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Hierarchy of Hamilton equations on Banach Lie–Poisson spaces related to restricted Grassmannian

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Abstract

We consider the Banach Lie–Poisson space $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$ and its complexification $\mathbb{C} \oplus L^1_{\mathrm{res}}$, where the first one of them contains the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ as a symplectic leaf. Using the Magri method we define an involutive family of Hamiltonians on these Banach Lie–Poisson spaces. The hierarchy of Hamilton equations given by these Hamiltonians is investigated. The operator equations of Ricatti-type are included in this hierarchy. For a few particular cases we give the explicit solutions.

Keywords: Integrable systems; Banach Lie–Poisson spaces; Restricted Grassmannian; Banach Lie groups and algebras

1. Introduction

The foundation of Banach Poisson differential geometry was developed in [11]. This theory, in particular, gives us a possibility to formulate geometrically and analytically rigorous language for the theory of infinite dimensional Hamiltonian systems. A special place in this theory is occupied by the Banach Lie–Poisson spaces. Recall that by definition $\mathfrak{b}$ is a Banach Lie–Poisson space if its dual $\mathfrak{b}^*$ is a Banach Lie algebra such that $\mathrm{ad}^*_x \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**}$ for $x \in \mathfrak{b}^*$, where $\mathrm{ad}^*_x : \mathfrak{b}^{**} \to \mathfrak{b}^{**}$ is dual to the adjoint representation $\mathrm{ad}_x := [x, \cdot] : \mathfrak{b}^* \to \mathfrak{b}^*$.

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Many infinite dimensional physical systems can be considered as systems on some Banach Lie–Poisson space $\mathfrak{b}$ in the Hamilton way

$$\frac{d}{dt}\rho = -\operatorname{ad}^*_{Dh(\rho)}\rho, \tag{1.1}$$

where $\rho \in \mathfrak{b}$ and $h \in C^\infty(\mathfrak{b})$, e.g. see [13].

The first aim of this paper is to investigate Banach Lie–Poisson spaces related to the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$, see [2,12]. The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ has its own long story as one of the most important infinite dimensional Kähler manifolds in mathematical physics. It is a set of Hilbert subspaces $W \subset \mathcal{H}$ of a polarized Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that the projectors $P_+ : W \to \mathcal{H}_+$ and $P_- : W \to \mathcal{H}_-$ are Fredholm and Hilbert–Schmidt operators respectively. The geometry of $\mathrm{Gr}_{\mathrm{res}}$ and its symmetry group play an important role in the quantum field theory [14,22,23,25,26], the loop group theory [15,21], and the integration of the KdV and KP hierarchies [8,17,20].

The second aim is to define and investigate a hierarchy of the Hamilton equations on the Banach Lie–Poisson space $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$ and on its complexification $\mathbb{C} \oplus L^1_{\mathrm{res}}$. This hierarchy is obtained from the involutive system of Hamiltonians constructed on $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$ and $\mathbb{C} \oplus L^1_{\mathrm{res}}$ by the Magri method [7].

The pair of coupled operator Ricatti equations, see (3.41), belongs to this hierarchy. As we show in Example 4.3 the finite dimensional version of the hierarchy provides the non-trivial example of an integrable Hamiltonian system.

In our considerations we use the functional analytical methods as well as Banach differential geometric methods. All the results we have obtained are also valid in finite dimension case.

Since the hierarchy consists of Hamilton equations, the flows preserve symplectic leaves of $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$ and $\mathbb{C} \oplus L^1_{\mathrm{res}}$. In particular, they preserve $\mathrm{Gr}_{\mathrm{res}}$, which is one of the symplectic leaves of $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$, see [2]. We show in Example 4.1 that after restriction to $\mathrm{Gr}_{\mathrm{res}}$ the flows linearize in natural complex coordinates.

The central place of the paper is occupied by Section 3, where (using the Magri method) we construct the infinite hierarchy under consideration. We also discuss its various realizations, see (3.20), (3.19), (3.38), (3.52) and (3.54).

In Section 2 we prepare the material necessary for the application of Magri method, i.e. we find explicit formulas for coadjoint representation of central extension $\widetilde{GL}_{\mathrm{res},0}$ of $GL_{\mathrm{res},0}$, the Poisson bracket and Casimirs of the Banach Lie–Poisson spaces $\mathbb{C} \oplus L^1_{\mathrm{res}}$ and $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$.

Finally Section 4 gives formulas for solutions in some particular cases.

We also include in the paper two appendices, where we present the Magri method and the theory of extensions of Banach Lie–Poisson spaces.

2. Banach Lie–Poisson spaces related to restricted Grassmannian

We investigate the extensions of Banach Lie groups, Banach Lie algebras and Banach Lie–Poisson spaces related to the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$. One of the aims of this section is to obtain explicit formulas for adjoint and coadjoint actions of constructed Banach Lie groups and Banach Lie algebras. To this end we apply the methods described in Appendix A.

Before that let us recall the definitions of objects we are going to use and fix the notation. For more information see [15,21,26].

2.1. Preliminary definitions and notation

Let us consider a complex separable Hilbert space with a fixed decomposition into two orthogonal Hilbert subspaces

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-. \tag{2.1}$$

Let P_+ and P_- denote the orthogonal projectors onto $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively. We assume that in general both Hilbert subspaces are infinite dimensional. However we also admit the case when one (or both) of them is finite dimensional. In this case many analytical problems are considerably simplified.

In what follows we omit the symbols $\mathcal{H}$ and $\mathcal{H}_\pm$ in the notation for various operator algebras and groups and put the subscript $\pm$ if we mean that the operators act in $\mathcal{H}_\pm$. In this way, for example instead of $L^2(\mathcal{H})$ or $L^2(\mathcal{H}_+)$ we write L^2 or L^2_+.

In order to simplify our notation we use the block decomposition

$$\begin{aligned} P_+AP_+ &= \begin{pmatrix} A_{++} & 0 \\ 0 & 0 \end{pmatrix}, \qquad & P_+AP_- &= \begin{pmatrix} 0 & A_{+-} \\ 0 & 0 \end{pmatrix}, \\ P_-AP_+ &= \begin{pmatrix} 0 & 0 \\ A_{-+} & 0 \end{pmatrix}, \qquad & P_-AP_- &= \begin{pmatrix} 0 & 0 \\ 0 & A_{--} \end{pmatrix} \end{aligned} \tag{2.2}$$

and we identify the operators $A_{++} : \mathcal{H}_+ \to \mathcal{H}_+$, $A_{--} : \mathcal{H}_- \to \mathcal{H}_-$, $A_{-+} : \mathcal{H}_+ \to \mathcal{H}_-$ and $A_{+-} : \mathcal{H}_- \to \mathcal{H}_+$ with P_+AP_+, P_-AP_-, P_-AP_+, P_+AP_- respectively when there is no risk of confusion.

By L^p we denote the Schatten classes of operators acting in $\mathcal{H}$ equipped with the norm $\|\cdot\|_p$. The L^p spaces are ideals in associative algebra L^∞ of bounded operators in $\mathcal{H}$. In particular L^1 denotes the ideal of trace-class operators and L^2 is the ideal of Hilbert–Schmidt operators. By $L^0 \subset L^\infty$ one denotes the ideal of compact operators, which is $\|\cdot\|_\infty$-norm closure $\overline{L^p} = L^0$ of any L^p ideal, see [4,18].

Let GL^∞ be the Banach Lie group of invertible bounded operators in $\mathcal{H}$. By $UL^\infty \subset GL^\infty$ we denote the real Banach Lie group of the unitary operators and its Lie algebra is denoted by $\mathcal{U}L^\infty$. By GL^1_+ we denote the group of invertible operators on $\mathcal{H}_+$ which have a determinant (i.e. they differ from identity by a trace-class operator) and by SL^1_+—its subgroup which consists of operators with the determinant equal to 1. See [5,16] for the definition and properties of the determinant for this case.

The **unitary restricted group** UL_{res} is defined as

$$UL_{\mathrm{res}} := \left\{ u \in UL^\infty \;\middle|\; [u, P_+] \in L^2 \right\}. \tag{2.3}$$

It possesses the Banach Lie group structure given by the embedding

$$UL_{\mathrm{res}} \ni u \mapsto (u, u_{-+}) \in UL^\infty \times L^2_{+-}. \tag{2.4}$$

This structure is not compatible with Banach Lie group structure of UL^∞. The Banach Lie algebra of UL_{res} is

$$\mathcal{U}L_{\mathrm{res}} := \left\{ x \in L^\infty \;\middle|\; x^+ = -x,\ [x, P_+] \in L^2 \right\} \tag{2.5}$$

with the norm

$$\|x\|_{\mathrm{res}} := \|x_{++}\|_\infty + \|x_{--}\|_\infty + \|x_{-+}\|_2 + \|x_{+-}\|_2, \tag{2.6}$$

where x^+ is the operator adjoint to x. Note that the topology of $\mathcal{U}L_{\mathrm{res}}$ is strictly stronger than the operator topology on L^∞.

The complexifications of UL_{res} and $\mathcal{U}L_{\mathrm{res}}$ are $UL^{\mathbb{C}}_{\mathrm{res}} = GL_{\mathrm{res}}$ and $\mathcal{U}L^{\mathbb{C}}_{\mathrm{res}} = L_{\mathrm{res}}$ respectively, where

$$GL_{\mathrm{res}} = \left\{ g \in GL^\infty \;\middle|\; [g, P_+] \in L^2 \right\} \tag{2.7}$$

and

$$L_{\mathrm{res}} = \left\{ x \in L^\infty \;\middle|\; [x, P_+] \in L^2 \right\}. \tag{2.8}$$

By definition the **restricted Grassmannian** $\mathrm{Gr}_{\mathrm{res}}$ consists of Hilbert subspaces $W \subset \mathcal{H}$ such that:

i) the projection P_+ restricted to W is a Fredholm operator;
ii) the projection P_- restricted to W is a Hilbert–Schmidt operator;

see e.g. [15,26].

The restricted Grassmannian is a Hilbert manifold modelled on the Hilbert space L^2_{+-}. The groups UL_{res} and GL_{res} act on it transitively. In this way the tangent space to the restricted Grassmannian in the point $\mathcal{H}_+$ can be described as follows

$$T_{\mathcal{H}_+}\mathrm{Gr}_{\mathrm{res}} \cong \mathcal{U}L_{\mathrm{res}} / \left(\mathcal{U}L^\infty_+ \times \mathcal{U}L^\infty_-\right). \tag{2.9}$$

Both UL_{res} and GL_{res} are disconnected and their connected components are $UL_{\mathrm{res},k}$ and $GL_{\mathrm{res},k}$, where $g \in UL_{\mathrm{res},k}$ and $g \in GL_{\mathrm{res},k}$ iff the Fredholm index $\operatorname{ind} g_{++}$ of the upper left block g_{++} of the operator g is equal to k, see [3,15]. The maximal connected subgroups $UL_{\mathrm{res},0}$ and $GL_{\mathrm{res},0}$ will be of special interest. In a similar fashion, connected components of the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ are the sets $\mathrm{Gr}_{\mathrm{res},k}$ consisting of elements of $\mathrm{Gr}_{\mathrm{res}}$ such that the index of the orthogonal projection P_+ restricted to that element is equal to k. Let us note that $UL_{\mathrm{res},0}$ acts transitively on $\mathrm{Gr}_{\mathrm{res},0}$.

2.2. *Extensions of* $GL_{\mathrm{res},0}$

The central object in the following construction is the group $\mathcal{E}$ defined as

$$\mathcal{E} := \left\{ (q, A) \in GL^\infty_+ \times GL_{\mathrm{res},0} \;\middle|\; A_{++} - q \in L^1_+ \right\} \tag{2.10}$$

with pairwise multiplication. The topology and Banach manifold structure on $\mathcal{E}$ is given by the embedding

$$(q, A) \mapsto (A_{++} - q, A) \in L^1_+ \times GL_{\mathrm{res}}, \tag{2.11}$$

see [15,26].

Let us consider the Banach Lie group extensions presented in the following commutative diagram:

$$\begin{tikzcd}
 & \{1\} & \{1\} & \{1\} & \\
\{1\} \arrow[r, hook] & \mathbb{C}^{\times} \arrow[u] \arrow[r, hook, "\iota"] & \widetilde{GL}_{\mathrm{res},0} \arrow[u] \arrow[r, "\pi"] & GL_{\mathrm{res},0} \arrow[u] \arrow[r] & \{1\} \\
\{1\} \arrow[r, hook] & GL^1_+ \arrow[u, "\det"] \arrow[r, hook, "\iota_1"] & \mathcal{E} \arrow[u, "\delta"] \arrow[r, "\pi_2"] & GL_{\mathrm{res},0} \arrow[u, "\mathrm{id}"] \arrow[r] & \{1\} \\
\{1\} \arrow[r, hook] & SL^1_+ \arrow[u, hook] \arrow[r, hook, "\iota_1"] & SL^1_+ \times \{\mathbb{1}\} \arrow[u, hook] \arrow[r, "\pi_2"] & \{1\} \arrow[u, hook] & \\
 & \{1\} \arrow[u, hook] & \{1\} \arrow[u, hook] & &
\end{tikzcd} \tag{2.12}$$

The map ι_1 is defined by $\iota_1(q) := (q, \mathbb{1})$ and the map π_2 is a projection onto the second component of the Cartesian product $GL^\infty_+ \times GL_{\mathrm{res},0}$. Thus $\iota_1(SL^1_+)$ is a normal subgroup of $\mathcal{E}$ and the group $\widetilde{GL}_{\mathrm{res},0}$ is defined as the quotient group

$$\widetilde{GL}_{\mathrm{res},0} := \mathcal{E}/\iota_1\big(SL^1_+\big). \tag{2.13}$$

The maps ι and π in upper row of diagram (2.12) are given as quotients of ι_1 and π_2 respectively. The map δ is the quotient map $\mathcal{E} \to \mathcal{E}/\iota_1(SL^1_+) = \widetilde{GL}_{\mathrm{res},0}$. In this way all rows and columns in diagram (2.12) are exact sequences of Banach Lie groups.

Using the approach described in Appendix A we define a local section (A.2) of the bundle $GL^1_+ \to \mathcal{E} \to GL_{\mathrm{res},0}$ by

$$\sigma(A) := (A_{++}, A) \tag{2.14}$$

for $A \in GL_{\mathrm{res},0}$ such that A_{++} is invertible. The Banach Lie group $\mathcal{E}$ can be locally identified with $GL^1_+ \times_{\Phi,\Omega} GL_{\mathrm{res},0}$ through the isomorphism $\Psi : GL^1_+ \times_{\Phi,\Omega} GL_{\mathrm{res},0} \to \mathcal{E}$ given in the properly chosen neighborhood of identity by

$$\Psi(n, A) = (nA_{++}, A). \tag{2.15}$$

The maps $\Phi : GL_{\mathrm{res},0} \to \operatorname{Aut} GL^1_+$ and $\Omega : GL_{\mathrm{res},0} \times GL_{\mathrm{res},0} \to GL^1_+$ defined in the general case by (A.7) and (A.8) in this case can be expressed locally as follows:

$$\Phi(A)(n) = A_{++} n A_{++}^{-1}, \tag{2.16}$$

$$\Omega(A_1, A_2) = A_{1++} A_{2++} (A_1 A_2)_{++}^{-1} \tag{2.17}$$

for $n \in GL^1_+$, $A, A_1, A_2 \in GL_{\mathrm{res},0}$ such that A_{++}, A_{1++}, A_{2++} and $(A_1 A_2)_{++}$ are invertible.

The map $\Phi(A)$ descends to the trivial automorphism of $\mathbb{C}^\times$. Thus the Banach Lie group $\widetilde{GL}_{\mathrm{res},0}$ can be identified with $\mathbb{C}^\times \times_{\mathrm{id},\tilde{\Omega}} GL_{\mathrm{res},0}$ for

$$\tilde{\Omega} := \det \circ\, \Omega \tag{2.18}$$

and it is a central extension of $GL_{\mathrm{res},0}$.

2.3. *Extensions of* L_{res}

The Banach Lie algebra counterpart of diagram (2.12) is the following:

$$\begin{tikzcd}
 & \{0\} & \{0\} & \{0\} & \\
\{0\} \arrow[r, hook] & \mathbb{C} \arrow[u] \arrow[r, hook] & \mathbb{C} \oplus L_{\mathrm{res}} \arrow[u] \arrow[r] & L_{\mathrm{res}} \arrow[u] \arrow[r] & \{0\} \\
\{0\} \arrow[r, hook] & L^1_+ \arrow[u, "\mathrm{Tr}"] \arrow[r, hook, "\iota_1"] & L^1_+ \oplus L_{\mathrm{res}} \arrow[u, "\mathrm{Tr}_1"] \arrow[r, "\pi_2"] & L_{\mathrm{res}} \arrow[u, "\mathrm{id}"] \arrow[r] & \{0\} \\
\{0\} \arrow[r, hook] & \mathscr{S}L^1_+ \arrow[u, hook] \arrow[r, hook, "\iota_1"] & \mathscr{S}L^1_+ \oplus \{0\} \arrow[u, hook] \arrow[r, "\pi_2"] & \{0\} \arrow[u, hook] & \\
 & \{0\} \arrow[u, hook] & \{0\} \arrow[u, hook] & &
\end{tikzcd} \tag{2.19}$$

where

$$\mathscr{S}L^1_+ := \left\{ \rho \in L^1_+ \;\middle|\; \mathrm{Tr}\,\rho = 0 \right\} \tag{2.20}$$

is the Banach Lie algebra of the group SL^1_+. The Banach Lie algebra $(L^1_+ \oplus L_{\mathrm{res}})/(\mathscr{S}L^1_+ \oplus \{0\})$ of the quotient group $\widetilde{GL}_{\mathrm{res},0}$ is naturally identified by $D\Psi(\mathbb{1}, \mathbb{1})$ with $\mathbb{C} \oplus L_{\mathrm{res}}$. The map Tr_1 is given by taking trace of the first component of $(\rho, X) \in L^1_+ \oplus L_{\mathrm{res}}$. The direct sums in diagram (2.19) are understood as direct sums of Banach spaces.

Similarly as in the group case, all rows and columns in diagram (2.19) are exact sequences of Banach Lie algebras.

By using formula (A.14) with functions (2.16) and (2.17) we obtain a local formula for the adjoint action

$$\mathrm{Ad}_{(n,A)}(\rho, X) = \big(nA_{++}(\rho + X_{++})(A_{++})^{-1}n^{-1} - \big(AXA^{-1}\big)_{++}, AXA^{-1}\big) \tag{2.21}$$

for (n, A) in some open set in $GL^1_+ \times_{\Phi,\Omega} GL_{\mathrm{res},0}$ and $(\rho, X) \in L^1_+ \oplus L_{\mathrm{res}}$. From (A.16) and (A.17) we get that

$$\varphi(X) := [X_{++}, \cdot], \tag{2.22}$$

$$\omega(X, Y) := -X_{+-}Y_{-+} + Y_{+-}X_{-+}. \tag{2.23}$$

Bracket (A.15) for φ and ω given by (2.22) and (2.23) assumes the form

$$\begin{aligned}\big[(\rho, X), \big(\rho', Y\big)\big] = \big(\big[\rho, \rho'\big] + \big[X_{++}, \rho'\big] - [Y_{++}, \rho] \\ - X_{+-}Y_{-+} + Y_{+-}X_{-+}, [X, Y]\big)\end{aligned} \tag{2.24}$$

where $(\rho, X), (\rho', Y) \in L^1_+ \oplus L_{\mathrm{res}}$. This Banach Lie algebra was presented in [12] (up to the sign conventions) as an example of extensions of Banach Lie algebras.

The structure of Banach Lie algebra on $\mathbb{C} \oplus L_{\mathrm{res}}$ is given by the function

$$\tilde{\varphi}(X) \equiv 0 \tag{2.25}$$

and the cocycle

$$\tilde{\omega}(X, Y) = -s(X, Y) = -\operatorname{Tr}(X_{+-}Y_{-+} - Y_{+-}X_{-+}), \tag{2.26}$$

where $s(X, Y)$ is called the Schwinger term, see [19,26]. Thus the adjoint representation of the Lie group $\mathbb{C}^\times \times_{\mathrm{id},\tilde{\Omega}} GL_{\mathrm{res},0}$ on $\mathbb{C} \oplus L_{\mathrm{res}}$ is given by

$$\mathrm{Ad}_{(\gamma,A)}(\lambda, X) = \big(\lambda + \operatorname{Tr}\big(P_+ - A^{-1}P_+A\big), AXA^{-1}\big) \tag{2.27}$$

for $(\gamma, A) \in \mathbb{C}^\times \times_{\mathrm{id},\tilde{\Omega}} GL_{\mathrm{res},0}$, $(\lambda, X) \in \mathbb{C} \oplus L_{\mathrm{res}}$. Moreover, the Lie bracket for $(\lambda, X), (\lambda', Y) \in \mathbb{C} \oplus L_{\mathrm{res}}$ is the following

$$\big[(\lambda, X), \big(\lambda', Y\big)\big] = \big(-s(X, Y), [X, Y]\big). \tag{2.28}$$

Let us note that formula (A.14) allows one to express Ad only locally. However the right-hand side of formula (2.27) defines some global representation of $\mathbb{C}^\times \times_{\mathrm{id},\tilde{\Omega}} GL_{\mathrm{res},0}$, which coincides with Ad on an open neighborhood of $(1, \mathbb{1})$. However every open neighborhood of the unit element in Banach Lie group generates a connected component, and $\mathbb{C}^\times \times_{\mathrm{id},\tilde{\Omega}} GL_{\mathrm{res},0}$ is connected. Thus the formula (2.27) is valid for all $(\gamma, A) \in \mathbb{C}^\times \times_{\mathrm{id},\tilde{\Omega}} GL_{\mathrm{res},0}$.

2.4. Extensions of complex Banach Lie–Poisson space L^1_{res}

In order to find a Banach space predual to the Banach Lie algebra L_{res}, we define the Banach space

$$L^1_{\mathrm{res}} := \left\{\mu \in L_{\mathrm{res}} \mid \mu_{++} \in L^1_+,\ \mu_{--} \in L^1_-\right\} \tag{2.29}$$

with the norm

$$\|\mu\|_* := \|\mu_{++}\|_1 + \|\mu_{--}\|_1 + \|\mu_{-+}\|_2 + \|\mu_{+-}\|_2. \tag{2.30}$$

Moreover we define the **restricted trace** $\mathrm{Tr}_{\mathrm{res}} : L^1_{\mathrm{res}} \to \mathbb{C}$ by

$$\mathrm{Tr}_{\mathrm{res}}\mu := \mathrm{Tr}(\mu_{++} + \mu_{--}), \tag{2.31}$$

for $\mu \in L^1_{\mathrm{res}}$. The domain of $\mathrm{Tr}_{\mathrm{res}}$ is larger than L^1 since $L^1 \subset L_{\mathrm{res}}$. However for trace-class operators the restricted trace $\mathrm{Tr}_{\mathrm{res}}$ coincides with the standard trace Tr. The properties of the restricted trace are similar to the properties of the standard trace but one needs to replace L^∞ with L_{res}.

Proposition 2.1. *The Banach space L^1_{res} is an ideal* (*in the sense of commutative algebras*) *in the Banach space L_{res}. Moreover for $\mu \in L^1_{\mathrm{res}}$, $\nu \in L_{\mathrm{res}}$ we have*

$$\mathrm{Tr}_{\mathrm{res}}(\mu\nu) = \mathrm{Tr}_{\mathrm{res}}(\nu\mu). \tag{2.32}$$

Proof. The conclusion that L^1_{res} is an ideal follows from the fact that L^1 and L^2 are ideals in L^∞ and a product of two operators from L^2 is trace-class. To prove formula (2.32) we expand its left-hand side

$$\mathrm{Tr}_{\mathrm{res}}(\mu\nu) = \mathrm{Tr}(\mu_{++}\nu_{++} + \mu_{+-}\nu_{-+} + \mu_{-+}\nu_{+-} + \mu_{--}\nu_{--}). \tag{2.33}$$

By assumptions of the proposition, we conclude that each term in the right-hand side is a trace-class operator. Since for $A \in L^\infty$ and $B \in L^1$ or for $A, B \in L^2$ one has

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \tag{2.34}$$

we conclude that

$$\mathrm{Tr}_{\mathrm{res}}(\mu\nu) = \mathrm{Tr}_{\mathrm{res}}(\nu\mu). \qquad \square \tag{2.35}$$

As a corollary to this proposition we get that for $g \in GL_{\mathrm{res}}$ and $\mu \in L^1_{\mathrm{res}}$, the operator μg^{-1} belongs to L^1_{res} and

$$\mathrm{Tr}_{\mathrm{res}}\left(g\mu g^{-1}\right) = \mathrm{Tr}_{\mathrm{res}}(\mu). \tag{2.36}$$

Using the pairing between $\mu \in L^1_{\mathrm{res}}$, $A \in L_{\mathrm{res}}$ given by

$$\langle \mu, A\rangle := \mathrm{Tr}_{\mathrm{res}}(\mu A) = \mathrm{Tr}(\mu_{++}A_{++}) + \mathrm{Tr}(\mu_{+-}A_{-+}) + \mathrm{Tr}(\mu_{-+}A_{+-}) + \mathrm{Tr}(\mu_{--}A_{--}), \tag{2.37}$$

we conclude that $(L^1_{\mathrm{res}})^* \cong L_{\mathrm{res}}$, i.e. the Banach space L^1_{res} is predual of L_{res}. This duality can be found in [2,12].

Proposition 2.2. *Space L^1_{res} with norm $\|\cdot\|_*$ is Banach $*$-algebra.*

Proof. The only point yet to be shown is the inequality

$$\|\mu\rho\|_* \leqslant \|\mu\|_*\|\rho\|_* \tag{2.38}$$

for $\mu, \rho \in L^1_{\mathrm{res}}$. In order to prove it we observe that

$$\begin{aligned}\|\mu\rho\|_* &= \big\|(\mu\rho)_{++}\big\|_1 + \big\|(\mu\rho)_{+-}\big\|_2 + \big\|(\mu\rho)_{-+}\big\|_2 + \big\|(\mu\rho)_{--}\big\|_1\\ &= \|\mu_{++}\rho_{++} + \mu_{+-}\rho_{-+}\|_1 + \|\mu_{++}\rho_{+-} + \mu_{+-}\rho_{--}\|_2\\ &\quad + \|\mu_{-+}\rho_{++} + \mu_{--}\rho_{-+}\|_2 + \|\mu_{--}\rho_{--} + \mu_{-+}\rho_{+-}\|_1.\end{aligned} \tag{2.39}$$

Next, applying the following inequalities

$$\|\rho\|_2 \leqslant \|\rho\|_1, \tag{2.40}$$

$$\|\rho\mu\|_1 \leqslant \|\rho\|_\infty\|\mu\|_1 \leqslant \|\rho\|_1\|\mu\|_1 \tag{2.41}$$

for $\rho, \mu \in L^1$ and the inequalities

$$\|\rho\mu\|_1 \leqslant \|\rho\|_2\|\mu\|_2, \tag{2.42}$$

$$\|\rho\mu\|_2 \leqslant \|\rho\|_\infty\|\mu\|_2 \leqslant \|\rho\|_2\|\mu\|_2 \tag{2.43}$$

for $\rho, \mu \in L^2$, we obtain

$$\begin{aligned}\|\mu\rho\|_* &\leqslant \|\mu_{++}\|_1\|\rho_{++}\|_1 + \|\mu_{+-}\|_2\|\rho_{-+}\|_2 + \|\mu_{++}\|_1\|\rho_{+-}\|_2 + \|\mu_{+-}\|_2\|\rho_{--}\|_2\\ &\quad + \|\mu_{-+}\|_2\|\rho_{++}\|_1 + \|\mu_{--}\|_1\|\rho_{-+}\|_2 + \|\mu_{--}\|_1\|\rho_{--}\|_1 + \|\mu_{-+}\|_2\|\rho_{+-}\|_2\\ &\leqslant \|\mu\|_*\|\rho\|_*\end{aligned} \tag{2.44}$$

The latter inequality in (2.44) follows directly from (2.30). □

The Banach space predual to extended Banach Lie algebra $L^1_+ \oplus L_{\mathrm{res}}$ is $L^0_+ \oplus L^1_{\mathrm{res}}$ with natural component-wise pairing

$$\big\langle (A, \mu), (\rho, X)\big\rangle = \mathrm{Tr}(A\rho) + \mathrm{Tr}_{\mathrm{res}}(\mu X). \tag{2.45}$$

It follows from the fact that the Banach space dual to the ideal of compact operators L^0_+ is L^1_+, see [24]. Analogously, the predual of $\mathbb{C} \oplus L_{\mathrm{res}}$ is $\mathbb{C} \oplus L^1_{\mathrm{res}}$ with the pairing given by

$$\big\langle(\gamma, \mu), (\lambda, X)\big\rangle = \gamma\lambda + \mathrm{Tr}_{\mathrm{res}}(\mu X), \tag{2.46}$$

for $\mu \in L^1_{\mathrm{res}}$, $X \in L_{\mathrm{res}}$, $\gamma, \lambda \in \mathbb{C}$.

The map $\mathrm{Tr}^* : \mathbb{C} \to L^\infty_+$ dual to $\mathrm{Tr} : L^1_+ \to \mathbb{C}$ is given by $\mathrm{Tr}^*(\lambda) = \lambda\mathbb{1}$. Since the ideal of compact operators L^0_+ is the Banach space predual to L^1_+ and Tr^* does not take values in L^0_+, we conclude that Tr^* cannot be restricted to predual spaces. Therefore only horizontal exact sequences in diagram (2.19) have their predual counterparts

$$\{0\} \hookrightarrow L^1_{\mathrm{res}} \xhookrightarrow{\pi_2^*} \mathbb{C} \oplus L^1_{\mathrm{res}} \xrightarrow{\iota_1^*} \mathbb{C} \longrightarrow \{0\} \tag{2.47}$$

$$\{0\} \hookrightarrow L^1_{\mathrm{res}} \xhookrightarrow{\pi_2^*} L^0_+ \oplus L^1_{\mathrm{res}} \xrightarrow{\iota_1^*} L^0_+ \longrightarrow \{0\} \tag{2.48}$$

where the map π_2^* is an injection into the second argument and the map ι_1^* is the projection onto the first component of the respective direct sums.

It follows from (A.25) and from (2.16), (2.17) that the coadjoint representation of the Banach Lie group $GL^1_+ \times_{\Phi,\Omega} GL_{\mathrm{res},0}$ on the predual Banach space $L^0_+ \oplus L^1_{\mathrm{res}}$ is given by

$$\mathrm{Ad}^*_{(n,A)}(\tau, \mu) = \big((A_{++})^{-1}n^{-1}\tau n A_{++}, (A_{++})^{-1}n^{-1}\tau n A_{++} - A^{-1}\tau A + A^{-1}\mu A\big) \tag{2.49}$$

for $(n, A) \in GL^1_+ \times_{\Phi,\Omega} GL_{\mathrm{res},0}$ and $(\tau, \mu) \in L^0_+ \oplus L^1_{\mathrm{res}}$. Similarly, the coadjoint representation of $\mathbb{C}^\times \times_{\mathrm{id},\tilde{\Omega}} GL_{\mathrm{res},0}$ on $\mathbb{C} \oplus L^1_{\mathrm{res}}$ is the following

$$\mathrm{Ad}^*_{(\lambda,A)}(\gamma, \mu) = \big(\gamma, A^{-1}\mu A + \gamma\big(P_+ - A^{-1}P_+A\big)\big) \tag{2.50}$$

where $(\lambda, A) \in \mathbb{C}^\times \times_{\mathrm{id},\tilde{\Omega}} GL_{\mathrm{res},0}$ and $(\gamma, \mu) \in \mathbb{C} \oplus L^1_{\mathrm{res}}$.

Let us also note that these coadjoint representations preserve the Banach subspaces $L^0_+ \oplus L^1_{\mathrm{res}} \subset (L^1_+ \oplus L_{\mathrm{res}})^*$ and $\mathbb{C} \oplus L^1_{\mathrm{res}} \subset \mathbb{C} \oplus L^*_{\mathrm{res}}$ respectively

$$\mathrm{Ad}^*_{(n,A)}\big(L^0_+ \oplus L^1_{\mathrm{res}}\big) \subset L^0_+ \oplus L^1_{\mathrm{res}}, \tag{2.51}$$

$$\mathrm{Ad}^*_{(\lambda,A)}\big(\mathbb{C} \oplus L^1_{\mathrm{res}}\big) \subset \mathbb{C} \oplus L^1_{\mathrm{res}}. \tag{2.52}$$

The coadjoint representation of the Banach Lie algebra $L^1_+ \oplus L_{\mathrm{res}}$ on its predual $L^0_+ \oplus L^1_{\mathrm{res}}$ is the following

$$\mathrm{ad}^*_{(\rho,X)}(\tau, \mu) = \big([-\rho, \tau] - [X_{++}, \tau], -[X, \mu] - [\rho, \tau] - \tau X_{+-} + X_{-+}\tau\big) \tag{2.53}$$

where $(\rho, X) \in L^1_+ \oplus L_{\mathrm{res}}$, $(\tau, \mu) \in L^0_+ \oplus L^1_{\mathrm{res}}$.

The coadjoint representation of the Banach Lie algebra $\mathbb{C} \oplus L_{\mathrm{res}}$ on $\mathbb{C} \oplus L^1_{\mathrm{res}}$ is given by

$$\mathrm{ad}^*_{(\lambda,X)}(\gamma, \mu) = \big(0, -[X, \mu] - \gamma(X_{+-} - X_{-+})\big) \tag{2.54}$$

where $(\lambda, X) \in \mathbb{C} \oplus L_{\mathrm{res}}$, $(\gamma, \mu) \in \mathbb{C} \oplus L^1_{\mathrm{res}}$.

We observe that the conditions (A.26) are satisfied for both extensions, thus the Banach spaces $L^0_+ \oplus L^1_{\mathrm{res}}$ and $\mathbb{C} \oplus L^1_{\mathrm{res}}$ are Banach Lie–Poisson spaces. The Poisson bracket for $F, G \in C^\infty(\mathbb{C} \oplus L^1_{\mathrm{res}})$ is obtained from the general formula (A.27) and is given by

$$\{F, G\}(\gamma, \mu) = \big\langle \mu, \big[D_2 F(\gamma, \mu), D_2 G(\gamma, \mu)\big]\big\rangle - \gamma s\big(D_2 F(\gamma, \mu), D_2 G(\gamma, \mu)\big), \quad (2.55)$$

where D_2 denotes the partial Fréchet derivative with respect to the second variable.

2.5. Extensions of real Banach Lie–Poisson space $\mathcal{U}L^1_{\mathrm{res}}$

In previous subsections we considered the extensions of the complex linear restricted group $GL_{\mathrm{res},0}$, its Banach Lie algebra L_{res} and the complex Banach Lie–Poisson space L^1_{res}, which is Banach predual of L_{res}. As we mentioned above they are complexifications of $UL_{\mathrm{res},0}, \mathcal{U}L_{\mathrm{res}}$ and

$$(\mathcal{U}L_{\mathrm{res}})_* \cong \mathcal{U}L^1_{\mathrm{res}} := \big\{\mu \in L^1_{\mathrm{res}} \bigm| \mu^+ = -\mu\big\} \quad (2.56)$$

respectively. The pairing between elements of $\mathcal{U}L_{\mathrm{res}}$ and $\mathcal{U}L^1_{\mathrm{res}}$ as in the complex case is given by (2.37).

By a construction similar to those for $GL_{\mathrm{res},0}$, we obtain the central extension of $UL_{\mathrm{res},0}$ by $U(1)$

$$\{1\} \hookrightarrow U(1) \hookrightarrow \widetilde{UL}_{\mathrm{res},0} \longrightarrow UL_{\mathrm{res},0} \longrightarrow \{1\}\ . \quad (2.57)$$

Namely we define the real Banach Lie group

$$U\mathcal{E} := \big\{(A, q) \in \mathcal{E} \bigm| A \in UL_{\mathrm{res},0},\ q \in UL^\infty_+\big\}, \quad (2.58)$$

which complexification $U\mathcal{E}^{\mathbb{C}}$ is $\mathcal{E}$. The group $\widetilde{UL}_{\mathrm{res},0}$ in (2.57) is defined as

$$\widetilde{UL}_{\mathrm{res},0} := U\mathcal{E}/\big(\iota_1\big(SUL^1_+\big)\big), \quad (2.59)$$

where $SUL^1_+ := SL^1_+ \cap UL^\infty$ and $\iota_1(q) := (q, \mathbb{1})$.

Restricting the Schwinger term (2.26) to $\mathcal{U}L_{\mathrm{res}}$ we obtain the central extension

$$\{0\} \hookrightarrow i\mathbb{R} \hookrightarrow i\mathbb{R} \oplus \mathcal{U}L_{\mathrm{res}} \longrightarrow \mathcal{U}L_{\mathrm{res}} \longrightarrow \{0\} \quad (2.60)$$

of the real Banach Lie algebra $\mathcal{U}L_{\mathrm{res}}$. The exact sequence of Banach Lie–Poisson spaces predual to (2.60) is

$$\{0\} \hookrightarrow i\mathbb{R} \hookrightarrow i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}} \longrightarrow \mathcal{U}L^1_{\mathrm{res}} \longrightarrow \{0\}\ . \quad (2.61)$$

The complexification of (2.61) gives (2.47). All expressions obtained above, including the ones for the Poisson bracket (2.55) and coadjoint representation (2.50), (2.54) are valid for the real case if one assumes that $\bar{\gamma} = -\gamma$ and $\mu^+ = -\mu$.

3. Hierarchy of Hamilton equations on Banach Lie–Poisson spaces $\mathbb{C} \oplus L^1_{\mathrm{res}}$ and $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$

In this section we use the Magri method (see [7]), to introduce the hierarchy of the Hamiltonian systems on the Banach Lie–Poisson spaces $\mathbb{C} \oplus L^1_{\mathrm{res}}$ and $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$, which were investigated in Section 2. Short description of Magri method is presented in Appendix B.

To this end we define for any $k \in \mathbb{N}$ the function

$$I^k(\gamma,\mu) := \mathrm{Tr}_{\mathrm{res}}\big((\mu - \gamma P_+)^{k+1} - (-\gamma)^k(\mu - \gamma P_+)\big) \tag{3.1}$$

on $\mathbb{C} \oplus L^1_{\mathrm{res}}$. Note that the expression under $\mathrm{Tr}_{\mathrm{res}}$ is a polynomial in variable μ without a free term, thus from Proposition 2.1 it follows that the function I^k is well defined.

We observe that I^k is invariant with respect to the coadjoint representation (2.50)

$$\begin{aligned} I^k\big(\mathrm{Ad}^*_{[n,A]}(\gamma,\mu)\big) &= I^k\big(\gamma, A^{-1}\mu A + \gamma\big(P_+ - A^{-1}P_+A\big)\big) \\ &= \mathrm{Tr}_{\mathrm{res}}\big(A^{-1}\big(\mu - \gamma P_+ + \gamma A P_+ A^{-1} - \gamma A P_+ A^{-1}\big)^{k+1} A \\ &\quad - A^{-1}(-\gamma)^k\big(\mu - \gamma P_+ + \gamma A P_+ A^{-1} - \gamma A P_+ A^{-1}\big) A\big) \\ &= \mathrm{Tr}_{\mathrm{res}}\big((\mu - \gamma P_+)^{k+1} - (-\gamma)^k(\mu - \gamma P_+)\big) = I^k(\gamma,\mu). \end{aligned} \tag{3.2}$$

Thus the functions I^k, $k \in \mathbb{N}$, are Casimirs

$$\big\{I^k, \cdot\big\} = 0 \tag{3.3}$$

for Poisson bracket (2.55). Note that the coordinate function γ is a Casimir too.

Observing that Poisson brackets for $F, G \in C^\infty(\mathbb{C} \oplus L^1_{\mathrm{res}})$ given by

$$\{F,G\}_1(\gamma,\mu) := \big\langle \mu, \big[D_2F(\gamma,\mu), D_2G(\gamma,\mu)\big]\big\rangle \tag{3.4}$$

and by

$$\{F,G\}_2(\gamma,\mu) := -\gamma s\big(D_2F(\gamma,\mu), D_2G(\gamma,\mu)\big) \tag{3.5}$$

are compatible, we introduce a Poisson pencil

$$\begin{aligned} \{F,G\}_\varepsilon(\gamma,\mu) &:= \{F,G\}_1(\gamma,\mu) + \varepsilon\{F,G\}_2(\gamma,\mu) \\ &= \big\langle \mu, \big[D_2F(\gamma,\mu), D_2G(\gamma,\mu)\big]\big\rangle - \varepsilon\gamma s\big(D_2F(\gamma,\mu), D_2G(\gamma,\mu)\big) \end{aligned} \tag{3.6}$$

on $\mathbb{C} \oplus L^1_{\mathrm{res}}$. Compatibility of $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ follows from the fact that the Poisson tensor for $\{\cdot,\cdot\}_2$ is constant with respect to the variable μ and $\{\cdot,\cdot\}_1$ depends only on the derivations with respect to the variable μ.

Due to the latter equality in (3.6) the Casimirs for $\{\cdot,\cdot\}_\varepsilon$ are:

$$I^k_\varepsilon(\gamma,\mu) = \mathrm{Tr}_{\mathrm{res}}\big((\mu - \varepsilon\gamma P_+)^{k+1} - (-\varepsilon\gamma)^k(\mu - \varepsilon\gamma P_+)\big) \tag{3.7}$$

where $k \in \mathbb{N}$. According to Magri method, we expand these Casimirs with respect to the parameter $-\varepsilon\gamma$

$$I_\varepsilon^k(\gamma,\mu) = \sum_{n=0}^{k-1}(-\varepsilon\gamma)^n \mathrm{Tr}_{\mathrm{res}} W_n^{k+1}(\mu) + (-\varepsilon\gamma)^k \mathrm{Tr}_{\mathrm{res}}\big(W_k^{k+1}(\mu) - \mu\big), \tag{3.8}$$

where the operators $W_n^k(\mu)$ are polynomials in operator arguments μ and P_+ defined by the equality

$$(\mu + \lambda P_+)^k = \sum_{n=0}^{k} \lambda^n W_n^k(\mu), \quad \lambda \in \mathbb{R}. \tag{3.9}$$

In this way from (B.7) it follows that we obtain a family

$$h_n^k(\gamma,\mu) = \gamma^n \mathrm{Tr}_{\mathrm{res}} W_n^{k+1}(\mu), \quad 0 \leqslant n \leqslant k-1, \tag{3.10a}$$

$$h_k^k(\gamma,\mu) = \gamma^k \mathrm{Tr}_{\mathrm{res}}\big(W_k^{k+1} - \mu\big) \tag{3.10b}$$

of Hamiltonians in involution

$$\big\{h_n^k, h_m^l\big\}_\varepsilon = 0 \tag{3.11}$$

with respect to the brackets $\{\cdot,\cdot\}_\varepsilon$ for $\varepsilon \in \mathbb{R}$. In the particular case $\varepsilon = 1$ they are in involution with respect to the bracket $\{\cdot,\cdot\}$ given by (2.55).

Let us now investigate the infinite system of the Hamilton equations on the Banach Lie–Poisson space $\mathbb{C} \oplus L^1_{\mathrm{res}}$

$$\frac{\partial}{\partial t_n^k}(\gamma,\mu) = -\operatorname{ad}^*_{(D_1 h_n^k(\gamma,\mu), D_2 h_n^k(\gamma,\mu))}(\gamma,\mu) \tag{3.12}$$

defined by the hierarchy of Hamiltonians h_n^k, $k \in \mathbb{N}$, $n = 0, 1, \ldots, k$. Using the explicit form of coadjoint action (2.54) one sees that observe that Eq. (3.12) take the form

$$\frac{\partial}{\partial t_n^k}\gamma = 0, \tag{3.13a}$$

$$\frac{\partial}{\partial t_n^k}\mu = -\big[\mu, D_2 h_n^k(\gamma,\mu)\big] + \gamma\big(P_+ D_2 h_n^k(\gamma,\mu) P_- - P_- D_2 h_n^k(\gamma,\mu) P_+\big). \tag{3.13b}$$

In (3.13) the real parameter t_n^k parametrizes the Hamiltonian flow generated by h_n^k. In order to compute $D_2 h_n^k(\gamma,\mu)$ we apply the partial derivative operator D_2 to both sides of equality (3.8). Since

$$D_2 I_\varepsilon^k(\gamma,\mu) = (k+1)(\mu - \varepsilon\gamma P_+)^k - (-\varepsilon\gamma)^k \mathbb{1} \tag{3.14}$$

we get that

$$D_2 h_n^k(\gamma,\mu) = (k+1)\gamma^n W_n^k(\mu) \tag{3.15a}$$

for $0 \leqslant n \leqslant k-1$, and

$$D_2 h_k^k(\gamma, \mu) = \gamma^k \big((k+1) W_k^k(\mu) - \mathbb{1}\big). \tag{3.15b}$$

Substituting (3.15) into (3.13) we obtain

$$\frac{\partial}{\partial t_n^k}\mu = -(k+1)\gamma^n \big[\mu - \gamma P_+, W_n^k(\mu)\big] \tag{3.16}$$

for $0 \leqslant n \leqslant k$.

From (3.9) we the following recurrence rules

$$\begin{aligned} W_n^{k+1}(\mu) &= W_n^k(\mu)\mu + W_{n-1}^k(\mu) P_+, \\ W_n^{k+1}(\mu) &= \mu W_n^k(\mu) + P_+ W_{n-1}^k(\mu) \end{aligned} \tag{3.17}$$

which yield the commutation relation

$$\big[\mu, W_n^k(\mu)\big] + \big[P_+, W_{n-1}^k(\mu)\big] = 0, \tag{3.18}$$

where $0 \leqslant n \leqslant k$ and we put $W_{-1}^k := 0$. Using (3.18) we can express the Hamilton equations (3.16) in the following two ways

$$\frac{\partial}{\partial t_n^k}\mu = -(k+1)\gamma^n \big[\mu, W_n^k(\mu) + \gamma W_{n+1}^k(\mu)\big], \tag{3.19}$$

$$\frac{\partial}{\partial t_n^k}\mu = (k+1)\gamma^n \big[P_+, \gamma W_n^k(\mu) + W_{n-1}^k(\mu)\big], \tag{3.20}$$

where $0 \leqslant n \leqslant k$. Rewriting (3.20) in the block form (2.2) we obtain

$$\frac{\partial}{\partial t_n^k}\mu_{++} = 0, \qquad \frac{\partial}{\partial t_n^k}\mu_{--} = 0 \tag{3.21}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t_n^k}\mu_{+-} = (k+1)\gamma^n P_+\big(\gamma W_n^k(\mu) + W_{n-1}^k(\mu)\big) P_-, \\ \dfrac{\partial}{\partial t_n^k}\mu_{-+} = -(k+1)\gamma^n P_-\big(\gamma W_n^k(\mu) + W_{n-1}^k(\mu)\big) P_+. \end{cases} \tag{3.22}$$

Let us observe that Eq. (3.19) is a Hamilton equation for the Poisson bracket $\{\cdot,\cdot\}_1$ and the Hamiltonian $h_n^k + h_{n+1}^k$, while Eq. (3.20) is a Hamilton equation for the Poisson bracket $\{\cdot,\cdot\}_2$ and the Hamiltonian $h_n^k + h_{n-1}^k$. From (3.21) we conclude that diagonal blocks μ_{++} and μ_{--} are invariants for all Hamiltonian flows under consideration.

The symplectic leaves for $\mathbb{C} \oplus L^1_{\mathrm{res}}$ and $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$ with Poisson bracket $\{\cdot,\cdot\}_2$ are the affine spaces obtained by shifting the vector spaces $L^2_{+-} \oplus L^2_{-+}$ or L^2_{+-} respectively by diagonal blocks μ_{++} and μ_{--}. This fact explain why we have obtained additional integrals of motion (3.21).

Eqs. (3.16) and (3.19) are in the Lax form. In the first case it is an equation on $\mu - \gamma P_+$, while it is on μ in the other.

Let us calculate several operators $W_n^k(\mu)$. We can do this by iterating the recurrence (3.17). The result is:

$$W_k^k = P_+, \tag{3.23}$$

$$W_{k-1}^k = \mu P_+ + P_+\mu + (k-2)P_+\mu P_+, \quad k \geqslant 2, \tag{3.24}$$

$$\begin{aligned} W_{k-2}^k &= \mu^2 P_+ + \mu P_+\mu + P_+\mu^2 + (k-3)\big(P_+\mu^2 P_+ + P_+\mu P_+\mu + \mu P_+\mu P_+\big) \\ &\quad + \frac{(k-3)(k-4)}{2} P_+\mu P_+\mu P_+, \quad k \geqslant 4, \end{aligned} \tag{3.25}$$

$$\vdots$$

$$W_1^k = P_+\mu^{k-1} + \mu P_+\mu^{k-2} + \cdots + \mu^{k-1}P_+, \tag{3.26}$$

$$W_0^k = \mu^k. \tag{3.27}$$

It is obvious that the Hamiltonians h_n^k are functionally interdependent and it implies the interdependence of t_n^k-flows given by (3.22). The above formulas suggest the introduction of the homogeneous polynomials

$$H_n^l(\mu) := \sum_{\substack{i_0, i_1, \ldots i_l = 0 \\ i_0 + \cdots + i_l = n}}^{1} P_+^{i_0}\mu P_+^{i_1}\mu \ldots \mu P_+^{i_l} \tag{3.28}$$

of the degree $l \in \mathbb{N}$ in the operator variable $\mu \in L^1_{\mathrm{res}}$, where $n \leqslant l+1$. These polynomials are linearly independent and they satisfy the recurrences

$$H_{n+1}^{l+1}(\mu) = P_+\mu H_n^l(\mu) + \mu H_{n+1}^l(\mu) \tag{3.29a}$$

for $n \leqslant l$, $l \in \mathbb{N}$ and

$$H_{l+2}^{l+1}(\mu) = P_+\mu H_{l+1}^l(\mu) \tag{3.29b}$$

for $l \in \mathbb{N}$.

Proposition 3.1. *Polynomials W_n^k are linear combinations of the homogeneous polynomials H_n^l*

$$W_{k-l}^k(\mu) = \sum_{n=1}^{l+1} \max\big\{0, p_n^l(k)\big\} H_n^l(\mu) \tag{3.30}$$

for $l < k$ and

$$W_0^k(\mu) = H_0^k(\mu), \tag{3.31}$$

where $p_n^l \in \mathbb{R}_{n-1}[x]$ are polynomials of degree $n-1$ that are defined by the recurrences:

$$p_{n+1}^{l}(k)=\sum_{i=l+1}^{k-1}\max\{0,\,p_n^l(i)\},\tag{3.32}$$

$$p_n^{l+1}(k)=p_n^l(k-1)\tag{3.33}$$

with initial condition $p_1^l(k)=1$.

Proof. We prove formula (3.30) by induction with respect to l. From recurrence (3.32) we infer that $p_2^1(k)=k-2$ and thus from (3.23) and (3.24) we see that formula (3.30) is satisfied for $l=0$ and $l=1$. From (3.17) we conclude that

$$W_{k-l}^{k}(\mu)=\mu W_{k-l}^{k-1}(\mu)+P_+\mu\sum_{i=1}^{k-l}W_{k-l-i}^{k-i-1}(\mu).\tag{3.34}$$

We apply (3.34) to W_{k-l}^k assuming that (3.30) is satisfied for $l-1$ and obtain

$$\begin{aligned}W_{k-l}^{k}(\mu)={}&\mu\sum_{n=1}^{l}\max\{0,\,p_n^{l-1}(k-1)\}H_n^{l-1}(\mu)\\&+P_+\mu\sum_{i=1}^{k-l-1}\sum_{n=1}^{l}\max\{0,\,p_n^{l-1}(k-i-1)\}H^{l-1}(\mu)+P_+\mu H_0^{l-1}(\mu).\end{aligned}\tag{3.35}$$

Changing the order of summation and using recurrences (3.32) and (3.33) we get

$$\begin{aligned}W_{k-l}^{k}(\mu)={}&\mu\sum_{n=0}^{l-1}\max\{0,\,p_{n+1}^{l-1}(k-1)\}H_{n+1}^{l-1}(\mu)\\&+P_+\mu\sum_{n=1}^{l}\max\{0,\,p_{n+1}^{l-1}(k-1)\}H_n^{l-1}(\mu)+P_+\mu H_0^{l-1}(\mu).\end{aligned}\tag{3.36}$$

By rearranging terms in the sums and applying (3.29) we end up with (3.30). Direct check shows that relations (3.32) and (3.33) are compatible. The fact that $\deg p_n^l=n-1$ follows from (3.32). □

From Proposition 3.1 we conclude:

Corollary 3.2.

i) *The dimension of the complex vector space spanned by* $\{W_{k-l}^k\}_{k=l+1}^{\infty}$ *is equal to* $l+1$ *and* $\{H_n^l\}_{n=1}^{l+1}$ *is a basis of this space*;

ii) *Using (3.30) one can express* H_n^l, $0\leqslant n\leqslant l+1$ *as a finite linear combination of* W_{k-l}^k, *where* $k\geqslant l+1$.

Proof. i) From (3.30) it follows that all elements of the set $\{W^k_{k-l}\}_{k=l+1}^{\infty}$ are linear combinations of $\{H^l_n\}_{n=1}^{l+1}$. Let us also note that the polynomials p^l_n assume positive values $p^l_n(k) > 0$ for k large enough and that the set $\{p^l_n\}_{n=1}^{l+1}$ spans an $(l+1)$-dimensional vector space. This concludes the proof.

ii) This statement is a consequence of i). $\square$

Introducing new variables $\tau^l_n \in \mathbb{R}$ through the linear combination

$$t^k_{k-l} = (k-1)\gamma^{k-l} \sum_{n=1}^{l+1} \max\{0, p^l_n(k)\}\tau^l_n, \tag{3.37}$$

we rewrite the hierarchy (3.16) in the equivalent form

$$\frac{\partial}{\partial \tau^l_n}\mu = \left[\mu - \gamma P_+, H^l_n(\mu)\right] \tag{3.38}$$

where $l \in \mathbb{N}$ and $n = 1, \ldots, l+1$.

Let us write out explicitly several equations from hierarchy (3.38). For H^k_0 and H^k_1 in block notation (2.2) we obtain

$$\begin{cases} \dfrac{\partial}{\partial \tau^k_0}\mu_{+-} = -\gamma\left(\mu^k\right)_{+-}, \\ \dfrac{\partial}{\partial \tau^k_0}\mu_{-+} = \gamma\left(\mu^k\right)_{-+} \end{cases} \tag{3.39}$$

and

$$\begin{cases} \dfrac{\partial}{\partial \tau^k_1}\mu_{+-} = -\left(\mu^{k+1}\right)_{+-} - \gamma\sum_{i=0}^{k-1}\left(\mu^i\right)_{++}\left(\mu^{k-i}\right)_{+-}, \\ \dfrac{\partial}{\partial \tau^k_1}\mu_{-+} = \left(\mu^{k+1}\right)_{-+} + \gamma\sum_{i=1}^{k}\left(\mu^i\right)_{-+}\left(\mu^{k-i}\right)_{++} \end{cases} \tag{3.40}$$

respectively. For $k = 1$ and $k = 2$ we get from (3.39) linear equations and for $k = 3$ we obtain from (3.39) a pair of coupled operator Ricatti-type equations

$$\begin{cases} \dfrac{\partial}{\partial \tau^3_0}\mu_{+-} = -\gamma\left((\mu_{++})^2\mu_{+-} + \mu_{+-}\mu_{-+}\mu_{+-} + \mu_{++}\mu_{+-}\mu_{--} + \mu_{+-}(\mu_{--})^2\right), \\ \dfrac{\partial}{\partial \tau^3_0}\mu_{-+} = \gamma\left(\mu_{-+}(\mu_{++})^2 + \mu_{--}\mu_{-+}\mu_{++} + \mu_{-+}\mu_{+-}\mu_{-+} + (\mu_{--})^2\mu_{-+}\right). \end{cases} \tag{3.41}$$

Let us recall that blocks μ_{++} and μ_{--} are constant with respect to all flows. Moreover if we assume that $\mu_{++} = 0$ or $\mu_{--} = 0$ then Eqs. (3.38) become linear.

After certain modifications we can also consider the hierarchy of Eqs. (3.38) on the real Banach Lie–Poisson $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$. To this end we have to modify the Hamiltonians h^k_n is such a way that they will take real values when restricted to $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$. In consequence we obtain the following equations

$$\frac{\partial}{\partial \tau_n^l}\mu = i^{l+1}\big[\mu - \gamma P_+, H_n^l(\mu)\big] \tag{3.42}$$

where $\mu \in \mathcal{U}L^1_{\mathrm{res}}$, $\gamma \in i\mathbb{R}$.

Now we express the Hamiltonian hierarchy (3.16) in a more compact and elegant form. To this end let us define the "generating" Hamiltonian for the Hamiltonians (3.10)

$$h_{\kappa,\lambda}(\gamma,\mu) := \sum_{k=1}^{\infty} \frac{1}{k+1}\kappa^k \sum_{n=0}^{k+1} \lambda^n h_n^k(\gamma,\mu), \tag{3.43}$$

where $\kappa, \lambda \in \mathbb{R}$. In order to show that the series of functions (3.43) is convergent on some non-empty open subset of $\mathbb{C} \oplus L^1_{\mathrm{res}}$ we observe that

$$\sum_{n=0}^{k+1} \lambda^n h_n^k(\gamma,\mu) = \mathrm{Tr}_{\mathrm{res}}\big((\mu + \gamma\lambda P_+)^{k+1} - (\gamma\lambda)^k(\mu + \gamma\lambda P_+)\big). \tag{3.44}$$

Equality (3.44) follows from (3.10) and (3.9). Next, let us prove the following lemma.

Lemma 3.3. *One has*

$$\big\|(\mu + \beta P_+)^{k+1} - \beta^k(\mu + \beta P_+)\big\|_* \leqslant \big(\|\mu\|_* + |\beta|\big)^{k+1} - |\beta|^k\big(\|\mu\|_* + |\beta|\big), \tag{3.45}$$

where $\beta \in \mathbb{C}$ *and* $\mu \in L^1_{\mathrm{res}}$.

Proof. We expand the left-hand side and apply the triangle inequality and Proposition 2.2. Moreover we note that $\|\nu P_+\|_* \leqslant \|\nu\|_*$ for $\nu \in L^1_{\mathrm{res}}$. In this way we get

$$\big\|(\mu + \beta P_+)^{k+1} - \beta^k(\mu + \beta P_+)\big\|_* \leqslant \sum_{i=1}^{k+1} \binom{k+1}{i} \|\mu\|_*^i |\beta|^{k-i+1} - |\beta|^k \|\mu\|_*. \tag{3.46}$$

By adding and subtracting the term $|\beta|^{k+1}$ and collecting terms we obtain the right-hand side. □

From this lemma and Proposition 2.2 we conclude:

Proposition 3.4. *One has*

$$h_{\kappa,\lambda}(\gamma,\mu) = \mathrm{Tr}_{\mathrm{res}}\left(-\frac{1}{\kappa}\log\big(\mathbb{1} - \kappa(\mu + \gamma\lambda P_+)\big) + (\mu + \gamma\lambda P_+)\frac{\log(1-\kappa\lambda\gamma)}{\kappa\lambda\gamma}\right) \tag{3.47}$$

and

$$\{h_{\kappa,\lambda}, h_{\kappa',\lambda'}\} = 0 \tag{3.48}$$

for $|\kappa|(\|\mu\|_* + |\lambda\gamma|) < 1$ *and* $|\kappa'|(\|\mu\|_* + |\lambda'\gamma|) < 1$, *where the Poisson bracket in* (3.48) *is given by* (2.55).

Now we find the explicit form of the Hamilton equation

$$\frac{\partial}{\partial t_{\kappa,\lambda}}(\gamma,\mu) = -\operatorname{ad}^*_{Dh_{\kappa,\lambda}(\gamma,\mu)}(\gamma,\mu) \tag{3.49}$$

generated by Hamiltonian (3.43). Using (2.54) we obtain

$$\frac{\partial}{\partial t_{\kappa,\lambda}}\gamma = 0, \tag{3.50a}$$

$$\frac{\partial}{\partial t_{\kappa,\lambda}}\mu = -\left[\mu, D_2 h_{\kappa,\lambda}(\gamma,\mu)\right] + \gamma\left(P_+ D_2 h_{\kappa,\lambda}(\gamma,\mu)P_- - P_- D_2 h_{\kappa,\lambda}(\gamma,\mu)P_+\right). \tag{3.50b}$$

Since

$$D_2 h_{\kappa,\lambda}(\gamma,\mu) = \left(\mathbb{1} - \kappa(\mu + \lambda\gamma P_+)\right)^{-1} + \frac{\log(1-\kappa\lambda\gamma)}{\kappa\lambda\gamma} \tag{3.51}$$

we get

$$\frac{\partial}{\partial t_{\kappa,\lambda}}x = -\alpha\left[P_+, (1-x)^{-1}\right], \tag{3.52}$$

where

$$x := \kappa(\mu + \lambda\gamma P_+), \tag{3.53}$$

and $\alpha := \kappa(1+\lambda)\gamma$. Replacing x in (3.52) by $y := (1-x)^{-1}$ we get the hierarchy of equations

$$\frac{\partial}{\partial t_{\kappa,\lambda}}y = \alpha[y, yP_+y], \quad \lambda,\kappa \in \mathbb{C} \tag{3.54}$$

equivalent to the hierarchy (3.16).

In this paper we don't intend to address the problem of finding general solutions for the considered Hamiltonian systems but in the next section we will present several examples of solutions.

Let us also observe that if $\mathcal{H}_+$ is finite dimensional, then the operator $\mu - \gamma P_+$ has a discrete spectrum. Formula (2.50) shows that orbits of coadjoint action of group $\widetilde{GL}_{\mathrm{res},0}$ coincide with orbits of standard coadjoint action of $\mathrm{GL}_{\mathrm{res},0} \subset GL^\infty$ shifted by γP_+. Thus one can use the spectrum $\operatorname{spec}(\mu - \gamma P_+)$ to distinguish partially symplectic leaves of the Banach Lie–Poisson space $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$, i.e. if $\operatorname{spec}(\mu_1 - \gamma P_+) \neq \operatorname{spec}(\mu_2 - \gamma P_+)$ then they belong to different symplectic leaves. If the dimension of $\mathcal{H}_+$ is infinite then the shift $\mu - \gamma P_+$ of the operator $\mu \in L^1_{\mathrm{res}}$ by the operator γP_+ is not an element of L^1_{res}, but of L_{res}. However from Weyl's criterion (see [16]) we deduce that the set $\operatorname{spec}(\mu - \gamma P_+) \setminus \{0, -\gamma\}$ is discrete. Therefore if $\mathcal{H}_+$ is infinite dimensional, one can use also elements of $\operatorname{spec}(\mu - \gamma P_+)$ for the partial indexation of the symplectic leaves. The problem of description of these leaves is complicated, see [2] for more information.

4. Examples of solutions

In this section we present several examples of explicit solutions to Eqs. (3.38) in some particular cases.

Example 4.1 *(Restricted Grassmannian).* The connected component $\mathrm{Gr}_{\mathrm{res},0}$ of the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ can be identified with the coadjoint orbit $\mathcal{O}_\gamma$ of the group $\widetilde{UL}_{\mathrm{res},0}$ in the Banach Lie–Poisson space $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$ generated from point $(\gamma, 0)$, see [2]. Namely from (2.50) we see that

$$\mathrm{Ad}^*_{(\lambda,g)}(\gamma, 0) = \big(\gamma, \gamma\big(P_+ - g^{-1}P_+g\big)\big) \tag{4.1}$$

for $(\lambda, g) \in U(1) \times_{\mathrm{id},\tilde{\Omega}} UL_{\mathrm{res},0}$ and $\gamma \in i\mathbb{R}$. This suggests to define the map $\iota_\gamma : \mathrm{Gr}_{\mathrm{res},0} \to \mathcal{O}_\gamma$ in the following way

$$\iota_\gamma(W) := \big(\gamma, \gamma(P_+ - P_W)\big). \tag{4.2}$$

Since $UL_{\mathrm{res},0}$ acts transitively on $\mathrm{Gr}_{\mathrm{res},0}$, we see that ι_γ maps $\mathrm{Gr}_{\mathrm{res},0}$ bijectively on $\mathcal{O}_\gamma$.

Let us introduce homogeneous coordinates on some open subset in $\mathrm{Gr}_{\mathrm{res}}$. To this end we fix a basis $\{|n\rangle\}$, $n \in \mathbb{Z}$ in $\mathcal{H}$, such that $|n\rangle$ for $n < 0$ spans $\mathcal{H}_-$ and for $n \geqslant 0$ spans $\mathcal{H}_+$. Let us fix a basis $w_1, w_2, \ldots$ in a subspace $W \in \mathrm{Gr}_{\mathrm{res}}$ and put the coefficients of w_k in the basis $\{|n\rangle\}_{n\in\mathbb{Z}}$ in the matrix form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \big(\langle n \mid w_k\rangle\big)_{n\in\mathbb{Z},k\in\mathbb{N}}, \tag{4.3}$$

where α, β are blocks obtained for $n \geqslant 0$ and $n < 0$ respectively. Let us consider a subspace $W \in \mathrm{Gr}_{\mathrm{res}}$ such that there exists an orthonormal basis $\{w_k\}_{k\in\mathbb{N}}$ such that α is invertible. Then we define

$$z := \beta\alpha^{-1}. \tag{4.4}$$

Definition of z is independent of the choice of the basis $\{w_k\}_{k\in\mathbb{N}}$.

The matrix of the projector P_W is

$$\big(\langle n|P_W k\rangle\big)_{n,k\in\mathbb{Z}} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \big(\alpha^+ \ \beta^+\big). \tag{4.5}$$

Thus $\iota_\gamma(W)$ takes the following form

$$\iota_\gamma(W) = \begin{pmatrix} (\mathbb{1} + z^+z)^{-1} - \mathbb{1} & (\mathbb{1} + z^+z)^{-1}z^+ \\ z(\mathbb{1} + z^+z)^{-1} & z(\mathbb{1} + z^+z)^{-1}z^+ \end{pmatrix}, \tag{4.6}$$

where we consider z as an operator $z : \mathcal{H}_+ \to \mathcal{H}_-$.

Hamilton equations (3.13) can be written in terms of z. Due to (3.21) we note that z^+z is constant. Thus any polynomial of $\iota_\gamma(W)$ has only constant and linear terms in z and Eqs. (3.13) are linear in homogeneous coordinates on $\mathcal{O}_\gamma$.

Example 4.2 *(Vector case).* Let us consider a particular case of the equations given above. We assume that $\dim \mathcal{H}_+ = 1$. We introduce the block notation for elements $\mu \in L^1_{\mathrm{res}}$

$$\mu = \begin{pmatrix} a & v^+ \\ w & A \end{pmatrix}, \tag{4.7}$$

where $a \in \mathbb{C}$, $A \in L^\infty(\mathcal{H}_-)$, $v, w \in L^2(\mathbb{C}, \mathcal{H}_-) \cong \ell^2$. We consider Eqs. (3.38) as non-linear equations for two vectors v, w coupled by interaction depending on constants a and A. Non-linear behavior is due to the terms of the type $\langle v \mid A^l w\rangle$.

First of all let us remark that in general all functions h^k_l are linear combinations of functions $\mathrm{Tr}_{\mathrm{res}}\mu^k = h^{k-1}_0(\gamma, \mu)$ and $\mathrm{Tr}_{\mathrm{res}}(\mu^{k_1} P_+ \mu^{k_2} P_+ \cdots + P_+ \mu^{k_n} P_+)$. However due to the fact that $\dim \mathcal{H}_+ = 1$ we have

$$\mathrm{Tr}_{\mathrm{res}}\big(\mu^{k_1} P_+ \mu^{k_2} P_+ \cdots + P_+ \mu^{k_n} P_+\big) = \frac{1}{\gamma^n} \frac{h^{k_1}_1(\gamma, \mu)}{k_1 + 1} \cdots \frac{h^{k_n}_1(\gamma, \mu)}{k_n + 1}. \tag{4.8}$$

Thus all integrals of motion h^k_l are functionally dependent on h^k_0 and h^k_1. Therefore one has only two independent families of Hamilton equations, i.e. (3.39) and (3.40).

In order to solve these families we note that

$$\big(\mu^k\big)_{-+} = M_k(\gamma, \mu) w, \tag{4.9}$$

$$\big(\mu^k\big)_{+-} = v^+ M_k(\gamma, \mu), \tag{4.10}$$

where

$$M_k(\gamma, \mu) := \left(\frac{h^{k-1}_1(\gamma, \mu)}{\gamma k} + \frac{h^{k-2}_1(\gamma, \mu)}{\gamma(k-1)} A + \cdots + \frac{h^2_1(\gamma, \mu)}{3\gamma} A^{k-3} + a A^{k-2} + A^{k-1}\right) \tag{4.11}$$

is a time independent operator.

Thus Eqs. (3.39) take the form

$$\begin{cases} \dfrac{\partial}{\partial \tau^k_0} v = -\gamma M_k\big(\overline{\gamma}, \mu^+\big) v, \\ \dfrac{\partial}{\partial \tau^k_0} w = \gamma M_k(\gamma, \mu) w. \end{cases} \tag{4.12}$$

In this way we have reduced system (3.39) to a linear system. Thus its solution is

$$v\big(\tau^2_0, \tau^3_0, \ldots\big) = \exp\left(-\gamma \sum_{k=2}^{\infty} M_k\big(\gamma, \mu^+(0, 0, \ldots)\big) \tau^k_0\right) v(0, 0, \ldots), \tag{4.13}$$

$$w\big(\tau^2_0, \tau^3_0, \ldots\big) = \exp\left(\gamma \sum_{k=2}^{\infty} M_k\big(\gamma, \mu(0, 0, \ldots)\big) \tau^k_0\right) w(0, 0, \ldots) \tag{4.14}$$

where $v(0, 0, \ldots), w(0, 0, \ldots) \in \ell^2$, $A \in L^\infty(\mathcal{H}_-)$, $a \in \mathbb{C}$ are initial conditions.

In the case of Eqs. (3.40) we get

$$\begin{cases} \dfrac{\partial}{\partial \tau_1^k} v = -\left(\displaystyle\sum_{j=1}^{k-1} \frac{h_1^j(\overline{\gamma}, \mu^+)}{j+1} M_{k-j}(\overline{\gamma}, \mu^+) + M_{k+1}(\overline{\gamma}, \mu^+) \right) v, \\ \dfrac{\partial}{\partial \tau_1^k} w = \left(\displaystyle\sum_{j=1}^{k-1} \frac{h_1^{k-j}(\gamma, \mu)}{k-j+1} M_j(\gamma, \mu) + M_{k+1}(\gamma, \mu) \right) w. \end{cases} \tag{4.15}$$

These equation are also linear and their solution can be obtained by exponentiation.

Example 4.3 *(4-dimensional case).* In this example we solve equation (3.41) in the $i\mathbb{R} \oplus \mathcal{U}L^1_{\mathrm{res}}$ case assuming $\dim \mathcal{H}_+ = \dim \mathcal{H}_- = 2$. We will use the following notation

$$\gamma = i\chi, \qquad \mu = i \begin{pmatrix} A & Z \\ Z^+ & D \end{pmatrix}, \tag{4.16}$$

where $\chi \in \mathbb{R}$ and $A = A^+$, $D = D^+$, $Z \in Mat_{2\times 2}(\mathbb{C})$.

Substituting (4.16) into (3.41) we obtain

$$\frac{d}{dt} A = 0, \qquad \frac{d}{dt} D = 0 \tag{4.17}$$

and

$$\frac{d}{dt} Z = -i\chi\left(A^2 Z + Z D^2 + AZD + ZZ^+ Z\right), \tag{4.18}$$

$$\frac{d}{dt} Z^+ = i\chi\left(Z^+ A^2 + D^2 Z^+ + DZ^+ A + Z^+ Z Z^+\right). \tag{4.19}$$

Let us note that Eqs. (4.18) do not change their form with respect to the transformation $A \mapsto UAU^+$, $D \mapsto VDV^+$, $Z \mapsto VZU^+$, where $UU^+ = \mathbb{1}$ and $VV^+ = \mathbb{1}$. So without loss of the generality we can assume

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.20}$$

where $a_1, a_2, d_1, d_2 \in \mathbb{R}$ are constants and a, b, c, d are complex-valued functions of $t \in \mathbb{R}$. From (4.18) we obtain

$$\begin{aligned} \frac{d}{dt} a &= i\chi\left(a_1^2 + a_1 d_1 + d_1^2 + |a|^2 + |b|^2 + |c|^2\right) a + i\chi bc\bar{d}, \\ \frac{d}{dt} b &= i\chi\left(a_1^2 + a_1 d_2 + d_2^2 + |a|^2 + |b|^2 + |d|^2\right) b + i\chi a\bar{c}d, \\ \frac{d}{dt} c &= i\chi\left(a_2^2 + a_2 d_1 + d_1^2 + |a|^2 + |c|^2 + |d|^2\right) c + i\chi a\bar{b}d, \\ \frac{d}{dt} c &= i\chi\left(a_2^2 + a_2 d_2 + d_2^2 + |b|^2 + |c|^2 + |d|^2\right) c + i\chi \bar{a}bc. \end{aligned} \tag{4.21}$$

Now we consider the generic case, i.e. $a_1 \neq a_2$ and $d_1 \neq d_2$. In order to solve this system of equations we calculate explicitly the integrals of motion $h_0^1(\mu) = \mathrm{Tr}_{\mathrm{res}}\mu^2$, $h_0^2(\mu) = \mathrm{Tr}_{\mathrm{res}}\mu^3$, $h_1^2(\mu) = \gamma\,\mathrm{Tr}(\mu^2 P_+)$, $h_1^3(\mu) = \gamma\,\mathrm{Tr}(\mu^3 P_+)$ and $h_0^3(\mu) = \mathrm{Tr}_{\mathrm{res}}\mu^4$. From that we conclude that

$$\begin{aligned}
|a|^2 + |b|^2 &=: p^2 = \text{const},\\
|a|^2 + |c|^2 &=: q^2 = \text{const},\\
|c|^2 + |d|^2 &=: r^2 = \text{const},\\
|b|^2 + |d|^2 &=: s^2 = \text{const},
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&|a|^2 a_1 d_1 + |b|^2 a_1 d_2 + |c|^2 a_2 d_1 + |d|^2 a_2 d_2 + |a|^2|c|^2 + |b|^2|d|^2 + 2\,\mathrm{Re}(a\bar{b}\bar{c}d)\\
&\quad =: \Delta = \text{const},
\end{aligned} \tag{4.23}$$

where $p^2 + r^2 = q^2 + s^2$. Using (4.22) and (4.21) we find

$$\frac{d}{dt}|a|^2 = -\frac{d}{dt}|b|^2 = -\frac{d}{dt}|c|^2 = \frac{d}{dt}|d|^2 = 2\chi\,\mathrm{Im}(a\bar{b}\bar{c}d). \tag{4.24}$$

Now from (4.23) and (4.24) we obtain the following equation

$$\frac{d}{dt}x = \pm\sqrt{w(x)} \tag{4.25}$$

on the function $x := |a|^2$, where

$$w(x) := 4x\big(p^2 - x\big)\big(q^2 - x\big)\big(r^2 - q^2 - x\big) - v^2(x) \tag{4.26}$$

and

$$\begin{aligned}
v(x) := {}& x(a_1 - a_2)(d_1 - d_2) - x\big(p^2 - x\big) - \big(q^2 - x\big)\big(r^2 - q^2 + x\big)\\
&- a_1 d_2 p^2 - a_2 d_1 q^2 - a_2 d_2\big(r^2 - q^2\big).
\end{aligned} \tag{4.27}$$

Since w is a polynomial of the fourth degree, this equation is solved by an elliptic integral of the first kind

$$t = \int \frac{dx}{\sqrt{w(x)}}. \tag{4.28}$$

This allows us to express $x(t)$ as an elliptic function of time parameter t.

By (4.22) we may calculate $|b|^2$, $|c|^2$ and $|d|^2$ in terms of $x(t)$. In order to find the functions $a(t)$, $b(t)$, $c(t)$ and $d(t)$ we substitute their polar decompositions $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$, $c = |c|e^{i\gamma}$ and $d = |d|e^{i\delta}$ into Eqs. (4.21). In this way we obtain

$$\frac{d}{dt}\alpha = \chi\left(a_1^2 + a_1 d_1 + d_1^2 + p^2 + q^2 - x + \frac{v(x)}{2x}\right),$$
$$\frac{d}{dt}\beta = \chi\left(a_1^2 + a_1 d_2 + d_2^2 + p^2 + r^2 - q^2 + x + \frac{v(x)}{2(p^2 - x)}\right),$$
$$\frac{d}{dt}\gamma = \chi\left(a_2^2 + a_2 d_1 + d_1^2 + r^2 + x + \frac{v(x)}{2(q^2 - x)}\right),$$
$$\frac{d}{dt}\delta = \chi\left(a_2^2 + a_2 d_2 + d_2^2 + p^2 + r^2 - x + \frac{v(x)}{2(r^2 - q^2 + x)}\right). \tag{4.29}$$

Since the right-hand sides of Eqs. (4.29) are known, we can find $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ by integration. In this way we have solved equations (4.18) in quadratures.

The Hamiltonian system solved in this example have applications for example in the nonlinear optics, see [6].

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Appendix A. Extensions of Banach Lie groups and related Banach Lie–Poisson spaces

Let us present an abbreviated description of extensions of Banach Lie groups, Banach Lie algebras and Banach Lie–Poisson spaces associated with them. Some of the results given below can be found in papers [1,9,10,12]. Our main aim is to compute the formulas for the adjoint and coadjoint actions of the extended Banach Lie group.

A.1. Extensions of Banach Lie groups

Let us consider an exact sequence of Banach Lie groups

$$\{e_N\} \hookrightarrow N \overset{\iota}{\hookrightarrow} G \xrightarrow{\pi} H \longrightarrow \{e_H\}\ . \tag{A.1}$$

We assume that $N \to G \to H$ is a smooth principal bundle, i.e. the maps ι and π are smooth and there exists a smooth local section $\sigma : U \to G$, where $U \subset H$ is an open neighborhood of identity. Additionally we impose on σ the normalization condition $\sigma(e_H) = e_G$. One can extend σ to a global section

$$\sigma : H \longrightarrow G, \tag{A.2}$$

but in general such extension will not be smooth. Let us define the map $\Psi : N \times H \longrightarrow G$ by

$$\Psi(n, h) := \iota(n)\sigma(h). \tag{A.3}$$

Since $G/N \cong H$, we get that for $g \in G$ there exists a unique $n \in N$ such that $g = \iota(n)\sigma(\pi(g))$. Thus Ψ is a locally smooth bijection with the inverse $\Psi^{-1} : G \longrightarrow N \times H$ given by

$$\Psi^{-1}(g) := \left(\iota^{-1}\left(g\left(\sigma\left(\pi(g)\right)\right)^{-1}\right), \sigma\left(\pi(g)\right)\right). \tag{A.4}$$

Using Ψ one defines the multiplication on $N \times H$ by

$$(n_1, h_1) \cdot (n_2, h_2) := \Psi^{-1}\left(\Psi(n_1, h_1)\Psi(n_2, h_2)\right) \tag{A.5}$$

and can express it as follows

$$(n_1, h_1) \cdot (n_2, h_2) = \left(n_1 \Phi(h_1)(n_2)\Omega(h_1, h_2), h_1 h_2\right), \tag{A.6}$$

where maps $\Phi : H \to \operatorname{Aut}(N)$ and $\Omega : H \times H \to N$ are defined by

$$\Phi(h)(n) := \iota^{-1}\left(\sigma(h)\iota(n)\sigma(h)^{-1}\right), \tag{A.7}$$

$$\Omega(h_1, h_2) := \iota^{-1}\left(\sigma(h_1)\sigma(h_2)\sigma(h_1 h_2)^{-1}\right). \tag{A.8}$$

Let us denote by $\overline{\Phi}$ the map

$$\overline{\Phi} : H \times N \ni (h, n) \mapsto \Phi(h)(n) \in N. \tag{A.9}$$

One has the following properties of Φ and Ω:

$$\Phi(e_H) = \mathrm{id}, \tag{A.10a}$$

$$\Omega(e_H, h) = \Omega(h, e_H) = e_N, \tag{A.10b}$$

$$\Omega(h_1, h_2)\Omega(h_1 h_2, h_3) = \Phi(h_1)\left(\Omega(h_2, h_3)\right)\Omega(h_1, h_2 h_3), \tag{A.10c}$$

$$\Omega(h_1, h_2)\Phi(h_1 h_2)(n) = \Phi(h_1) \circ \Phi(h_2)(n)\Omega(h_1, h_2). \tag{A.10d}$$

Forgetting about definitions (A.7), (A.8) we can consider Φ and Ω as abstract maps satisfying conditions (A.10). We assume that the map $\overline{\Phi}$ is smooth on $U \times N$ and Ω is smooth on some neighborhood of (e_H, e_H). Moreover we have to assume that the map $H \ni x \mapsto \Omega(h, x)\Omega(hxh^{-1}, h)^{-1}$ is smooth for all $h \in H$ on a neighborhood of e_H (if H is connected then this condition is automatically satisfied). Under these conditions there exists on $N \times H$ a structure of Banach Lie group defined by (A.6), see [10]. One denotes this Banach Lie group by $N \times_{\Phi,\Omega} H$.

We get that the inverse of (n, h) in $N \times_{\Phi,\Omega} H$ is given by

$$(n, h)^{-1} = \left(\Omega\left(h^{-1}, h\right)^{-1}\Phi\left(h^{-1}\right)\left(n^{-1}\right), h^{-1}\right) \tag{A.11}$$

and the inner automorphism $I_{(n,h)}(m, g) := (n, h) \cdot (m, g) \cdot (n, h)^{-1}$ can be expressed in terms of Φ and Ω as

$$I_{(n,h)}(m, g) = \left(n\Phi(h)(m)\Omega(h, g)\Omega\left(hgh^{-1}, h\right)^{-1}\Phi\left(hgh^{-1}\right)\left(n^{-1}\right), hgh^{-1}\right). \tag{A.12}$$

Now, let us pass to the extensions of related Banach Lie algebras.

A.2. Extensions of Banach Lie algebras

We will denote the Banach Lie algebras of G, H, N by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$ respectively. Taking derivatives of the maps in (A.1) we obtain the exact sequence of Banach Lie algebras

$$\{0\} \hookrightarrow \mathfrak{n} \xhookrightarrow{D\iota(e_N)} \mathfrak{g} \xrightarrow{D\pi(e_G)} \mathfrak{h} \longrightarrow \{0\} . \tag{A.13}$$

The derivative $D\Psi^{-1}(e_G) : \mathfrak{g} \to \mathfrak{n} \oplus \mathfrak{h}$ of Ψ^{-1} at the point e_G allows us to identify the Banach space $\mathfrak{g}$ with $\mathfrak{n} \oplus \mathfrak{h}$.

The adjoint representation of $N \times_{\Phi,\Omega} H$ on $\mathfrak{g} \cong \mathfrak{n} \oplus \mathfrak{h}$ can be locally computed from (A.12) and it is the following

$$\begin{aligned} \mathrm{Ad}_{(n,h)}(\zeta, \eta) = \big(&\mathrm{Ad}_n\big(D_2\overline{\Phi}(h, e_N)(\zeta) + D_2\Omega(h, e_H)(\eta) - D_1\Omega(e_H, h)(\mathrm{Ad}_h\, \eta)\big) \\ &+ \big(DL_n\big(n^{-1}\big) \circ D_1\overline{\Phi}\big(e_H, n^{-1}\big) \circ \mathrm{Ad}_h\big)(\eta), \mathrm{Ad}_h\, \eta\big) \end{aligned} \tag{A.14}$$

for $(n, h) \in N \times_{\Phi,\Omega} U$, $(\eta, \zeta) \in \mathfrak{n} \oplus \mathfrak{h}$, where $\overline{\Phi}$ is defined by (A.9), and D_i denotes partial derivative with respect to the ith argument. We denote by L_n the left group action $L_n m = nm$, $n, m \in N$, on itself.

Differentiating (A.14) we obtain the formula for the Lie bracket

$$\big[(\zeta, \eta), (\nu, \xi)\big] := \big([\zeta, \nu] + \varphi(\eta)(\nu) - \varphi(\xi)(\zeta) + \omega(\eta, \xi), [\eta, \xi]\big), \tag{A.15}$$

for $(\zeta, \eta), (\nu, \xi) \in \mathfrak{n} \oplus \mathfrak{h}$, where $\varphi : \mathfrak{h} \to \mathrm{Aut}(\mathfrak{n})$ is the linear continuous map and $\omega : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{n}$ is the continuous bilinear skew symmetric map defined by Φ and Ω as follows:

$$\varphi(\eta)(\zeta) := D_1 D_2 \overline{\Phi}(e_H, e_N)(\eta, \zeta), \tag{A.16}$$

$$\omega(\eta, \xi) := D_1 D_2 \Omega(e_H, e_H)(\eta, \xi) - D_1 D_2 \Omega(e_H, e_H)(\xi, \eta). \tag{A.17}$$

In these formulas $D_1 D_2$ is the second mixed partial derivative.

The maps φ and ω satisfy the following infinitesimal version of conditions (A.10):

$$\begin{aligned} &\omega\big(\big[\eta, \eta'\big], \eta''\big) + \omega\big(\big[\eta', \eta''\big], \eta'\big) + \omega\big(\big[\eta'', \eta\big], \eta'\big) \\ &\quad - \varphi(\eta)\big(\omega\big(\eta', \eta''\big)\big) - \varphi\big(\eta'\big)\big(\omega\big(\eta'', \eta\big)\big) - \varphi\big(\eta''\big)\big(\omega\big(\eta, \eta'\big)\big) = 0, \end{aligned} \tag{A.18}$$

and

$$\mathrm{ad}_{\omega(\eta,\eta')} + \varphi\big(\big[\eta, \eta'\big]\big) - \big[\varphi(\eta), \varphi\big(\eta'\big)\big] = 0 \tag{A.19}$$

for all $\eta, \eta', \eta'' \in \mathfrak{h}$.

If we forget about the underlying Banach Lie groups and consider their Banach Lie algebras only, then the maps φ and ω satisfying conditions (A.18)–(A.19) with additional smoothness conditions, define the structure of Banach Lie algebra on $\mathfrak{n} \oplus \mathfrak{h}$, see [1,12].

A.3. Extensions of Banach Lie–Poisson spaces

According to [11] the **Banach Lie–Poisson space** is a Banach space $\mathfrak{b}$ such that its dual $\mathfrak{b}^*$ is Banach Lie algebra with the property

$$\mathrm{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**} \tag{A.20}$$

for all $x \in \mathfrak{b}^*$. This property allows us to define the Poisson bracket on $\mathfrak{b}$

$$\{f, g\}(b) = \big\langle \big[Df(b), Dg(b)\big], b\big\rangle, \tag{A.21}$$

where $Df(b), Dg(b) \in \mathfrak{b}^*$ are Fréchet derivatives at point $b \in \mathfrak{b}$. The bracket makes the Banach space $\mathfrak{b}$ a Banach Poisson space in the sense of [11].

Let us assume that Banach Lie algebras $\mathfrak{n}$, $\mathfrak{h}$ and $\mathfrak{g}$ possess predual Banach spaces $\mathfrak{n}_*$, $\mathfrak{h}_*$ and $\mathfrak{g}_*$ satisfying condition (A.20). We also assume that maps $(D\iota(e_N))^*$, $(D\pi(e_G))^*(h_*)$ dual to $D\iota(e_N)$ and $D\pi(e_G)$ preserve predual spaces, i.e.

$$\big(D\iota(e_N)\big)^*(\mathfrak{g}_*) \subset \mathfrak{n}_*, \qquad \big(D\pi(e_G)\big)^*(\mathfrak{h}_*) \subset \mathfrak{g}_*. \tag{A.22}$$

In that situation one obtains the exact sequence of predual Banach spaces

$$\{0\} \hookrightarrow \mathfrak{h}_* \xhookrightarrow{(D\pi(e_G))^*} \mathfrak{g}_* \xrightarrow{(D\iota(e_G))^*} \mathfrak{n}_* \longrightarrow \{0\}\ , \tag{A.23}$$

see Lemma 3.7 in [12].

We can identify $\mathfrak{g}_*$ with $\mathfrak{n}_* \oplus \mathfrak{h}_*$ by the map dual to the derivative $D\Psi^{-1}(e_G)$ at the point e_G. This identification allows us to compute coadjoint actions of $N \times_{\Phi,\Omega} H$ and $\mathfrak{n} \oplus \mathfrak{h}$ on $\mathfrak{n}_* \oplus \mathfrak{h}_*$ as follows:

$$\begin{aligned}\mathrm{Ad}^*_{(n,h)}(\tau,\mu) = \Big(&\big(D_2\overline{\Phi}(h,e_N)\big)^* \mathrm{Ad}_n^* \tau, \Big(\big(D_2\Omega(h,e_H)\big)^* \mathrm{Ad}_n^* - \mathrm{Ad}_h^*\big(D_1\Omega(e_H,h)\big)^* \mathrm{Ad}_n^* \\ &+ \mathrm{Ad}_h^*\big(D_1\overline{\Phi}\big(e_H,n^{-1}\big)\big)^*\Big)\tau + \mathrm{Ad}_h^* \mu\Big),\end{aligned} \tag{A.24}$$

where $(n,h) \in N \times_{\Phi,\Omega} U$, $(\tau,\mu) \in \mathfrak{n}_* \oplus \mathfrak{h}_*$ and

$$\mathrm{ad}^*_{(\zeta,\eta)}(\tau,\mu) = \big(\mathrm{ad}_\zeta^* \tau + \big(\varphi(\eta)\big)^*\tau, -\big(\varphi(\,\cdot\,)(\zeta)\big)^*\tau + \big(\omega(\eta,\cdot)\big)^*\tau + \mathrm{ad}_\eta^* \mu\big) \tag{A.25}$$

for $(\zeta,\eta) \in \mathfrak{n} \oplus \mathfrak{h}$, $(\tau,\mu) \in \mathfrak{n}_* \oplus \mathfrak{h}_*$.

The coadjoint representation (A.25) satisfies condition (A.20) if and only if

$$\big(\varphi(\eta)\big)^*(\mathfrak{n}_*) \subset \mathfrak{n}_*, \qquad \big(\varphi(\cdot)(\zeta)\big)^*(\mathfrak{n}_*) \subset \mathfrak{h}_*, \qquad \big(\omega(\eta,\cdot)\big)^*(\mathfrak{n}_*) \subset \mathfrak{h}_*. \tag{A.26}$$

So, under these conditions the Banach space $\mathfrak{n} \oplus \mathfrak{h}$ is a Banach Lie–Poisson space. Using definition (A.21) and Lie bracket (A.15) we obtain the Poisson bracket on $\mathfrak{n}_* \oplus \mathfrak{h}_*$:

$$\begin{aligned}\{f,g\}(\tau,\mu) = &\big\langle [D_1 f, D_1 g] + \varphi(D_2 f)(D_1 g) - \varphi(D_2 g)(D_1 f) + \omega(D_2 f, D_2 g), \tau\big\rangle \\ &+ \big\langle [D_2 f, D_2 g], \mu\big\rangle\end{aligned} \tag{A.27}$$

for $f, g \in C^\infty(\mathfrak{n}_* \oplus \mathfrak{h}_*)$.

Further investigation of extensions of Lie groups and Lie algebras is beyond the scope of this paper, so for more information we refer to [1,9,10,12].

Appendix B. Magri method

We briefly recall the Magri method of constructing integrals of motion in involution. For more details see e.g. [7].

Let us consider a pencil of compatible Poisson brackets

$$\{\cdot,\cdot\}_\varepsilon := \{\cdot,\cdot\}_1 + \varepsilon\{\cdot,\cdot\}_2 \tag{B.1}$$

where $\varepsilon \in \mathbb{R}$. Compatibility of Poisson brackets means that $\{\cdot,\cdot\}_\varepsilon$ is also a Poisson bracket for any parameter ε. Let I^k_ε be a family of Casimirs for Poisson bracket $\{\cdot,\cdot\}_\varepsilon$ indexed by $k \in \mathbb{N}$, i.e.

$$\{I^k_\varepsilon, \cdot\}_\varepsilon = 0. \tag{B.2}$$

Assuming that I^k_ε depends analytically on the parameter ε one expands the equality (B.2) and computes the coefficients in front of ε^n. Thus one obtains that $\{h^k_0, \cdot\}_1 = 0$ and

$$\{h^k_l, \cdot\}_1 = \{h^k_{l+1}, \cdot\}_2, \quad l = 0, 1, \ldots \tag{B.3}$$

where h^k_l are defined by

$$I^k_\varepsilon = \sum_{l=0}^{\infty} h^k_l \varepsilon^l. \tag{B.4}$$

Due to relation (B.3), the sequence $\{h^k_l\}_{l\in\mathbb{N}\cup\{0\}}$ is called a Magri chain.

By using (B.3) twice one gets that

$$\{h^k_l, h^{k'}_n\}_1 = \{h^k_{l-1}, h^{k'}_{n+1}\}_1. \tag{B.5}$$

Next, by iterating this procedure one concludes that

$$\{h^k_l, h^{k'}_n\}_1 = \{h^k_0, h^{k'}_{n+l}\}_1 = 0. \tag{B.6}$$

Thus functions h^k_l are in involution

$$\{h^k_l, h^{k'}_n\}_\varepsilon = \{h^k_l, h^{k'}_n\}_1 = \{h^k_l, h^{k'}_n\}_2 = 0 \tag{B.7}$$

for all Poisson brackets under consideration.

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