JOURNAL OF APPROXIMATION THEORY 20, 70–76 (1977)

# Distributional Inequalities and Landau's Proof of the Weierstrass Theorem

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*Communicated by G. G. Lorentz*

Received September 15, 1975

Distributional inequalities are shown to determine analytic, geometric, and convergence properties of the "Landau type polynomials" for a continuous function on a compact interval.

## INTRODUCTION

After slightly altering the approximating polynomials in Landau's proof of the Weierstrass theorem we show that each of the redefined *Landau polynomials* $\tilde{f}_n$ for a continuous function $f$ on $[a, b]$ has any nice property on $[a, b]$ that $f$ has. This is done using distributional inequalities. Also, we prove that if the $m + 2$th distributional derivative of $f$ is a measure, then $f \in C^m(a, b)$ and the sequence of $m$th derivatives $\{\tilde{f}_n^{(m)}\}$ converges uniformly to $f^{(m)}$ on compact subintervals of $[a, b]$.

## 1. NOTATION AND PRELIMINARIES

The space of test functions on the open interval $(a, b)$ is denoted by $\mathscr{D}(a, b)$ and its dual space, $\mathscr{D}'(a, b)$, is the set of distributions in $(a, b)$. For $f \in L^1_{\mathrm{loc}}(a, b)$ (the space of locally Lebesgue integrable functions on $(a, b)$), $T_f$ is its associated distribution in $\mathscr{D}'(a, b)$ and $\mathscr{D}^n T_f$ is the $n$th distributional derivative of $f$. For $T \in \mathscr{D}'(a, b)$, $T \geqslant 0$ in $(a, b)$ means that $T$ is positive semidefinite (when $T$ is positive definite we write $T > 0$). For regular distributions $T_f$ and $T_g$ in $\mathscr{D}'(a, b)$, $\mathscr{D}^m T_f \geqslant T_g$ in $(a, b)$ means $\mathscr{D}^m T_f - T_g \geqslant 0$ in $(a, b)$.

$C^n(a, b)$ is the space of $n$-times continuously differentiable functions on $(a, b)$ and $C[a, b]$ denotes the space of continuous functions on $[a, b]$.

A function $f$ is said to be locally Lipschitzian on $(a, b)$ if for each compact subinterval $[c, d]$ of $(a, b)$ there exists a constant $M$ (depending on $[c, d]$) such that $|f(x) - f(y)| \leqslant M |x - y|$ for all $x$ and $y$ in $[c, d]$.





ISSN 0021-9045

Let $Q_n(x)$ be $(1 - x^2)^n$ for $|x| \leqslant 1$ and 0 elsewhere. Set $1/c_n = \int_{-1}^{1} Q_n(x)\,dx$. Then $K_n = c_n Q_n$ forms a *Dirac sequence* $\{K_n\}$ and the functions $K_n$ are called the *Landau kernels.*

For $f \in C[a, b]$, let $L_f$ be the function determined by the line through the points $(a, f(a))$ and $(b, f(b))$, and let $\bar{f} = f - L_f$ on $[a, b]$ and vanish outside of $[a, b]$. Landau's proof of the Weierstrass theorem originally given in [5] (compare [2, p. 214]) essentially states: (1) If $f \in C[a, b]$, then $L_f + \bar{f} * \tilde{K}_n \to f$ uniformly on $[a, b]$; and (2) for each $n$, $L_f + \bar{f} * \tilde{K}_n$ is a polynomial of degree $\leqslant 2n$ on $[a, b]$, where $\tilde{K}_n(t) = (b - a)^{-1} K_n(t/(b - a))$ and $\bar{f} * \tilde{K}_n$ is the convolution

$$\int_{-\infty}^{+\infty} \bar{f}(t)\,\tilde{K}_n(x - t)\,dt = \int_{a}^{b} \bar{f}(t)\,\tilde{K}_n(x - t)\,dt.$$

Define $\tilde{f}$ by

$$\begin{aligned} \tilde{f}(x) &= f(x), && \text{if } x \in [a, b], \\ &= L_f(x), && \text{if } x \notin [a, b]. \end{aligned}$$

Since $L_f * \tilde{K}_n = L_f$, one can easily show that $\tilde{f} * \tilde{K}_n = L_f + (\tilde{f} - L_f) * \tilde{K}_n = L_f + \bar{f} * \tilde{K}_n$, and we shall denote the polynomial $\tilde{f} * \tilde{K}_n$ on $[a, b]$ by $f_n$, and the polynomial $f_n$ is called the $n$th *Landau polynomial* for $f$.

For any polynomials $p$ and $q$ such that $p(a) = f(a)$ and $q(b) = f(b)$, let $f(p; q)$ be the extension of $f$ defined by

$$\begin{aligned} f(p; q)(x) &= p(x), && \text{if } x < a \\ &= f(x), && \text{if } x \in [a, b] \\ &= q(x), && \text{if } x > b. \end{aligned}$$

Let $f_n(p; q)$ denote the convolution $f(p; q) * \tilde{K}_n$. It is easy to see that $f_n(p; q)$ is a polynomial of degree $\leqslant 2n + 1 + \max \operatorname{degr}(p, q)$. Also, since $\{\tilde{K}_n\}$ is a *Dirac sequence* and since $\tilde{K}_n(x - t)$ as a function of $t$ has support in $[2a - b, 2b - a]$ for each $x \in [a, b]$, $f_n(p; q) \to f$ uniformly on $[a, b]$ and

$$f_n(p; q)(x) = \int_{2a-b}^{2b-a} f(p; q)(t)\,\tilde{K}_n(x - t)\,dt.$$

We shall call $f_n(p; q)$ a redefined *Landau polynomial.* Note that when $p = q = L_f$, then $f_n(p; q)$ is the original $n$th *Landau polynomial* $f_n$. However, when $f$ is a polynomial of degree $i$, $f_n$ is usually of degree $2n$, but for each $n$ the redefined Landau polynomial $f_n(f; f)$(with $p = q = f$) is of degree $i$ and has the same $i$th and $(i - 1)$th derivatives as $f$.

## 2.

The following lemma is well known (see [1, p. 106] or [3, p. 54]).

LEMMA 1. *Let $T$ be in $\mathscr{D}'(a, b)$.*

(i) *$\mathscr{D}T \geqslant 0$ in $(a, b)$ if and only if $T$ is defined by a nondecreasing function in $(a, b)$.*

(ii) *$\mathscr{D}^2 T \geqslant 0$ in $(a, b)$ if and only if $T$ is defined by a convex function in $(a, b)$.*

*Remark.* Strict inequality (i.e., positive definiteness) in Lemma 1 characterizes the increasing functions and the strictly convex functions.

Let $L$ be defined by $L(t) = (b - a)t + a$.

THEOREM 2. *Let $f$ and $g$ be in $C(a, b)$ and suppose that for some integer $m \geqslant 0$,*

$$\mathscr{D}^m T_f \geqslant T_g \qquad \text{in } (a, b).$$

(i) *If $m = 1$, then $f$ is locally of bounded variation in $(a, b)$.*

(ii) *If $m \geqslant 2$, then $f \in C^{m-2}(a, b)$ and $f^{(m-2)}$ is locally Lipschitzian on $(a, b)$.*

(iii) *If $m$ is any nonnegative integer, then*

$$\mathscr{D}^m T_{f \circ L} \geqslant (b - a)^m \, T_{g \circ L} \qquad \text{in } (0, 1).$$

*Proof.* (i) Let $\zeta$ be such that $\zeta' = g$ in $(a, b)$. Then $\mathscr{D}T_{f-\zeta} \geqslant 0$ in $(a, b)$, and by Lemma 1(i) we see that $f - \zeta$ is nondecreasing. The result now follows from the monotonicity of $f - \zeta$ since $\zeta \in C^1(a, b)$.

(ii) Let $H$ be any solution to the differential equation $y^{(m)} = g$ in $(a, b)$. Then $\mathscr{D}^m T_{f-H} \geqslant 0$ since $\mathscr{D}^m T_H = T_g$. By Lemma 1(ii), we can conclude that the distribution $\mathscr{D}^{m-2} T_{f-H}$ is defined by a convex function $\kappa$ on $(a, b)$. Then the continuity of $f$ implies that $(f - H)^{(m-2)} = \kappa$. Since convex functions are locally Lipschitzian and $H^{(m)}$ is continuous, we obtain that $f^{(m-2)}$ exists and is locally Lipschitzian.

(iii) $\mathscr{D}^m T_f \geqslant T_g$ in $(a, b)$ implies that for any nonnegative test function $\phi$ on $(0, 1)$,

$$\begin{aligned}
\mathscr{D}^m T_{f \circ L}(\phi) &= (-1)^m \int_0^1 f((b - a)\, t + a)\, \phi^{(m)}(t)\, dt \\
&= \frac{(-1)^m}{(b - a)} \int_a^b f(u)\, \phi^{(m)} \left( \frac{u - a}{b - a} \right) du \\
&= (b - a)^{m-1}\, \mathscr{D}^m T_f \left( \phi \left( \frac{u - a}{b - a} \right) \right) \\
&\geqslant (b - a)^{m-1}\, T_g \left( \phi \left( \frac{u - a}{b - a} \right) \right),
\end{aligned}$$

since the function defined by $u \to \phi((u-a)/(b-a)) \geqslant 0$ is in $\mathscr{D}(0, 1)$. Then because $T_g(\phi((u-a)/(b-a))) = (b-a)\, T_{g\circ L}(\phi)$, we have that $\mathscr{D}^m T_{f\circ L} \geqslant (b-a)^m\, T_{g\circ L}$ in $(0, 1)$. The proof is complete.

THEOREM 3. *Let $f$ be in $C[a, b]$ and suppose that for some integer $m \geqslant 0$, there exists $g \in C[a, b]$ such that*

$$\mathscr{D}^m T_{\tilde{f}} \geqslant T_{\tilde{g}} \qquad \text{in } (2a-b, 2b-a).$$

*Then for each integer $n > m$, $f_n^{(m)} \geqslant g_n$ holds in $(a, b)$, where $\{f_n\}$ and $\{g_n\}$ are the* Landau polynomials *for $f$ and $g$, respectively, on $[a, b]$. Furthermore, $f' \in L^1_{\text{loc}}(a, b)$ when $m = 1$ and when $m \geqslant 2$, $f \in C^{m-2}(a, b)$ and $f_n^{(i)} \to f^{(i)}$ uniformly on compact subintervals of $(a, b)$ for each $i$ $(1 \leqslant i \leqslant m-2)$.*

*Proof.* First assume that $[a, b] = [0, 1]$ and let $K_{n,\epsilon}(x-t)$ be the regularizations (as defined in [1, p. 56]) of $K_n(x-t)$ for fixed $x \in (0, 1)$. For $\epsilon$ sufficiently small $K_{n,\epsilon}(x-t)$ is a nonnegative test function in $\mathscr{D}(-1, 2)$ as a function of $t$ for $x \in (0, 1)$ because $K_n(x-t) \geqslant 0$ and has support $= [x-1, x+1] \subset (-1, 2)$. Furthermore, for $n > m$ and fixed $x \in (0, 1)$ we have that $K_{n,\epsilon}^{(m)}(x-t) \to K_n^{(m)}(x-t)$ uniformly on $[-1, 2]$ as $\epsilon \to 0$. Now we can conclude that for $n > m$,

$$\begin{aligned}
f_n^{(m)}(x) &= \tilde{f} * K_n^{(m)}(x) = \int_{-1}^{2} \tilde{f}(t)\, K_n^{(m)}(x-t)\, dt \\
&= (-1)^m \int_{-1}^{2} \tilde{f}(t)\, D_t^{(m)}(K_n(x-t))\, dt \\
&= \lim_{\epsilon\to 0} (-1)^m \int_{-1}^{2} \tilde{f}(t)\, D_t^{(m)}(K_{n,\epsilon}(x-t))\, dt \\
&= \lim_{\epsilon\to 0} \mathscr{D}^m T_{\tilde{f}}(K_{n,\epsilon}(x-t)) \geqslant \lim_{\epsilon\to 0} T_{\tilde{g}}(K_{n,\epsilon}(x-t)) \\
&= \tilde{g} * K_n(x) = g_n(x) \text{ for } x \in (0, 1).
\end{aligned}$$

The differentiability properties follow from Theorem 2 and permit us to conclude that $f_n^{(i)} = \tilde{f}^{(i)} * K_n \to f^{(i)}$ uniformly on compact subintervals of $(0, 1)$ because $f^{(i)}$ is in $C(0, 1)$(see [2, p. 212]).

In the general case when $[a, b]$ is any compact interval, we have from parts (ii) and (iii) of Theorem 2 that $f \in C^{m-2}(a, b)$ (when $m \geqslant 2$), and

$$\mathscr{D}^m T_{\tilde{f}\circ L} \geqslant (b-a)^m\, T_{\tilde{g}\circ L} \qquad \text{in } (-1, 2).$$

Let

$$L^{-1}(x) = \frac{x-a}{b-a}\,;$$

then $(f \circ L)_n(L^{-1}(x)) = f_n(x) = (\tilde{f} * \tilde{K}_n)(x)$ for $x \in [a, b]$, and it follows from the part of the theorem already proved that

$$f_n^{m}(x) = (f \circ L)_n^{(m)}(L^{-1}(x)) \frac{1}{(b-a)} m \geqslant (g \circ L)_n(L^{-1}(x))$$
$$= g_n(x) = \tilde{g} * \tilde{K}_n(x).$$

Since $f_n^{(i)} = \tilde{f}^{(i)} * \tilde{K}_n$, we also obtain from the case $[a, b] = [0, 1]$ that $f_n \to f$ uniformly on $[a, b]$ and $f_n^{(i)} \to f^{(i)}$ uniformly on compact subintervals of $(a, b)$ for each $i$ $(1 \leqslant i \leqslant m - 2)$.

*Remark.* Note that $\tilde{f}$ is monotone on $(2a - b, 2b - a)$ whenever $f$ is monotone on $(a, b)$. So Lemma 1 and Theorem 3 imply that each of the original Landau polynomials $f_n$ is monotone on $[a, b]$ when $f$ is.

THEOREM 4. *Let $f$ be in $C[a, b]$ and suppose that for some integer $m \geqslant 0$, there exists $g \in C[a, b]$ such that*

$$\mathscr{D}^m T_f \geqslant T_g \qquad \text{in } (a, b).$$

*Then there exists a sequence of redefined Landau polynomials $\{\tilde{f}_n\}$ depending on $m$ and $g$ such that $\tilde{f}_n^{(m)} \geqslant \tilde{g}_n$ holds in $(a, b)$ for each $n$, where $\{\tilde{g}_n\}$ is a redefined sequence of Landau polynomials which converges uniformly to $g$ on $[a, b]$. Furthermore, $f' \in L^1_{\text{loc}}(a, b)$ when $m = 1$ and when $m \geqslant 2$, $f \in C^{m-2}(a, b)$ and $\tilde{f}_n^{(i)} \to f^{(i)}$ uniformly on compact subintervals of $(a, b)$ for each $i(0 \leqslant i \leqslant m - 2)$.*

*Proof.* The differentiability properties of $f$ follow from Theorem 2. From the proof of (ii) of that theorem we know that $\kappa = (f - H)^{(m-2)}$ is convex on $(a, b)$. Since $\kappa$ is convex, it is differentiable a.e., and the left- and right-hand derivatives exist everywhere in $(a, b)$ and are nondecreasing. Thus $\kappa''$ is defined a.e. in $(a, b)$. This implies that $f^{(m)}$ exists a.e. in $(a, b)$ because $H \in C^m(a, b)$.

Let $\{(a_k, b_k)\}$ be a sequence of intervals which increase to $(a, b)$ and such that $f^{(m)}$ exists at each $a_k$ and $b_k$. For each $k$, define $\bar{f}(p_k; q_k)$ by

$$\bar{f}(p_k; q_k)(x) = p_k(x), \qquad \text{if } x < a_k,$$
$$= f(x), \qquad \text{if } x \in [a_k, b_k],$$
$$= q_k(x), \qquad \text{if } x > b_k,$$

where $p_k$ and $q_k$ are the $m$th Taylor polynomials for $f$ at $a_k$ and $b_k$, respectively. Let $g_k$ be $g(a_k)$ for $x < a_k$, $g(b_k)$ for $x > b_k$, and agree with $g$ on $[a_k, b_k]$. If $H_k$ is a $C^m$ function such that $H_k^{(m)} = g_k$, then $\kappa_k = (\bar{f}(p_k; q_k) - H_k)^{(m-2)}$ is convex on $(-\infty, +\infty)$ because $\kappa_k'' \geqslant 0$ outside of $[a_k, b_k]$, and because $\kappa_k$ equals $(f - H)^{(m-2)}$ on $(a_k, b_k)$ and is differentiable at the endpoints. Now it follows from Lemma 1 and $(m - 2)$ integrations by parts that the $m$th distributional derivative of $\tilde{f}(p_k ; q_k)$ is greater than or equal to (in the sense of distributions) the distribution defined by $g_k$ for each integer $k \geqslant 0$. From this we obtain, just as in the proof of Theorem 3, that $(\tilde{f}(p_k; q_k) * \tilde{K}_n)^{(m)} \geqslant g_k * \tilde{K}_n$ holds in $(a, b)$ for any $n \geqslant 0$. Hence the sequence defined by $\tilde{f}_n = \tilde{f}(p_n ; q_n) * \tilde{K}_n$ satisfies $\tilde{f}_n^{(m)} \geqslant \tilde{g}_n$ in $(a, b)$ and has the desired convergence properties since $\tilde{f}(p_n; q_n) = f$ (for $n$ sufficiently large) on each compact subinterval of $(a, b)$ and $\tilde{g}_n = g_n * \tilde{K}_n \to g$ uniformly on $[a, b]$. This completes the proof.

*Remark.* Lemma 1 and Theorems 2, 3, and 4 remain valid when we reverse the inequalities and replace the words "nondecreasing" and "convex" by "nonincreasing" and "concave" wherever they appear.

Theorem 3 shows that the original Landau polynomials inherit their properties from $\tilde{f}$, so, for example, they might not be convex on $[a, b]$ when $f$ is convex on $[a, b]$. Theorem 4 shows how to approximate by redefined Landau polynomials having one prescribed property on $[a, b]$ possessed by $f$. The following theorem treats (simultaneously) special prescribed properties.

THEOREM 5. *Let $f$ be in $C[a, b]$. If $f$ has any of the properties*

(i) *odd or even where $b = -a > 0$,*

(ii) *nondecreasing or nonincreasing,*

(iii) *convex or concave*

*on $[a, b]$, then there exists a sequence of redefined Landau polynomials $\{\tilde{f}_n\} \to f$ uniformly on $[a, b]$ such that each $\tilde{f}_n$ has the corresponding property on $[a, b]$.*

*Proof.* For $2/k \leqslant b - a$, let $p_k$ be the function determined by the line through the points $(a, f(a))$ and $(a_k, f(a_k))$, and let $q_k$ be determined by the line through $(b, f(b))$ and $(b_k, f(b_k))$, where $a_k = a + 1/k$ and $b_k = b - 1/k$. Then it is easy to see that each $\tilde{f}(p_k ; q_k)$ (as defined in Theorem 4) inherits on $[2a - b, 2b - a]$ any of the properties (i), (ii), and (iii) from $f$. (By renumbering we may assume that $\tilde{f}(p_k ; q_k)$ is defined for each positive integer $k$.) Now it follows, from Lemma 1, Theorem 4, and obvious facts about the convolutions of even or odd functions with even functions, that $\tilde{f}(p_k ; q_k) * \tilde{K}_n$ inherits on $[a, b]$ any of these properties from $f$ for any positive integers $n$ and $k$. Since $\tilde{f}(p_k ; q_k) \to f$ uniformly on $[a, b]$ and $\{\tilde{K}_n\}$ is a *Dirac sequence*, we can select from $\{\tilde{f}(p_k ; q_k) * \tilde{K}_n\}$ a sequence $\{\tilde{f}_n\}$ which converges uniformly to $f$ on $[a, b]$.

*Remarks.* (1) It is obvious that if each of the polynomials satisfies the hypotheses of either Theorem 3 or Theorem 4 or has any of the properties in Theorem 5, then $f$ does also.

(2) The results given here easily generalize

(i) when $f \in L^1[a, b]$ and convergence is in $L^1$ norm;

(ii) in higher dimensions, when the functions are continuous on a finite product of compact intervals (see [4, p. 123]).

(3) Since the function $-x^{1/2}$ is convex on [0, 1] but cannot be extended to a convex function on $[a, 1]$ for any $a < 0$, we see why the convolutions $\bar{f}(p_k; q_k) * \tilde{K}_n$ were used in the proof of Theorem 5.

(4) The degree of approximation of continuous functions by certain convolution-type operators (including the Landau kernel) can be found in [6].

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