Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 261 (2011) 408–450

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Regular singular Sturm–Liouville operators and their zeta-determinants ☆

Matthias Lesch, Boris Vertman *

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Received 4 June 2010; accepted 18 March 2011

Available online 9 April 2011

Communicated by I. Rodnianski

Abstract

We consider Sturm–Liouville operators on the line segment [0, 1] with general regular singular potentials and separated boundary conditions. We establish existence and a formula for the associated zeta-determinant in terms of the Wronski-determinant of a fundamental system of solutions adapted to the boundary conditions. This generalizes the earlier work of the first author, treating general regular singular potentials but only the Dirichlet boundary conditions at the singular end, and the recent results by Kirsten–Loya–Park for general separated boundary conditions but only special regular singular potentials.

Keywords: Regular singular Sturm–Liouville operators; Zeta-determinants; Boundary conditions; Spectral theory

Contents

☆ Both authors were partially supported by the Hausdorff Center for Mathematics. The second author thanks Stanford University for hospitality.

* Corresponding author.

E-mail addresses: ml@matthiaslesch.de, lesch@math.uni-bonn.de (M. Lesch), vertman@math.uni-bonn.de (B. Vertman).

URLs: http://www.matthiaslesch.de, http://www.math.uni-bonn.de/people/lesch (M. Lesch), http://www.math.uni-bonn.de/people/vertman (B. Vertman).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.03.011

1. Introduction and formulation of the result

In this paper we will investigate the zeta-determinant of Sturm–Liouville operators of the form

$$H = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2} + \frac{1}{x} V(x), \quad \operatorname{Re} \nu \geqslant 0,\ x \in (0,1),$$

the regularity assumptions on V will be minimal.

Such operators are sometimes also called Bessel operators. H is the prototype of a differential expression with one regular singularity and hence it appears naturally in the classical theory of ordinary differential equations with regular singularities [12]. The physical relevance of H stems from the fact that it arises when separation of variables is used for the radial Schrödinger operator in Euclidean space.

Quite recently there has been a lot of interest in the inverse spectral theory of H, see e.g. [24,23,8] and the references therein.

Our motivation for looking at the zeta-determinant of H comes from geometry: Spectral geometry on manifolds with singularities has been initiated by Cheeger in his seminal papers [10,11]. Manifolds with conical singularities are an important case study for this general programme. Separation of variables for the Laplacian on a cone leads to an infinite sum of Bessel type operators like H above. Recently, there has been a revived interest in extending the celebrated Cheeger–Müller Theorem [9,29] on the equality of the analytic torsion and the Reidemeister torsion to manifolds with conic singularities [13,14,17,34,35].

The separation of variables mentioned above leads naturally to the problem of determining the zeta-determinant of a single regular singular Sturm–Liouville operator on the line segment [0, 1] with separated boundary conditions. We only make minimal regularity assumptions on the potential. Nevertheless, we establish existence and a formula for the associated zeta-determinant in terms of the Wronskian of a fundamental system of solutions adapted to the boundary conditions, see Theorem 1.5 below.

We emphasize that for the calculation of the analytic torsion or the zeta-determinant on a cone additional considerations are necessary. This is because on a cone one has to deal with an *infinite* direct sum of operators like H.

The fundamental results of Brüning and Seeley in [4,5] guarantee the existence of zeta-determinants for regular singular Sturm–Liouville operators with Dirichlet boundary conditions at the singularity. However, Brüning and Seeley in [4,5] require the potential to be of the form $a(x)/x^2$ with $a(x)$ smooth up to 0. For such operators Theorem 1.5 was proved in [26] by the first author, generalizing earlier results by Burghelea, Friedlander and Kappeler [2] to the regular singular setting.

The method of [26] is limited to the Friedrichs extension at the singularity. In a recent series of papers Kirsten, Loya, and Park [22,21,20] were able to calculate the zeta-determinant for an explicit example of a regular singular Sturm–Liouville operator with general self-adjoint boundary conditions; cf. also the subsequent discussion by the second author in [34] and in the appendix to [21]. Their method, however, is based on an intricate analysis of Bessel functions and is therefore limited to their explicit potential.

The main result of this paper combines and generalizes these two results, however only for scalar valued potentials. Since we deal with a rather general class of potentials it is natural that our method is closer to that of [26]. Special functions are used in this paper only implicitly as we are using the formula from [26] for the zeta-determinant of the Friedrichs extension of the regular singular model operator $l_\nu = -\frac{d^2}{dx^2} + (\nu^2 - 1/4)/x^2$.

The paper is organized as follows. In the remainder of this section we will introduce some notation, explain the basic concepts of regularized integrals and zeta-determinants, and we will formulate our main result. In Section 2 we derive the asymptotic behavior of a fundamental system for H, slightly generalizing a result due to Bôcher [3].

In Section 3 we study the maximal domain of H and its closed extensions with separated boundary conditions. Let $H(\theta_0, \theta_1)$ be such an extension, θ_0, θ_1 stand for the boundary conditions at 0, 1 respectively. We give criteria under which it is possible to factorize $H(\theta_0, \theta_1)$ into a product $D_1 D_2$ of closed extensions of first order regular singular differential operators. We prove a comparison result for the Wronskians of normalized fundamental solutions for $D_1 D_2$ and $D_2 D_1$.

In Section 4 we discuss the asymptotic expansion of the resolvent trace. We start with the Friedrichs extension. The resolvent of the Friedrichs extension L_ν of the regular singular model operator l_ν is explicitly known and rather well behaved with respect to perturbations of the form $X^{-1}V$. From [4] we only use the result that the resolvent of the model operator L_ν has a complete asymptotic expansion. The expansion of the resolvent of $L_\nu + X^{-1}V$ then follows by a perturbation analysis. Boundary conditions other than the Friedrichs extension at 0 are more subtle since the resolvent does not absorb high enough negative powers of x. For the resolvent of general boundary conditions we therefore use the factorization results of Section 3. In addition one needs to treat compactly supported L^2-perturbations of factorizable operators. For this we employ a standard method of pasting together local resolvents, cf. [27, Appendix].

In Section 5 we derive a variational formula for the dependence of the zeta-determinant under variation of the potential. The method is well known [2,26]. However, due to the low regularity assumptions on the potential and due to the singularity of the operator the analysis becomes a little delicate. In particular we have to analyze the dependence of a normalized fundamental system (and its asymptotic behavior near 0) on the parameter. At the end of Section 5 we compile the established results to a proof of the main Theorem 1.5.

1.1. Function and distribution spaces

Following the requests of several of the referees we are going to specify in detail the notation for function and distribution spaces used throughout the paper.

Let $I \subset \mathbb{R}$ be an interval, which may be of any of the possible forms (a,b), $[a,b)$, $(a,b]$ or $[a,b]$ for real numbers $a<b$. Let $I^\circ = I \setminus \{a,b\}$ denote the interior of I.

For a map $f : I \to E$ into some vector space E the support of f, denoted by $\operatorname{supp} f$, is defined as the closure in I of $\{x \in I \mid f(x) \neq 0\}$,

$$\operatorname{supp} f := \overline{\{x \in I \mid f(x) \neq 0\}}^{I}, \tag{1.1}$$

$\operatorname{supp} f$ is always closed in I but not necessarily compact, since I might be non-compact itself.

For spaces of continuous, respectively differentiable complex-valued functions we use the standard notation $C(I)$, $C^k(I)$, $C^\infty(I)$, cf. e.g. [18, Secs. 1.1, 1.2]. The space $C_0^k(I)$, $0 \leqslant k \leqslant \infty$, denotes the subspace of those $f \in C^k(I)$ with compact support.

The space $C_0^\infty(I^\circ)$ carries a natural locally convex topology and its dual space $\mathscr{D}'(I^\circ)$ is called the space of distributions on I°.

For $T \in \mathscr{D}'(I^\circ)$ one can define $\operatorname{supp} T$ [18, Sec. 2.2]. For an *arbitrary* subset $A \subset \mathbb{R}$ one now writes $\mathscr{E}'(A) = \{T \in \mathscr{D}'(\mathbb{R}) \mid \operatorname{supp} T \subset A\}$, cf. [18, Sec. 2.3]. For the half open interval $I = (a,b]$, e.g., $T \in \mathscr{E}'((a,b])$ if there is a $\delta > 0$ such that $\operatorname{supp} T \subset (a+\delta, b]$.

For distributions it also makes sense to talk about restrictions. If $J \subset I$ are intervals and $T \in \mathscr{D}'(I^\circ)$, we put $T|_J := T \restriction C_0^\infty(J^\circ)$.

Let F be a map which assigns to each interval $I \subset \mathbb{R}$ a subspace $F(I) \subset \mathscr{D}'(I^\circ)$. Furthermore, assume that F is compatible with restrictions in the following sense: if $J \subset I$ are intervals and $f \in F(I)$, then $f|_J \in F(J)$. Then $F_{\mathrm{loc}}(I)$ denotes the space of $T \in \mathscr{D}'(I^\circ)$ such that $T|_K \in F(K)$ for each compact interval $K \subset I$. Furthermore, $F_{\mathrm{comp}}(I) := F_{\mathrm{loc}}(I) \cap \mathscr{E}'(I)$.

Example 1.1. For an interval $I \subset \mathbb{R}$ we denote by $L^p(I)$, $1 \leqslant p \leqslant \infty$, the Banach space of p-summable (equivalence classes modulo equality almost everywhere) functions with respect to Lebesgue measure; for $f \in L^p(I)$ the norm is given by

$$\|f\|_{L^p} := \left(\int_I |f(x)|^p \, dx \right)^{1/p},$$

$L^p(I)$ is naturally embedded into $\mathscr{D}'(I^\circ)$ by identifying $f \in L^p(I)$ with the distribution

$$C_0^\infty(I^\circ) \ni \phi \mapsto \int_I f \cdot \phi; \tag{1.2}$$

needless to say that (1.2) is independent of the choice of a function representative of the class f. The support of $f \in L^p(I)$ is now defined as the *closure in* $\overline{I}$ of the support of the corresponding distribution in $\mathscr{D}'(I^\circ)$. For continuous functions $C(I) \subset L^p(I)$ (each L^p-class has at most one continuous representative) the latter definition of support coincides with (1.1), assuming that I is closed.

The assignment $I \mapsto L^p(I)$ is an example for the map F discussed above. Hence, $L^p_{\mathrm{comp}}(I)$ and $L^p_{\mathrm{loc}}(I)$ are defined. Note that although $L^p(I) = L^p(\bar{I})$, we only have

$$L^p_{\mathrm{comp}}(I) \subset L^p_{\mathrm{comp}}(\bar{I}), \qquad L^p_{\mathrm{loc}}(\bar{I}) \subset L^p_{\mathrm{loc}}(I).$$

Sobolev spaces will only be used in the Hilbert space setting $p = 2$. We write $H^k(I)$ for the Sobolev space $W^{k,2}(I)$ of those $f \in L^2(I) \subset \mathscr{D}'(I^\circ)$, for which all weak distributional derivatives $\partial^j f$, $1 \leqslant j \leqslant k$, taken a priori in $\mathscr{D}'(I^\circ)$, are actually in $L^2(I)$.

1.1.1. The Schatten ideals

For a Hilbert space $\mathcal{H}$ we denote by $\mathcal{B}(\mathcal{H})$ the space of bounded and by $\mathcal{K}(\mathcal{H})$ the space of compact operators on $\mathcal{H}$. For $1 \leqslant p < \infty$ let $\mathcal{B}^p(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ be the von Neumann–Schatten ideal of p-summable operators, cf. e.g. [30, Sec. 3.4]. For $T \in \mathcal{B}^p(\mathcal{H})$ the p-norm is given by

$$\|T\|_p := \left(\operatorname{Tr}\left(T^*T\right)^{p/2}\right)^{1/p} = \left(\sum_{\lambda \in \operatorname{spec} T^*T} \lambda^{p/2}\right)^{1/p}.$$

Tr denotes the trace [30, Sec. 3.4]. We will only need $p = 1$ and $p = 2$. Operators in $\mathcal{B}^1(\mathcal{H})$ are called trace class operators and elements of $\mathcal{B}^2(\mathcal{H})$ are called Hibert–Schmidt operators. To avoid possible confusion with the L^p-norm of functions, we write $\|\cdot\|_{\mathrm{tr}}$ for the trace norm $\|\cdot\|_1$ and $\|\cdot\|_{\mathrm{HS}}$ for the Hilbert–Schmidt norm $\|\cdot\|_2$.

1.1.2. Regularized integrals

Let us briefly recall the partie finie regularization, cf. [25, Sec. 2.1], [26] and [28], of integrals on $\mathbb{R}_+ := [0, \infty)$. Let $f : (0, \infty) \to \mathbb{C}$ be a locally integrable function. Assume furthermore, that for $x \geqslant x_1$ we have a representation

$$f(x) = \sum_{j=1}^{N} f_j^\infty x^{\alpha_j} + g(x), \tag{1.3}$$

with real numbers α_j, numbered in descending order with $\alpha_N = -1$, and $g \in L^1[x_1, \infty)$. Then

$$\int_{x_1}^{R} f(x)\,dx =: \sum_{j=1}^{N-1} \frac{f_j^\infty}{\alpha_j + 1} R^{\alpha_j+1} + f_N^\infty \log R + \fint_{x_1}^{\infty} f(x)\,dx + o(1), \quad \text{as } R \to \infty, \tag{1.4}$$

$o(1)$ is the usual Landau notation for a function of R whose limit as $R \to \infty$ is zero; here we have explicitly $o(1) = \int_R^\infty g$. The regularized integral $\fint_{x_1}^\infty f(x)\,dx$ is therefore defined as the constant term in the asymptotic expansion of $\int_{x_1}^R f(x)\,dx$ as $R \to \infty$.

If for $0 < x \leqslant x_0$ we have a representation

$$f(x) = \sum_{j=1}^{M} f_j^0 x^{\beta_j} + h(x), \tag{1.5}$$

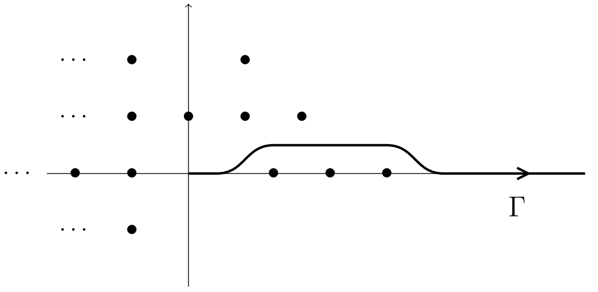

Fig. 1. The contour of integration Γ.

with real numbers $\beta_1 < \beta_2 < \cdots < \beta_M = -1$, and $h \in L^1[0, x_0]$, then

$$\int_\delta^{x_0} f(x)\,dx =: -\sum_{j=1}^{M-1} \frac{f_j^0}{\beta_j + 1}\delta^{\beta_j+1} - f_M^0 \log\delta + \fint_0^{x_0} f(x)\,dx + o(1), \quad \text{as } \delta \to 0, \tag{1.6}$$

and the regularized integral $\fint_0^{x_0} f(x)\,dx$ is defined as the constant term in the asymptotic expansion of $\int_\delta^{x_0} f(x)\,dx$ as $\delta \to 0$.

Now assume that f satisfies (1.3) and (1.5). Since f is locally integrable, it is clear that (1.4) holds for any $x_1 > 0$ and (1.6) holds for any $x_0 > 0$. One then puts for any $c > 0$

$$\fint_0^\infty f(x)\,dx := \fint_0^c f(x)\,dx + \fint_c^\infty f(x)\,dx, \tag{1.7}$$

and in fact the right-hand side is independent of $c > 0$.

1.2. The zeta-determinant

Let H be a closed not necessarily self-adjoint operator acting on some Hilbert space with $\operatorname{spec}(-H) \cap \mathbb{R}_+$ finite, $0 \notin \operatorname{spec} H$. We assume that the resolvent of H is trace class, and that for $z \in \mathbb{R}$, $z \geqslant z_0 > \max(\operatorname{spec}(-H) \cap \mathbb{R}_+)$

$$\operatorname{Tr}(H+z)^{-1} = \frac{a}{\sqrt{z}} + \frac{b}{z} + R(z) \tag{1.8}$$

with

$$\lim_{z\to\infty} zR(z) = 0, \tag{1.9}$$

$$\int_{z_0}^\infty \bigl|R(z)\bigr|\,dz < \infty. \tag{1.10}$$

Let Γ be the contour as sketched in Fig. 1, where the bullets indicate the eigenvalues of $-H$. Fix a branch of the logarithm in the simply connected domain $\mathbb{C} \setminus \{-\Gamma\}$. Note that the previous definition (1.7) of the regularized integral can easily be adapted to functions defined on the contour Γ, since there are $0 < x_0 < x_1 < \infty$ such that $[0, x_0]$ and $[x_1, \infty)$ are contained in (the image of) Γ. Consider for fixed $s \in \mathbb{C}$ the function

$$f_s(x) := x^{-s} \operatorname{Tr}(H + x)^{-1}.$$

In view of (1.8) and (1.10) it satisfies (1.3) if $\operatorname{Re} s \geqslant 0$. Furthermore, since H is assumed to be invertible, the function $x \mapsto \operatorname{Tr}(H + x)^{-1}$ is smooth up to $x = 0$ and its Taylor expansion at $x = 0$ shows that f_s satisfies (1.5) for all $s \in \mathbb{C}$.

Exploiting the definition of $⨍_\Gamma$ it is now not hard to see, cf. [28, (2.30)], that for $1 < \operatorname{Re} s < 2$ the zeta-function is given by

$$\zeta_H(s) := \sum_{\lambda \in \operatorname{spec} H \setminus \{0\}} \lambda^{-s} = \frac{\sin \pi s}{\pi} ⨍_\Gamma x^{-s} \operatorname{Tr}(H + x)^{-1} \, dx. \tag{1.11}$$

Furthermore using the asymptotic expansions as $x \to \infty$ and $x \to 0$ of $x^{-s} \operatorname{Tr}(H + x)^{-1}$ one deduces that the right-hand side of (1.11) extends meromorphically to the half plane $\operatorname{Re} s > 0$, [28, Proposition 2.1.2]. The identity (1.11) persists except for the poles of the function $s \mapsto \frac{\pi}{\sin \pi s} \zeta_H(s)$. Thanks to (1.10) the function ζ_H is differentiable from the right at $s = 0$ and one puts

$$\log \det{}_\zeta H := -\zeta_H'(0) = -⨍_\Gamma \operatorname{Tr}(H + x)^{-1} \, dx. \tag{1.12}$$

$\det_\zeta H$ is called the *zeta-regularized determinant* of H. For non-invertible H one puts $\det_\zeta H = 0$. With this setting the function $z \mapsto \det_\zeta(H + z)$ is an entire holomorphic function with zeroes exactly at the eigenvalues of $-H$. The multiplicity of a zero z equals the algebraic multiplicity of the eigenvalue z.

1.3. A regular singular operator

We now introduce the class of operators we are going to study in this paper. Put

$$l_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \quad \operatorname{Re} \nu \geqslant 0, \tag{1.13}$$

acting as an operator in the Hilbert space $L^2[0, 1]$, a priori with domain $C_0^\infty(0, 1)$. We will study perturbations of l_ν of the form

$$H = l_\nu + X^{-1} V, \tag{1.14}$$

with suitable conditions on the operator V to be specified below. X denotes the function $X(x) = x$. We view V respectively $X^{-1} V$ as a multiplication operator on functions on the unit interval. In order not to overburden the notation we will in general not distinguish between a function f and the operator of multiplication by f.

Definition 1.2. 1. For an interval $I \subset \mathbb{R}$ we denote by $AC^k(I)$, $k \geqslant 1$, the space of $f \in C(I) \subset \mathscr{D}'(I^\circ)$ such that $\partial^j f \in C(I)$, $0 \leqslant j \leqslant k-1$, and $\partial^k f \in L^1_{\mathrm{loc}}(I)$. $AC^1(I) = AC(I)$ is the well-known space of absolutely continuous functions. Note that for this definition it matters whether a boundary point $p \in \partial I$ belongs to I or not.

2. Denote by $\mathscr{V}_\nu$ the space of those $V \in L^2_{\mathrm{loc}}(0,1)$ such that

$$V \cdot \log\left(\frac{\cdot}{2}\right) \in L^1[0,1], \quad \text{if } \nu \neq 0, \tag{1.15}$$

$$V \cdot \log^2\left(\frac{\cdot}{2}\right) \in L^1[0,1], \quad \text{if } \nu = 0. \tag{1.16}$$

A natural norm on $\mathscr{V}_\nu$ is given by

$$\|V\|_{\mathscr{V}_\nu} = \left\| \log\left(\frac{\cdot}{2}\right) V \right\|_{L^1}, \quad \text{if } \nu \neq 0, \tag{1.17}$$

$$\|V\|_{\mathscr{V}_\nu} = \left\| \log\left(\frac{\cdot}{2}\right)^2 V \right\|_{L^1}, \quad \text{if } \nu = 0, \tag{1.18}$$

$\mathscr{V}_\nu$ is a Fréchet space with seminorms $\|\cdot\|_{\mathscr{V}_\nu}$ and $\|V_{|[1/n,1-1/n]}\|_{L^2}$, $n = 2, 3, \ldots$.

3. Finally, let $\mathscr{A}$ be the space of those $f \in AC^2(0,1)$ such that $f', Xf'' \in \log(\frac{\cdot}{2})^{-1} L^1[0,1]$.

Some of the results will hold under the weaker hypothesis $V \in \log(\frac{\cdot}{2})\mathscr{V}_\nu \supset \mathscr{V}_\nu$. Unless said otherwise, function spaces consist of complex valued functions.

In Section 2 we will prove the following refinement of the theorem of Bôcher [3] (Theorem 2.1 and Proposition 2.6):

Theorem 1.3. *Let $V \in \mathscr{V}_{\mathrm{Re}\,\nu}$, $H = l_\nu + X^{-1}V$, $\mathrm{Re}\,\nu \geqslant 0$, and let $\nu_1 = \nu + 1/2$, $\nu_2 = -\nu + 1/2$ be the characteristic exponents of the regular singular point 0 of the differential equation $Hg = 0$. Then there is a fundamental system g_1, g_2 of solutions to the equation $Hg = 0$ such that*

$$g_1(x) = x^{\nu_1} \widetilde{g}_1(x), \tag{1.19}$$

$$g_2(x) = \begin{cases} -\frac{1}{2\nu} x^{\nu_2} \widetilde{g}_2(x), & \text{if } \nu \neq 0, \\ \sqrt{x}\log(x)\widetilde{g}_2(x), & \text{if } \nu = 0, \end{cases} \tag{1.20}$$

where $\widetilde{g}_j \in \mathscr{A}$.

The spectra and fundamental system of solutions to Bessel type Sturm–Liouville differential expressions on finite intervals have also been studied (mainly in connection with the inverse spectral problem) in a number of recent publications [1,8,15,23,24,32].

1.4. Separated boundary conditions

Denote by H_0 the differential expression H restricted to $C_0^\infty(0,1) \subset L^2[0,1]$. Let H_0^t be the formal adjoint of H_0. This is the differential expression $-\frac{d^2}{dx^2} + \frac{\overline{\nu}^2-1/4}{x^2} + \frac{1}{x}\overline{V}(x)$ acting on $C_0^\infty(0,1)$. H_0 is symmetric if both $\nu \in \mathbb{R}$ and V is real valued.

As usual we denote by $H_{\min} = \overline{H_0}$ the closure of H_0 and by $H_{\max} = (H_0^t)^* = (H_{\min}^t)^*$. For convenience we introduce the *left minimal and right maximal domain* $\mathscr{D}_L(H)$ as the domain of the closure of $H \upharpoonright C_0^\infty(0,1]$. The *left maximal and right minimal domain* $\mathscr{D}_R(H)$ is defined accordingly with $C_0^\infty(0,1]$ replaced by $\{f \in \mathscr{D}(H_{\max}) \mid \operatorname{supp} f \subset [0,1) \text{ compact}\}$. Note that by the definition of support in (1.1), compactness of $\operatorname{supp} f \subset [0,1)$ means that $\operatorname{supp} f$ has a positive distance from the point $x = 1$.

Although there is no simple Weyl alternative in the non-self-adjoint context, we say that $x = 0$ (resp. $x = 1$) is in the limit point case for H if $\mathscr{D}_L(H) = \mathscr{D}(H_{\max})$ (resp. $\mathscr{D}_R(H) = \mathscr{D}(H_{\max})$). Otherwise, we say that it is in the limit circle case.

We will see in Section 3 that there are continuous linear functionals c_j, $j = 1, 2$, on $\mathscr{D}(H_{\max})$ such that for $f \in \mathscr{D}(H_{\max})$,

$$f = c_1(f)g_1 + c_2(f)g_2 + \widetilde{f}, \quad \widetilde{f} \in \mathscr{D}_L(H), \tag{1.21}$$

where g_1, g_2 are defined in (1.19) and (1.20).

A boundary condition at the left end point is therefore of the form

$$B_{0,\theta} f := \sin\theta \cdot c_1(f) + \cos\theta \cdot c_2(f) = 0, \quad 0 \leqslant \theta < \pi. \tag{1.22}$$

$\theta = 0$ gives the Dirichlet boundary condition (Friedrichs extension near 0).

It should be noted here that 0 is in the limit point case for H if and only if $\operatorname{Re}\nu \geqslant 1$. In this case $g_2 \notin L^2[0,1]$, $c_2 = 0$, and hence $\mathscr{D}_L(H) = \mathscr{D}(H_{\max})|$. Thus if $\operatorname{Re}\nu \geqslant 1$ we consider only the case $\theta = 0$. Boundary conditions such as (1.22) at the singular end point have been studied in depth by Rellich [31] and extended by Bulla and Gesztesy [7].

From the well-known fact that a linear second order ODE with L^1-coefficients has AC^2-solutions it follows in view of our assumptions on V that $\mathscr{D}(H_{\max}) \subset AC^2(0,1]$ and hence 1 is always in the limit circle case for H. At the right end-point we therefore impose boundary conditions of the form

$$B_{1,\theta} f := \sin\theta \cdot f'(1) + \cos\theta \cdot f(1) = 0, \quad 0 \leqslant \theta < \pi. \tag{1.23}$$

For each admissible pair $(\theta_0, \theta_1) \in [0,\pi)^2$ ($0 \leqslant \theta_0 < \pi$ if $\operatorname{Re}\nu < 1$, $\theta_0 = 0$ if $\operatorname{Re}\nu \geqslant 1$) we obtain a closed realization $H(\theta_0, \theta_1)$ of the operator with separated boundary conditions B_{0,θ_0}, B_{1,θ_1}. All eigenvalues of $H(\theta_0, \theta_1)$ are therefore simple.

Under the technical assumption that V is of determinant class, see Definition 4.4, which is satisfied for all real valued potentials $V \in \mathscr{V}_\nu$ we can prove (Theorem 4.3 and Theorem 4.10).

Theorem 1.4. *Let $\nu \geqslant 0$, $V \in \mathscr{V}_\nu$, and assume that $\theta_0 = 0$ or that V is of determinant class and $\nu > 0$. Then the resolvent of $H(\theta_0, \theta_1)$ is trace class. Moreover, there is a $z_0 > 0$ such that $H(\theta_0, \theta_1) + z$ is invertible for $z \geqslant z_0$ and there is an asymptotic expansion*

$$\operatorname{Tr}\big(H(\theta_0,\theta_1)+z\big)^{-1} = \frac{a}{\sqrt{z}} + \frac{b}{z} + R(z), \quad z \geqslant z_0,\ z \in \mathbb{R}, \tag{1.24}$$

with

$$\lim_{z\to\infty} zR(z) = 0, \tag{1.25}$$

$$\int_{z_0}^{\infty} \big|R(z)\big|\,dz < \infty. \tag{1.26}$$

In view of this theorem we may define $\det_\zeta(H(\theta_0,\theta_1))$ according to (1.12).

1.5. The main result

To explain our result we need to introduce the notion of a normalized solution at one of the end points. First, we define an invariant of the boundary operator $B_{j,\theta}$:

$$\mu_0 := \mu(B_{0,\theta}) := \begin{cases} \nu, & \text{if } \theta = 0, \\ -\nu, & \text{if } 0 < \theta < \pi; \end{cases} \tag{1.27}$$

respectively

$$\mu_1 := \mu(B_{1,\theta}) := \begin{cases} 1/2, & \text{if } \theta = 0, \\ -1/2, & \text{if } 0 < \theta < \pi. \end{cases} \tag{1.28}$$

To explain the $\pm 1/2$ we note that the right end point may artificially be viewed as a regular singular point with $\nu = 1/2$. Hence μ_j depend in fact on θ and the characteristic exponent of the regular singular point.

A solution of the homogeneous differential equation $Hg = 0$ is called *normalized at* 0 with respect to the boundary operator $B_{0,\theta}$ if $B_{0,\theta}g = 0$ and if $g(x) \sim x^{\mu_0+1/2}$, as $x \to 0$; here we use the notation

$$f(x) \sim h(x), \quad \text{as } x \to x_0 \quad :\Leftrightarrow \quad \lim_{x\to x_0} \frac{f(x)}{h(x)} = 1. \tag{1.29}$$

Similarly, g is called *normalized at* 1 with respect to the boundary operator $B_{1,\theta}$ if $B_{1,\theta}g = 0$ and if $g(x) \sim (1-x)^{\mu_1+1/2}$ as $x \to 1$. It is straightforward to check that there is always a unique normalized solution.

Theorem 1.5. *Let B_{j,θ_j}, $j = 0, 1$ be admissible boundary operators for H. Under the same assumptions as in Theorem* 1.4 *the zeta-regularized determinant of $H(\theta_0,\theta_1)$ is given by*

$$\det_\zeta\big(H(\theta_0,\theta_1)\big) = \frac{\pi}{2^{\mu_0+\mu_1}\Gamma(\mu_0+1)\Gamma(\mu_1+1)} W(\psi,\varphi). \tag{1.30}$$

Here, φ, ψ are solutions to the homogeneous differential equation $Hg = 0$ such that φ is normalized for B_{0,θ_0} (at 0*) and ψ is normalized for B_{1,θ_1} (at* 1*). Furthermore, $W(\psi,\varphi) = \psi\varphi' - \psi'\varphi$ denotes the Wronskian of ψ, φ.*

Theorem 1.5 has a relatively straightforward extension to the case where the potential has regular singularities at both end points. The proof does not require any essentially new idea; the details, however, are a bit tedious and are therefore left to the reader, cf. Remark 5.6.

The case $\nu = 0$ and $\theta_0 > 0$, which is not covered by Theorem 1.5, requires specific analysis of unusual singular phenomena in the trace expansion of H, as observed first by Falomir, Muschietti, Pisani, and Seeley [16]; see also the nice elaboration by Kirsten, Loya and Park in [20]. The discussion of the zeta-determinant in this case therefore requires another publication.

To outline the proof of Theorem 1.5 we first observe that if D_1, D_2 are closed operators in a Hilbert space then $\operatorname{spec} D_1 D_2 \cup \{0\} = \operatorname{spec} D_2 D_1 \cup \{0\}$ and, even more, non-zero eigenvalues of $D_1 D_2$ and $D_2 D_1$ have the same multiplicity. Hence if both $D_1 D_2$ and $D_2 D_1$ satisfy the general assumptions of Section 1.2 then for $z \in \mathbb{C}$,

$$\det_\zeta(D_1 D_2 + z) = z^d \det_\zeta(D_2 D_1 + z), \qquad d := \dim\ker D_1 D_2 - \dim\ker D_2 D_1. \tag{1.31}$$

We will show in Proposition 3.5 that $H(\theta_0, \theta_1)$ can always be written in the form

$$H(\theta_0, \theta_1) = D_1 D_2 + W \tag{1.32}$$

with a compactly supported L^2-potential W and D_1, D_2 suitable closed extensions of the operators

$$d_1 = \frac{d}{dx} + \frac{\omega'}{\omega}, \qquad d_2 = -\frac{d}{dx} + \frac{\omega'}{\omega} \tag{1.33}$$

with a certain function ω which is singular at 0; its properties will be described in detail in the text. The crucial point is that for the interesting case $\theta_0 > 0$ one can choose D_1, D_2 in such a way that $D_2 D_1$ also is an operator to which Theorem 1.4 applies and such that the boundary condition at 0 is the Friedrichs extension. The Friedrichs extension at 0 is much better behaved and can be treated for our class of operators basically as in [26]. The proof is completed then by employing a variation result for the behavior of the zeta-determinant under variation of the potential W (Theorem 5.4).

2. The fundamental system of a regular singular equation – Bôcher's Theorem

Consider the following regular singular model operator

$$l_\nu := -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \quad \nu \in \mathbb{C}, \tag{2.1}$$

acting on $C_0^\infty(0,1) \subset L^2[0,1]$. ν is a complex number for which without loss of generality we may assume $\operatorname{Re}\nu \geqslant 0$.

We are interested in perturbations of the form

$$H := l_\nu + X^{-1} V, \tag{2.2}$$

with $V \in L^1_{\mathrm{loc}}(0,1)$ and X denoting the function $X(x) = x$. In this section we are concerned with the description of the asymptotic behavior as $x \to 0$ of a fundamental system of solutions to the equation $Hf = 0$.

If V is *analytic*, then the classical theory of ordinary differential equations with regular singularities, cf. e.g. [12, Chap. 5], applies and the characteristic exponents of the regular singular point at $x = 0$ are given by

$$\nu_1 = \nu + 1/2, \qquad \nu_2 = -\nu + 1/2.$$

Furthermore, there is a fundamental system of solutions to $Hf = 0$ of the form

$$f_1(x) = x^{\nu_1}\widetilde{f}_1(x), \qquad f_2(x) = \begin{cases} -\frac{1}{2\nu}x^{\nu_2}\widetilde{f}_2(x), & \text{if } \nu \neq 0, \\ \sqrt{x}\log(x)\ \widetilde{f}_2(x), & \text{if } \nu = 0, \end{cases} \tag{2.3}$$

where $\widetilde{f}_j$, $j = 1, 2$, are analytic functions with $\widetilde{f}_j(0) = 1$. The normalization of solutions is chosen so that

$$W(f_1, f_2) = f_1 f_2' - f_1' f_2 = 1. \tag{2.4}$$

It is less known that already M. Bôcher [3] investigated regular singular points of ordinary differential equations with non-analytic coefficients. For Bessel operators with L^2 potentials a thorough analysis of the fundamental system of solutions was made e.g. by Carlson [8]. Bôcher's result reads as follows.

Theorem 2.1 *(M. Bôcher). Let*

$$H = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2} + \frac{1}{x}V(x),$$

where $\nu \in \mathbb{C}$, $\operatorname{Re}\nu \geqslant 0$, and $V \in \log(\frac{\cdot}{2})\mathscr{V}_\nu$. Then the differential equation $Hg = 0$ has a fundamental system of solutions g_1, g_2, such that

$$g_1(x) = x^{\nu_1}\widetilde{g}_1(x), \tag{2.5}$$

$$g_2(x) = \begin{cases} -\frac{1}{2\nu}x^{\nu_2}\widetilde{g}_2(x), & \text{if } \nu \neq 0, \\ \sqrt{x}\log(x)\widetilde{g}_2(x), & \text{if } \nu = 0, \end{cases} \tag{2.6}$$

where $\widetilde{g}_j \in C[0,1]$, $\widetilde{g}_j(0) = 1$ for $j = 1, 2$.

Furthermore,

$$g_1'(x) = \nu_1 x^{\nu_1 - 1}h_1(x), \tag{2.7}$$

$$g_2'(x) = \begin{cases} -\frac{\nu_2}{2\nu}x^{\nu_2 - 1}h_2(x), & \text{if } \nu \neq 0, \\ \frac{1}{2\sqrt{x}}\log(x)h_2(x), & \text{if } \nu = 0, \end{cases} \tag{2.8}$$

where $h_j \in C[0,1]$, $h_j(0) = 1$ for $j = 1, 2$.

Finally, with these normalizations

$$W(g_1, g_2) = g_1 g_2' - g_1' g_2 = 1.$$

Remark 2.2. In the case $\nu = 0$ the theorem as stated is slightly more general than [3], where $V \log^2 \in L^1[0,1]$ is assumed. Moreover, note that the conditions on the potential V in the theorem are satisfied whenever $V \in L^p[0,1]$, $p > 1$, or more generally $V \in \mathscr{V}_\nu$.

We briefly sketch a proof of Theorem 2.1 in modern language. Being self-contained is not the only reason for presenting the proof in some detail: the method of proof will allow a more precise analysis of the regularity properties of $\widetilde{g}_j$ (see Proposition 2.5 and Proposition 2.6 below) which will be needed later on. Furthermore, the method will be needed for deriving the variation formula for the zeta-determinant under variation of the potential (Section 5).

Proof. The regular singular operator l_ν has the following fundamental system of solutions to $l_\nu f = 0$:

$$f_1(x) = x^{\nu_1}, \qquad f_2(x) = \begin{cases} -\frac{1}{2\nu} x^{\nu_2}, & \text{if } \nu \neq 0, \\ \sqrt{x}\log x, & \text{if } \nu = 0. \end{cases}$$

For the Wronskian we have $W(f_1, f_2) = f_1 f_2' - f_1' f_2 = 1$. For a solution to $Hg = 0$ we make the ansatz

$$g_1(x) = f_1(x) + x^{\nu_1}\phi(x) = x^{\nu_1}\bigl(1 + \phi(x)\bigr).$$

Plugging this ansatz into the ordinary differential equation $Hg = 0$ yields for $\psi(x) = x^{\nu_1}\phi(x)$,

$$-\psi''(x) + \frac{\nu^2 - 1/4}{x^2}\psi(x) = -\frac{1}{x}V(x)\bigl[f_1(x) + \psi(x)\bigr], \tag{2.9}$$

thus

$$\begin{aligned} \phi(x) &= -x^{-\nu_1} f_1(x) \int_0^x f_2(y)\frac{1}{y}V(y)y^{\nu_1}\bigl[1 + \phi(y)\bigr]\,dy \\ &\quad + x^{-\nu_1} f_2(x) \int_0^x f_1(y)\frac{1}{y}V(y)y^{\nu_1}\bigl[1 + \phi(y)\bigr]\,dy \\ &=: (K_\nu V\mathbf{1})(x) + (K_\nu V\phi)(x), \end{aligned} \tag{2.10}$$

where K_ν is the Volterra operator with the kernel

$$k_\nu(x,y) = \frac{1}{2\nu}\bigl(1 - x^{-2\nu}y^{2\nu}\bigr), \quad y \leqslant x, \text{ if } \nu \neq 0, \tag{2.11}$$

$$k_0(x,y) = -\log(y) + \log(x), \quad y \leqslant x, \text{ if } \nu = 0. \tag{2.12}$$

We view $K_\nu V$ as an operator on the Banach space $C[0,1]$. Indeed, for any $\phi \in C[0,1]$ one easily checks

$$\left|K_\nu V\phi(x)\right| \leqslant \frac{1}{|\nu|}\int_0^x \left|V(y)\right|\left|\phi(y)\right| dy, \quad \text{if } \nu \neq 0, \tag{2.13}$$

$$\left|K_0 V\phi(x)\right| \leqslant 2\int_0^x \left|\log(y)\right|\left|V(y)\right|\left|\phi(y)\right| dy, \quad \text{if } \nu = 0. \tag{2.14}$$

From (2.13) and (2.14) one infers by induction

$$\left|(K_\nu V)^n\phi(x)\right| \leqslant \frac{1}{|\nu|^n n!}\|\phi\|_{\infty,[0,x]}\left(\int_0^x \left|V(y)\right| dy\right)^n, \quad \text{if } \nu \neq 0, \tag{2.15}$$

$$\left|(K_0 V)^n\phi(x)\right| \leqslant \frac{2^n}{n!}\|\phi\|_{\infty,[0,x]}\left(\int_0^x \left|V(y)\log y\right| dy\right)^n, \quad \text{if } \nu = 0. \tag{2.16}$$

Hence for any $\nu \in \mathbb{C}$, $\operatorname{Re}\nu \geqslant 0$, the Volterra operator $K_\nu V$ is a bounded operator on $C[0,1]$ with spectral radius zero. Consequently Eq. (2.10) has a unique solution in $C[0,1]$ given by

$$\phi = (I - K_\nu V)^{-1} K_\nu V \mathbf{1}. \tag{2.17}$$

Moreover, by (2.15) and (2.16) one has

$$\left|\phi(x)\right| \leqslant \begin{cases} C_1 \int_0^x |V(y)|\, dy, & \text{if } \nu \neq 0, \\ C_2 \int_0^x |V(y)\log y|\, dy, & \text{if } \nu = 0, \end{cases} \tag{2.18}$$

for some constants C_1, C_2, not depending on V. This proves that

$$g_1(x) = x^{\nu_1}\big(1 + \phi(x)\big) = x^{\nu_1}\widetilde{g}_1(x),$$

is indeed a non-trivial solution to $Hg = 0$ with $\widetilde{g}_1 \in C[0,1]$ and $\widetilde{g}_1(0) = 1$. To see (2.7), note that by (2.10) ϕ is absolutely continuous in $(0,1)$ with its derivative given by

$$\phi'(x) = x^{-2\nu-1}\int_0^x y^{2\nu} V(y)\big(1 + \phi(y)\big)\, dy, \quad \text{for all } \operatorname{Re}\nu \geqslant 0. \tag{2.19}$$

This implies

$$\left|\phi'(x)\right| \leqslant \frac{C}{x}\int_0^x \left|V(y)\right| dy$$

and thus (2.7) and the claims about g_1 are proved.

The second solution g_2 can now be constructed as usual by putting near $x=0$,

$$g_2(x) = C(x) g_1(x), \tag{2.20}$$

where

$$C(x) = -\int_x^{x_0} g_1^{-2}(y)\,dy = \begin{cases} -\frac{1}{2\nu} x^{-2\nu_1+1} \widetilde{C}(x), & \text{if } \nu \neq 0, \\ \log(x)\widetilde{C}(x), & \text{if } \nu = 0, \end{cases} \tag{2.21}$$

$\widetilde{C}$ is continuous over $[0, x_0)$ and $x_0 \in (0,1]$ is chosen so that $\widetilde{g}_1(y) \neq 0$ for $0 < y \leqslant x_0$. Such an x_0 exists, since $\widetilde{g}_1(0) = 1$ and $\widetilde{g}_1 \in C[0,1]$. It is then straightforward to check that g_2 extends to a solution to $lg = 0$ on $(0,1]$ which has the claimed properties. □

Now we come to the aforementioned improvement of the regularity properties of $\widetilde{g}_j(x)$ as $x \to 0$.

Lemma 2.3. *Let $f \in AC^2(0,1)$ with $f', Xf'' \in L^1[0,1]$. Then $(Xf')(0) = 0$. This holds in particular for $f \in \mathscr{A}$ (cf. Definition* 1.2*).*

Proof. By assumption the function $F(x) := xf'(x) - \int_0^x h(s)\,ds$, $h := (Xf')'$, is locally absolutely continuous and $F' = 0$, hence $F = (Xf')(0) =: c$ is constant. Thus

$$f'(x) = \frac{c}{x} + \frac{1}{x}\int_0^x h(s)\,ds. \tag{2.22}$$

By assumption we have $f', h \in L^1[0,1]$. Thus

$$\begin{aligned} f(1) - f(x) &= \int_x^1 f'(s)\,ds \\ &= -c\log x - \log x \int_0^x h(s)\,ds - \int_x^1 h(s)\log s\,ds \\ &= -c\log x + o(\log x), \quad \text{as } x \to 0, \end{aligned} \tag{2.23}$$

since for $0 < \delta < 1$ we have $|\int_x^1 h(s)\log s\,ds| \leqslant C_\delta + |\log x| \int_0^\delta |h|$. Because the left-hand side of (2.23) is bounded it follows that $c = 0$.

The last claim follows, since $f \in \mathscr{A}$ implies $f', (Xf')' \in L^1[0,1]$. □

Lemma 2.4. *Let $\alpha \in \mathbb{C}$ and $\rho \in \log(\frac{\cdot}{2})\mathscr{V}_{\operatorname{Re}\alpha}$. Put*

$$f(x) := \begin{cases} x^{-\alpha-1}\int_0^x y^\alpha \rho(y)\,dy, & \textit{if } \operatorname{Re}(\alpha) \geqslant 0, \\ -x^{-\alpha-1}\int_x^1 y^\alpha \rho(y)\,dy, & \textit{if } \operatorname{Re}(\alpha) < 0. \end{cases} \tag{2.24}$$

Then we have

$$f \in L^1[0,1] \cap AC(0,1), \qquad Xf' \in L^1[0,1], \qquad (Xf)(0) = 0.$$

If $\rho \in \mathscr{V}_{\mathrm{Re}\,\alpha}$ then $f, Xf' \in \log(\frac{\cdot}{2})^{-1} L^1[0,1]$, that is $\int_0^{\cdot} f \in \mathscr{A}$.

Proof. Integration by parts shows easily that $f \in L^1[0,1]$ (resp. $f \in \log(\frac{\cdot}{2})^{-1} L^1[0,1]$ if $\rho \in \mathscr{V}_{\mathrm{Re}\,\alpha}$). Moreover, clearly f and hence also Xf are both locally absolutely continuous in the interval $(0,1]$. Furthermore, we have

$$Xf' = (Xf)' - f = -(\alpha+1)f + \rho \in L^1[0,1]$$
$$\left(\text{resp. } \in \log\left(\frac{\cdot}{2}\right)^{-1} L^1[0,1] \text{ if } \rho \in \mathscr{V}_{\mathrm{Re}\,\alpha}\right).$$

$(Xf)(0) = 0$ follows from Lemma 2.3 applied to $\int_0^{\cdot} f$. □

Proposition 2.5. *In the setup and notation of Theorem* 2.1 *we have for $j = 1, 2$,*

$$\widetilde{g}_j \in AC[0,1], \qquad X\widetilde{g}_j' \in AC[0,1], \qquad \big(X\widetilde{g}_j'\big)(0) = 0.$$

Proof. We have for the first fundamental solution

$$g_1(x) = x^{\nu_1}\widetilde{g}_1 = x^{\nu_1}\big(1 + \phi(x)\big),$$

where by ϕ is given by (2.10). The claim about $\widetilde{g}_1$ now follows from Lemma 2.4 and the explicit form of the derivative (2.19). To prove the claim for $\widetilde{g}_2$, recall that for some $x_0 \in (0,1]$, such that $g_1(y) \neq 0$ for $0 < y \leqslant x_0$, the second fundamental solution is given by

$$g_2(x) = -g_1(x) \int_x^{x_0} g_1(y)^{-2}\, dy.$$

If $\nu \neq 0$, then

$$\begin{aligned}
\widetilde{g}_2(x) &= -2\nu x^{\nu-1/2} g_2(x) \\
&= 2\nu \widetilde{g}_1(x) x^{2\nu} \int_x^{x_0} y^{-2\nu-1} \widetilde{g}_1(y)^{-2}\, dy \\
&=: \widetilde{g}_1(x) f(x).
\end{aligned} \tag{2.25}$$

In view of the statement being proved for $\widetilde{g}_1$ before and since the claimed properties are preserved under multiplication, it suffices to prove the claim for f. Integration by parts gives

$$f(x) = -x^{2\nu}\big(y^{-2\nu}\widetilde{g}_1(y)^{-2}\big)\big|_x^{x_0} + x^{2\nu}\int_x^{x_0} y^{-2\nu}\rho(y)\,dy$$
$$= c(x_0)x^{2\nu} + \widetilde{g}_1(x)^{-2} + x^{2\nu}\int_x^{x_0} y^{-2\nu}\rho(y)\,dy, \tag{2.26}$$

where $\rho = (\widetilde{g}_1^{-2})' \in L^1[0, x_0]$.

The first two summands are a priori in $AC[0, x_0]$. The last one is $AC[0, x_0]$ by Lemma 2.4. Furthermore, from the definition we infer

$$Xf' = 2\nu f - 2\nu\widetilde{g}_1^{-2} \in AC[0, x_0].$$

$(Xf')(0) = 0$ then follows from Lemma 2.3.

Clearly, $\widetilde{g}_2$ and $X\widetilde{g}_2'$ are locally absolutely continuous in the whole interval $(0, 1]$ and hence the claim is proved for $\widetilde{g}_2$ and $\nu \neq 0$.

Finally, for $\nu = 0$ we have

$$\widetilde{g}_2(x) = -\frac{1}{\sqrt{x}\log x} g_1(x)\int_x^{x_0} y^{-1}\widetilde{g}_1(y)^{-2}\,dy$$
$$= -\frac{1}{\log x}\widetilde{g}_1(x)\int_x^{x_0} y^{-1}\widetilde{g}_1(y)^{-2}\,dy$$
$$=: \widetilde{g}_1(x) f(x).$$

Again, it suffices to prove the claim for f. We compute with $\rho := (\widetilde{g}_1^{-2})' \in L^1[0, x_0]$ as before

$$f(x) = -\frac{1}{\log x}\int_x^{x_0} y^{-1}\widetilde{g}_1(y)^{-2}\,dy$$
$$= -\frac{1}{\log x}\log(y)\widetilde{g}_1(y)^{-2}\big|_x^{x_0} + \frac{1}{\log x}\int_x^{x_0}\log(y)\rho(y)\,dy$$
$$= \frac{c(x_0)}{\log x} + \widetilde{g}_1(x)^{-2} + \frac{1}{\log x}\int_x^{x_0}\log(y)\rho(y)\,dy.$$

From this one checks that $f \in AC[0, x_0]$ and hence $\widetilde{g}_2 \in AC[0, 1]$. Furthermore

$$xf'(x) = -\frac{c(x_0)}{\log^2 x} - \frac{1}{\log^2(x)}\int_x^{x_0}\log(y)\rho(y)\,dy,$$

and differentiating this again shows $(Xf')' \in L^1[0,1]$. The remaining claims now follow as in the case $\nu \neq 0$. $\square$

Finally we prove the following refinement of the properties of $\widetilde{g}_j$, $j = 1, 2$, which will be crucial for the rest of the paper.

Proposition 2.6. *Let $V \in \mathscr{V}_{\operatorname{Re}\nu}$. Then, in the notation of Theorem* 2.1 *we have for $j = 1, 2$,*

$$\widetilde{g}_j' \log\left(\frac{\cdot}{2}\right), X\widetilde{g}_j'' \log\left(\frac{\cdot}{2}\right) \in L^1[0,1],$$

i.e. $\widetilde{g}_j \in \mathscr{A}$.

Proof. We prove the result only for $\nu \neq 0$ and leave the case $\nu = 0$ to the reader. The result for $\nu = 0$ will not be used in the rest of the paper.

Recall that $g_1(x) = x^{\nu_1}\widetilde{g}_1(x) = x^{\nu_1}(1 + \phi(x))$, where $\phi \in AC[0,1]$ (Proposition 2.5) is given by (2.10) and observe that by (2.19) we have

$$\phi'(x) = x^{-2\nu-1} \int_0^x y^{2\nu} r(y)\,dy, \tag{2.27}$$

with $r := V \cdot (1 + \phi)$, $r \log(\frac{\cdot}{2}) \in L^1[0,1]$ since $V \in \mathscr{V}_{\operatorname{Re}\nu}$. The claims about $\widetilde{g}_1$ now follow from (2.27) and Lemma 2.4.

Recall from (2.25) $\widetilde{g}_2(x) = \widetilde{g}_1(x) f(x)$ with f given by (2.26). Differentiating the latter we find

$$f'(x) = \widetilde{c}(x_0)x^{2\nu-1} + 2\nu x^{2\nu-1} \int_x^{x_0} y^{-2\nu}\big(\widetilde{g}_1^{-2}\big)'(y)\,dy. \tag{2.28}$$

The first summand is still in $L^1[0,x_0]$ after multiplying by $\log(\frac{\cdot}{2})$. For the second summand note that $(\widetilde{g}_1^{-2})' \log(\frac{\cdot}{2}) = -2\widetilde{g}_1^{-3}\widetilde{g}_1' \log(\frac{\cdot}{2}) \in L^1[0,x_0]$ by the inclusion $\widetilde{g}_1' \log(\frac{\cdot}{2}) \in L^1[0,1]$, proved above. Hence we can apply Lemma 2.4 to the second summand to conclude $f' \log(\frac{\cdot}{2}) \in L^1[0,x_0]$. Differentiating (2.28) we infer similarly $Xf'' \log(\frac{\cdot}{2}) \in L^1[0,x_0]$. Then one easily checks the claimed properties for the product $\widetilde{g}_2 = \widetilde{g}_1 f$. Since $\widetilde{g}_2' \log(\frac{\cdot}{2})$ and $X\widetilde{g}_2'' \log(\frac{\cdot}{2})$ are locally integrable in the interval $(0,1]$ we reach the conclusion. $\square$

3. The maximal domain of regular singular operators

We continue in the notation of the preceding section and consider the regular singular Sturm–Liouville operator H with the fundamental system (g_1, g_2) of solutions to the differential equation $Hg = 0$ (cf. Theorem 2.1). We will freely use the notation introduced in Section 1.4.

We have the following characterization of the maximal domain of H, compare [6] and [10] and the basic discussion of the second author in [34, Proposition 2.10]. Note that it holds under a slightly weaker assumption on the potential ($V \in \log(\frac{\cdot}{2})\mathscr{V}_\nu$) than the one imposed in the rest of the paper.

Theorem 3.1. *Let l_ν be the operator* (2.1) *and let $H = l_\nu + X^{-1}V$ with $V \in \log(\frac{\cdot}{2})\mathscr{V}_\nu$, $\operatorname{Re}\nu \geqslant 0$, and let g_1, g_2 be the fundamental system to $Hg = 0$ of Theorem* 2.1. *Let f be a solution of the ordinary differential equation*

$$Hf = -f'' + qf = g \in L^2[0,1],$$
$$q(x) = \frac{\nu^2 - 1/4}{x^2} + \frac{1}{x}V(x). \tag{3.1}$$

Then $f \in AC^2(0,1]$ and

$$f(x) = c_1(f)g_1(x) + c_2(f)g_2(x) + \widetilde{f}(x), \tag{3.2}$$

for some constants $c_j(f)$, $j = 1, 2$, depending only of f,

$$\widetilde{f}(x) = O\big(x^{3/2}\log(x)\big), \qquad \widetilde{f}'(x) = O\big(x^{1/2}\log(x)\big), \quad x \to 0+. \tag{3.3}$$

Remark 3.2. We emphasize that the solution g_1 is completely determined by Eq. (2.5) and therefore canonical. However this is not so for g_2. Surely, any function $g_2 + \lambda g_1$ also satisfies (2.6). We mention this because as a consequence the functional c_2 (!) is canonically given while c_1 depends on the choice of g_2.

Proof. We first note that it is well known that solutions to linear differential equations with L^1_{loc} coefficients are locally absolutely continuous. Therefore a solution f to (3.1) is absolutely continuous in the interval $(0,1]$ and from $f'' = g - qf$ one then infers that f' is also absolutely continuous in $(0,1]$.

For $x_0 \in \{0, 1\}$,

$$\widetilde{f}(x) = g_1(x)\int_{x_0}^{x} g_2(y)g(y)\,dy - g_2(x)\int_0^x g_1(y)g(y)\,dy \tag{3.4}$$

is a solution to (3.1); note $W(g_1, g_2) = 1$. Depending on ν we will choose x_0 such that (3.3) is satisfied. We first note that by applying the Cauchy–Schwarz inequality

$$\left|g_2(x)\int_0^x g_1(y)g(y)\,dy\right| \leqslant \left|g_2(x)\right|\left(\int_0^x \left|g_1(y)\right|^2 dy\right)^{1/2}\|g\|_{L^2}$$
$$= \begin{cases} O(x^{3/2}), & \text{if } \nu \neq 0, \\ O(x^{3/2}\log(x)), & \text{if } \nu = 0, \end{cases} \quad x \to 0+. \tag{3.5}$$

Furthermore, if $\operatorname{Re}\nu \geqslant 1$, we put $x_0 = 1$ and find

$$\left|g_1(x)\right|\left|\int_x^1 g_2(y)g(y)\,dy\right| \leqslant Cx^{\operatorname{Re}\nu + 1/2}\left(\int_x^1 y^{-2\operatorname{Re}\nu + 1}\,dy\right)^{1/2}\|g\|_{L^2}$$
$$= \begin{cases} O(x^{3/2}|\log(x)|^{1/2}), & \text{if } \operatorname{Re}\nu = 1, \\ O(x^{3/2}), & \text{if } \operatorname{Re}\nu > 1, \end{cases} \quad x \to 0+. \tag{3.6}$$

Finally, if $0 \leqslant \operatorname{Re} \nu < 1$, we put $x_0 = 0$ and estimate

$$\left|g_1(x)\right| \left| \int_0^x g_2(y) g(y)\, dy \right| \leqslant C x^{\operatorname{Re} \nu + 1/2} \left(\int_0^x \left|g_2(y)\right|^2 dy \right)^{1/2} \|g\|_{L^2}$$
$$= \begin{cases} O(x^{3/2}), & \text{if } \nu \neq 0, \\ O(x^{3/2} \log(x)), & \text{if } \nu = 0, \end{cases} \quad x \to 0+. \tag{3.7}$$

This proves the estimates for $\widetilde{f}(x)$. Differentiating (3.4) we find

$$\widetilde{f}'(x) = g_1'(x) \int_{x_0}^x g_2(y) g(y)\, dy - g_2'(x) \int_0^x g_1(y) g(y)\, dy,$$

and (2.7), (2.8) together with (3.5), (3.6) and (3.7) immediately give the claimed estimate for $\widetilde{f}'$. Thus $\widetilde{f}$ is a solution to (3.1) satisfying (3.3). Eq. (3.2) is now clear. □

Remark 3.3. The above proof shows that for $\nu \neq 0$, $\operatorname{Re} \nu \neq 1$, the estimate (3.3) can actually be replaced by

$$\widetilde{f}(x) = O\big(x^{3/2}\big), \qquad \widetilde{f}'(x) = O\big(x^{1/2}\big), \quad x \to 0+, \tag{3.8}$$

and for $\operatorname{Re} \nu = 1$ by

$$\widetilde{f}(x) = O\big(x^{3/2}\left|\log(x)\right|^{1/2}\big), \qquad \widetilde{f}'(x) = O\big(x^{1/2}\left|\log(x)\right|^{1/2}\big), \quad x \to 0+. \tag{3.9}$$

Corollary 3.4. *Under the assumptions of Theorem* 3.1 *there are continuous linear functionals* c_j, $j = 1, 2$ *on* $\mathscr{D}(H_{\max})$ *such that for* $f \in \mathscr{D}(H_{\max})$,

$$f = c_1(f) g_1 + c_2(f) g_2 + \widetilde{f}, \tag{3.10}$$

with $\widetilde{f} \in \mathscr{D}_L(H)$ (*cf. Section* 1.4). *Let* c_j^t *be the corresponding functionals for* H_0^t. *Then we have for* $f \in \mathscr{D}(H_{\max})$, $g \in \mathscr{D}(H_{\max}^t)$,

$$\langle H_{\max} f, g\rangle - \big\langle f, H_{\max}^t g\big\rangle = -\overline{f'(1)} g(1) + \overline{f(1)} g'(1) + \overline{c_2(f)} c_1^t(g) - \overline{c_1(f)} c_2^t(g). \tag{3.11}$$

Finally, 0 *is in the limit point case for* H *if and only if* $\operatorname{Re} \nu \geqslant 1$. *In this case* $c_2 = 0$.

Proof. (3.11) follows easily from (3.2), (3.3) and the Lagrange formula. The formulas (3.11) and (3.3) show that $\widetilde{f} \in \mathscr{D}_L(H)$; the latter was defined in Section 1.4. Now, it follows from (3.2) that the quotient space $\mathscr{D}(H_{\max})/\mathscr{D}_L(H)$ is spanned by $g_1 + \mathscr{D}_L(H)$, $g_2 + \mathscr{D}_L(H)$ if $g_2 \in L^2[0,1]$ and by $g_1 + \mathscr{D}_L(H)$ if $g_2 \notin L^2[0,1]$. This implies the continuity of the functionals c_1, c_2 on $\mathscr{D}(H_{\max})$. Finally, $c_2 = 0$ if and only if $g_2 \notin L^2[0,1]$. The latter is equivalent to $\operatorname{Re} \nu \geqslant 1$ and the claim is proved. □

As already outlined in Section 1.4 we obtain a closed extension of H_0 with separated boundary conditions by choosing boundary operators B_{j,θ_j}

$$\begin{aligned} B_{0,\theta_0} f &:= \sin\theta_0 \cdot c_1(f) + \cos\theta_0 \cdot c_2(f), \\ B_{1,\theta_1} f &:= \sin\theta_1 \cdot f'(1) + \cos\theta_1 \cdot f(1), \quad (\theta_0, \theta_1) \in [0,\pi)^2. \end{aligned} \tag{3.12}$$

B_{0,θ_0} depends on the choice of a fundamental system, cf. Remark 3.2. To treat the limit point and limit circle cases at 0 in a unified way we call a pair of boundary operators *admissible* if $\theta_1 \in [0,\pi)$ and either ($\theta_0 = 0$ and $\operatorname{Re}\nu \geqslant 1$) or ($\theta_0 \in [0,\pi)$ and $0 \leqslant \operatorname{Re}\nu < 1$).

Given an admissible pair B_{j,θ_j}, $j = 0, 1$, of boundary operators we denote by $H(\theta_0, \theta_1)$ the closed extension of H_0 with domain

$$H(\theta_0, \theta_1) := \{ f \in \mathscr{D}(H_{\max}) \mid B_{j,\theta_j} f = 0, \ j = 0, 1 \}. \tag{3.13}$$

If $\nu \in \mathbb{R}$ and V is real valued then H_0 is symmetric and $H(\theta_0, \theta_1)$ is self-adjoint. If $\operatorname{Re}\nu \geqslant 1$ then all self-adjoint extensions are obtained in this way. If $\operatorname{Re}\nu < 1$ there also exist self-adjoint extensions with non-separated boundary conditions. These extensions will not be studied in this paper.

3.1. Factorizable operators

Next we investigate when H can be factorized as $d_1 d_2$ with $d_j = \pm\frac{d}{dx} + \frac{\omega'}{\omega}$. For simplicity we confine ourselves to the case $\nu \neq 0$. Clearly, with some modifications one has similar results for $\nu = 0$.

We have seen in Proposition 2.6 that if $V \in \mathscr{V}_{\operatorname{Re}\nu}$ and ω is a solution to the differential equation $H\omega = 0$ then $\omega(x) = x^{\mu+1/2}\widetilde{\omega}(x)$ with $\widetilde{\omega} \in \mathscr{A}$ and $\mu = \pm\nu$.

Conversely, let $\mu \in \mathbb{C}$ and $\widetilde{\omega} \in \mathscr{A}$ with $\widetilde{\omega}(x) \neq 0$, $0 \leqslant x \leqslant 1$, be given. Put

$$\begin{aligned} d_1 &:= \frac{d}{dx} + \frac{\omega'}{\omega} = \frac{d}{dx} + \frac{\mu + 1/2}{x} + \frac{\widetilde{\omega}'}{\widetilde{\omega}}, \\ d_2 &:= -\frac{d}{dx} + \frac{\omega'}{\omega}. \end{aligned} \tag{3.14}$$

Then

$$H_{12} := d_1 d_2 = -\frac{d^2}{dx^2} + \frac{\omega''}{\omega} = -\frac{d^2}{dx^2} + \frac{\mu^2 - 1/4}{x^2} + 2\frac{\mu + 1/2}{x}\frac{\widetilde{\omega}'}{\widetilde{\omega}} + \frac{\widetilde{\omega}''}{\widetilde{\omega}}, \tag{3.15}$$

$$\begin{aligned} H_{21} := d_2 d_1 &= -\frac{d^2}{dx^2} + 2\left(\frac{\omega'}{\omega}\right)^2 - \frac{\omega''}{\omega} \\ &= -\frac{d^2}{dx^2} + \frac{(\mu+1)^2 - 1/4}{x^2} + 2\frac{\mu + 1/2}{x}\frac{\widetilde{\omega}'}{\widetilde{\omega}} + 2\left(\frac{\widetilde{\omega}'}{\widetilde{\omega}}\right)^2 - \frac{\widetilde{\omega}''}{\widetilde{\omega}}, \end{aligned} \tag{3.16}$$

thus we have $H_{12} = l_\mu + X^{-1}V_{12}$, $H_{21} = l_{\mu+1} + X^{-1}V_{21}$ and using $\widetilde{\omega} \in \mathscr{A}$ one directly checks $V_{12}, V_{21} \in \mathscr{V}_\nu$ (see Definition 1.2).

Hence for a given $V \in \mathscr{V}_{\operatorname{Re}\nu}$ the operator $H = l_\nu + X^{-1}V$ can be factorized in the form $H = d_1 d_2$ with d_1, d_2 as above if and only if there is a solution to the homogeneous equation $H\omega = 0$ with $\omega(x) \neq 0$ for $0 < x \leqslant 1$. Indeed, if ω solves $H\omega = 0$ and is nowhere vanishing in $(0, 1]$, one directly verifies from the first line of (3.15) that $H_{12} = d_1 d_2$ coincides with H. Conversely, given a factorization of $H = d_1 d_2$ with d_1, d_2 as in (3.14) it is immediate that $d_2\omega = 0$ and thus $H\omega = 0$.

By Proposition 2.6, ω is then of the form $\omega(x) = x^{\pm\nu+1/2}\widetilde{\omega}(x)$ with $\omega \in \mathscr{A}$.

The problem is that such a nowhere vanishing solution does not necessarily exist. However, we will be able to reduce the calculation of the zeta-determinant to the calculation for *factorizable* operators. In fact, the main essence of Proposition 3.5 below is that although H itself may not be factorizable, it becomes factorizable after adding a suitable $L^2_{\mathrm{comp}}(0, 1)$ potential. For such perturbations a variational formula for the zeta-determinant will be established subsequently.

Next we investigate the separated boundary conditions for d_j. For d_j we can choose four possibly different closed extensions with *separated* boundary conditions: For $p, q \in \{a, r\}$ denote by $d_{j,pq}$ the closed extension of d_j with boundary condition p at the left end point and boundary condition q at the right end point. Here, r stands for the relative boundary condition and a for the absolute boundary condition. More concretely, $d_{j,rr} = d_{j,\min}$ is the closure of d_j on $C_0^\infty(0, 1)$, $d_{j,aa} = d_{j,\max} = (d^t_{j,\min})^*$ is the maximal extension. The domains of the mixed extensions can be characterized by

$$\begin{aligned} \mathscr{D}(d_{j,ar}) &= \bigl\{ f \in \mathscr{D}(d_{j,\max}) \bigm| f(1) = 0 \bigr\}, \\ \mathscr{D}(d_{j,ra}) &= \bigl\{ f \in \mathscr{D}(d_{j,\max}) \bigm| f(x) = O\bigl(\sqrt{x}\,\bigl|\log(x)\bigr|^{1/2}\bigr) \text{ as } x \to 0+ \bigr\}. \end{aligned} \tag{3.17}$$

For each choice of a closed extension D_1 of d_1 with boundary condition of the form aa, rr, ar, ra we choose D_2 to be the closed extension with dual boundary condition, i.e. rr, aa, ra, ar, for d_2. If ω is real then this means that $D_2 = D_1^t$. We summarize case by case the corresponding boundary conditions for H_{12}, H_{21}:

Note that ω is a solution to the homogeneous differential equation $H_{12}g = 0$. In the notation of Section 1.4 we assume that

$$\omega = \cos\vartheta_0 \cdot g_1 - \sin\vartheta_0 \cdot g_2, \tag{3.18}$$

with $0 \leqslant \vartheta_0 < \pi$, thus $B_{0,\vartheta_0}\omega = 0$. Moreover, we assume that $B_{1,\vartheta_1}\omega = 0$ for a $0 < \vartheta_1 \leqslant \pi$. We have to exclude $\vartheta_1 = 0$ because in that case $\omega(1) = 0$ and hence there would be a regular singularity also at the right end point. But see Remark 5.6.

For future reference we now list the

3.2. Separated boundary conditions for the factorized operator $D_1 D_2$

Case I: $D_1 = d_{1,rr}$, $D_2 = d_{2,aa}$. $f \in \mathscr{D}(D_1 D_2)$ if and only if $f \in \mathscr{D}(d_{2,\max})$ and $d_2 f \in \mathscr{D}(d_{1,\min})$. Thus one checks that $D_1 D_2 = H_{12}(\vartheta_0, \vartheta_1)$ and $D_2 D_1 = H_{21}(0, 0)$ is the Friedrichs extension of H_{21}.

Case II: $D_1 = d_{1,ra}$, $D_2 = d_{2,ar}$. Then

$$D_1 D_2 = H_{12}(\vartheta_0, 0),\ D_2 D_1 = H_{21}(0, \pi - \vartheta_1).$$

Case III: $D_1 = d_{1,ar}$, $D_2 = d_{2,ra}$. Then

$$D_1 D_2 = H_{12}(0, \vartheta_1),\ D_2 D_1 = H_{21}(\pi - \vartheta_0, 0).$$

Case IV: $D_1 = d_{1,aa}$, $D_2 = d_{2,rr}$. Then

$$D_1 D_2 = H_{12}(0, 0),\ D_2 D_1 = H_{21}(\pi - \vartheta_0, \pi - \vartheta_1).$$

We summarize the previous considerations in the following

Proposition 3.5. *Let $V \in \mathscr{V}_{\operatorname{Re}\nu}$ be given and let $H = l_\nu + X^{-1}V$ be the corresponding regular singular Sturm–Liouville operator. Suppose that we are given admissible separated boundary conditions B_{j,θ_j}, $j = 0, 1$, for H. Then for $0 < \vartheta_1 < \pi$ there exists a function $\omega(x) = x^{\mu+1/2}\widetilde{\omega}$ such that $\widetilde{\omega} \in \mathscr{A}$ (see Definition 1.2), $\widetilde{\omega}(x) \neq 0$ for $0 \leqslant x \leqslant 1$ and $B_{0,\theta_0}\omega = 0$, $B_{1,\vartheta_1}\omega = 0$. Moreover, $H\omega = 0$ in a neighborhood of 0 and in a neighborhood of 1.*

If $\theta_1 = 0$ we choose any $0 < \vartheta_1 < \pi$ and if $0 < \theta_1 < \pi$ we let $\vartheta_1 = \theta_1$.

Putting $D_1 = d_{1,ra}$, $D_2 = d_{2,ar}$ if $0 < \theta_0 < \pi$, $\theta_1 = 0$; $D_1 = d_{1,rr}$, $D_2 = d_{2,aa}$ if $0 < \theta_0 < \pi$, $\theta_1 > 0$; $D_1 = d_{1,aa}$, $D_2 = d_{2,rr}$ if $\theta_0 = \theta_1 = 0$ and $D_1 = d_{1,ar}$, $D_2 = d_{2,ra}$ if $\theta_0 = 0$, $\theta_1 > 0$ we have

$$H(\theta_0, \theta_1) = D_1 D_2 + W \tag{3.19}$$

with $W \in L^2_{\mathrm{comp}}(0, 1)$.

If V is real then ω can be chosen to be real.

Proof. We can certainly find an $\varepsilon > 0$ and solutions ω_j, $j = 0, 1$, to the differential equation $Hg = 0$ on the intervals $(0, \varepsilon)$ respectively $(1 - \varepsilon, 1]$ such that $\omega_0(x) \neq 0$ for $0 < x < \varepsilon$, $\omega_1(x) \neq 0$ for $1 - \varepsilon \leqslant x \leqslant 1$ and $B_{0,\theta_0}\omega_0 = 0$, $B_{1,\vartheta_1}\omega_1 = 0$. If V is real we may choose ω_j to be positive. In any case we may choose a nowhere vanishing extension ω to the whole interval with the claimed regularity properties.

By construction $D_1 D_2$ has the same boundary conditions as $H(\theta_0, \theta_1)$ and there are neighborhoods of $0, 1$ respectively on which the potential of $D_1 D_2$ coincides with that of H, whence (3.19). □

3.3. Comparison of Wronskians

For a factorizable operator $H = D_1 D_2$ and given admissible boundary conditions we are now able to compare the Wronskians of normalized fundamental solutions of $D_1 D_2$ and $D_2 D_1$.

Proposition 3.6. *Let* $\operatorname{Re}\nu \geqslant 0$ *and let $\omega(x) = x^{-\nu+1/2}\widetilde{\omega}(x)$ with $\widetilde{\omega} \in \mathscr{A}$, $\widetilde{\omega}(x) \neq 0$ for $0 \leqslant x \leqslant 1$. Put d_1, d_2 as in* (3.14) *with $\mu = -\nu$. Assume that $B_{0,\vartheta_0}\omega = 0$, $B_{1,\vartheta_1}\omega = 0$ with admissible boundary conditions B_{j,ϑ_j}, $0 < \vartheta_j < \pi$ for $H_{12} = d_1 d_2$. Choose D_1, D_2 as in Proposition* 3.5 *with $\theta_0 = \vartheta_0$ and $\theta_1 = 0$ or $\theta_1 = \vartheta_1$. Let φ_z, ψ_z be normalized solutions for $(D_2 D_1 + z)g = 0$.*

Case I: $0 < \vartheta_1 = \theta_1 < \pi$. *Then $\widetilde{\varphi}_z := \frac{1}{2-2\nu} d_1 \varphi$, $\widetilde{\psi}_z = -d_1 \psi_z$ are normalized solutions for $(D_1 D_2 + z)g = 0$. Furthermore, we have*

$$W(\widetilde{\psi}_z, \widetilde{\varphi}_z) = \frac{z}{2 - 2\nu} W(\psi_z, \varphi_z). \tag{3.20}$$

D_2D_1 is invertible and the kernel of D_1D_2 is one-dimensional, hence $\operatorname{spec} D_1D_2 = \operatorname{spec} D_2D_1 \cup \{0\}$.

Case II: $\theta_1 = 0$. *Then* $\widetilde{\varphi}_z := (2-2\nu)^{-1} d_1\varphi$, $\widetilde{\psi}_z = -z^{-1} d_1\psi_z$ *are normalized solutions for* $(D_1D_2 + z)g = 0$. *Furthermore, we have*

$$W(\widetilde{\psi}_z, \widetilde{\varphi}_z) = \frac{1}{2-2\nu} W(\psi_z, \varphi_z). \tag{3.21}$$

D_2D_1, D_1D_2 are both invertible and $\operatorname{spec} D_2D_1 = \operatorname{spec} D_1D_2$.

Proof. In view of (3.15), (3.16) the characteristic exponents of d_1d_2 are $\pm\nu + 1/2$ and the characteristic exponents of d_2d_1 are $\pm(-\nu + 1) + 1/2$. $d_1\varphi_z$ satisfies $(d_1d_2 + z)d_1\varphi_z = d_1(d_2d_1 + z)\varphi_z = 0$. Furthermore, since φ_z is normalized at 0 for $D_2D_1 + z$ we have in the notation of (1.29) $\varphi_z(x) \sim x^{3/2-\nu}$ as $x \to 0$ and using (3.14) we obtain

$$d_1\varphi_z(x) \sim (3/2 - \nu + 1/2 - \nu)x^{1/2-\nu} = (2-2\nu)x^{-\nu+1/2}, \tag{3.22}$$

hence $\frac{1}{2-2\nu} d_1\varphi_z$ is normalized at 0 for $D_1D_2 + z$ as claimed. This applies to both Cases I and II.

Now consider $d_1\psi_z$ which also solves $(d_1d_2 + z)d_1\psi_z = 0$.

Case I: $0 < \vartheta_1 = \theta_1 < \pi$. Then $D_2D_1 = H_{21}(0,0)$ and hence ψ_z being normalized at 1 means $\psi_z(1) = 0$, $\psi_z'(1) = -1$. Then $d_1\psi_z(1) = \psi_z'(1) + (\frac{\omega'}{\omega}\psi_z)(1) = -1$ and hence $-d_1\psi_z$ is normalized at 1.

We now find for the Wronskian

$$\begin{aligned}
W(\widetilde{\psi}_z, \widetilde{\varphi}_z) &= \frac{-1}{2-2\nu} W(d_1\psi_z, d_1\varphi_z) \\
&= \frac{-1}{2-2\nu}\big(d_1\psi_z(d_1\varphi_z)' - (d_1\psi_z)' d_1\varphi_z\big) \\
&= \frac{1}{2-2\nu}\big(d_1\psi_z(d_2d_1\varphi_z) - (d_2d_1\psi_z)d_1\varphi_z\big) \\
&= \frac{-z}{2-2\nu}\big((d_1\psi_z)\varphi_z - \psi_z d_1\varphi_z\big) \\
&= \frac{z}{2-2\nu} W(\psi_z, \varphi_z).
\end{aligned} \tag{3.23}$$

Case II: $\theta_1 = 0$. Then $D_2D_1 = H_{21}(0, \pi - \vartheta_1)$ with $0 < \pi - \vartheta_1 < \pi$. Thus ψ_z being normalized at 1 means $\psi_z(1) = 1$ and $B_{1,\pi-\vartheta_1}\psi_z = 0$. Then $\psi_z \in \mathscr{D}(D_1 = d_{1,ra})$ thus $0 = d_1\psi_z(1)$. Hence $(d_1\psi_z)'(1) = -(d_2d_1\psi_z)(1) = z\psi_z(1) = z$. Thus $-z^{-1}d_1\psi_z$ is normalized at 1 for H_{12}. Now the same calculation as in (3.23) yields (3.21) and the proposition is proved. □

4. The asymptotic expansion of the resolvent trace

Standing assumptions. Let $l_\nu := -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}$ be the regular singular model operator. In this section we assume ν to be real and non-negative.

4.1. The Dirichlet condition at 0

Let $B_{1,\theta} = \sin\theta \cdot f'(1) + \cos\theta \cdot f(1)$ be a boundary operator for the right end point and let $L_\nu = L_\nu(0,\theta)$ be the closed extension of l_ν with domain

$$\mathscr{D}(L_\nu) = \{ f \in \mathscr{D}(l_{\nu,\max}) \mid c_2(f) = 0,\ B_{1,\theta} f = 0 \}. \tag{4.1}$$

The following perturbation result will be crucial for establishing the resolvent trace expansion Theorem 1.4 (cf. [26, Lemma 3.1]).

Proposition 4.1. *Let $\nu \geqslant 0$. Then L_ν is self-adjoint and bounded below.*

Let W be a measurable function on $[0,1]$ *such that*

$$\begin{cases} W^2 \in L^1[0,1], & \text{if } \nu > 0, \\ W^2 \log\left(\frac{\cdot}{2}\right) \in L^1[0,1], & \text{if } \nu = 0. \end{cases} \tag{4.2}$$

Then for $z_0^2 > \max \operatorname{spec}(-L_\nu)$ there is a constant $C(z_0)$ such that for $z \in \mathbb{R}$, $z \geqslant z_0$, we have for the Hilbert–Schmidt norms

$$\begin{aligned} &\left\| x^{-1/2} W (L_\nu + z^2)^{-1/2} \right\|_{\mathrm{HS}}^2 + \left\| (L_\nu + z^2)^{-1/2} x^{-1/2} W \right\|_{\mathrm{HS}}^2 \\ &\quad \leqslant C(z_0) \begin{cases} (\frac{1}{z} + \int_0^{1/z} |W(x)|^2 \, dx + \frac{1}{z} \int_{1/z}^1 \frac{1}{x} |W(x)|^2 \, dx), & \text{if } \nu > 0, \\ (\frac{1}{z} + \int_0^{1/z} |W(x)|^2 |\log(xz)| \, dx + \frac{1}{z} \int_{1/z}^1 \frac{1}{x} |W(x)|^2 \, dx), & \text{if } \nu = 0, \end{cases} \\ &\quad =: R(z). \end{aligned} \tag{4.3}$$

If (4.2) *is replaced by $W^2 \in \mathscr{V}_\nu$, i.e.*

$$\begin{cases} W^2 \log\left(\frac{\cdot}{2}\right) \in L^1[0,1], & \text{if } \nu > 0, \\ W^2 \log^2\left(\frac{\cdot}{2}\right) \in L^1[0,1], & \text{if } \nu = 0, \end{cases} \tag{4.4}$$

then

$$\lim_{z \to \infty} R(z) = 0, \tag{4.5}$$

$$\int_{z_0}^{\infty} \frac{1}{z} |R(z)| \, dz < \infty. \tag{4.6}$$

Proof. The boundary conditions for L_ν are separated and admissible. Therefore, L_ν is self-adjoint. We will see below that the resolvent is Hilbert–Schmidt. Thus L_ν has a pure point spectrum. An eigenfunction satisfying $L_\nu f = \lambda^2 f$, $\lambda \in \mathbb{R} \cup i\mathbb{R}$, is therefore a multiple of $\sqrt{x} J_\nu(\lambda x)$, where J_ν denotes the Bessel function of order ν [36]. From the known asymptotic behavior of the Bessel functions with imaginary argument one deduces that L_ν has at most finitely many negative eigenvalues and hence is bounded below.

The kernel $k_\nu(x,y;z)$ of the resolvent $(L_\nu + z^2)^{-1}$ is given in terms of the modified Bessel functions I_ν, K_ν

$$k_\nu(x,y;z) = \sqrt{xy} I_\nu(xz)\big(K_\nu(yz) - \beta(z) I_\nu(yz)\big), \quad x \leqslant y, \tag{4.7}$$

where $\beta(z)$ is determined by the requirement $B_{1,\theta} k(x,\cdot;z) = 0$ (cf. [4]). One finds

$$\begin{aligned}\beta(z) &= \frac{(\cos\theta + \frac{1}{2}\sin\theta) K_\nu(z) + z K_\nu'(z)\sin\theta}{(\cos\theta + \frac{1}{2}\sin\theta) I_\nu(z) + z I_\nu'(z)\sin\theta} \\ &= \frac{(\cos\theta + (\frac{1}{2}+\nu)\sin\theta) K_\nu(z) - z K_{\nu+1}(z)\sin\theta}{(\cos\theta + (\frac{1}{2}+\nu)\sin\theta) I_\nu(z) + z I_{\nu+1}(z)\sin\theta},\end{aligned} \tag{4.8}$$

where in the last equation we used the recursion relations [36, 3.71]

$$z I_\nu'(z) = z I_{\nu+1}(z) + \nu I_\nu(z), \qquad z K_\nu'(z) = -z K_{\nu+1} + \nu K_\nu(z). \tag{4.9}$$

Recall the following asymptotic relations for the modified Bessel functions [36, 7.23]

$$\begin{aligned} I_\nu(z) &= \frac{1}{\sqrt{2\pi z}} e^z\big(1 + O\big(z^{-2}\big)\big), \\ K_\nu(z) &= \sqrt{\frac{\pi}{2z}} e^{-z}\big(1 + O\big(z^{-2}\big)\big), \quad z \to \infty, \end{aligned} \tag{4.10}$$

and [36, Sec. 3.7]

$$I_\nu(z) \sim \frac{1}{2^\nu \Gamma(\nu+1)} z^\nu, \qquad K_\nu(z) \sim \begin{cases} 2^{\nu-1}\Gamma(\nu) z^{-\nu}, & \text{if } \nu \neq 0, \\ -\log z, & \text{if } \nu = 0, \end{cases} \quad \text{as } z \to 0; \tag{4.11}$$

for the notation $\sim$ see (1.29). From the asymptotics as $z \to \infty$ one infers

$$\beta(z) = O\big(e^{-2z}\big), \quad z \to \infty. \tag{4.12}$$

To prove the estimate (4.3) we fix $z_0 > \max \operatorname{spec}(-L_\nu)$ and find for $z \geqslant z_0$,

$$\begin{aligned} \big\|x^{-1/2} W\big(L_\nu + z^2\big)^{-1/2}\big\|_{\mathrm{HS}}^2 &= \big\|\big(L_\nu + z^2\big)^{-1/2} x^{-1/2} W\big\|_{\mathrm{HS}}^2 \\ &= \operatorname{Tr}\big(x^{-1/2} W \big(L_\nu + z^2\big)^{-1} \overline{W} x^{-1/2}\big) \\ &= \int_0^1 x^{-1}\big|W(x)\big|^2 k_\nu(x,x;z)\,dx \\ &= \int_0^1 \big|W(x)\big|^2 I_\nu(xz) K_\nu(xz)\,dx - \beta(z) \int_0^1 \big|W(x)\big|^2 \big|I_\nu(xz)\big|^2\,dx. \end{aligned} \tag{4.13}$$

In the following C denotes a generic constant depending only on z_0 and ν. We split the integrals into an integration from 0 to $1/z$ and from $1/z$ to 1. In the first regime (4.11) yields

$$\left|I_\nu(xz)K_\nu(xz)\right| \leqslant \begin{cases} C, & \text{if } \nu \neq 0, \\ C|\log(xz)|, & \text{if } \nu = 0, \end{cases} \tag{4.14}$$

and $|I_\nu(xz)| \leqslant C$. Thus,

$$\int_0^{1/z} x^{-1}\left|W(x)\right|^2 k_\nu(x,x;z)\,dx \leqslant \begin{cases} C\int_0^{1/z}|W(x)|^2\,dx, & \text{if } \nu \neq 0, \\ C\int_0^{1/z}|W(x)|^2|\log(xz)|\,dx, & \text{if } \nu = 0. \end{cases} \tag{4.15}$$

For $1/z \leqslant x \leqslant 1$ we apply (4.10) and find $|I_\nu(xz)K_\nu(xz)| \leqslant \frac{C}{xz}$, $|I_\nu(xz)|^2 \leqslant \frac{C}{xz}e^{2xz}$. Thus

$$\int_{1/z}^{1} \left|W(x)\right|^2\left|I_\nu(xz)K_\nu(xz)\right|dx \leqslant C\frac{1}{z}\int_{1/z}^{1}\frac{1}{x}\left|W(x)\right|^2dx, \tag{4.16}$$

and in view of (4.12)

$$\begin{aligned} \left|\beta(z)\right|\int_{1/z}^{1}\left|W(x)\right|^2\left|I_\nu(xz)\right|^2dx &\leqslant Ce^{-2z}\int_{1/z}^{1}\left|W(x)\right|^2\frac{1}{xz}e^{2xz}\,dx \\ &\leqslant C\left(e^{-z}\frac{1}{z}\int_{1/z}^{1/2}\frac{1}{x}\left|W(x)\right|^2dx + \frac{1}{z}\int_{1/2}^{1}\left|W(x)\right|^2dx\right) \\ &\leqslant C\frac{1}{z}, \quad z \geqslant z_0. \end{aligned} \tag{4.17}$$

The estimate (4.3) now follows from (4.13), (4.15), (4.16), (4.17).

Under the assumptions (4.4) we apply Lemma 2.4 to $\frac{1}{x}R(\frac{1}{x})$ since $\int_{z_0}^{\infty}\frac{1}{z}|R(z)|\,dz = \int_0^{1/z_0}\frac{1}{x}|R(\frac{1}{x})|\,dx$ and conclude (4.5), (4.6). □

We return to the discussion of the operator $H = l_\nu + X^{-1}V$; recall that X denotes the function $X(x) = x$. We have seen in the previous Proposition that L_ν is a bounded below self-adjoint operator. In fact L_ν is the Friedrichs extension of l_ν restricted to the domain

$$\mathscr{D}(l_\nu) = \left\{f \in C_0^\infty(0,1] \;\middle|\; B_{1,\theta}f = 0\right\}. \tag{4.18}$$

We now want to construct the Friedrichs extension of H on $\mathscr{D}(l_\nu)$ and compare its resolvent to that of L_ν; cf. [19, VI, 2.3]. The problem is that the domains of L_ν and of the Friedrichs extension of H on $\mathscr{D}(l_\nu)$ are not necessarily equal. This is because the domain of L_ν contains functions $f(x)$ with $f(x) \sim x^{\nu+1/2}$ as $x \to 0$. For such a function, $X^{-1}Vf$ is not necessarily in $L^2[0,1]$.

Proposition 4.2. *Let $H = l_\nu + X^{-1}V$ with $V \in \mathscr{V}_\nu$ (cf. Definition* 1.2*). Moreover, let $\mathscr{D}(l_\nu)$ be given by* (4.18)*, $0 \leqslant \theta < \pi$, and let $q(f,g) := \langle l_\nu f, g\rangle$ be the form of the operator l_ν. Then the form*

$$v(f,g) := \langle X^{-1}Vf, g\rangle_{L^2[0,1]}, \quad f,g \in \mathscr{D}(l_\nu) \tag{4.19}$$

is q-bounded with arbitrarily small q-bound b.

Proof. We compute for any $g \in \mathscr{D}(l_\nu)$ and $z \geqslant z_0$,

$$\begin{aligned}|v(g,g)| &= \|x^{-1/2}|V|^{1/2}g\|^2_{L^2} \\ &\leqslant \|x^{-1/2}|V|^{1/2}(L_\nu + z^2)^{-1/2}\|^2 \langle (L_\nu + z^2)g, g\rangle.\end{aligned}$$

Now Proposition 4.1 implies, that for any $b < 1$ there exists $z \in \mathbb{R}_+$ sufficiently large, such that

$$|v(g,g)| \leqslant b\langle (L_\nu + z^2)g, g\rangle = bz^2\|g\|^2_{L^2} + bq(g,g). \qquad \square$$

The quadratic form q is bounded below and closable with closure Q. By the second representation theorem [19, IV, 2.6 Theorem 2.23], we have

$$\mathscr{D}(Q) = \mathscr{D}\big((L_\nu + z_0^2)^{1/2}\big). \tag{4.20}$$

As a consequence of Proposition 4.2 we find in view of [19, VI, 1.6, Theorem 1.33] that $(q + v)$ is a sectorial form with

$$\mathscr{D}(\overline{q+v}) = \mathscr{D}(Q) = \mathscr{D}\big((L_\nu + z_0^2)^{1/2}\big). \tag{4.21}$$

By the first representation theorem ([19, VI, 2.1, Theorem 2.1]) it determines uniquely a closed m-sectorial extension $H(0,\theta)$ of $H = l_\nu + X^{-1}V$, with domain given by

$$\begin{aligned}\mathscr{D}\big(H(0,\theta)\big) &= \big\{f \in \mathscr{D}\big((L_\nu + z_0^2)^{1/2}\big) \;\big|\; (l_\nu + X^{-1}V + z_0^2)f \in L^2[0,1]\big\}, \\ &= \big\{f \in \mathscr{D}(H_{\max}) \;\big|\; c_2(f) = 0,\ B_{1,\theta}f = 0\big\}.\end{aligned} \tag{4.22}$$

Note that the functional c_2 (as well as c_1) depends on the potential and the c_2 in (4.22) is the one associated to H.

Theorem 4.3. *The operator $H(0,\theta)$ is m-sectorial, in particular* $\operatorname{spec} H(0,\theta)$ *is a subset of a sector $\{\xi \in \mathbb{C} \mid |\arg(\xi - \eta)| \leqslant \alpha\}$, for some fixed angle $\alpha \in (0, \pi/2)$ and $\eta \in \mathbb{R}$. Its resolvent is trace class and*

$$R_1(z) := \big\|\big(H(0,\theta) + z\big)^{-1} - (L_\nu + z)^{-1}\big\|_{\mathrm{tr}}, \quad z \in \mathbb{R}_+,\ z > \max(-\eta, 0) \tag{4.23}$$

satisfies

$$\lim_{z\to\infty} zR_1(z)=0, \quad z\in\mathbb{R}_+, \tag{4.24}$$

$$\int_{z_0}^{\infty}\left|R_1(z)\right|dx<\infty. \tag{4.25}$$

Furthermore,

$$\operatorname{Tr}\big(H(0,\theta)+z\big)^{-1}=\operatorname{Tr}(L_\nu+z)^{-1}+R_2(z)=\frac{a}{\sqrt{z}}+\frac{b}{z}+R_3(z), \tag{4.26}$$

where

$$a=\frac{1}{2},\qquad b=-\frac{1}{2}\big(\nu+\mu_1(B_{1,\theta})\big)=\begin{cases}-\frac{1}{2}(\nu+\frac{1}{2}), & \textit{if } \theta_1=0,\\ -\frac{1}{2}(\nu-\frac{1}{2}), & \textit{if } 0<\theta_1<\pi\end{cases} \quad \big(\textit{cf.}\ (1.28)\big). \tag{4.27}$$

The remainders $R_2(z)$, $R_3(z)$ satisfy (4.24) *and* (4.25) *and therefore the zeta-determinant of $H(0,\theta)$ is well defined by the formula* (*see* (1.12) *and Fig.* 1)

$$\log\det_\zeta H(0,\theta)=-\,⨍_{\Gamma}\operatorname{Tr}\big(\big(H(0,\theta)+z\big)^{-1}\big)\,dz. \tag{4.28}$$

Proof. The operator $H(0,\theta)$ is m-sectorial, as it arises from the sectorial form $(q+v)$, see [19, VI.2, Theorem 2.1]. Since by Proposition 4.1 we have

$$\lim_{z\to\infty}\big\|x^{-1/2}|V|^{1/2}(L_\nu+z)^{-1/2}\big\|_{\mathrm{HS}}=0,$$

we may invoke the Neumann series to obtain

$$\begin{aligned}&\big(H(0,\theta)+z\big)^{-1}-\big(L_\nu+z^2\big)^{-1}\\ &\quad=\sum_{n\geqslant 1}(-1)^n(L_\nu+z)^{-\frac{1}{2}}\big[(L_\nu+z)^{-\frac{1}{2}}x^{-1}V(L_\nu+z)^{-\frac{1}{2}}\big]^n(L_\nu+z)^{-\frac{1}{2}}.\end{aligned} \tag{4.29}$$

There is a little subtlety here since $\mathscr{D}(H(0,\theta))$ does not necessarily equal $\mathscr{D}(L_\nu)$. However, by Proposition 4.2 the forms of $H(0,\theta)$ and L_ν have the same domain. This is used decisively by writing $(L_\nu+z^2)^{-1/2}$ at the beginning and at the end of (4.29).

We estimate the trace norm of the individual summands by

$$\begin{aligned}&\big\|(L_\nu+z)^{-1/2}\big\|^2\cdot\big\|(L_\nu+z)^{-1/2}x^{-1}V(L_\nu+z)^{-1/2}\big\|_{\mathrm{tr}}^n\\ &\quad\leqslant\big\|(L_\nu+z)^{-1}\big\|\big\|x^{-1/2}|V|^{1/2}(L_\nu+z)^{-1/2}\big\|_{\mathrm{HS}}^n\cdot\big\|(L_\nu+z)^{-1/2}x^{-1/2}|V|^{1/2}\big\|_{\mathrm{HS}}^n\\ &\quad\leqslant Cz^{-1}\widetilde{R}(z)^n,\end{aligned}$$

where $\widetilde{R}(z)=\|x^{-1/2}|V|^{1/2}(L_\nu+z)^{-1/2}\|_{\mathrm{HS}}\cdot\|(L_\nu+z)^{-1/2}x^{-1/2}|V|^{1/2}\|_{\mathrm{HS}}$. The claim about $R_1(z)$ now follows from Proposition 4.1.

The first line of (4.26) follows since $|R_2(z)| \leqslant R_1(z)$. As for the second line of (4.26) we note that $\operatorname{Tr}(L_\nu + z)^{-1}$ has a complete asymptotic expansion as $z \to \infty$ [4], in particular

$$\operatorname{Tr}(L_\nu + z)^{-1} = \frac{a}{\sqrt{z}} + \frac{b}{z} + O\big(z^{-3/2}\log z\big),$$

with a, b as in (4.27).

For the claim about the zeta-determinant see Section 1.2. □

4.2. *General boundary conditions*

We now extend Theorem 4.3 to general boundary conditions at 0. Recall that 0 is in the limit point case if and only if $\nu \geqslant 1$. So the following discussion is of relevance only in the case $\nu < 1$. The case $\nu = 0$ bears more difficulties (see [16,20]) and therefore we assume from now on $0 < \nu (< 1)$. The difficulty then is that for $0 < \theta_0 < \pi$ the resolvent of $l_\nu(\theta_0, \theta_1)$ does not absorb negative x powers as the operator $l_\nu(0, \theta_1)$ does. Therefore, we do not have (4.3) at our disposal and hence the resolvent of $H(\theta_0, \theta_1)$ cannot be constructed as a perturbation of the resolvent of $l_\nu(\theta_0, \theta_1)$. Instead we will employ the results about factorizable operators in Section 3.1. However we have to impose a slight restriction on the class of potentials:

Definition 4.4. Let $V \in \mathscr{V}_\nu$ and let $H = l_\nu + X^{-1}V$ be the corresponding regular singular Sturm–Liouville operator. V is called of *determinant class* if for any pair of admissible boundary conditions B_{j,θ_j} the operator $H(\theta_0, \theta_1)$ satisfies for $z \geqslant z_0$, $z \in \mathbb{R}_+$,

$$\big\| \big(H(\theta_0, \theta_1) + z\big)^{-1} \big\| = O\big(|z|^{-1}\big), \tag{4.30}$$

$$\big\| \big(H(\theta_0, \theta_1) + z\big)^{-1} \big\|_{\mathrm{tr}} = O\big(|z|^{-1/2}\big), \tag{4.31}$$

and for any $\varphi \in C_0^\infty(0, 1]$,

$$\big\| \varphi\big(H(\theta_0, \theta_1) + z\big)^{-1} \big\|_{L^2 \to H^1} = O\big(|z|^{-1/2}\big). \tag{4.32}$$

Here, $\| \cdot \|_{L^2 \to H^1}$ denotes the norm of a map from $L^2[0, 1]$ into the first Sobolev space $H^1[0, 1]$. We denote the set of determinant class potentials by $\mathscr{V}_\nu^{\det}$.

We note some consequences and give some criteria for V being of determinant class.

Lemma 4.5. *Let $V \in \mathscr{V}_\nu^{\det}$ and let $W \in L^2_{\mathrm{comp}}(0, 1]$ with* $\operatorname{supp} W \subset [\delta, 1]$*, $\delta > 0$. Then*

$$\big\| W\big(H(\theta_0, \theta_1) + z\big)^{-1} \big\| \leqslant C_\delta \|W\|_{L^2} |z|^{-2/3}, \quad z \geqslant z_0. \tag{4.33}$$

For $W \in L^\infty[0, 1]$ we have

$$\big\| W\big(H(\theta_0, \theta_1) + z\big)^{-1} \big\| \leqslant C \|W\|_\infty |z|^{-1}, \quad z \geqslant z_0. \tag{4.34}$$

Proof. Choose a cut-off function $\varphi \in C_0^\infty(0,1]$ with $\varphi(x) = 1$ for $x \geqslant \delta$. Then (4.30), (4.32) and the complex interpolation method [33, Sec. 4.2] yield for $0 \leqslant s \leqslant 1$

$$\big\|\varphi\big(H(\theta_0,\theta_1)+z\big)^{-1}\big\|_{L^2\to H^s} \leqslant C_s|z|^{-1+s/2}. \tag{4.35}$$

By Sobolev embedding we have $H^s[0,1] \subset C[0,1]$ for $s > 1/2$ and thus for these s multiplication by W is continuous $H^s \to L^2$ with norm bounded by $C_s\|W\|_{L^2}$. Combining this with (4.35) gives

$$\big\|W\big(H(\theta_0,\theta_1)+z\big)^{-1}\big\|_{L^2\to L^2} \leqslant C_{s,\delta}\|W\|_{L^2}|z|^{-1+s/2}. \tag{4.36}$$

(4.33) follows by putting $s = 2/3$, (4.34) is obvious from (4.30). □

Lemma 4.6. *Let $V \in \mathscr{V}_\nu^{\det}$. If $W = W_1 + W_2$, $W_1 \in L^\infty[0,1]$, $W_2 \in L^2_{\mathrm{comp}}(0,1]$ then $V + XW \in \mathscr{V}_\nu^{\det}$.*[1]

Consequently, if $V_1 \in \mathscr{V}_\nu^{\det}$, $V_2 \in \mathscr{V}_\nu$ and $V_1(x) = V_2(x)$ for almost all x in a neighborhood of 0 then $V_2 \in \mathscr{V}_\nu^{\det}$. Furthermore, there is a constant depending only on $H(\theta_0,\theta_1)$ and the support of W_2 such that for $z \geqslant z_0$,

$$\big\|\big(H(\theta_0,\theta_1)+W+z\big)^{-1} - \big(H(\theta_0,\theta_1)+z\big)^{-1}\big\|_{\mathrm{tr}} \leqslant C\big(\|W_1\|_\infty + \|W_2\|_{L^2}\big)|z|^{-7/6}. \tag{4.37}$$

Proof. It follows from Lemma 4.5 that for z large enough we can employ the Neumann series

$$\begin{aligned}&\big(H(\theta_0,\theta_1)+W+z\big)^{-1} - \big(H(\theta_0,\theta_1)+z\big)^{-1}\\ &\quad= \sum_{n=1}^{\infty}(-1)^n\big(H(\theta_0,\theta_1)+z\big)^{-1}\big(W\big(H(\theta_0,\theta_1)+z\big)^{-1}\big)^n\end{aligned} \tag{4.38}$$

and (4.30), (4.31), (4.32) follow for $H(\theta_0,\theta_1)+W$; also (4.37) immediately follows.

The second claim follows from the first with $W = X^{-1}(V_2 - V_1) \in L^2_{\mathrm{comp}}(0,1]$. □

Proposition 4.7. *Let $V \in \mathscr{V}_\nu$ be real valued in a neighborhood of 0. Then $V \in \mathscr{V}_\nu^{\det}$.*

Together with Lemma 4.6 this shows that at least potentials of the form $V + \lambda$, where $V \in \mathscr{V}_\nu$ is real valued and $\lambda \in \mathbb{C}$, are of determinant class.

Proof. In view of Lemma 4.6 and Proposition 3.5 we may change V outside a neighborhood of 0 such that V becomes real valued everywhere and such that $H(\theta_0,\theta_1) = D^*D$, where D is a closed extension of $d = -\frac{d}{dx} + \omega'/\omega$. For the properties of ω see Proposition 3.5. Note that since V is real valued we may choose ω to be real valued, too and hence, in the notation of Proposition 3.5, $D_1 = D_2^*$.

Since D^*D is self-adjoint, elliptic and non-negative (4.30), (4.32) follow immediately from the spectral theorem. If $\theta_0 = 0$ then (4.31) follows from Theorem 4.3. If $\theta_0 \neq 0$ then by Proposition 3.5 the operator DD^* has Dirichlet boundary condition at 0. Hence by Theorem 4.3 the

[1] Note that then $H + W = l_\nu + X^{-1}(V + XW)$.

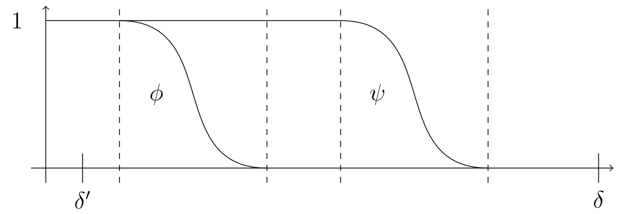

Fig. 2. The cut-off functions ϕ and ψ.

estimate (4.31) holds for DD^*. Since for a non-negative operator the estimate (4.31) depends only on the spectrum and since $\operatorname{spec} DD^* \cup \{0\} = \operatorname{spec} D^*D \cup \{0\}$ we reach the conclusion. □

Next we prove two comparison results for the asymptotics of the resolvent in the trace norm. These will then lead to an asymptotic expansion of the trace of the resolvent for $H(\theta_0, \theta_1)$ for arbitrary admissible boundary conditions and all determinant class potentials. The technique used in the first comparison result is well known for elliptic operators with smooth coefficients on manifolds (cf. e.g. [27, Appendix B]). We have to be slightly more careful here due to the low regularity assumptions on the potential.

Proposition 4.8. *Let $V_j \in \mathscr{V}_\nu^{\det}$, $j = 1, 2$ with $V_2 - V_1 \in L^2_{\mathrm{comp}}(0, 1]$, that is there is a $\delta > 0$ such that $V_1(x) = V_2(x)$ for $0 \leqslant x \leqslant \delta$. Let $H_j = l_\nu + X^{-1} V_j$ be the corresponding regular singular Sturm–Liouville operators and let B_{θ_0}, B_{θ_1} respectively $B_{\widetilde{\theta}_1}$, be admissible boundary conditions for H_j. Then there is a $z_0 \geqslant 0$ such for any $\delta' < \delta$ and $z \geqslant z_0$ the difference $(H_1(\theta_0, \theta_1) + z)^{-1} - (H_2(\theta_0, \widetilde{\theta}_1) + z)^{-1}$ restricted to $L^2[0, \delta']$ is of trace class and*

$$\left\| \left(\left(H_1(\theta_0, \theta_1) + z \right)^{-1} - \left(H_2(\theta_0, \widetilde{\theta}_1) + z \right)^{-1} \right) \right|_{L^2[0,\delta']} \Big\|_{\mathrm{tr}} = O\left(|z|^{-3/2} \right), \quad z \geqslant z_0,\ z \in \mathbb{R}_+.$$

Proof. We choose cut-off functions $\phi, \psi \in C_0^\infty[0, \delta)$, cf. Fig. 2, such that they are identically one over $[0, \delta']$ and

- $\operatorname{supp}(\phi) \subset \operatorname{supp}(\psi)$,
- $\operatorname{supp}(\phi) \cap \operatorname{supp}(d\psi) = \emptyset$.

In particular these conditions yield $\psi\phi = \phi$. In this proof we will write for brevity H_1 instead of $H_1(\theta_0, \theta_1)$ and H_2 instead of $H_2(\theta_0, \widetilde{\theta}_1)$.

We now consider

$$R(z) := \psi \left[(H_1 + z)^{-1} - (H_2 + z)^{-1} \right] \phi.$$

$R(z)$ maps into the domain of H_j and on the support of ψ the differential expressions H_1 and H_2 coincide; moreover $\psi \mathscr{D}(H_1) = \psi \mathscr{D}(H_2)$. Thus $(H_1 + z)R(z) = [H_1, \psi]((H_1 + z)^{-1} - (H_2 + z)^{-1})$. Arguing similarly for $R(z)^*$ and taking adjoints one then finds

$$(H_1 + z) R(z) (H_2 + z) = \left[-\partial_x^2, \psi \right] \left((H_1 + z)^{-1} - (H_2 + z)^{-1} \right) \left[\partial_x^2, \phi \right],$$

where $[\cdot,\cdot]$ denotes the commutator of the corresponding operators and any function is viewed as a multiplication operator. Hence

$$R(z) = (H_1+z)^{-1}\big[-\partial_x^2, \psi\big]\big((H_1+z)^{-1} - (H_2+z)^{-1}\big)\big[\partial_x^2, \phi\big](H_2+z)^{-1}$$

and thus

$$\begin{aligned}\big\|R(z)\big\|_{\mathrm{tr}} \leqslant \big\|(H_1+z)^{-1}\big\|_{\mathrm{tr}}\big(\big\|\big[\partial_x^2,\psi\big](H_1+z)^{-1}\big\| + \big\|\big[\partial_x^2,\psi\big](H_2+z)^{-1}\big\|\big)\\ \cdot \big\|\big[\partial_x^2,\phi\big](H_2+z)^{-1}\big\|.\end{aligned}$$

By (4.31) we have $\|(H_1+z)^{-1}\|_{\mathrm{tr}} = O(|z|^{-1/2})$. Let f denote ψ or ϕ. Then $[\partial_x^2, f]$ is a first order differential operator whose coefficients are compactly supported in $(0,1)$, hence it maps $H^1[0,1]$ continuously into $L^2_{\mathrm{comp}}(0,1)$. Therefore by (4.32), with a cut-off function $\chi \in C_0^\infty(0,1)$ with $\chi = 1$ in a neighborhood of $\operatorname{supp}([\partial_x^2, f])$,

$$\big\|\big[\partial_x^2, f\big](H_j+z)^{-1}\big\| \leqslant \big\|\big[\partial_x^2, f\big]\big\|_{H^1\to L^2}\big\|\chi(H_j+z)^{-1}\big\|_{L^2\to H^1} = O\big(|z|^{-1/2}\big)$$

for $z \geqslant z_0$. Hence $\|[\partial_x^2,\phi](H_j+z)^{-1}\| = O(|z|^{-1/2})$ and $\|[\partial_x^2,\psi](H_j+z)^{-1}\| = O(|z|^{-1/2})$ and the proposition is proved. □

We note that in this proof the estimate (4.31) was used only for H_1.

Proposition 4.9. *Let $V_j \in \mathscr{V}_\nu^{\det}$, $H_j = l_\nu + X^{-1}V_j$, $j = 1, 2$, and let $B_{0,\widetilde{\theta}_0}$, B_{0,θ_0}, B_{1,θ_1} be admissible boundary conditions. Then for any $\delta > 0$,*

$$\big\|\big(\big(H_1(\theta_0,\theta_1)+z\big)^{-1} - \big(H_2(\widetilde{\theta}_0,\theta_1)+z\big)^{-1}\big)\big|_{L^2[\delta,1]}\big\|_{\mathrm{tr}} = O\big(|z|^{-3/2}\big), \quad z \geqslant z_0,\ z \in \mathbb{R}_+.$$

Proof. Fix $\delta > 0$ and put

$$H_3 := -\frac{d^2}{dx^2} + \left(\frac{\nu^2 - 1/4}{X^2} + \frac{1}{X}V_1\right)1_{[\delta/2,1]} =: \Delta + q, \tag{4.39}$$

with $q \in L^2_{\mathrm{comp}}(0,1]$, $\Delta := -\frac{d^2}{dx^2}$.

Exactly as in Proposition 4.8 one now shows

$$\big\|\big(\big(H_1(\theta_0,\theta_1)+z\big)^{-1} - \big(H_3(\widetilde{\theta}_0,\theta_1)+z\big)^{-1}\big)\big|_{L^2[\delta,1]}\big\|_{\mathrm{tr}} = O\big(|z|^{-3/2}\big), \tag{4.40}$$

for $z \geqslant z_0$, $z \in \mathbb{R}_+$. Furthermore, an elementary calculation involving the explicitly computable resolvent kernel of $\Delta(\widetilde{\theta}_0,\theta_1)$ shows $\|q(\Delta(\widetilde{\theta}_0,\theta_1)+z)^{-1}\|_{\mathrm{tr}} = O(|z|^{-1/2})$. A Neumann series argument then gives

$$\big\|\big(\big(H_3(\widetilde{\theta}_0,\theta_1)+z\big)^{-1} - \big(\Delta(\widetilde{\theta}_0,\theta_1)+z\big)^{-1}\big)\big|_{L^2[\delta,1]}\big\|_{\mathrm{tr}} = O\big(|z|^{-3/2}\big). \tag{4.41}$$

Eqs. (4.40) and (4.41) imply for $z \geqslant z_0$, $z \in \mathbb{R}_+$,

$$\big\|\big(\big(H_1(\theta_0,\theta_1)+z\big)^{-1} - \big(\Delta(\widetilde{\theta}_0,\theta_1)+z\big)^{-1}\big)\big|_{L^2[\delta,1]}\big\|_{\mathrm{tr}} = O\big(|z|^{-3/2}\big). \tag{4.42}$$

The same line of reasoning applies to H_2 and hence (4.42) also holds with $H_2(\widetilde{\theta}_0, \theta_1)$ instead of $H_1(\theta_0, \theta_1)$, whence the result. □

Theorem 4.10. *Let $V \in \mathscr{V}_\nu^{\det}$, $\nu > 0$, and let $H = l_\nu + X^{-1}V$ be the corresponding regular singular Sturm–Liouville operator. Let $0 \leqslant \theta_j < \pi$ ($\theta_0 = 0$ if $\nu \geqslant 1$).*

Then the resolvent of $H(\theta_0, \theta_1)$ is trace class. Moreover, there is a $z_0 \geqslant 0$ such that $H(\theta_0, \theta_1) + z$ is invertible for $z \geqslant z_0$ and

$$\operatorname{Tr}\bigl(H(\theta_0, \theta_1) + z\bigr)^{-1} = \frac{a}{\sqrt{z}} + \frac{b}{z} + R_3(z), \quad z \geqslant z_0,$$

where $a = \frac{1}{2}$,

$$b = -\frac{1}{2}\bigl(\mu(B_{0,\theta_0}) + \mu(B_{1,\theta_1})\bigr) = \begin{cases} -\frac{1}{2}\nu - \frac{1}{4}, & \text{if } \theta_0 = \theta_1 = 0, \\ -\frac{1}{2}\nu + \frac{1}{4}, & \text{if } \theta_0 = 0,\ 0 < \theta_1 < \pi, \\ \frac{1}{2}\nu - \frac{1}{4}, & \text{if } 0 < \theta_0 < \pi,\ \theta_1 = 0, \\ \frac{1}{2}\nu + \frac{1}{4}, & \text{if } 0 < \theta_0,\ \theta_1 < \pi, \end{cases} \tag{4.43}$$

(*cf.* (1.28)) *is independent of V, and the remainder $R_3(z)$ satisfies*

$$\lim_{z\to\infty} zR_3(z) = 0, \quad z \in \mathbb{R}_+, \tag{4.44}$$

$$\int_{z_0}^{\infty} \bigl|R_3(z)\bigr|\,dx < \infty, \quad z_0 > \max \operatorname{spec}\bigl(-H(\theta_0, \theta_1)\bigr) \cap \mathbb{R}. \tag{4.45}$$

In particular the zeta-determinant of $H(\theta_0, \theta_1)$ is well defined by the formula (1.12),

$$\log \det\nolimits_\zeta L = -\,⨍_{\Gamma} \operatorname{Tr}\bigl(H(0, \theta) + z\bigr)^{-1}\,dz. \tag{4.46}$$

Proof. By Proposition 3.5 we may choose a factorizable operator $H_1(\theta_0, \theta_1) = D_1 D_2$ such that there is a $\delta > 0$ such that the coefficients of H_1 and H coincide on the interval $[0, \delta]$. Here, D_1, D_2 are appropriate closed extensions of the operators d_1, d_2 in (3.14) with $\mu = -\nu$. Then by Propositions 4.8, 4.9 we find

$$\bigl\|\bigl(\bigl(H(\theta_0, \theta_1) + z\bigr)^{-1} - (D_1 D_2 + z)^{-1}\bigr)\bigr\|_{\mathrm{tr}} = O\bigl(|z|^{-3/2}\bigr), \quad z \geqslant z_0,\ z \in \mathbb{R}_+, \tag{4.47}$$

and hence

$$\operatorname{Tr}\bigl(H(\theta_0, \theta_1) + z\bigr)^{-1} = \operatorname{Tr}(D_1 D_2 + z)^{-1} + O\bigl(|z|^{-3/2}\bigr). \tag{4.48}$$

We now have to discuss the four possible cases listed in Section 3.2, see also (3.15), (3.16):

Case I: $D_1 = d_{1,rr}$, $D_2 = d_{2,aa}$. Then $D_1 D_2$ has a one-dimensional null space and $D_2 D_1$ is invertible. Applying Theorem 4.3 to $D_2 D_1$ we obtain

$$\operatorname{Tr}(D_1 D_2 + z)^{-1} = \operatorname{Tr}(D_2 D_1 + z)^{-1} + z^{-1} = \frac{a}{\sqrt{z}} + \frac{b}{z} + R_3(z), \tag{4.49}$$

where $R_3(z)$ has the claimed properties (4.44) and (4.45) and $a = 1/2$, $b = -1/2(1 - \nu + 1/2 - 2) = 1/2(\nu + 1/2)$. Note that in formulas involving $D_2 D_1$, according to (3.16), the ν has to be replaced by $1 - \nu$.

Case II: $D_1 = d_{1,ra}$, $D_2 = d_{2,ar}$. Then $D_1 D_2$ and $D_2 D_1$ are both invertible and hence $\operatorname{Tr}(D_1 D_2 + z)^{-1} = \operatorname{Tr}(D_2 D_1 + z)^{-1}$, and we can proceed as in Case I.

In the remaining Cases III ($D_1 = d_{1,ar}$, $D_2 = d_{2,ra}$) and IV ($D_1 = d_{1,aa}$, $D_2 = d_{2,rr}$) one can apply Theorem 4.3 directly to $D_1 D_2$. □

5. Variation of the regular singular potential

In this section we discuss the behavior of the fundamental system of solutions under a certain variation of the potential and derive a variational formula for the zeta-determinant.

Standing assumptions. Let $\nu \geqslant 0$, $V \in \mathscr{V}_\nu^{\det}$ and let $W_\eta \in L^\infty[0,1] + L^2_{\text{comp}}(0,1]$ be a family of functions depending on a real or complex parameter η. To avoid unnecessary technicalities we assume that W_η is of the form $W_\eta = W_{1,\eta} + W_{2,\eta}$ where $W_{1,\eta} \in L^\infty[0,1]$, $W_{2,\eta} \in L^2_{\text{comp}}(0,1]$ satisfy

(i) $\eta \mapsto W_{1,\eta}$ is differentiable as a map into the Banach space $L^\infty[0,1]$,
(ii) there is a fixed $\delta > 0$ such that $\operatorname{supp} W_{2,\eta} \subset [\delta, 1]$ and $\eta \mapsto W_{2,\eta}|_{[\delta,1]}$ is differentiable as a map into the Banach space $L^2[\delta, 1]$.

For notational convenience we assume $W_0 = 0$ and put $V_\eta := V + X W_\eta$ and

$$H_\eta := l_\nu + X^{-1} V_\eta = l_\nu + X^{-1}(V + X W_\eta) =: H_0 + W_\eta =: -\frac{d^2}{dx^2} + q_\eta. \tag{5.1}$$

$\eta_0 = 0$ serves as a base point for a perturbative construction of a fundamental system.

5.1. Fundamental solutions and their asymptotics

According to Theorem 2.1 let $g_{1,\eta}$ be the unique solution of the ODE $H_\eta g_{1,\eta} = 0$ with $g_{1,\eta}(x) \sim x^{\nu_1}$, as $x \to 0+$. Note that the second solution $g_{2,\eta}$ in Theorem 2.1 is not uniquely determined by the requirement $g_{2,\eta}(x) = -\frac{1}{2\nu} x^{\nu_2}$, cf. Remark 3.2. Since the solutions now depend on the parameter η, the choice of $g_{2,\eta}$ becomes important. Before we specify $g_{2,\eta}$ we discuss the dependence of $g_{1,\eta}$ on η. To do so recall the operator K_ν from (2.11) in Section 2. For $\alpha \geqslant 0$ consider the Banach space $X^\alpha C[0,1]$ with norm

$$\|f\|_\alpha := \sup_{0 \leqslant x \leqslant 1} \left| x^{-\alpha} f(x) \right|, \tag{5.2}$$

and the Banach space $C^1_\alpha[0,1]$ consisting of those functions in $f \in (X^\alpha C[0,1]) \cap C^1(0,1]$ with $f' \in X^{\alpha-1} C[0,1]$ and norm

$$\|f\|_{C^1_\alpha} := \|f\|_\alpha + \left\|f'\right\|_{\alpha-1}. \tag{5.3}$$

For $f \in X^\alpha C[0,1]$ the inequality (2.15) gives

$$\left|(K_\nu V)^n f(x)\right| \leqslant x^\alpha \frac{1}{|\nu|^n n!}\|f\|_\alpha \left(\int_0^x |V(y)|\,dy\right)^n, \quad \nu \neq 0, \tag{5.4}$$

thus $K_\nu V$ is a bounded operator on $X^\alpha C[0,1]$ with spectral radius zero. Furthermore, for $f \in X^\alpha C[0,1]$ (cf. (2.19)),

$$\left|(K_\nu V f)'(x)\right| \leqslant x^{\alpha-1}\|f\|_\alpha \int_0^x \left|V(y)\right| dy, \tag{5.5}$$

hence $K_\nu V$ maps $X^\alpha C[0,1]$ continuously into $C^1_\alpha[0,1]$.

Recall from (2.17) that $g_{1,\eta}(x) = x^{\nu_1}(1+\phi_\eta(x))$ with

$$\phi_\eta = (I - K_\nu V_\eta)^{-1} K_\nu V_\eta \mathbf{1}. \tag{5.6}$$

Consequently ϕ_η is differentiable in η and

$$\begin{aligned}\partial_\eta \phi_\eta &= (I - K_\nu V_\eta)^{-1} K_\nu (X \partial_\eta W_\eta)\mathbf{1} \\ &\quad + (I - K_\nu V_\eta)^{-1} K_\nu (X\partial_\eta W_\eta)(I - K_\nu V_\eta)^{-1} K_\nu V_\eta \mathbf{1}.\end{aligned} \tag{5.7}$$

Since $\partial_\eta W_\eta$ is bounded near 0, the operator $K_\nu(X\partial_\eta W_\eta)$ maps $C^1_\alpha[0,1]$ continuously into $C^1_{\alpha+2}[0,1]$ and hence we have proved

Lemma 5.1. *Under the assumptions stated at the beginning of this section, $g_{1,\eta}$ is differentiable in η with $\partial_\eta g_{1,\eta}(x) = O(x^{\nu_1+2})$, $\partial_\eta g'_{1,\eta}(x) = O(x^{\nu_1+1})$ as $x \to 0+$. Moreover, the O-constants are locally uniform in η and hence $g_{1,\eta}(x) - g_{1,\eta_0}(x) = O(x^{\nu_1+2})$.*

After these preparations we can discuss the second fundamental solution $g_{2,\eta}$. For η in a neighborhood of 0 we can fix $x_0 \in (0,1)$ such that $g_{1,\eta}(x) \neq 0$ for $0 \leqslant x \leqslant x_0$. For these x we note

$$\begin{aligned} g_{1,\eta}(x)^{-2} - g_{1,0}(x)^{-2} &= \frac{[g_{1,\eta}(x) + g_{1,0}(x)][g_{1,0}(x) - g_{1,\eta}(x)]}{(g_{1,\eta}(x))^2 (g_{1,0}(x))^2} \\ &= O\left(x^{2-2\nu_1}\right), \quad x \to 0+, \end{aligned} \tag{5.8}$$

where the O-constant is independent of η. Hence $g_{1,\eta}^{-2} - g_{1,0}^{-2}$ is integrable over $(0, x_0]$ and we put for $x \in (0, x_0)$,

$$g_{2,0}(x) = g_{1,0}(x) \int_x^{x_0} g_{1,0}(y)^{-2}\,dy, \tag{5.9}$$

$$g_{2,\eta}(x) = -g_{1,\eta}(x) \int_0^x \left[g_{1,\eta}(y)^{-2} - g_{1,0}(y)^{-2}\right] dy + \frac{g_{1,\eta}(x)}{g_{1,0}(x)} g_{2,0}(x). \tag{5.10}$$

From (5.8), (5.9) and (5.10) we immediately get

Lemma 5.2. *$g_{1,\eta}, g_{2,\eta}$ is a fundamental system of solutions for the ODE $H_\eta g = 0$ satisfying* (2.5)*,* (2.6)*. Moreover, $g_{2,\eta}$ is also differentiable in η and we have for $\nu > 0$,*

$$g_{2,\eta}(x) = g_{2,0}(x) + O\left(x^{\nu_2+2}\right), \tag{5.11}$$

$$\partial_\eta g_{2,\eta}(x) = O\left(x^{\nu_2+2}\right), \qquad \partial_\eta g_{2,\eta}'(x) = O\left(x^{\nu_2+1}\right), \tag{5.12}$$

as $x \to 0$. For $\nu = 0$ the estimates are $O(x^{5/2-\nu}\log x) = O(x^{5/2}\log x)$, $O(x^{\nu_2+2}\log x) = O(x^{5/2}\log x)$, $O(x^{\nu_2+1}\log x) = O(x^{3/2}\log x)$, respectively.

Lemmas 5.1 and 5.2 imply:

Corollary 5.3. *For $\nu > 0$ we have the following asymptotics for the Wronskians $W(g_{j,\eta}, \partial_\eta g_{k,\eta}) = g_{j,\eta}g_{k,\eta}' - g_{j,\eta}' g_{k,\eta}$, $j, k = 1, 2$, as $x \to 0+$:*

$$W(g_{1,\eta}, \partial_\eta g_{1,\eta})(x) = O\left(x^{2\nu_1+1}\right), \tag{5.13}$$

$$W(g_{2,\eta}, \partial_\eta g_{1,\eta})(x) = O\left(x^{\nu_1+\nu_2+1}\right) = O\left(x^2\right), \tag{5.14}$$

$$W(g_{1,\eta}, \partial_\eta g_{2,\eta})(x) = O\left(x^{\nu_1+\nu_2+1}\right) = O\left(x^2\right), \tag{5.15}$$

$$W(g_{2,\eta}, \partial_\eta g_{2,\eta})(x) = O\left(x^{2\nu_2+1}\right). \tag{5.16}$$

If $\nu = 0$ then the estimates are $O(x^2 \log x)$ in all four cases.

Hence for $\nu \geqslant 0$ and all $j, k = 1, 2$, we have

$$\lim_{x\to 0} W(g_{j,\eta}, \partial_\eta g_{k,\eta})(x) = 0.$$

Now we are ready to state the variational result which generalizes [26, Proposition 3.4] to arbitrary boundary conditions and to more general potentials:

Theorem 5.4. (1) *Let $0 < \nu < 1$, $V \in \mathscr{V}_\nu^{\det}$ and let $\eta \mapsto W_\eta \in L^\infty[0,1] + L^2_{\text{comp}}(0,1]$ be differentiable in the sense described at the beginning of this section. Furthermore, let $0 \leqslant \theta_j < \pi$, $j = 0, 1$ and let $H_\eta = l_\nu + X^{-1}V + W_\eta$. Fix η_0 and let $g_{j,\eta}$ be the fundamental system constructed above, relative to the base point η_0 (g_{j,η_0} plays the role of the $g_{j,0}$ above).*

Then we have $H_\eta(\theta_0,\theta_1) = H_{\eta_0}(\theta_0,\theta_1) + W_\eta - W_{\eta_0}$. Moreover, if $H_{\eta_0}(\theta_0,\theta_1)$ is invertible then $\eta \mapsto \log\det_\zeta H_\eta(\theta_0,\theta_1)$ is differentiable at η_0 and if φ_η, ψ_η denotes a fundamental system which is normalized for the boundary conditions B_{j,θ_j}, $j = 0, 1$ we have

$$\left.\frac{d}{d\eta}\right|_{\eta_0} \log\det_\zeta H_\eta(\theta_0,\theta_1) = \left.\frac{d}{d\eta}\right|_{\eta_0} \log W(\psi_\eta, \varphi_\eta). \tag{5.17}$$

(2) *Let* $\nu \geqslant 0$ *and let* $\eta \mapsto V_\eta \in \mathscr{V}_\nu$ *be differentiable* (*recall from Definition* 1.2 *that* $\mathscr{V}_\nu$ *is naturally a Fréchet space*). *Let* $H_\eta = l_\nu + X^{-1}V_\eta$, $0 \leqslant \theta < \pi$. *If* $H_{\eta_0}(0,\theta_0)$ *is invertible then* $\eta \mapsto \log\det_\zeta H_\eta(0,\theta)$ *is differentiable at* η_0 *and formula* (5.17) *holds accordingly.*

Proof. (1) Let $0 < \nu < 1$. By Lemma 5.1 and Lemma 5.2 we have

$$\begin{aligned} g_{1,\eta}(x) - g_{1,\eta_0}(x) &= O\big(x^{5/2}\big), \\ g_{2,\eta}(x) - g_{2,\eta_0}(x) &= O\big(x^{3/2}\big), \quad x \to 0. \end{aligned} \tag{5.18}$$

Hence by Theorem 3.1 the domain of $H_{\eta,\max}$ as well as the functionals c_1, c_2 are independent of η. Thus we have indeed $H_\eta(\theta_0,\theta_1) = H_{\eta_0}(\theta_0,\theta_1) + W_\eta - W_{\eta_0}$. The proof of Lemma 4.5 shows that W_η is $H_{\eta_0}(\theta_0,\theta_1)$-bounded and the assumptions on the map $\eta \mapsto W_\eta$ then imply that $\eta \mapsto H_\eta(\theta_0,\theta_1)$ is a graph continuous family of self-adjoint operators; in particular there is an $\varepsilon > 0$ such that $H_\eta(\theta_0,\theta_1)$ is invertible for $|\eta - \eta_0| < \varepsilon$. From now on we assume $|\eta - \eta_0| < \varepsilon$.

From the estimate (4.37) we conclude that

$$\begin{aligned} &\log\det_\zeta H_\eta(\theta_0,\theta_1) - \log\det_\zeta H_{\eta_0}(\theta_0,\theta_1) \\ &\quad = -\int_\Gamma \operatorname{Tr}\big(\big(H_\eta(\theta_0,\theta_1)+z\big)^{-1} - \big(H_{\eta_0}(\theta_0,\theta_1)+z\big)^{-1}\big)\,dz \end{aligned} \tag{5.19}$$

where the integrand on the right is absolutely summable as it is $O(|z|^{-7/6})$, $z \to \infty$.

Furthermore, according to our assumptions on W_η we have

$$\begin{aligned} &\frac{d}{d\eta}\big(\big(H_\eta(\theta_0,\theta_1)+z\big)^{-1} - \big(H_{\eta_0}(\theta_0,\theta_1)+z\big)^{-1}\big) \\ &\quad = -\big(H_\eta(\theta_0,\theta_1)+z\big)^{-1}(\partial_\eta W_\eta)\big(H_\eta(\theta_0,\theta_1)+z\big)^{-1}. \end{aligned} \tag{5.20}$$

By (4.37) the trace norm of the right-hand side is $O(|z|^{-7/6})$ where the O-constant is locally independent of η. By the Dominated Convergence Theorem we may thus differentiate under the integral and find

$$\begin{aligned} \frac{d}{d\eta}\log\det_\zeta H_\eta(\theta_0,\theta_1) &= \int_\Gamma \operatorname{Tr}\big(\big(H_\eta(\theta_0,\theta_1)+z\big)^{-1}(\partial_\eta W_\eta)\big(H_\eta(\theta_0,\theta_1)+z\big)^{-1}\big)\,dz \\ &= -\int_\Gamma \frac{d}{dz}\operatorname{Tr}\big((\partial_\eta W_\eta)\big(H_\eta(\theta_0,\theta_1)+z\big)^{-1}\big)\,dz \\ &= \operatorname{Tr}\big((\partial_\eta W_\eta)H_\eta(\theta_0,\theta_1)^{-1}\big). \end{aligned} \tag{5.21}$$

Having established this identity we can now proceed as in the proof of [26, Proposition 3.4], making essential use of Corollary 5.3.

The kernel $G_\eta(x,y)$ of $H_\eta(\theta_0,\theta_1)$ is given by

$$G_\eta(x,y) = W(\psi_\eta,\varphi_\eta)^{-1}\varphi_\eta(x)\psi_\eta(y), \quad x \leqslant y, \tag{5.22}$$

$W(\psi_\eta,\varphi_\eta)\neq 0$ since $H_\eta(\theta_0,\theta_1)$ is invertible by assumption. Since $g_{j,\eta}$ are differentiable in η (Lemma 5.1 and Lemma 5.2) so are φ_η,ψ_η. In fact, the normalization condition implies

$$\begin{aligned}\varphi_\eta &= \begin{cases} -\cot\theta_0\cdot g_{1,\eta}+g_{2,\eta}, & \text{if } 0<\theta_0<\pi,\\ g_{1,\eta}, & \text{if } \theta_0=0.\end{cases}\\ \psi_\eta &= a_\eta g_{1,\eta}+b_\eta g_{2,\eta},\end{aligned} \tag{5.23}$$

where a_η, b_η depend differentiably on η. Differentiating the formula $\varphi''_{\theta,\eta}=q_\eta\varphi_{\theta,\eta}$ with respect to η gives

$$\partial_\eta\varphi''_\eta=(\partial_\eta q_\eta)\varphi_\eta+q_\eta\partial_\eta\varphi_\eta=(\partial_\eta W_\eta)\varphi_\eta+q_\eta\partial_\eta\varphi_\eta, \tag{5.24}$$

and hence

$$\begin{aligned}(\partial_\eta W_\eta)\varphi_\eta\psi_\eta &= \big(\partial_\eta\varphi''_\eta\big)\psi_\eta-q_\eta(\partial_\eta\varphi_\eta)\psi_\eta\\ &=(\partial_\eta\varphi_\eta)''\psi_\eta-(\partial_\eta\varphi_\eta)\psi''_\eta\\ &=\frac{d}{dx}\big((\partial_\eta\varphi_\eta)'\psi_\eta-(\partial_\eta\varphi_\eta)\psi'_\eta\big)\\ &=\frac{d}{dx}W(\psi_\eta,\partial_\eta\varphi_\eta).\end{aligned} \tag{5.25}$$

Thus we find

$$\begin{aligned}\frac{d}{d\eta}\log\det{}_\zeta H_\eta(\theta_0,\theta_1) &= \operatorname{Tr}\big((\partial_\eta W_\eta)H_\eta(\theta_0,\theta_1)^{-1}\big)\\ &= W(\psi_\eta,\varphi_\eta)^{-1}\int_0^1\frac{d}{dx}W(\psi_\eta,\partial_\eta\varphi_\eta)(x)\,dx\\ &= W(\psi_\eta,\varphi_\eta)^{-1}\Big(W(\psi_\eta,\partial_\eta\varphi_\eta)(1)-\lim_{x\to 0+}W(\psi_\eta,\partial_\eta\varphi_\eta)(x)\Big).\end{aligned}$$

By Corollary 5.3 we have

$$\lim_{x\to 0+}W(\psi_\eta,\partial_\eta\varphi_\eta)(x)=0. \tag{5.26}$$

On the other hand

$$W(\partial_\eta\psi_\eta,\varphi_\eta)(1)=0, \tag{5.27}$$

since ψ_η is normalized with $\psi_\eta(1)=0$ and $\psi'_\eta(1)=-1$ in case of Dirichlet boundary conditions and with $\psi_\eta(1)=1$ in case of generalized Neumann boundary conditions.

Note that in contrast to [26], the proof of relation (5.26) requires a careful asymptotic analysis of the fundamental solutions as summarized in Corollary 5.3.

In view of (5.26) and (5.27) we arrive at

$$\begin{aligned}\frac{d}{d\eta}\log\det_\zeta H_\eta(\theta_0,\theta_1) &= W(\psi_\eta,\varphi_\eta)^{-1}\big(W(\psi_\eta,\partial_\eta\varphi_\eta)(1)+W(\partial_\eta\psi_\eta,\varphi_\eta)(1)\big)\\ &= W(\psi_\eta,\varphi_\eta)^{-1}\frac{d}{d\eta}W(\psi_\eta,\varphi_\eta)\\ &= \frac{d}{d\eta}\log W(\psi_\eta,\varphi_\eta)\end{aligned}$$

and the proof of (1) is complete.

(2) For the proof of (2) we only have to note that by Proposition 4.1 we can estimate the trace norm of $(H_\eta(0,\theta)+z)^{-1}(\partial_\eta V_\eta)(H_\eta(0,\theta)+z)^{-1}$ by $C|z|^{-1}R(z)$ where $R(z)$ satisfies (4.6) and the constant C is locally independent of η. Thus we conclude the variation formula (5.21). The remaining arguments are then completely analogous to the proof of (1) $\square$

Remark 5.5. One can also prove a variation formula for the dependence of the zeta-determinant on the boundary conditions θ_0, θ_1. For the variation of θ_1 at the regular end this is standard, see e.g. [26, Proposition 3.6]. For the variation of θ_0 the proof is much more delicate. Due to our approach via factorizable operators the result is not needed and therefore omitted. However, the factorization method does not extend to matrix valued potentials in a straightforward way. So, if one would like to generalize the results of this paper to matrix valued potentials with regular singularities then one would probably need to establish a formula for the variation of the zeta-determinant under the variation of the boundary conditions at the singular end.

5.2. *Proof of the main Theorem 1.5*

We are now finally ready to prove the main Theorem 1.5. As in [26, Sec. 4] we first note that (1.30) is obviously true if $H(\theta_0,\theta_1)$ is not invertible. Furthermore, if $\varphi(\cdot,z)$, $\psi(\cdot,z)$ denote the normalized solutions for $H(\theta_0,\theta_1)+z$ it follows from Theorem 5.4 (surely, for $V\in\mathscr{V}_\nu^{\det}$ the family $z\mapsto V+zX$ satisfies the standing assumptions of the beginning of this section) that $\det_\zeta(H(\theta_0,\theta_1)+z)$ and $W(\psi(\cdot,z),\varphi(\cdot,z))$ are holomorphic functions in $\mathbb{C}$ with the same logarithmic derivative. Hence it suffices to prove the formula for $H(\theta_0,\theta_1)+z$ for one $z\in\mathbb{C}$.

Let us now first assume that $\theta_0=0$, i.e. at the left end point we have the Dirichlet boundary condition. Except for the low regularity assumptions on the potential this case was treated in [26]. From [26] we will only use the result that the formula (1.30) holds for $\theta_0=0$ and $V(x)=xz$, i.e. for the operator $l_\nu(0,\theta_1)+z$. To reduce the claim to this case we consider $V_\eta:=\eta V$. By Proposition 4.1, $L_\nu:=l_\nu(0,\theta_1)$ is self-adjoint and bounded below and from (4.3) we infer that $H_\eta(0,\theta_1)+z:=L_\nu+\eta X^{-1}V+z$ is invertible for $0\leqslant\eta\leqslant 1$ and $z\geqslant z_0$. Hence we may apply the variation result Theorem 5.4(2) and we are reduced to the case $V=0$ and thus to [26].

Next we consider the case $0<\theta_0<1$. As noted before this necessarily means $\nu<1$, since for $\nu\geqslant 1$ the left end point is in the limit point case. The case $\nu=0$ is beyond the scope of this paper and so we assume $0<\nu<1$. By Proposition 3.5 we have $H(\theta_0,\theta_1)=D_1D_2+W$ with $W\in L^2_{\mathrm{comp}}(0,1]$ and $D_1=d_{1,ra}$, $D_2=d_{2,ar}$ if $\theta_1=0$, and $D_1=d_{1,rr}$, $D_2=d_{2,aa}$ if $\theta_1>0$. Putting $W_\eta=\eta W$, $0\leqslant\eta\leqslant 1$, we infer from Lemma 4.5 that $D_1D_2+\eta W+z$ is invertible for $0\leqslant\eta\leqslant 1$ and $z\geqslant z_0$, thus invoking again the variation result Theorem 5.4(1) we are reduced to prove the formula (1.30) for the operator D_1D_2+z. Note that D_2D_1 has the Dirichlet boundary condition at 0 and hence (1.30) holds for D_2D_1+z by the first part of this proof. We now look at the cases already discussed in the proof of Proposition 3.6. We use the notation from [26], in particular φ_z, ψ_z denote a pair of normalized solutions for $(D_2D_1+z)g=0$ and $\widetilde{\varphi}_z = \frac{1}{2-2\nu} d_1\varphi_z$, $\widetilde{\psi}_z = -d_1\psi_z$ the corresponding pair of normalized solutions for $(D_1D_2+z)g=0$.

Denote by μ_0, μ_1 the invariants defined in (1.27), (1.28) of the boundary conditions for D_1D_2. Denote by μ'_j the corresponding invariants for D_2D_1.

Case I: $0<\theta_1<\pi$. The kernel of D_1D_2 is one-dimensional and D_2D_1 is invertible. We have $\mu_0=-\nu$, $\mu_1=-1/2$, $\mu'_0=1-\nu$, $\mu'_1=1/2$. Thus, using the proven formula (1.30) for D_2D_1 and Proposition 3.6

$$\begin{aligned}\det{}_\zeta(D_1D_2+z) &= z\det{}_\zeta(D_2D_1+z)\\ &= \frac{z\pi}{2^{\mu'_0+\mu'_1}\Gamma(\mu'_0+1)\Gamma(\mu'_1+1)}W(\psi_z,\varphi_z)\\ &= \frac{2(1-\nu)\pi}{2^{\mu_0+\mu_1+2}\Gamma(\mu_0+2)\Gamma(\mu_1+2)}W(\widetilde{\psi}_z,\widetilde{\varphi}_z)\\ &= \frac{\pi}{2^{\mu_0+\mu_1}\Gamma(\mu_0+1)\Gamma(\mu_1+1)}W(\widetilde{\psi}_z,\widetilde{\varphi}_z). \end{aligned} \tag{5.28}$$

Case II: $\theta_1=0$. Here D_1D_2 and D_2D_1 are both invertible and we have $\mu_0=-\nu$, $\mu_1=1/2$, $\mu'_0=1-\nu$, $\mu'_1=-1/2$. Thus

$$\begin{aligned}\det{}_\zeta(D_1D_2+z) &= \det{}_\zeta(D_2D_1+z)\\ &= \frac{\pi}{2^{\mu'_0+\mu'_1}\Gamma(\mu'_0+1)\Gamma(\mu'_1+1)}W(\psi_z,\varphi_z)\\ &= \frac{2(1-\nu)\pi}{2^{\mu_0+\mu_1}\Gamma(\mu_0+2)\Gamma(\mu_1)}W(\widetilde{\psi}_z,\widetilde{\varphi}_z)\\ &= \frac{\pi}{2^{\mu_0+\mu_1}\Gamma(\mu_0+1)\Gamma(\mu_1+1)}W(\widetilde{\psi}_z,\widetilde{\varphi}_z). \end{aligned} \tag{5.29}$$

The proof is complete.

Remark 5.6. We conclude by mentioning that Theorem 1.5 can be extended to potentials with regular singularities at both end points (and otherwise having the same regularity properties as the class $\mathscr{V}_\nu^{\det}$). The formula (1.30) remains the same. For the proof one first employs the factorization method we used here to arrange that, say at the left end point, one has Dirichlet boundary conditions. For this boundary condition a variation formula for the variation of the singular potential was proved in [26, Proposition 3.7]. This variation formula is still valid for our class of potentials and it allows to deform the parameter ν to $\nu=1/2$. Now one is basically in the situation with one regular end point and one singular end point and Theorem 1.5 can be applied. The details are left to the reader.

Acknowledgments

We are indebted to the referees who went far beyond the call of duty and provided a comprehensive list of suggestions for improvements. We think the paper has benefited a lot from these suggestions and we would like to express our gratitude.

References

[1] S. Albeverio, R. Hryniv, Y. Mykytyuk, Inverse spectral problems for Bessel operators, J. Differential Equations 241 (1) (2007) 130–159, MR 2356213 (2008k:34035).
[2] D. Burghelea, L. Friedlander, T. Kappeler, On the determinant of elliptic boundary value problems on a line segment, Proc. Amer. Math. Soc. 123 (10) (1995) 3027–3038, MR 1301012 (95m:58131).
[3] M. Bôcher, On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic, Trans. Amer. Math. Soc. 1 (1) (1900) 40–52, MR 1500523.
[4] J. Brüning, R. Seeley, Regular singular asymptotics, Adv. Math. 58 (2) (1985) 133–148, MR 814748 (87b:41032).
[5] J. Brüning, R. Seeley, The resolvent expansion for second order regular singular operators, J. Funct. Anal. 73 (2) (1987) 369–429, MR 899656 (88g:35151).
[6] J. Brüning, R. Seeley, An index theorem for first order regular singular operators, Amer. J. Math. 110 (4) (1988) 659–714, MR 955293 (89k:58271).
[7] W. Bulla, F. Gesztesy, Deficiency indices and singular boundary conditions in quantum mechanics, J. Math. Phys. 26 (10) (1985) 2520–2528, MR 803795 (87d:35097).
[8] R. Carlson, A Borg–Levinson theorem for Bessel operators, Pacific J. Math. 177 (1) (1997) 1–26, MR 1444769 (98b:34029).
[9] J. Cheeger, Analytic torsion and the heat equation, Ann. of Math. (2) 109 (2) (1979) 259–322, MR 528965 (80j:58065a).
[10] J. Cheeger, On the spectral geometry of spaces with cone-like singularities, Proc. Natl. Acad. Sci. USA 76 (5) (1979) 2103–2106, MR 530173 (80k:58098).
[11] J. Cheeger, Spectral geometry of singular Riemannian spaces, J. Differential Geom. 18 (4) (1983) 575–657 (1984), MR 730920 (85d:58083).
[12] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw–Hill Book Company, Inc., New York, Toronto, London, 1955, MR 0069338 (16,1022b).
[13] X. Dai, X. Huang, The intersection R-torsion for finite cone, downloaded from http://www.math.ucsb.edu/~dai/preprints.html, 2010.
[14] T. de Melo, L. Hartmann, M. Spreafico, Reidemeister torsion and analytic torsion of discs, Boll. Unione Mat. Ital. (9) 2 (2) (2009) 529–533, MR 2537286 (2010f:58043).
[15] W.N. Everitt, H. Kalf, The Bessel differential equation and the Hankel transform, J. Comput. Appl. Math. 208 (1) (2007) 3–19, MR 2347733 (2008g:34118).
[16] H. Falomir, M.A. Muschietti, P.A.G. Pisani, R. Seeley, Unusual poles of the ζ-functions for some regular singular differential operators, J. Phys. A 36 (39) (2003) 9991–10010, MR 2024508 (2004k:58049).
[17] L. Hartmann, M. Spreafico, The analytic torsion of a cone over a sphere, J. Math. Pures Appl. (9) 93 (4) (2010) 408–435, MR 2609036.
[18] L. Hörmander, The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis, second ed., Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 256, Springer-Verlag, Berlin, 1990, MR 1065993 (91m:35001a).
[19] T. Kato, Perturbation Theory for Linear Operators, Classics Math., Springer-Verlag, Berlin, 1995, reprint of the 1980 edition, MR 1335452 (96a:47025).
[20] K. Kirsten, P. Loya, J. Park, The very unusual properties of the resolvent, heat kernel, and zeta function for the operator $-d^2/dr^2 - 1/(4r^2)$, J. Math. Phys. 47 (4) (2006) 043506, 27, MR 2226343 (2007c:58050).
[21] K. Kirsten, P. Loya, J. Park, Exotic expansions and pathological properties of ζ-functions on conic manifolds, with an appendix by B. Vertman, J. Geom. Anal. 18 (3) (2008) 835–888, MR 2420767 (2009j:58051).
[22] K. Kirsten, P. Loya, J. Park, Functional determinants for general self-adjoint extensions of Laplace-type operators resulting from the generalized cone, Manuscripta Math. 125 (1) (2008) 95–126, MR 2357751 (2009h:58073).
[23] A. Kostenko, A. Sakhnovich, G. Teschl, Inverse eigenvalue problems for perturbed spherical Schrödinger operators, Inverse Problems 26 (2010).

[24] A. Kostenko, A. Sakhnovich, G. Teschl, Weyl–Titchmarsh theory for Schrödinger operators with strongly singular potentials, arXiv:1007.0136.
[25] M. Lesch, Operators of Fuchs Type, Conical Singularities, and Asymptotic Methods, Teubner-Texte Math. (Teubner Texts in Mathematics), vol. 136 , B.G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997, MR 1449639 (98d:58174).
[26] M. Lesch, Determinants of regular singular Sturm–Liouville operators, Math. Nachr. 194 (1998) 139–170, MR 1653090 (99j:58220).
[27] M. Lesch, H. Moscovici, M. Pflaum, Connes–Chern character and η-cochains, arXiv:0912.0194 [math.OA], 2009.
[28] M. Lesch, J. Tolksdorf, On the determinant of one-dimensional elliptic boundary value problems, Comm. Math. Phys. 193 (3) (1998) 643–660, MR 1624851 (2000b:58060).
[29] W. Müller, Analytic torsion and R-torsion of Riemannian manifolds, Adv. Math. 28 (3) (1978) 233–305, MR 498252 (80j:58065b).
[30] G.K. Pedersen, Analysis Now, Grad. Texts in Math., vol. 118, Springer-Verlag, New York, 1989, MR 971256 (90f:46001).
[31] F. Rellich, Die zulässigen Randbedingungen bei den singulären Eigenwertproblemen der mathematischen Physik (Gewöhnliche Differentialgleichungen zweiter Ordnung), Math. Z. 49 (1944) 702–723, MR 0013183 (7,118d).
[32] F. Serier, The inverse spectral problem for radial Schrödinger operators on [0, 1], J. Differential Equations 235 (1) (2007) 101–126, MR 2309569 (2008b:34015).
[33] M.E. Taylor, Partial Differential Equations. I Basic Theory, Appl. Math. Sci., vol. 115, Springer-Verlag, New York, 1996, MR 1395148 (98b:35002b).
[34] B. Vertman, Zeta determinants for regular-singular Laplace-type operators, J. Math. Phys. 50 (8) (2009) 083515, 23, MR 2554443.
[35] B. Vertman, The metric anomaly at the regular boundary of the analytic torsion of a bounded generalized cone, I, II, arXiv:1004.2067, arXiv:1004.2069.
[36] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge Math. Lib., Cambridge University Press, Cambridge, 1995, reprint of the second (1944) edition, MR 1349110 (96i:33010).