_1^{\mathrm{T}} & \widehat{H}_{q_{m,k}} & \cdots & O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hat{h}_{q_{m,k}} u_{k-r_{m,k}}^{\mathrm{T}} & O & \cdots & \widehat{H}_{q_{m,k}} & O & \cdots & O \\ \hat{h}_{q_{m,k}+1} u_{k-r_{m,k}+1}^{\mathrm{T}} & O & \cdots & O & \widehat{H}_{q_{m,k}+1} & \cdots & O \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \hat{h}_{q_{m,k}+1} u_k^{\mathrm{T}} & O & \cdots & O & O & \cdots & \widehat{H}_{q_{m,k}+1} \end{bmatrix}.$$

[1] The existence of such a matrix is obvious, as we may take $U = I - (2/k)J$ where J is the matrix of all ones.

It is easy to verify that $F_{m,k}$ is an m by m row-orthogonal matrix whose first k columns are full.

To compute $\#(F_{m,k})$ we consider two cases. First suppose that $q_{m,k} \neq 2^{\ell_{m,k}+1} - 1$. Note that $\lfloor \lg(q_{m,k}+1) \rfloor = \lfloor \lg q_{m,k} \rfloor = \ell_{m,k}$. Since $F_{m,k}$ consists of k full columns, $(k - r_{m,k})$ $\hat{H}_{q_{m,k}}$'s, and $r_{m,k}$ $\hat{H}_{q_{m,k}+1}$'s, (3) implies that

$$\begin{aligned}\#(F_{m,k}) &= mk + (k - r_{m,k}) \left[(\ell_{m,k}+2) q_{m,k} - 2^{\ell_{m,k}+1} \right] \\ &\quad + r_{m,k} \left[(\ell_{m,k}+2)(q_{m,k}+1) - 2^{\ell_{m,k}+1} \right] \\ &= mk + k \left[(\ell_{m,k}+2) q_{m,k} - 2^{\ell_{m,k}+1} \right] + r_{m,k}(\ell_{m,k}+2) \\ &= mk + (kq_{m,k} + r_{m,k})(\ell_{m,k}+2) - k2^{\ell_{m,k}+1} \\ &= mk + m(\ell_{m,k}+2) - k2^{\ell_{m,k}+1} \\ &= (\ell_{m,k} + k + 2)m - k2^{\ell_{m,k}+1}. \end{aligned} \tag{4}$$

Next suppose that $q_{m,k} = 2^{\ell_{m,k}+1} - 1$. Note that $\lfloor \lg(q_{m,k}+1) \rfloor = \lfloor \lg q_{m,k} \rfloor + 1 = \ell_{m,k} + 1$. Thus,

$$\begin{aligned}\#(F_{m,k}) &= mk + (k - r_{m,k}) \left[(\ell_{m,k}+2) q_{m,k} - 2^{\ell_{m,k}+1} \right] \\ &\quad + r_{m,k} \left[(\ell_{m,k}+3)(q_{m,k}+1) - 2^{\ell_{m,k}+2} \right] \\ &= mk + (k - r_{m,k})(q_{m,k})(\ell_{m,k}+2) + r_{m,k} q_{m,k}(\ell_{m,k}+2) \\ &\quad - \left[(k - r_{m,k}) 2^{\ell_{m,k}+1} + r_{m,k} 2^{\ell_{m,k}+1} \right] \\ &\quad + r_{m,k}(\ell_{m,k} + 3 + q_{m,k} - 2^{\ell_{m,k}+1}) \\ &= mk + (kq_{m,k})(\ell_{m,k}+2) - k2^{\ell_{m,k}+1} + r_{m,k}(\ell_{m,k}+2) \\ &= mk + (kq_{m,k} + r_{m,k})(\ell_{m,k}+2) - k2^{\ell_{m,k}+1} \\ &= (\ell_{m,k} + k + 2)m - k2^{\ell_{m,k}+1}. \end{aligned} \tag{5}$$

Based on (4) and (5) we define

$$f(m,k) = (\ell_{m,k} + k + 2)m - k2^{\ell_{m,k}+1}.$$

Thus, $\#(F_{m,k}) = f(m,k)$.

We now construct sparse m by n row-orthogonal matrices with no column of zeros and k full columns. For $n \geqslant m$, set $g(m,n,k) = f(m,k) + (n-m)$. Let D be an m by $(n-m)$ matrix with exactly one nonzero entry in each column. Then

$$\begin{bmatrix} F_{m,k} & D \end{bmatrix}$$

is an m by n row-orthogonal matrix whose first k columns are full, none of whose columns is the zero column, and with $g(m,n,k)$ nonzero entries. In the next section, we will show that these m by n matrices are the sparsest such row-orthogonal matrices.

3. Lower bounds on sparsity

We begin this section by deriving some useful facts about $g(m, n, k)$.

Lemma 3.1. *Assume that $m > k$. Then*

$$g(m, n, k) = g(m-1, n, k) + k + \ell_{m,k} + \begin{cases} 0 & \text{if } r_{m,k} = 0 \text{ and } q_{m,k} = 2^{\ell_{m,k}}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. First suppose that $r_{m,k} \neq 0$. Then $q_{m-1,k} = q_{m,k}$, and $\ell_{m-1,k} = \ell_{m,k}$. The desired equality now follows from the definition of the function g.

Next suppose that $r_{m,k} = 0$ and $q_{m,k} \neq 2^{\ell_m}$. Although $q_{m-1,k} = q_{m,k} - 1$, we still have $\ell_{m-1,k} = \ell_{m,k}$. The equality again follows from the definition of the function g.

Finally suppose that $r_{m,k} = 0$ and $q_{m,k} = 2^{\ell_{m,k}}$. Then $m = k2^{\ell_{m,k}}$, and $\ell_{m-1,k} = \ell_{m,k} - 1$. Thus,

$$\begin{aligned} g(m-1, n, k) &= (\ell_{m,k} + k + 1)(m-1) - k2^{\ell_{m,k}} + (n - (m-1)) \\ &= (\ell_{m,k}m + km + m - \ell_{m,k} - k - 1) \\ &\quad - \left(k2^{\ell_{m,k}+1} - m\right) + n - m + 1 \\ &= g(m, n, k) - k - \ell_{m,k} \end{aligned}$$

and the desired equality readily follows. □

Lemma 3.2. *Assume that $m > k$. Then*

$$\left\lceil \frac{g(m-1, n, k)}{m-1} \right\rceil \geqslant \ell_{m,k} + k + \begin{cases} 0 & \text{if } r_{m,k} = 0 \text{ and } q_{m,k} = 2^{\ell_{m,k}}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. First suppose that $r_{m,k} = 0$ and $q_{m,k} = 2^{\ell_{m,k}}$. Then $m = k2^{\ell_{m,k}}$, and

$$\begin{aligned} g(m-1, n, k) &= (\ell_{m,k} + k + 1)(m-1) - m + (n - (m-1)) \\ &= (\ell_{m,k} + k)(m-1) + (n - m). \end{aligned}$$

The desired inequality now follows from the fact that $n \geqslant m$.

Next suppose that either $r_{m,k} \neq 0$ or $q_{m,k} \neq 2^{\ell_{m,k}}$. Then

$$\begin{aligned} g(m-1, n, k) &= (\ell_{m,k} + k + 2)(m-1) - k2^{\ell_{m,k}+1} + (n - (m-1)) \\ &= (\ell_{m,k} + k + 1)(m-1) - 2k\left(2^{\ell_{m,k}}\right) + n \\ &= (\ell_{m,k} + k + 1)(m-1) - 2kq_{m,k} + 2k\left(q_{m,k} - 2^{\ell_{m,k}}\right) + n \\ &= (\ell_{m,k} + k+1)(m-1) - 2m + 2r_{m,k} + 2k\left(q_{m,k} - 2^{\ell_{m,k}}\right) + n \\ &= (\ell_{m,k} + k)(m-1) + (n - m) - 1 + 2r_{m,k} + 2k\left(q_{m,k} - 2^{\ell_{m,k}}\right). \end{aligned}$$

Since either $r_{m,k}$ or $q_{m,k} - 2^{\ell_{m,k}}$ is a positive integer, we conclude that

$$g(m-1, n, k) > (\ell_{m,k} + k)(m-1) + (n-m).$$

The inequality now follows from the fact that $n \geqslant m$. □

We are now ready to establish a lower bound on the sparsity of a row-orthogonal matrices with k full columns.

Theorem 3.3. *Let A be an m by n row-orthogonal matrix which has no column of zeros and whose first k columns are full. Then $\#(A) \geqslant g(m, n, k)$.*

Proof. The proof is by induction on $m + n$. The base case is when $m = k$. In this case, since each of the last $n - k = n - m$ columns of A have at least one nonzero entry, $\#(A) \geqslant km + n - k$. Also, $q_m = 1$, and $\ell_m = 0$, and hence $g(m, n, k) = (k + 2)m - 2m + (n - m) = km + (n - m)$. Thus, $\#(A) \geqslant g(m, n, k)$ in the base case.

Proceeding by induction, we assume that $m > k$ and that the result holds for all such m' by n' matrices with $m' + n' < m + n$.

Suppose that one of the last $n - k$ columns of A has one nonzero entry. Then the m by $(n - 1)$ matrix obtained from A by deleting such a column satisfies the inductive hypothesis, and hence $\#(A) \geqslant g(m, n - 1, k) + 1 = g(m, n, k)$.

Thus we may assume that no column of A has exactly one nonzero entry. Without loss of generality we may assume that the last row, y, of A has the maximum number of nonzero entries among the rows of A. Let B be the matrix obtained from A by deleting the last row. Then B is an $(m - 1)$ by n row-orthogonal matrix which has no column of zeros and whose first k columns are full. Hence by induction, $\#(B) \geqslant g(m - 1, n, k)$. Lemma 3.2 now implies that some row of B, and thus y, has at least

$$\ell_{m,k} + k + \begin{cases} 0 & \text{if } r_{m,k} = 0 \text{ and } q_{m,k} = 2^{\ell_{m,k}}, \\ 1 & \text{otherwise.} \end{cases}$$

nonzero entries. This, coupled with Lemma 3.1, implies that

$$\begin{aligned} \#(A) &= \#(B) + \#(y) \\ &\geqslant g(m-1, n, k) + \#(y) \\ &\geqslant g(m, n, k). \end{aligned}$$

The proof is now complete. □

4. Consequences

In this section we discuss some consequences of the results in the preceding sections. The first two follow immediately from the construction of sparse orthogonal matrices in Section 2 and Theorem 3.3.

Corollary 4.1. *Let k, m, n be integers with $0 < k \leqslant m \leqslant n$. Then the minimum number of nonzero entries in an m by n row-orthogonal matrix with k full columns, and no column of all zeros is*

$$\left(\left\lfloor \lg\left(\frac{m}{k}\right)\right\rfloor + k + 2\right)m - k2^{\lfloor \lg(m/k)\rfloor+1} + (n-m).$$

Corollary 4.2. *Let k and m be integers with $0 < k \leqslant m$. Then the minimum number of nonzero entries in an m by m orthogonal matrix with k full columns is*

$$\left(\left\lfloor \lg\left(\frac{m}{k}\right)\right\rfloor + k + 2\right)m - k2^{\lfloor \lg(m/k)\rfloor+1}.$$

In [3] it is shown that the minimum number of nonzero entries in an m by n row-orthogonal matrix with no column of zeros and with a full column is

$$(\lfloor \lg(m)\rfloor + 3)m - 2^{\lfloor \lg(m)\rfloor+1} + (n-m).$$

The next corollary determines the minimum number of nonzero entries in an m by n row-orthogonal matrix with a full row.

Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. A *Given's rotation* in $\mathbb{R}^n$ is an n by n orthogonal matrix, R, such that there exist integers i and j and an angle θ with

$$Re_k = \begin{cases} e_k & \text{if } k \notin \{i, j\},\\ (\cos\theta)e_i + (\sin\theta)e_j & \text{if } k = i,\\ -(\sin\theta)e_i + (\cos\theta)e_j & \text{if } k = j. \end{cases}$$

It is clear that if Q is an n by n orthogonal matrix whose first row is full, and $j \neq 1$, then there exists a Given's rotation, R, such that both the first and jth row of RQ are full.

Corollary 4.3. *The minimum number of nonzero entries in an m by n row-orthogonal matrix with a full row is*

$$\left(\left\lfloor \lg \frac{n}{n-m+1}\right\rfloor + 3\right)n - (n-m+1)2^{\lfloor \lg(n/(n-m+1))\rfloor+1}. \tag{6}$$

Proof. The transpose of the matrix obtained from $F_{n,n-m+1}$ by deleting the first $n-m$ columns is an m by n row-orthogonal matrix with a full row and (6) nonzero entries.

To show that (6) is a lower bound on the sparsity of such matrices, let A be an m by n row-orthogonal matrix whose last row is full. Without loss of generality we may assume that the rows of A each have Euclidean length 1. Clearly, there exists an orthogonal n by n matrix B whose first m rows are the rows of A. By the observation proceeding the corollary, we may pre-multiply B by a sequence of Given's rotations which involve the full row of A and the last $n-m$ rows of B, to obtain an orthogonal matrix $\widehat{B}$ whose last $n-m+1$ rows are full, and whose first $m-1$ rows are those of A. Hence $\widehat{B}^{\mathrm{T}}$ is an n by n orthogonal matrix with $n-m+1$ full columns. It follows from Theorem 3.3 that $\#(A) = \#(\widehat{B}) - (n-m)n \geqslant (\lfloor \lg(n/(n-m+1)) \rfloor + 3)n - (n-m+1)2^{\lfloor \lg(n/(n-m+1)) \rfloor + 1}$. $\square$

The next corollary shows that if a row-orthogonal matrix is sufficiently sparse, then no vector orthogonal to its rows is full.

Corollary 4.4. *Let A be an m by n row-orthogonal matrix with $n > m$. Suppose*

$$\#(A) < \left(\left\lfloor \lg \frac{n}{n-m} \right\rfloor + 2\right) n - (n-m)2^{\lfloor \lg(n/(n-m)) \rfloor + 1}.$$

Then each vector orthogonal to the rows of A has a zero entry.

Proof. Suppose to the contrary that v^{T} is a full vector orthogonal to the rows of A. Then the $(m+1)$ by n matrix obtained from A by appending v^{T} on the top, is an $(m+1)$ by n row-orthogonal matrix with a full row and less than

$$\left(\left\lfloor \lg \frac{n}{n-m} \right\rfloor + 3\right) n - (n-m)2^{\lfloor \lg(n/(n-m)) \rfloor + 1}$$

nonzero entries. This contradicts Corollary 4.3. $\square$

We conclude this section by pointing out a connection between this work and Pothen's [6] dissertation. In Chapter 3 of his dissertation, Pothen studied the problem of determining the sparsest orthogonal basis of a null space of an k by n matrix A. Under the assumptions that A is generic (that is, every i by i submatrix of A is invertible for $i = 1, 2, \ldots, k$) and k divides n, Pothen shows that every orthogonal basis of the null space of A has at least

$$nk\,(\lfloor \lg(n/k) \rfloor + 2) - k^2 2^{\lfloor \lg(n/k) \rfloor + 1} \tag{7}$$

nonzero entries. Corollary 4.2 implies a different answer when the assumption that A is generic is weakened. Namely, the minimum number of nonzero entries in an orthogonal basis for the null space of a full n by k matrix of rank k, is

$$n(\lfloor \lg(n/k) \rfloor + 2) - k2^{\lfloor \lg(n/k) \rfloor + 1}. \tag{8}$$

Thus, the assumption that A is generic causes a k-fold increase in the number of nonzero entries in a sparse orthgonal basis for the null space of A. Note that the first k columns of the sparse orthogonal matrices, $F_{n,k}$ constructed in Section 2, are far from generic.

5. Related problems

In this section we raise some problems for future research, and give some partial results. Let M be a p by q (0, 1)-matrix, and let n be an integer with $n \geqslant \max\{p, q\}$.

Problem 1. Does there exist an n by n orthogonal matrix which has a submatrix whose zero pattern is M?

Problem 2. If the answer to Problem 1 is yes, then what is the minimum number of nonzero entries in an orthogonal matrix which has a submatrix with zero pattern M?

For example, consider the case that $M = J_{p,q}$, the all ones matrix. Clearly, the existence of an n by n orthogonal matrix with a full p by q submatrix requires $n \geqslant \max\{p, q\}$. Without loss of generality assume that $p \geqslant q$. For $n \geqslant p$, the direct sum of the matrix $F_{p,q}$ and I_{n-p} is an n by n orthogonal matrix which contains a full p by q submatrix, and has

$$p\left(\left\lfloor \lg \frac{p}{q} \right\rfloor + q + 2\right) - q2^{\lfloor \lg(p/q)\rfloor+1} + (n-p)$$

nonzero entries.

Suppose that Q is an n by n orthogonal matrix with a full p by q submatrix with $p \geqslant q$. Without loss of generality we may assume that the p by q submatrix occurs in the upper left corner of Q. By Corollary 4.1, the first p rows of Q have at least

$$p\left(\left\lfloor \lg \frac{p}{q} \right\rfloor + 2 + q\right) - q2^{\lfloor \lg(p/q)\rfloor+1}$$

nonzero entries, and by the orthogonality of Q the last $n-p$ rows have at least $n-p$ nonzero entries. Hence we have proven the following result.

Corollary 5.1. *Let n, p, and q be integers with $p \geqslant q$. There exists an n by n orthogonal matrix with a full p by q submatrix if and only if $n \geqslant p$. Furthermore, the minimum number of nonzero entries in such an n by n orthogonal matrix is*

$$p\left(\left\lfloor \lg \frac{p}{q} \right\rfloor + 2 + q\right) - q2^{\lfloor \lg(p/q)\rfloor+1} + (n-p).$$

One can ask the analogous questions under the assumption that the orthogonal matrix is not a direct sum (no matter how you permute its rows and columns). More precisely, an m by n matrix A is *disconnected*, if the rows and columns of A can be permuted to obtain a matrix of the form

$$\begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}.$$

Here, either of the matrices A_1 or A_2 may be vacuous by virtue of having no rows or no columns. But neither A_1 nor A_2 is allowed to be the 0 by 0 matrix. A matrix which is not disconnected is *connected*. If A is an n by n orthogonal matrix, then it is easy to verify that if A contains a zero submatrix whose dimensions sum to n, then the submatrix complementary to it is also a zero submatrix. Hence an n by n orthogonal matrix is disconnected if and only if there exists an r by s zero submatrix of A for some positive integers r and s with $r + s = n$.

Our last result answers the sparsity question for connected, orthogonal matrices that have a column of p nonzero entries. We use $A[\alpha, \beta]$ to denote the submatrix of A whose rows are indexed by the set α, and whose columns are indexed by the set β.

Theorem 5.2. *Let n and p be integers with $n \geqslant p \geqslant 2$. The minimum number of nonzero entries in a connected, n by n orthogonal matrix whose first column has p nonzero entries is*

$$(\lfloor \lg p \rfloor + 3)p - 2^{\lfloor \lg p \rfloor + 1} + 4(n - p). \tag{9}$$

Proof. We first recursively define a family Q_n, $n \geqslant p$, of orthogonal matrices. Let Q_p be a matrix in $\mathscr{H}_p$ whose first column has p nonzero entries, and whose last column has two nonzero entries and these are in rows $p-1$ and p. For $n > p$, define Q_n as follows. If $n-p$ is odd, then

$$Q_n = (Q_{n-1} \oplus [1]) \left(I_{n-2} \oplus \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \right).$$

If $n-p$ is even, then

$$Q_n = \left(I_{n-2} \oplus \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \right) (Q_{n-1} \oplus [1]).$$

It is easy to verify that Q_n is a connected, orthogonal matrix that has p nonzero entries in its first column, and

$$\#(Q_n) = (\lfloor \lg p \rfloor + 3)p - 2^{\lfloor \lg p \rfloor + 1} + 4(n - p).$$

Next, let A be a connected, n by n orthogonal matrix whose first column has p nonzero entries. We show, by induction on $n-p$, that A has at least (9) nonzero entries,

If $n - p = 0$, then this follows from Theorem 3.3. Assume that $n - p > 0$ and proceed by induction. Without loss of generality we may assume that the last $n-p$ entries in the first column of A are 0.

If each of the last $n-p$ rows of A has at least four nonzero entries, then by Theorem 3.3,

$$\#(A) \geqslant (\lfloor \lg p \rfloor + 3)p - 2^{\lfloor \lg p \rfloor + 1} + 4(n - p).$$

Since A is connected, each row of A has at least two nonzero entries. Suppose that one of the last $n-p$ rows, say row n, of A has two nonzero entries, say $a_{n,n-1} \neq 0$ and $a_{n,n} \neq 0$. Then the orthogonality of A implies that the last two columns of A have nonzero entries in the same set of rows, and that

$$A = (\widehat{A} \oplus [1]) \left(I_{n-2} \oplus \begin{bmatrix} a_{n,n} & a_{n,n-1} \\ -a_{n,n-1} & a_{n,n} \end{bmatrix} \right),$$

where $\widehat{A}$ is the matrix obtained from A by deleting its last row and column, and then scaling column $n-1$ by $1/a_{n,n}$. Since A is connected, $\widehat{A}$ is connected, and since the first column of A has p nonzero entries, so does the first column of $\widehat{A}$. Thus, by induction,

$$\#(A) = \widehat{A} + 2 + \#(\widehat{A}_{\cdot,n-1}) \geqslant (\lfloor \lg p \rfloor + 3)p - 2^{\lfloor \lg p \rfloor + 1}$$
$$+ 4(n-p-1) + 2 + \#(\widehat{A}_{\cdot,n-1}).$$

The assumption that A is connected implies that $\#(\widehat{A}_{\cdot,n-1}) \geqslant 2$. Hence, the desired inequality has been established in this case.

Finally, suppose that one of the last $n-p$ rows of A has three nonzero entries. Without loss of generality we may assume row n has three nonzero entries and that $a_{n,n-2}$, $a_{n,n-1}$ and $a_{n,n}$ are nonzero. Thus, A has the form

$$\begin{bmatrix} X & u & v & w \\ O & a & b & c \end{bmatrix},$$

where X is an $(n-1)$ by $(n-3)$ matrix. We may further assume that $\#(u) \geqslant \#(v)$.

The orthogonality of A implies that the null space of $[u\ v\ w]$ is one-dimensional and is spanned by the vector $(a, b, c)^{\mathrm{T}}$. In particular, u and v are linearly independent. Since u and v are orthogonal to each column of X,

$$Q' = \begin{bmatrix} u' & v' & X \end{bmatrix}$$

is an orthogonal matrix of order $n-1$, where u' and v' are the vectors obtained from u and v by applying the Gram–Schmidt process.

The assumption that $\#(u) \geqslant \#(v)$ and the independence of u and v, imply that there exists an i such that the ith entries of u' and v' are both nonzero. Suppose that Q' can be written as a direct sum of two matrices. Then, since u'_i and v'_i are nonzero, there exists an r by s zero submatrix, $Q'[\alpha, \beta]$, for some integers r and s such that $r + s = n - 1$, which intersects column $n-2$ and column $n-1$. But then $Q[\alpha, \beta \cup \{n\}]$ is an r by $(s+1)$ zero submatrix of Q, and we contradict the assumption that Q is connected.

By the induction hypothesis,

$$\#(Q') \geqslant (\lfloor \lg p \rfloor + 3)p - 2^{\lfloor \lg p \rfloor + 1} + 4(n-p-1)$$

Thus

$$\#(Q) \geqslant (\lfloor \lg p \rfloor + 3)p - 2^{\lfloor \lg p \rfloor + 1} + 4(n-p) - 1 + \#(v) + \#(w) - \#(v').$$

Since rows $1, 2, \ldots, n-1$ of Q are orthogonal to the last row of Q, no row of $[u\ \ v\ \ w]$ contains exactly one nonzero entry. Thus, each row of $[v\ \ w]$ contains at least as many nonzero entries as the corresponding row of v'. Since the $(n-1)$th and nth columns of Q are orthogonal, some row of $[v\ \ w]$ has no zero entries. Thus, for some i, row i of $[v\ \ w]$ has more nonzero entries than row i of v'. It follows that

$$\#(Q) \geqslant (\lfloor \lg p \rfloor + 3)p - 2^{\lfloor \lg p \rfloor + 1} + 4(n-p). \qquad \square$$

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