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Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces

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Abstract

In this paper, we introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Then, we prove a strong convergence theorem which is connected with Combettes and Hirstoaga's result [P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136] and Wittmann's result [R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486–491]. Using this result, we obtain two corollaries which improve and extend their results.

Keywords: Viscosity approximation method; Equilibrium problem; Fixed point; Nonexpansive mapping

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geqslant 0 \quad \text{for all } y \in C. \tag{1.1}$$

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The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T: C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geqslant 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [2,3]. Recently, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

A mapping S of C into H is called nonexpansive if

$$\|Sx - Sy\| \leqslant \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by $F(S)$ the set of fixed points of S. If $C \subset H$ is bounded, closed and convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty; for instance, see [8]. There are some methods for approximation of fixed points of a nonexpansive mapping. In 2000, Moudafi [4] proved the following strong convergence theorem.

Theorem 1.1. (Moudafi [4]) *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a nonexpansive mapping of C into itself such that $F(S)$ is nonempty. Let f be a contraction of C into itself and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \frac{1}{1+\varepsilon_n} Sx_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n)$$

for all $n \in \mathbf{N}$, where $\{\varepsilon_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \varepsilon_n = 0, \quad \sum_{n=1}^{\infty} \varepsilon_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0.$$

Then, $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Such a method for approximation of fixed points is called the viscosity approximation method.

In this paper, motivated Combettes and Hirstoaga [2], Moudafi [4], and Tada and Takahashi [7], we introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Then, we prove a strong convergence theorem which is connected with Combettes and Hirstoaga's result [2] and Wittmann's result [11]. Using this result, we obtain two corollaries which improve and extend their results.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. When $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ implies that $\{x_n\}$ converges weakly to x and $x_n \to x$ means the strong convergence. In a real Hilbert space H, we have

$$\left\|\lambda x + (1-\lambda)y\right\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbf{R}$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\left\|x - P_C(x)\right\| \leqslant \|x - y\| \quad \text{for all } y \in C.$$

Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \Leftrightarrow \quad \langle x - z, z - y \rangle \geqslant 0 \quad \text{for all } y \in C.$$

We also know that for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$; see [5,8] for more details.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leqslant 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F\big(tz + (1 - t)x, y\big) \leqslant F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.1. [1] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into* $\mathbf{R}$ *satisfying* (A1)–(A4). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geqslant 0 \quad \textit{for all } y \in C.$$

The following lemma was also given in [2].

Lemma 2.2. [2] *Assume that $F : C \times C \to \mathbf{R}$ satisfies* (A1)–(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C\colon\ F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geqslant 0,\ \forall y \in C \right\}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single-valued*;
(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leqslant \langle T_r x - T_r y, x - y \rangle;$$

(3) *$F(T_r) = EP(F)$*;
(4) *$EP(F)$ is closed and convex.*

3. Strong convergence theorem

In this section, we deal with an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

As in the proof of [9, p. 171] or [6, Lemma 1], we can prove the following lemma; see also [10].

Lemma 3.1. [10] *Let* $\{a_n\} \subset [0, \infty)$, $\{b_n\} \subset [0, \infty)$ *and* $\{c_n\} \subset [0, 1)$ *be sequences of real numbers such that*

$$a_{n+1} \leqslant (1 - c_n)a_n + b_n \quad \text{for all } n \in \mathbf{N},$$
$$\sum_{n=1}^{\infty} c_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Theorem 3.2. *Let* C *be a nonempty closed convex subset of* H. *Let* F *be a bifunction from* $C \times C$ *to* $\mathbf{R}$ *satisfying* (A1)–(A4) *and let* S *be a nonexpansive mapping of* C *into* H *such that* $F(S) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* H *into itself and let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in H$ *and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n \end{cases}$$

for all $n \in \mathbf{N}$, *where* $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in F(S) \cap EP(F)$, *where* $z = P_{F(S)\cap EP(F)} f(z)$.

Proof. Let $Q = P_{F(S)\cap EP(F)}$. Then Qf is a contraction of H into itself. In fact, there exists $a \in [0, 1)$ such that $\|f(x) - f(y)\| \leqslant a\|x - y\|$ for all $x, y \in H$. So, we have that

$$\big\|Qf(x) - Qf(y)\big\| \leqslant \big\|f(x) - f(y)\big\| \leqslant a\|x - y\|$$

for all $x, y \in H$. So, Qf is a contraction of H into itself. Since H is complete, there exists a unique element $z \in H$ such that $z = Qf(z)$. Such a $z \in H$ is an element of C.

Let $v \in F(S) \cap EP(F)$. Then from $u_n = T_{r_n} x_n$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leqslant \|x_n - v\|$$

for all $n \in \mathbf{N}$. Put $M = \max\{\|x_1 - v\|, \frac{1}{1-a}\|f(v) - v\|\}$. It is obvious that $\|x_1 - v\| \leqslant M$. Suppose $\|x_n - v\| \leqslant M$. Then, we have

$$\begin{aligned} \|x_{n+1} - v\| &= \big\|\alpha_n f(x_n) + (1 - \alpha_n)Su_n - v\big\| \\ &\leqslant \alpha_n \big\|f(x_n) - v\big\| + (1 - \alpha_n)\|Su_n - v\| \\ &\leqslant \alpha_n \big(\big\|f(x_n) - f(v)\big\| + \big\|f(v) - v\big\|\big) + (1 - \alpha_n)\|u_n - v\| \\ &\leqslant \alpha_n \big(a\|x_n - v\| + \big\|f(v) - v\big\|\big) + (1 - \alpha_n)\|u_n - v\| \\ &\leqslant \alpha_n \big(a\|x_n - v\| + \big\|f(v) - v\big\|\big) + (1 - \alpha_n)\|x_n - v\| \\ &= \big(1 - \alpha_n(1 - a)\big)\|x_n - v\| + \alpha_n(1 - a)\frac{1}{1 - a}\big\|f(v) - v\big\| \\ &\leqslant \big(1 - \alpha_n(1 - a)\big)M + \alpha_n(1 - a)M = M. \end{aligned}$$

So, we have that $\|x_n - v\| \leqslant M$ for any $n \in \mathbf{N}$ and hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{Sx_n\}$ and $\{f(x_n)\}$ are bounded. Next, we show that $\|x_{n+1} - x_n\| \to 0$. We have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \big\|\alpha_n f(x_n) + (1-\alpha_n)Su_n - \alpha_{n-1} f(x_{n-1}) - (1-\alpha_{n-1})Su_{n-1}\big\| \\ &= \big\|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1} f(x_{n-1}) \\ &\quad + (1-\alpha_n)Su_n - (1-\alpha_n)Su_{n-1} + (1-\alpha_n)Su_{n-1} \\ &\quad - (1-\alpha_{n-1})Su_{n-1}\big\| \\ &\leqslant \alpha_n a\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K + (1-\alpha_n)\|u_n - u_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|K, \end{aligned} \tag{3.1}$$

where $K = \sup\{\|f(x_n)\| + \|Su_n\|\colon n \in \mathbf{N}\}$. On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0 \quad \text{for all } y \in C \tag{3.2}$$

and

$$f(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geqslant 0 \quad \text{for all } y \in C. \tag{3.3}$$

Putting $y = u_{n+1}$ in (3.2) and $y = u_n$ in (3.3), we have

$$f(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geqslant 0$$

and

$$f(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geqslant 0.$$

So, from (A2) we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geqslant 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geqslant 0.$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbf{N}$. Then, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leqslant \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\ &\leqslant \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1} - u_n\| &\leqslant \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leqslant \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|L, \end{aligned} \tag{3.4}$$

where $L = \sup\{\|u_n - x_n\|\colon n \in \mathbf{N}\}$. So, from (3.1) we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leqslant \alpha_n a\|x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|K \\ &\quad + (1-\alpha_n)\left(\|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|L\right) \\ &= (1 - \alpha_n + \alpha_n a)\|x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|K \\ &\quad + (1-\alpha_n)\frac{1}{b}|r_n - r_{n-1}|L \\ &= \big(1 - \alpha_n(1-a)\big)\|x_n - x_{n-1}\| + 2K|\alpha_n - \alpha_{n-1}| + \frac{L}{b}|r_n - r_{n-1}|.\end{aligned}$$

Using Lemma 3.1, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From (3.4) and $|r_{n+1} - r_n| \to 0$, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Since $x_n = \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Su_{n-1}$, we have

$$\begin{aligned}\|x_n - Su_n\| &\leqslant \|x_n - Su_{n-1}\| + \|Su_{n-1} - Su_n\| \\ &\leqslant \alpha_{n-1}\big\|f(x_{n-1}) - Su_{n-1}\big\| + \|u_{n-1} - u_n\|.\end{aligned}$$

From $\alpha_n \to 0$, we have $\|x_n - Su_n\| \to 0$. For $v \in F(S) \cap EP(F)$, we have

$$\begin{aligned}\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \\ &\leqslant \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle \\ &= \langle u_n - v, x_n - v\rangle \\ &= \frac{1}{2}\big(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - v_n\|^2\big)\end{aligned}$$

and hence

$$\|u_n - v\|^2 \leqslant \|x_n - v\|^2 - \|x_n - u_n\|^2.$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \big\|\alpha_n f(x_n) + (1-\alpha_n)Su_n - v\big\|^2 \\ &\leqslant \alpha_n\big\|f(x_n) - v\big\|^2 + (1-\alpha_n)\|Su_n - v\|^2 \\ &\leqslant \alpha_n\big\|f(x_n) - v\big\|^2 + (1-\alpha_n)\|u_n - v\|^2 \\ &\leqslant \alpha_n\big\|f(x_n) - v\big\|^2 + (1-\alpha_n)\big(\|x_n - v\|^2 - \|x_n - u_n\|^2\big) \\ &\leqslant \alpha_n\big\|f(x_n) - v\big\|^2 + \|x_n - v\|^2 - (1-\alpha_n)\|x_n - u_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_n)\|x_n - u_n\|^2 &\leqslant \alpha_n\big\|f(x_n) - v\big\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\ &\leqslant \alpha_n\big\|f(x_n) - v\big\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - v\| + \|x_{n+1} - v\|\big).\end{aligned}$$

So, we have $\|x_n - u_n\| \to 0$. From

$$\|Su_n - u_n\| \leqslant \|Su_n - x_n\| + \|x_n - u_n\|,$$

we also have $\|Su_n - u_n\| \to 0$. Next, we show that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leqslant 0,$$

where $z = P_{F(S)\cap EP(F)}f(z)$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle = \limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From $\|Su_n - u_n\| \to 0$, we obtain $Su_{n_i} \rightharpoonup w$. Let us show $w \in EP(F)$. By $u_n = T_{r_n}x_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant f(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geqslant f(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, from (A4) we have

$$0 \geqslant f(y, w)$$

for all $y \in C$. For t with $0 < t \leqslant 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $f(y_t, w) \leqslant 0$. So, from (A1) and (A4) we have

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leqslant tf(y_t, y) + (1-t)f(y_t, w) \\ &\leqslant tf(y_t, y) \end{aligned}$$

and hence $0 \leqslant f(y_t, y)$. From (A3), we have

$$0 \leqslant f(w, y)$$

for all $y \in C$ and hence $w \in EP(F)$. We shall show $w \in F(S)$. Assume $w \notin F(S)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Sw$, from Opial's theorem [5] we have

$$\begin{aligned} \liminf_{i\to\infty}\|u_{n_i} - w\| &< \liminf_{i\to\infty}\|u_{n_i} - Sw\| \\ &\leqslant \liminf_{i\to\infty}\bigl\{\|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\|\bigr\} \\ &\leqslant \liminf_{i\to\infty}\|u_{n_i} - w\|. \end{aligned}$$

This is a contradiction. So, we get $w \in F(S)$. Therefore $w \in F(S) \cap EP(F)$. Since $z = P_{F(T)\cap EP(F)}f(z)$, we have

$$\limsup_{n\to\infty}\langle f(z)-z, x_n - z\rangle = \lim_{i\to\infty}\langle f(z)-z, x_{n_i}-z\rangle = \langle f(z)-z, w-z\rangle \leqslant 0. \tag{3.5}$$

From $x_{n+1}-z=\alpha_n(f(x_n)-z)+(1-\alpha_n)(Su_n-z)$, we have

$$(1-\alpha_n)^2\|Su_n-z\|^2 \geqslant \|x_{n+1}-z\|^2 - 2\alpha_n\langle f(x_n)-z, x_{n+1}-z\rangle.$$

So, we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leqslant (1-\alpha_n)^2\|Su_n-z\|^2 + 2\alpha_n\langle f(x_n)-z, x_{n+1}-z\rangle \\ &\leqslant (1-\alpha_n)^2\|u_n-z\|^2 + 2\alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle \\ &\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\ &\leqslant (1-\alpha_n)^2\|x_n-z\|^2 + 2\alpha_n a\|x_n-z\|\|x_{n+1}-z\| \\ &\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\ &\leqslant (1-\alpha_n)^2\|x_n-z\|^2 + \alpha_n a\{\|x_n-z\|^2+\|x_{n+1}-z\|^2\} \\ &\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leqslant \frac{(1-\alpha_n)^2+\alpha_n a}{1-\alpha_n a}\|x_n-z\|^2 + \frac{2\alpha_n}{1-\alpha_n a}\langle f(z)-z, x_{n+1}-z\rangle \\ &= \frac{1-2\alpha_n+\alpha_n a}{1-\alpha_n a}\|x_n-z\|^2 + \frac{\alpha_n^2}{1-\alpha_n a}\|x_n-z\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n a}\langle f(z)-z, x_{n+1}-z\rangle \\ &\leqslant \left(1-\frac{2(1-a)\alpha_n}{1-\alpha_n a}\right)\|x_n-z\|^2 \\ &\quad + \frac{2(1-a)\alpha_n}{1-\alpha_n a}\left\{\frac{\alpha_n M}{2(1-a)} + \frac{1}{1-a}\langle f(z)-z, x_{n+1}-z\rangle\right\},\end{aligned}$$

where $M=\sup\{\|x_n-z\|^2\colon n\in\mathbf{N}\}$. Put $\beta_n=\frac{2(1-a)\alpha_n}{1-a\alpha_n}$. Then, we have $\sum_{n=1}^{\infty}\beta_n=\infty$ and $\lim_{n\to\infty}\beta_n=0$. Let $\varepsilon>0$. From (3.5), there exists $m\in\mathbf{N}$ such that

$$\frac{\alpha_n M}{2(1-a)} \leqslant \frac{\varepsilon}{2} \quad\text{and}\quad \frac{1}{1-a}\langle f(z)-z, x_{n+1}-z\rangle \leqslant \frac{\varepsilon}{2}$$

for all $n\geqslant m$. Then, we have

$$\|x_{n+1}-z\|^2 \leqslant (1-\beta_n)\|x_n-z\|^2 + \big(1-(1-\beta_n)\big)\varepsilon.$$

Similarly, we have

$$\|x_{m+n}-z\|^2 \leqslant \prod_{k=m}^{m+n-1}(1-\beta_k)\|x_m-z\|^2 + \left(1-\prod_{k=m}^{m+n-1}(1-\beta_k)\right)\varepsilon.$$

From $\sum_{k=m}^{\infty}\beta_k=\infty$, we know that $\prod_{k=m}^{\infty}(1-\beta_k)=0$. Therefore, we have

$$\limsup_{n\to\infty}\|x_n-z\|^2 = \limsup_{n\to\infty}\|x_{m+n}-z\|^2 \leqslant \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{n\to\infty} \|x_n - z\|^2 \leqslant 0.$$

So, we conclude that $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$. □

As direct consequences of Theorem 3.2, we obtain two corollaries.

Corollary 3.3. *Let C be a nonempty closed convex subset of H and let S be a nonexpansive mapping of C into H such that $F(S) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C x_n$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then, $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbf{N}$ in Theorem 3.2. Then, we have $u_n = P_C x_n$. So, from Theorem 3.2, the sequence $\{x_n\}$ generated by $x_1 \in H$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C x_n$$

for all $n \in \mathbf{N}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$. □

Corollary 3.4. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying* (A1)–(A4) *such that $EP(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) u_n \end{cases}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in EP(F)$, where $z = P_{EP(F)} f(z)$.

Proof. Put $Sx = x$ for all $x \in C$ and $r_n = 1$ in Theorem 3.2. Then, from Theorem 3.2 the sequences $\{x_n\}$ and $\{u_n\}$ generated in Corollary 3.4 converge strongly to $z \in EP(F)$, where $z = P_{EP(F)} f(z)$. □

We obtain Wittmann's theorem [11] in the case when $f(y) = x_1 \in C$ for all $y \in H$ and S is a nonexpansive mapping of C into itself in Corollary 3.3. We also obtain Combettes and Hirstoaga's theorem [2] in the case when $f(y) = x_1 \in H$ for all $y \in H$ in Corollary 3.4.

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